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Common Errors and Misconceptions in Mathematical Proving by Education Undergraduates

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Abstract

Ninety-seven education students majoring or minoring in mathematics had their math homework examined in a Number Theory or Abstract Algebra course. Each student's homework was observed for the purpose of identifying common errors and misconceptions when writing mathematical proofs. The results showed that students collectively made four recurring errors: assuming the conclusion in order to prove the conclusion, proving general statements using specific examples, not proving both conditions in a biconditional statement, and misusing definitions. In the same courses taken subsequently by 91 new students, we informed them about these common errors prior to assigning their homework to see how the students' proving processes would differ. The results showed that more exercises were left blank with comments such as "I'm not sure how to start the proof", and that many students provided unnecessary examples to supplement their valid proofs.

Key Words: College Mathematics; Mathematical Proof; Reasoning; Algebra; Number Theory; Proof Frameworks.

Introduction

The concept of mathematical proofs and their role in mathematics learning are popular topics of research and discussion. The precise definition of a proof and its role vary by context and scholar (Reid, 2002; 2005), but the general purpose of proving is to verify, explain, communicate, and systematize statements into deductive systems (Almeida, 2000; Hersh, 1993). Between 1990 and 1999, the leading journals of mathematics education published over one hundred research papers on various topics related to proofs and proving in mathematics education (Hanna, 2000).

There are a variety of issues addressed in the literature concerning the topic of proofs in mathematics education, including: how students learn and solve proofs (Herbst, 2002; Hazzan and Zazkis, 2003; Kuchemann and Hoyles, 1999; Balacheff, 1988), teaching techniques of proving (Marty, 1986; Lamport, 2012; Hanna and de Villiers, 2008), how proofs are validated (Selden and Selden, 2003; Weber, 2008; Weber and Alcock, 2005), how students and teachers perceive proofs (Patkin, 2011; Knuth, 2002; Varghese, 2009; Jones, 1997; Healy and Hoyles, 2000), how proofs relate to convincing and refutation (Stylianides, 2009; Stylianides and Al-Murani, 2010), difficulties in the transition of high school to undergraduate mathematics (Moore, 1994; Almeida, 2000; Blanton, 2003; Raman, 2002; Tall, 2008), and the extent to which proofs are important in educational settings (de Villiers, 1990; Hanna, 1995; Volminik, 1990; Tucker, 1999; Pfeiffer, 2010).

Looking at the processes involved in proving and how students and teachers understand proofs are thoroughly addressed topics. Of particular interest to us is how

students and teachers verify and validate proofs and the misconceptions in proving. Stylianides (2009) reports students' misconception that empirical arguments constitute valid proofs. That is, students use specific examples to prove general statements. Stylianides and Al-Murani (2009) examine the misconception students have about the coexistence of a proof and a counterexample for the same assertion. Pfeiffer (2010) conducted a study in which she presented mathematical proofs containing errors to first year mathematics undergraduates and asked them to evaluate and criticize the statements. Pfeiffer notes that many students identified proving using examples to be invalid. This is interesting to us because it is one of the common errors that we notice, which we will discuss shortly. Send (1985) considers how students write geometry proofs, and points out that errors often occur in their notation. We also address this later, as it is related to misunderstanding and misusing definitions. Selden and Selden (2003) also examine undergraduate students' ability to determine when an argument properly proves a theorem. They present different versions of proofs containing errors and investigate how students reason through the arguments. Despite the growing literature emphasizing how proofs are validated and what constitutes a formal and rigorous proof, there is still a gap in the field's understanding of what the common errors and misconceptions involved in proving are. Our work seeks to enumerate these.

This paper considers undergraduate education students who are planning to teach mathematics in the K-12 school system. We examine common errors and misconceptions that these students make when proving statements in their course work, and rank them by frequency of occurrence. After gathering and defining these common mistakes, we explain them to an independent group of students enrolled in the course at a later time. Our purpose then is to see how students change their proving habits when they are aware of these errors. This paper was guided by the following research questions:

1. What are the common errors that math education students make when writing proofs?
2. How do students' proving habits differ when they are explicitly aware of these common errors?

Methodology

Data Sources. We observed the homework of 188 education undergraduate students who chose mathematics as their major or minor. These students are required to take a Number Theory and Abstract Algebra course at our university. These courses are proof-based, and require that students be able to write and understand mathematical proofs. The students plan to teach mathematics in the K-12 school system, and are at least in their second year of study. Every student has taken a minimum of two first-year math courses prior to enrolling in Number Theory or Abstract Algebra.

Procedure. The students were given regularly graded homework that included proving routine statements covering basic Number Theory or Abstract Algebra. As their assignments were being graded, we compiled a list of errors made by the students. From this list, the most common errors were noted and described. These observations included homework from 97 different students. The purpose of this is to answer the first research question of what the common errors are that education students make when proving.

In subsequent Number Theory and Abstract Algebra courses taken by 91 new students, we use the data collected previously to inform the new students of these common errors and misconceptions prior to distributing their homework. The purpose of this is to determine how their proving habits differ from the first group who were not necessarily aware of these errors. This allows us to answer the second research question about how students prove when they are explicitly told about invalid proving methods.

Results and Discussion

What are the common errors that math education students make when writing proofs? The purpose of this research question was to gain insight into the proving habits and techniques of math education students. The most common error made was proving general statements using specific examples. The literature refers to this as using empirical evidence in place of a valid proof (Stylianides, 2009). This error was most common in statements of the form "If P then Q" and statements requiring a proof by mathematical induction (see Cupillari (2011) for an explanation on mathematical induction and other proof techniques). Below are sample examples of students using examples in place of valid proofs. In these examples, we insert parenthetical remarks for clarity or further explanation.

A) If a|b (a divides b) and b|c (b divides c) then show that a|c (a divides c).

A sample solution is: Let a = 2, b = 4, and c = 8. We see that 2|4, 4|8, and 2|8, so it is true.

B) If x is any real number greater than -1, then show $(1+x)^n \geq 1+nx$ for all positive integers n.

A sample solution is: Let x=0. We can choose this because 0>-1. If n=1 then the statement holds. Similarly if n=2, 3, 4, … (we omit the calculation performed for each value of n) it always holds no matter what we pick.

C) Prove the sum of any two primes larger than 2 is even.

A sample solution is: The first few primes are 1, 2, 3, 5, 7, 11, 13, 17, 19, … but we don't count 1 or 2 since they are not larger than 2. So if we take any pair like 3+5=8 or 3+7=10 or 11+13=24 we always get an even answer no matter what pair we take.

The second most common error was that the students assumed the conclusion of the statement holds in order to prove the conclusion. This error was most prevalent in statements of the form "If P then Q". Mathematicians refer to this error as "begging the question". In this case, students would start the proof by assuming Q is true, and then creating circular arguments in order to conclude that Q is true. Clearly, if we assume Q holds then there would be nothing to prove! Below are sample examples:

D) If g:A->B and f:B->C are surjective maps then show the composition map f(g) is also surjective.

A sample solution is: Suppose f(g) is a surjective map. Then there exists an element *a* in A such that f(g(*a*)) = c, where c is in C. Thus f(g) is surjective (the student assumed f(g) is surjective in order to prove that it is surjective).

E) If x and y are even positive integers then prove x+y is even.

A sample solution is: If x+y is even then x+y is a multiple of 2 and so we can write x+y=2k for some positive integer k. Since x+y=2k, x+y is even.

The third most common error was that students did not prove both conditions in a biconditional (if and only if) statement. These statements have the form "P if and only if Q", and are equivalent to the statements "If P then Q" and "If Q then P". For example, students were asked to prove that "A is symmetric if and only if I – 2A is symmetric". Here, the majority of students would use the fact that A is symmetric to eventually conclude that I – 2A is symmetric too, but did not prove the converse: if I – 2A is symmetric then A is too.

The fourth most common error was that students did not apply the definitions correctly, if at all. Edwards and Ward (2004) have observed the misuse of mathematical definitions, noting that students did not understand the role formal definitions play in mathematics. In particular, they observed the following key themes. Firstly, students do not understand that definitions are stipulated and context-dependent. Secondly, students that can correctly state definitions cannot necessarily apply them correctly, if at all, "even in the apparent absence of any other course of action" (p.417). Moore (1994) also remarks that students do not necessarily understand the content of relevant definitions or how to apply them in writing proofs. Below are sample examples:

F) Prove the sum of any two primes larger than 2 is even.

A sample solution is: The first few primes are 1, 2, 3, 5, 7, 11, 13, 17, 19, … but we don't count 1 or 2 (the student forgets that, by definition, 1 is not prime). So if we take any pair like 3+5=8 or 3+7=10 or 11+13=24 we always get an even answer no matter what pair we take.

G) Prove that the center of the group G, denoted C(G), is a subgroup of G.

In this case, students did not know how to use the definition of a subgroup with the definition of the center of a group to show this.

How do students' proving habits differ when they are aware of these common errors? In the same courses taken subsequently by 91 different students, we provided them with the list of common proving errors and explained that they are invalid. We do not explain how to avoid these errors or what students should do instead. Rather, we only emphasize that these common errors do not constitute formal proofs. Consequently, we observed three striking patterns. Firstly, students chose to leave more questions unanswered than the previous group of 97 students who were not aware of these common

proving errors. This occurred in exercises in which the former group typically proved using examples or assumed the conclusion to prove the conclusion. A possible explanation for this is that students may already realize that these approaches are invalid, and since they realize such a solution will not warrant additional marks, they choose to leave the answer blank. A second possible explanation is that students simply cannot write a proof that avoids these errors, and thus they choose to write nothing.

The second pattern observed is that students began supplementing their valid proofs with empirical evidence. It is common for students to generate examples to convince themselves or to further understand the statement. However, the examples should not be included with the formal proof.

The third pattern was that students often wrote "I'm not sure how to start the proof" for their answer. That is to say, the students were unsure what method of proof to use. Proving methods, or proof frameworks as they are also called in the literature, have been discussed by many researchers including Selden and Selden (2003), Martin and Harel (1989), and Marty (1986). These authors highlight the part of the proving process that involves deciding what frameworks can be applied and why.

As for the other errors, we observed that students attempted to prove both implications in a biconditional proof, which we expected since they were told it is wrong otherwise. As for the error of misusing definitions, this was still an issue and did not appear to get any better with the new group of students. This is not surprising since we did not explain to students how to correctly use the definitions, we only pointed out that misusing definitions was a common error.

Conclusion and Further Research

This paper was an investigation into the conceptions of proofs and proving held by undergraduate education teachers who are interested in teaching K-12 mathematics. By observing a sample of 97 students' homework exercises, we determined that there was a list of common errors that repeatedly surfaced. We saw that students often replaced formal proofs with specific examples, used the conclusion of a statement in order to prove the conclusion, forgot to prove both implications in a biconditional statement, and misused definitions. A few of these errors have been discussed previously in the literature, but our purpose was to catalog the most prevalent errors. Further, we presented these errors and misconceptions to a subsequent group of 91 students to determine how their proving methods would differ. We saw that students left more exercises unanswered, which is something that should be investigated more closely in further work. For instance, one might wonder if the first group of students knew that proving by example was invalid, but hoped to receive partial marks. In contrast, the second group was explicitly told such methods were invalid and thus knew they would not receive credit for these answers. We also observed that many students stated their lack of knowing how to start the proof. How mathematicians prove is an important aspect of mathematics, and thus it would be useful to look in more detail at how mathematicians decide which proving methods to use. Weber (2008) conducted a study that looks at how mathematicians validate proofs, but more research needs to consider how mathematicians create and structure proofs. We hope that this discussion inspires future research in this area.

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