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Introduction

The ATCM 2024 (atcm.mathandtech.org) returned to Yogyakarta after ATCM 2014 and ATCM 2018. The ATCM 2024 is in a hybrid format. We accepted 12 invited papers and 27 contributed papers. There are about 35% of the presentations done virtually this year and we are happy to see many old friends and we are equally thrilled to see many new faces too.

Given a Ph.D. or BS math degree program has been eliminated in recent years, the combination of ChatGPT with the Wolfram plug-in scored 96% in a UK Maths A-level paper, we certainly need to ponder why students must choose math as a major. It is exciting to see some presentations this year providing some partial solutions to this question.

All authors and readers are encouraged to contribute their favorite ideas to the next ATCM or publish interesting articles in the Electronic Journal of Mathematics and Technology (eJMT: https://ejmt.mathandtech.org) Selected eJMT papers will be published in the Research Journal for Mathematics and Technology (RJMT: http://rjmt.mathandtech.org).

It is always nice to renew old friendships, and also exciting to make more new friends at an ATCM. We hope you will invite your colleagues to experience future ATCMs for themselves. We look forward to seeing everyone in person at ATCM 2025 in the Philippines, more details will be announced later.

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November 2024

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Invited Papers

Contributed Papers

Connecting Trigonometry to Its Geometric Roots: An Introduction to Trigonometric Values

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Abstract: *This research explores two methods through which ten preservice math teachers develop an understanding of trigonometric values. Using the unit circle, preservice math teachers engage in knowledge-building activities such as paper folding and GeoGebra application. Grounded in Altman and Kidron's 2016 didactical design research, this study examines the cognitive processes ten preservice math teachers undergo during knowledge acquisition procedure. Employing the dynamically nested epistemic action model for abstraction, we analyze how different tasks facilitate ten preservice math teachers' comprehension of unit circle representations for trigonometric expressions and their associated values. Furthermore, we apply the abstraction in context framework to observe how ten preservice math teachers' knowledge progresses from traditional 'triangle' trigonometry to 'circle' trigonometry, aiding in determining trigonometric values.*

1. Introduction

Trigonometry is rooted in geometry, particularly in studying right-angled triangles and trigonometric functions. The fundamental trigonometric functions—sine, cosine, and tangent—are defined based on the ratio of the sides of a right triangle. So, trigonometric values and functions are the foundation of geometry, relating angles to sides of a right-angled triangle. The measurements of trigonometric ratios have captivated mathematicians from the inception of trigonometry and remain a topic of intense fascination today. Studies on the pedagogy of trigonometric functions have identified challenges faced by learners in grasping these mathematical concepts [1-3]. Learning trigonometry involves understanding these relationships through functions and/or ratios like sine, cosine, and tangent. These functions and/or ratios are essential for solving math, physics, and engineering problems. Teaching trigonometry effectively requires clear explanations, visual aids, and real-world examples. It is about guiding learners to see beyond numbers and equations to the patterns and principles that govern the physical world [4-5]. Whether through interactive tools such as GeoGebra or Desmos applications or hands-on activities such as paper folding, the goal is to make trigonometry understandable and connect to its root with relevant exploration [6-7].

With the gap identified above, in this study, we propose a didactical design to fill the gap by connecting trigonometry to its geometric roots using two methods (1) paper folding method and (2) visual representation using the GeoGebra application for finding the trigonometric values. This study is grounded on the previous research proposed by Altman and Kidron that describes how students' cognitive process through constructing knowledge [3]. Hence, this research presents a didactical approach to teaching trigonometry, transitioning from right-angle triangle principles to the unit circle, thereby facilitating a comprehensive finding of trigonometry values. The study aimed to explore effective instructional designs that teachers could adopt to connect the discovery of trigonometric values from right-angle triangles through hands-on activities like paper folding and digital visualization with the GeoGebra application. Specifically, the research sought to answer the question:

How can the proposed lesson design enable preservice math teachers to integrate the paper folding method and GeoGebra visual tools to elucidate trigonometric values derived from right-angle triangles and unit circles?

2. Abstraction in Context and Construction of Knowledge

Mathematical principles and even trigonometric concepts find their application across various domains such as engineering, astronomy, finance, economics, and statistics, making them instrumental in navigating real-world scenarios. These principles are often elucidated through tangible means—be it physical, digital, or graphical representations. This methodological approach, supported by educational psychology (as noted by Bruner in 1966 and Piaget in 1970), underscores the significance of concrete materials in fostering a deeper understanding of theoretical knowledge for the construction of knowledge. The advantage of employing physical objects such as paper folding lies in their ability to offer a hands-on context, which not only aids learners in grasping practical knowledge but also helps the exploration and construction of abstract mathematical ideas in trigonometry [6-7]. In this context, the integration of ICT tools like GeoGebra in mathematics while finding the value of trigonometry is noteworthy. GeoGebra bridges the gap between the abstract nature of geometric transformations and their tangible counterparts through visual aids and animations. This interactivity encourages the use of other software that allows for the manipulation of trigonometry values by altering variables, thereby simplifying the visualization process for students. Consequently, GeoGebra transforms the inherent abstractness of mathematics into something more concrete, enhancing the learning experience through improved visualization capabilities.

3. The Learning Experience and Methodological Issues

3.1 Research Context

For this purpose, the design is based on didactical design research. The study focuses on the learning experiences of ten preservice math teachers teaching at the secondary level, each boasting over a decade of teaching in the field. Our research, as outlined in the introductory section, was craft to eliminate the need for prior knowledge of trigonometry. The intent was to guide preservice math teachers through a novel method of knowledge construction by engaging in the geometric creation of the unit circle. This was achieved through two distinct approaches: traditional paper folding and the modern GeoGebra software. Emphasis was placed on observing the progression of knowledge construction rather than the ultimate learning outcomes of the students. Consequently, aligned with our research purpose, this investigation adopts a qualitative approach, incorporating a singular case study and an in-depth micro-analysis of the preservice math teachers' knowledge construction processes.

3.2 The Learning Setting and a-Priori Analysis

The journey is structured into two distinct stages. Initially, the preservice math teachers engage with activities centered around paper folding, which revisit the right-angle triangle concept. Subsequent exercises prompt the preservice math teachers to embark on crafting a unit circle, which in turn aids in discerning trigonometric values. This initial stage aims to acquaint the learner with the unit circle's framework. While no significant challenges were predicted for triangles within the first quadrant, the dynamics altered for other quadrants where triangles assume various orientations. It is envisaged that grappling with these complexities shall fortify the preservice math teacher's understanding, paving the way for the subsequent stage that involves broadening the scope of trigonometric values within the GeoGebra application. The second stage's tasks are designed to draw a correlation between trigonometric values and the geometric representation of the unit circle established earlier, with a reflective analysis of these findings within the GeoGebra application. The purpose of these bifurcated stages is to facilitate the preservice math teacher's comprehension of the geometric significance of trigonometric values as they pertain to the unit circle. Our preliminary analysis posited that learners would draw parallels with prior knowledge from 'triangle' trigonometry, recognizing the interplay between the sides of a right triangle and its angles. The tasks are methodically crafted to promote a seamless shift from 'triangle' trigonometry to 'circle' trigonometry, with the goal of enabling learners to deduce the trigonometric values of the unit circle and show their relevance. While we observed several anticipated outcomes in our retrospective analysis, we also encountered a range of intriguing and unforeseen constructs.

4. Reporting and Analysis of Findings

4.1 Reporting and Analysis of Findings in the First Phase of the Preservice Math Teacher Experiences

Prior to starting the activity, we ensured items: a single sheet of unmarked A4 paper, an item with a round shape, like a bowl or a compact disc, and a bold marker or a pencil with an HB lead for clear markings. Drawn the circle. See Figure 1. Using a circular object, such as a bowl or CD, carefully draw a circle in the lower left portion of your A4 paper, as shown in Figure 1. The circle has a radius of about 6 cm and about 2 cm from the bottom and left edges of the paper. Throughout this activity, we thought that the circle as the unit circle $x^2 + y^2 = 1$.

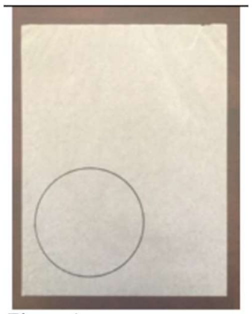

Figure 1

Fold 1. The x-axis was created as the horizontal diameter of the circle. See Figure 2. Holding the paper up to a light source so we can see through it, fold the circle onto itself, matching semicircle to semicircle. Be sure to line up the left edge of the paper. The two halves of the circle should line up perfectly, and the left edge of the paper should line up with itself. Ensured that the fold line is parallel to the paper's bottom edge. Creased the paper on the fold line (shown in red).

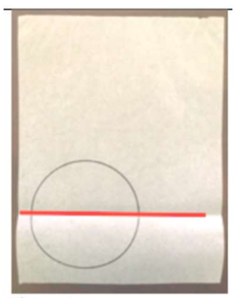

Figure 2

Located the first 2 of the 16 points. See Figure 3. We have marked the intersection points between the fold line and the circle.

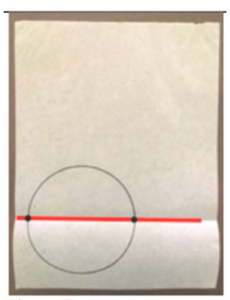

Figure 3

Fold 2. The y-axis was created as the vertical diameter of the circle. See Figure 4. Holding the paper up to a light source, fold the circle onto itself while lining up the bottom edge of the paper. The two halves of the circle should line up perfectly, and the bottom edge of the paper line up with itself. (The two marked points should fold exactly onto each other. Also, the fold line parallel the paper's left edge.) Creased the paper on the fold line.

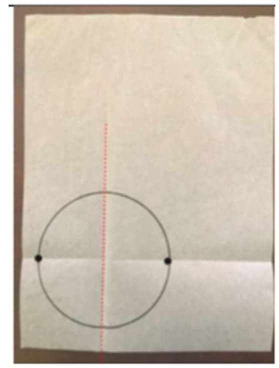
Figure 4

Locate the next 2 of the 16 points and the center of the circle. See Figure 5. We have marked the new intersection points between the fold line and the circle. It also, marked the intersection of the two folds, which is the center of the circle. (The circle's center is marked because it is a useful reference point, but it is NOT one of the 16 points we seek on the unit circle itself.)

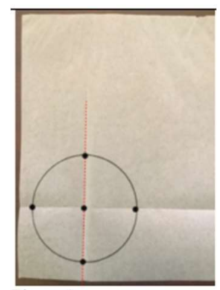
Figure 5

Fold 3. Create the line y = x.
See Figure 6. Folded the positive x-axis onto the positive y-axis. Make sure that the fold passes through the origin (0, 0), and that red point (1, 0) folds onto red point (0, 1). The red line illustrates the location of the fold line. Using light to see through the paper, fold the circle onto itself, matching semicircle to semicircle.

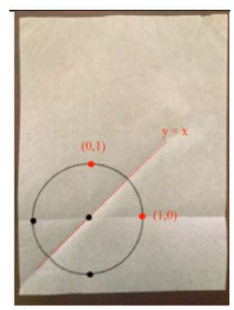

Figure 6

Locate the 2 new points on the line y = x.
See Figure 7. Creased the paper and marked the intersection points between the new fold line and the circle.

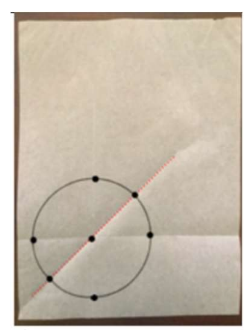
Figure 7

Fold 4. Create the line y = –x.
See Figure 8. Folded the negative x-axis onto the positive y-axis. Make sure that the fold passes through the origin (0, 0), and that red point (–1, 0) folds onto red point (0, 1). The red line illustrates the location of the fold line. Using light to see through the paper, fold the circle onto itself, matching semicircle to semicircle.

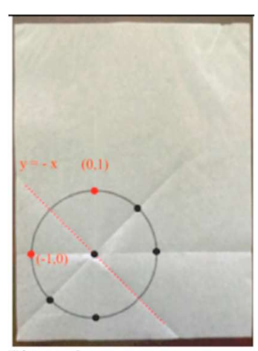

Figure 8

Locate the 2 new points on the line y = –x.
See Figure 9. Creased the paper and marked the intersection points between the new fold line and the circle.

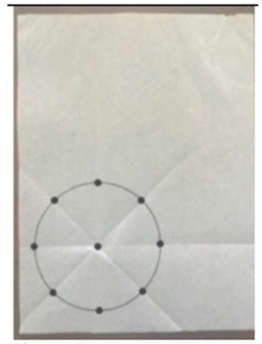
Figure 9

Fold 5. Create the vertical line x = 1/2.
See Figure 10. Folded the red point (1, 0) onto the red origin (0, 0). The red line showed the location of the fold line.

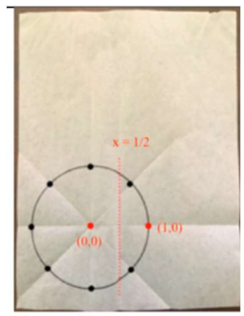

Figure 10

Locate the 2 new points on the line x = 1/2.
See Figure 11. Creased the paper and marked the intersection points between the new fold line and the circle.

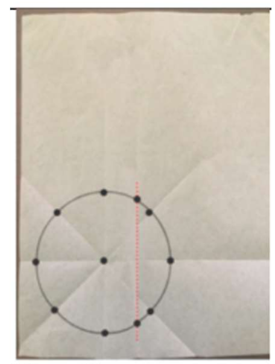
Figure 11

Fold 6. Create the horizontal line y = 1/2.
See Figure 12.
Folded the red point (0, 1) onto the red origin (0, 0). The red line showed the location of the fold line.

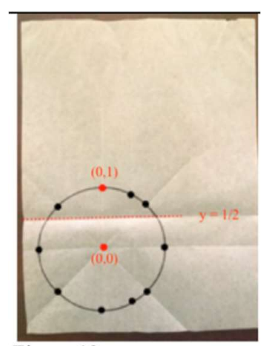

Figure 12

Locate the 2 new points on the line y = 1/2.
See Figure 13.

Creased the paper and marked the intersection points between the new fold line and the circle.

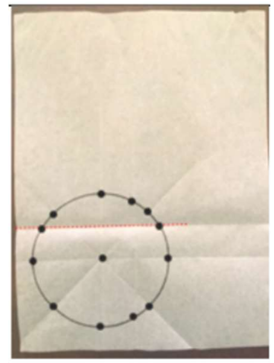
Figure 13

Fold 7. Create the vertical line x = –1/2.
See Figure 14.
Folded the red point (–1, 0) onto the red origin (0, 0). The red line showed the location of the fold line.

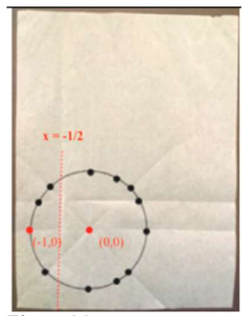

Figure 14

Locate the 2 new points on the line x = –1/2.
See Figure 15.
Creased the paper and marked the intersection points between the new fold line and the circle.

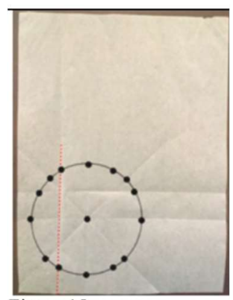
Figure 15

Fold 8. Create the horizontal line y = –1/2.
See Figure 16.
Folded the red point (0, –1) onto the red origin (0, 0). The red line showed the location of the fold line.

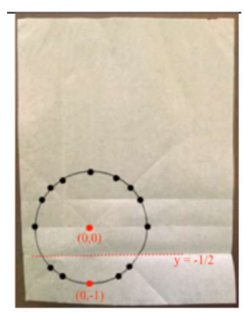

Figure 16

Locate the 2 new points on the line y = –1/2.
See Figure 17.

Creased the paper and marked the intersection points between the new fold line and the circle. Your 16-point circle was completed!

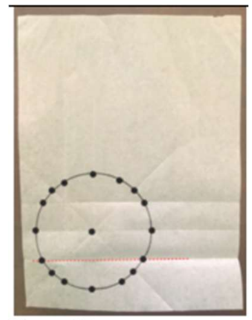

Figure 17

Label the 16 points on the unit circle.
Ensure that your paper is oriented so that the circle is in the bottom left corner, as shown in Figure 18. Begin labeling at the 0° position on the unit circle and label the degrees for each of the 16 points. The first 5 points were labeled in Figure 18; continue labeling the other 11 points.

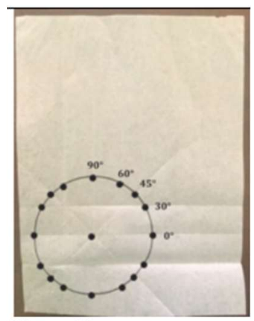

Figure 18

Bonus. Two additional folds...
See Figure 19.
Folded to create the line x = 1, tangent to the unit circle at 0°. Also, folded to create the line that passes through both the origin (0, 0) and the point on the unit circle at the 60° position. Drawn lines along these new creases. Compute the exact values of the following trig functions by any method you wish:

- sin (60°)
- cos (60°)
- tan (60°)
- sec (60°)

Find line segments in Figure 19 that match these values.

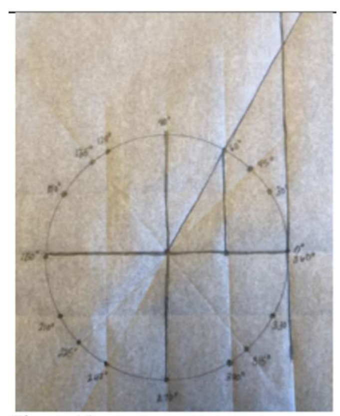

Figure 19

The discussion was initiated based on the questions (1) are all of the angle measures shown in Figure 18 correct?, how can you be sure? Explain your thinking. For example, how do you know that the point labeled 60° actually forms a 60° angle with the positive x-axis? (2) what are the coordinates of each of the 16 points? (3) what are the sine (sin) and cosine (cos) values associated with each point? And (4) how can you be sure about your answers to Questions 2 and 3?

4.2 Reporting and Analysis of Findings in the Second Phase of the Learning Experience

While exploring a dynamic point on a unit circle, Trigonometric situation 1 (GeoGebra Configuration 1, click to open the following link: https://www.geogebra.org/m/vqzuvuyy, See figure 20 in the next page).

The directions and discussion questions for configuration 1 were as follows:

1. Grab point P and move it around the Unit Circle. Consider all four quadrants as you move point P. Notice how the numerical values change. Does any value stay fixed?
2. Describe the line segments that make up the right-angle triangle OBP. How do the side lengths of triangle OBP relate to the coordinates of point P?
3. How is the angle a related to the x-coordinate of point P? How is the angle a related to the y-coordinate of point P?
4. Describe the trigonometric relationships within the dynamic configuration. Can you locate the 16 points that you found during the Paper-Folding Activity?
5. For any particular location of point P, how would you calculate sin θ, cos θ, and tan θ?

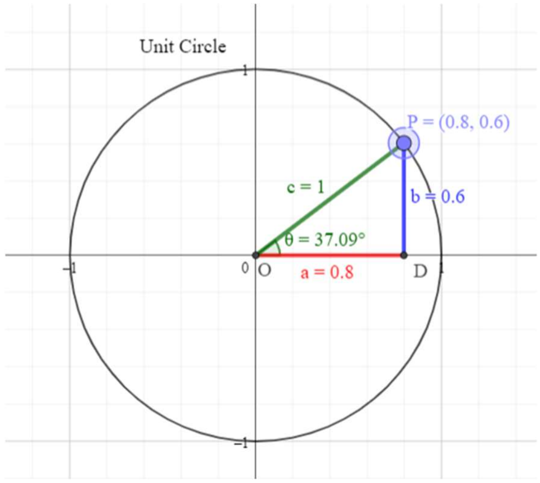

Figure 20

Next is trigonometric situation 2 (GeoGebra Configuration 2, click to open the link: https://www.geogebra.org/m/jh4qrzdg, See figure 21). The directions and discussion questions were for configuration 2 as following:

1. Grab point P and move it around the Unit Circle. Consider all four quadrants as you move point P. Notice how the numerical values change. Does any value stay fixed?
2. Describe the line segments that make up the right-angle triangle OBA. How do the side lengths of triangle OBA relate to the measure of angle θ?
3. Describe the trigonometric relationships within the dynamic configuration. Can you locate the 16 points that you found during the Paper-Folding Activity?
4. For any particular location of point P, how would you determine the values of tan θ and sec θ?

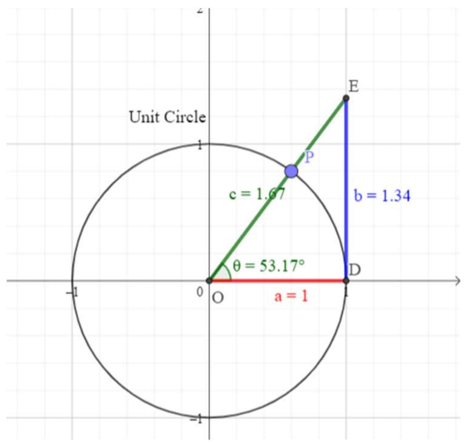

Figure 21

Finally, in trigonometric situation 3 (GeoGebra Configuration 3, see Figure 22). The directions and discussion questions for configuration 3 as follows:

1. Grab point P and move it around the Unit Circle. Consider all four quadrants as you move point P. Notice how the numerical values change. Does any value stay fixed?
2. Describe the line segments that make up the right-angle triangles OPA and OPB. How do the side lengths of these two triangles relate to the measure of angle θ?
3. Describe the trigonometric relationships within the dynamic configuration. Can you locate the 16 points that you found during the Paper-Folding Activity?
4. For any particular location of point P, how would you determine the values of

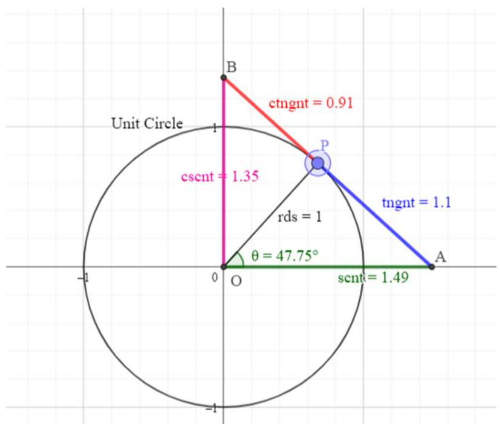

Figure 22

- $\tan \theta$?
- $\sec \theta$?
- $\cot \theta$?
- $\csc \theta$?

4.3. Reflection on Analysis of Findings in the two Phases

In the first phase of the learning experience, preservice math teachers are involved in a hands-on paper-folding activity designed to illustrate the geometric and trigonometric properties of the unit circle. Preservice math teachers began by preparing their materials, ensuring each had an A4 paper, a round object for tracing, and a bold marker or HB pencil for clear markings. The activity meticulously guided participants through multiple folds, each aimed at establishing creases of the lines, such as the x-axis and y-axis, and various angular bisectors represented by lines y = x and y = -x. This meticulous process helped locate 16 points on the unit circle, each corresponding to specific angle degrees. The folding activity served as a practical introduction to circle geometry and set the stage for understanding more complex trigonometric relationships. The completion of the task was marked by labeling the angles, reinforcing the angular concepts, and ensuring comprehension of the geometric placements of the points on the circle. In the second phase, the experience of preservice math teachers transitioned to a digital exploration using the GeoGebra software, which allowed for a dynamic engagement of preservice math teachers with the trigonometric concepts previously approached through paper folding. Preservice math teachers manipulated a point on a digital unit circle, observing the changes in trigonometric values as the point moved through various quadrants in the GeoGebra application. This interactive module was divided into multiple scenarios, prompting preservice math teachers to explore different trigonometric relationships and calculate values such as sine, cosine, tangent, and their reciprocals. This phase significantly enriched the understanding of preservice math teachers by linking the tangible insights from the paper-folding exercise with the abstract mathematical functions, providing a comprehensive learning experience emphasizing trigonometry and geometry's practical and theoretical dimensions. The discussion points and questions raised during this phase aimed at deepening the analytical skills of the preservice math teachers, ensuring a thorough grasp of the trigonometric identities and their real-world applications.

5. Discussion and Concluding Remarks

Our strategy was designed to integrate two methodologies: the time-honored practice of paper folding and the innovative GeoGebra software, both essential for preservice math teachers and learners to grasp trigonometric concepts and their corresponding values. We incorporated hands-on paper folding activities to engage preservice math teachers in connecting these methods, followed by a reflective analysis of their work. This approach guided preservice math teachers to a new understanding of trigonometry values within different frameworks, such as the unit circle, using both traditional and modern techniques. Our study validated this approach, demonstrating its effectiveness in comprehensively understanding trigonometry values. Preservice math teachers were able to locate points on a circle’s coordinates with fewer difficulties and misconceptions than previously reported in academic studies [8-17, 24]. Additionally, it cultivated a holistic grasp of trigonometry [7], both as ratios and within the context of the unit circle. We identified errors in preservice math teachers’ responses related to the coordinate system, though it was uncertain whether these were due to oversight or underlying misconceptions. Further in-depth research is necessary to clarify this issue. Upon reviewing our didactical design, we recognized the critical factor of time management, particularly concerning technical aspects [18-21]. To optimize the design, we have highlighted some areas for revisions to minimize time expenditure: 1) Ensuring all preservice math teachers or groups have sufficient resources throughout the learning process, 2) Providing clear, detailed instructions, and 3) Confirming preservice math teachers' foundational knowledge, such as understanding the coordinate system, paper-folding skills and handling GeoGebra application. In terms of group work evaluation, we recommended that educators assign specific tasks to each group member of preservice math teachers to enhance collaboration and performance. Lastly, these adjustments aim to facilitate smoother cooperation among preservice math teachers and learners.

6. Acknowledgements and Funding

We extend our heartfelt gratitude to the ten secondary-level preservice math teachers from different schools in the Kathmandu Valley, Nepal, for their invaluable participation in this journey. We sincerely appreciate their voluntary involvement. Likewise, we extend our special thanks to Prof. Gregory Foley for sharing his insights on trigonometry and its geometric roots during one of the webinars during COVID-19 via the Zoom platform. We greatly appreciate his valuable contribution as the foundation of this work. Additionally, we acknowledge the partial funding support from the Kathmandu University School of Education (KUSOED), Department of STEAM Education, Hattiban, Lalitpur, Nepal.

References

[1] Ahmad, N. I., & Khotimah, R. P. (2023). Analysis of student's errors in solving trigonometry story problems. In *AIP Conference Proceedings* (Vol. 2886, No. 1). AIP Publishing.

[2] Allouche, J. P. (2024). Note on a simple trigonometric equality. *The Ramanujan Journal*, *64*(2), 529-535.

[3] Altman, R., & Kidron, I. (2016). Constructing knowledge about the trigonometric functions and their geometric meaning on the unit circle. *International Journal of Mathematical Education in Science and Technology*, *47*(7), 1048-1060. https://doi.org/10.1080/0020739X.2016.1189005

[4] Bressoud, D. M. (2010). Historical reflections on teaching trigonometry. *The Mathematics Teacher*, *104*(2), 106-112.

[5] Dahal, N. (2019). Integration of GeoGebra in teaching mathematics: Insights from teaching experiment. In *Proceeding of Seventh National Conference on Mathematics and Its Applications* (pp. 60-71), Butwal, Nepal.

[6] Dahal, N., Pant, B. P., Shrestha, I. M. ., & Manandhar, N. K. (2022). Use of GeoGebra in teaching and learning geometric transformation in school mathematics. *International Journal of Interactive Mobile Technologies*, *16*(8), 65–78. https://doi.org/10.3991/ijim.v16i08.29575

[7] Dahal, N., Shrestha, D., & Pant, B. P. (2019). Integration of GeoGebra in teaching and learning geometric transformation. *Journal of Mathematics and Statistical Science, 5*(12), 323–332

[8] Demir, Ö., & Heck, A. (2013). A new learning trajectory for trigonometric functions. In *Proceedings of the 11th international conference on technology in mathematics teaching* (pp. 119-124).

[9] Dikovic, L. (2009). Applications of GeoGebra into teaching some topics of mathematics at the college level. *Computer Science and Information Systems, 6*(2), 191-203.

[10] Kendal, M., & Stacey, K. (1996). Trigonometry: Comparing ratio and unit circle methods. In *Technology in Mathematics Education. Proceedings of the 19th annual conference of the Mathematics Education Research Group of Australasia* (pp. 322-329).

[11] Khan, S. A. (2021). Trigonometric ratios using algebraic methods. *Mathematics and Statistics, 9*(6), 899-907. https://doi.org/10.13189/ms.2021.090605
[12] Moore, K. C. (2013). Making sense by measuring arcs: A teaching experiment in angle measure. *Educational Studies in Mathematics*, *83*(2), 225–245. https://doi.org/10.1007/s10649-012-9450-6
[13] Mosese, N., & Ogbonnaya, U. I. (2021). GeoGebra and students' learning achievement in trigonometric functions graphs representations and interpretations. *Cypriot Journal of Educational Sciences*, *16*(2), 827-846.
[14] Niranjan, C., & Brijlall, D. (2024). The use of manipulatives in experimental learning in solving three-dimensional tasks in trigonometry. *Gulf Journal of Mathematics*, *16*(2), 183-195.
[15] Presmeg, N. (2008). Trigonometric connections through a semiotic lens. In *Semiotics in Mathematics Education*. Leiden, The Netherlands: Brill. https://doi.org/10.1163/9789087905972_007
[16] Saxena, S. (Ed.). (2015). Geometry and trigonometry. Britannica Educational Publishing.
[17] Tallman, M. A. (2021). Investigating the transformation of a secondary teacher's knowledge of trigonometric functions. *The Journal of Mathematical Behavior*, *62*, 100869.
[18] Thapa, R., Dahal, N., & Pant, B. P. (2022). GeoGebra integration in high school mathematics: An experiential exploration on concepts of circle. *Mathematics Teaching-Research Journal*, *14*(5), 16-33.
[19] Tyata, R. K., Dahal, N., Pant, B. P., & Luitel, B. C. (2021). Exploring project-based teaching for engaging students' mathematical learning. *Mathematics Education Forum Chitwan*, *6*(6), 30–49. https://doi.org/10.3126/mefc.v6i6.42398
[20] Weber, K. (2005). Students' understanding of trigonometric functions. *Mathematics Education Research Journal*, *17*(3), 91-112.
[21] Weber, K. (2008). Connecting research to teaching: Teaching trigonometric functions: Lessons Learned from Research. *The Mathematics Teacher*, *102*(2), 144-150. https://doi.org/10.5951/MT.102.2.0144
[22] Bruner, J. S. (1966). From communication to language a psychological perspective. *Cognition, 3*(3), 255-287.
[23] Piaget, J. (1970). Genetic epistemology. *American Behavioral Scientist, 13*(3), 459-480.
[24] Lamichhane, B. R., Dahal, N., & Luitel, B. C. (2023). Secondary level mathematics teachers' critical reflections on the use of GeoGebra for teaching trigonometry. The Electronic Proceedings of the *28th Asian Technology Conference in Mathematics* (ATCM 2023), *28*, 418-428. https://atcm.mathandtech.org/EP2023/regular/30100.pdf