Building Fraction Magnitude Knowledge With Number Lines: Partitioning Versus Analogy

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All deidentified data, analysis code, and materials are available at [https://osf.io/3e7q9]. This study's design, hypotheses, and analyses were preregistered [https://osf.io/6zvkd].

Abstract

Understanding fraction magnitudes is foundational for later math achievement. To represent a fraction *x/y*, children are often taught to use *partitioning*: break the whole into *y* parts, and shade in *x* parts. Past research has shown that partitioning on number lines supports children's fraction magnitude knowledge more than partitioning on area models. However, partitioning may not take full advantage of children's prior knowledge or the structure of the number line. We tested an alternative fraction number line lesson that leveraged children's pre-existing whole number knowledge using a domain-general learning tool: *analogy*. In a preregistered online experiment, 2nd and 3rd graders (N = 84, Mean = 8.83 years), were randomly assigned to an Analogy lesson (e.g., if I know how big 3 is on a 0-4 line, I know how big ¾ is on a 0-1 line), a Partitioning lesson on number lines, or a Control lesson using square area models. Results showed that the Analogy lesson was more effective for promoting fraction magnitude understanding than the Control lesson, and it was at least as effective as the Partitioning lesson. The analogy group, but not the partitioning group, significantly outperformed the control group with large-denominator fractions at retention (i.e., one-week delayed posttest) and on transfer tests (i.e., fraction comparison). We also replicated past findings that fraction partitioning lessons are more effective on number lines than on area models, and this advantage was partially sustained after a one-week delay. Overall, these findings highlight the power of domain-general analogy to support mathematical development.

Keywords: analogy; visual models; fractions; number lines; partitioning

Public Significance Statement

Fractions are notoriously difficult, but this study suggests that number line lessons can lead to immediate and sustained improvements in second and third graders' understanding of fraction sizes, even when delivered online. For children in our study, an analogy lesson (e.g., if you know how big 3 is compared to 4, you know how big 3/4 is) had the most consistent benefits over a non-number line control lesson.

Building Fraction Magnitude Knowledge with Number Lines: Partitioning versus Analogy

Fraction magnitude knowledge provides an important foundation for learning fraction arithmetic, algebra, and other higher math skills (Bailey et al., 2017; Booth et al., 2014; Torbeyns et al., 2015). Being able to estimate the size of fractions is also helpful for financial and health decisions in everyday life, such as choosing which of two differently-sized products is a better deal (Lamon, 2007). Unfortunately, many children (e.g., Resnick et al., 2016) and even adults (e.g., Obersteiner et al., 2013) struggle to extract a single magnitude from the multiplicative relation between the numerator and denominator, which can lead to errors like thinking 10/18 is larger than 5/9 because 10 and 18 are larger than 5 and 9 (Braithwaite & Siegler, 2018; Ni & Zhou, 2005).

Fortunately, number line models of fractions can ameliorate some of these misconceptions and help students understand fraction magnitudes (Barbieri et al., 2020; Fazio et al., 2016; Fuchs et al., 2017; Gunderson et al., 2019; Siegler et al., 2011). However, little is known about which instructional strategies with number lines are most helpful for fraction magnitude learning and retention. Previous studies have shown that a fraction magnitude lesson that teaches partitioning, or segmenting, on a number line is more effective than a lesson teaching partitioning with other visual models (Gunderson et al., 2019; Hamdan & Gunderson, 2017). Our study extends this research, comparing partitioning lessons to a fraction number line lesson that instead leverages a domain-general learning tool, analogy, to improve children's fraction magnitude knowledge. Findings from this experiment can inform early fraction instruction to support later math achievement.

Number line models help students visualize fraction magnitudes

Many visual models are used in early fraction instruction, including area models like pizzas or squares, fraction bars, Cuisenaire rods, discrete set models, and number lines (Alajmi, 2012; Hodges et al., 2008; Jayanthi et al., 2021). Research suggests number line models are particularly effective for fostering students' fraction magnitude knowledge (Barbieri et al., 2020; Fazio et al., 2016; Fuchs et al., 2017; Gunderson et al., 2019; Hamdan & Gunderson, 2017). Large-scale classroom experiments have shown that fraction instruction highlighting number lines and measurement interpretations of fractions is more effective for promoting students' conceptual and procedural fractions knowledge than instruction emphasizing area models and part-whole interpretations of fractions (Barbieri et al., 2020; Fuchs et al., 2017). Two lab-based experiments showed that even brief 15-minute instruction on fraction magnitudes with number lines led to gains in fraction comparison accuracy, compared to circular (Hamdan & Gunderson, 2017) and square (Gunderson et al., 2019) area models (but see Tian et al., 2021).

Number lines are effective for multiple reasons. They may help students see that a two-part fraction has one holistic magnitude because they represent a fraction as one position (e.g., rendering 3/4 as a single place that is 75 percent of the way from 0 to 1). Indeed, Gunderson et al. (2019) demonstrated that unidimensional number lines support fraction magnitude understanding better than two-dimensional area models do. This is consistent with the hypothesis that number lines may be effective because they represent fractions as a single magnitude on a single dimension. Number lines can also help children see how fraction magnitudes relate to whole number magnitudes. The *integrated theory of numerical cognition* (Siegler et al., 2011) argues that the number line is a central, unifying structure of numerical development, because it represents all real numbers. Finally, number lines show fraction magnitudes in a way that may align with children's intuitions about spatial proportions (Matthews & Ellis, 2018; Matthews & Hubbard, 2017; Sidney et al., 2017). Even infants and preschoolers can judge the approximate magnitude of proportions in spatial formats, such as a ratio of line-lengths (McCrink & Wynn, 2007; Park et al., 2021; Boyer et al., 2008). Number lines map these nonsymbolic visual proportions to symbolic proportions (e.g., 3 on a 0-10 line is placed 3/10 of the way along the line), which may allow children to take advantage of their approximate nonsymbolic proportional reasoning when learning about fractions.

Partitioning

In experimental studies and in many curricular materials, the most common strategy for representing fraction magnitudes on visual models remains via *partitioning*: in representing a fraction (e.g., ¾ of a square), this method first partitions or segments the whole into 4 equal parts, and then counts or shades 3 of those parts. Partitioning with number lines and area models is recommended by U.S. Common Core State Standards (2010) and other widely known research-based recommendations (Siegler et al., 2010).

Partitioning has many strengths for fraction learning. It helps children see the relations between unit fractions (e.g., 1/4, 1/3, 1/3, etc.), non-unit fractional parts, and wholes (Steffe & Olive, 2010). Many researchers argue that understanding unit fractions and part-whole conceptions of fractions (i.e., thinking about a fraction x/y as x parts of a whole that is partitioned into y parts) provides a foundation for understanding other meanings of fractions like ratio, measure, or quotient (e.g., Behr et al., 1983; Charalambous & Pitta-Panzatti, 2007). Over time, partitioning objects or number lines may help students build magnitude knowledge as they see how the sizes of the numerator and denominator relate to the size of the overall fraction. For example, students may eventually reason that large denominators correspond to dividing a whole into many parts, so each part is smaller. Eventually, students can also learn to construct any fraction by first partitioning to make a unit fraction $1/y$ and then iterating that unit fraction x times to produce x/y.

Two experiments show that representing fractions by partitioning on number lines, but not area models, boosts children's fraction magnitude knowledge. Children's fraction estimation and comparison improved more after they learned to partition number lines to show fraction magnitudes, relative to students who learned to partition circular area models and students in a crossword-puzzle control group (Hamdan & Gunderson, 2017). A follow-up study showed that unidimensionality was key to the number line's effectiveness: lessons using a thin number line or a slightly thicker "hybrid" line were more effective for second and third graders' fraction magnitude knowledge than square models (Gunderson et al., 2019). These studies offer compelling evidence that number lines are more effective than other visual models for teaching young children fraction magnitudes.

However, by using a partitioning approach, Gunderson and colleagues (Hamdan & Gunderson, 2017; Gunderson et al, 2019) might have underutilized the affordances of the number line visual model (Matthews & Ellis, 2018). For instance, the process of applying a partitioning approach to generate fraction representations on number lines may overemphasize counting and allow children to maintain thinking about a fraction as two separate numbers. That is, a person using a partitioning approach to place x/y on a number line attends first to breaking the line into y parts, then separately to counting x of these parts. This sequence of steps may encourage children to focus on the magnitudes of the numerator and denominator separately, rather than on the single magnitude generated by their relation (x/y). Separately counting denominators and numerators might be particularly ineffective for estimating large-denominator fractions, as a child tries to envision partitioning a line into many smaller parts. Similarly, partitioning may not allow children to take full advantage of their approximate proportional reasoning abilities. Children as young as 6-8 years old can make approximate judgments about proportions with whole numbers (Szkudlarek & Brannon, 2021) and nonsymbolic visual models (Park et al., 2021), but children are less likely to use this approximate reasoning when proportional magnitudes are discrete or countable (Begolli et al., 2020; Boyer et al., 2008).

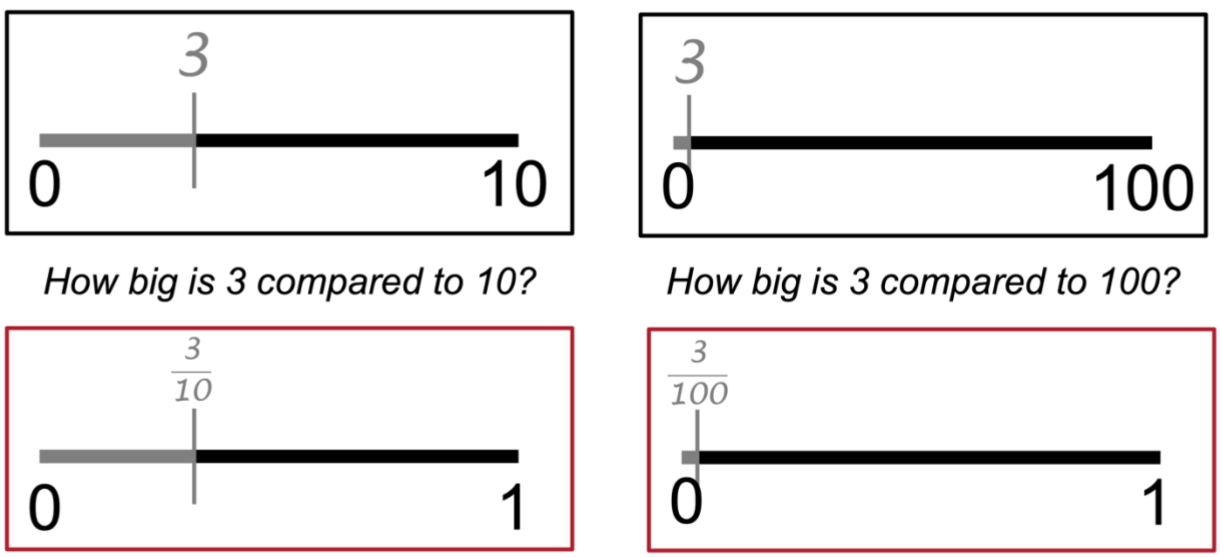
Analogy: An alternative approach to fraction estimation

Analogy offers an alternative strategy for teaching about fraction magnitudes that may more fully leverage the structure of number lines by building on children's knowledge of whole numbers (Thompson & Opfer, 2010). A long line of research has firmly established the power of analogy to support children's and adult's learning in many domains, as learners apply what is known about a familiar "base" domain to a similar, but unfamiliar "target" domain (Gentner, 1983; Goswami, 1986; Loewenstein & Gentner, 2001; Novick & Holyoak, 1991; Richland et al., 2004). Analogies are easier to use when familiar and unfamiliar domains are aligned spatially or temporally, making the relevant similarities between the domains more salient (e.g., Gentner et al., 2016). We designed a lesson featuring alignment to leverage the domain-general learning tool of analogy to teach children about fraction magnitudes.

Long before children learn to represent fractions on number lines, they can already represent proportions on number lines using whole numbers. Six-year-olds can accurately estimate whole numbers along 0-10 and 0-20 number lines (Berteletti et al., 2010), and by eight years of age they can accurately estimate on 0-100 number lines (Siegler & Booth, 2004). This shows that children can judge the proportional size of the target number relative to the size of the endpoint (Barth & Paladino, 2011). For example, to correctly estimate 3 on both 0-10 and 0-100 lines, a child must consider the magnitude of 3 relative to the magnitude of the endpoint; 3 must be placed three-tenths of the line length from 0 when the endpoint is 10, but three-hundredths of the line length from 0 when the endpoint is 100 (Figure 1). Because of this proportional structure of number lines, whole number estimation can serve as a base for learning about fraction estimation through analogy (Matthews & Hubbard, 2017; Sidney et al., 2017).

Figure 1

Example of the analogy from whole number to fraction estimation



Note. Children as young as six-year-olds can judge the magnitude of whole numbers relative to different endpoints (top). Aligning these estimation problems with their equivalent fraction counterparts (bottom) may help children use the analogy from their whole number knowledge to judge fraction magnitudes.

As shown in Figure 1, children who correctly estimate 3 on a number line from 0-10 are making a judgment about the magnitude of 3 relative to 10, which is *analogous* to the magnitude of 3/10 relative to 1. Young children do not use the analogy spontaneously; children in elementary school are much more accurate when estimating whole numbers than when estimating fraction magnitudes (Hamdan & Gunderson, 2017; Resnick et al., 2016; Yu et al., 2022; see also Thompson & Opfer, 2010). Visually aligning similar whole number and fraction estimation problems, which has been shown to support analogical learning in many domains (e.g., Gentner et al., 2016; Mason, 2004), may help children use the analogy to support their understanding of fraction magnitude.

One recent experiment tested this hypothesis. Yu et al. (2022) asked 3rd-5th grade students to complete 80 number line estimation (NLE) problems that aligned whole number and fraction problems (e.g., 3 on a 0-8 number line and 3/8 on a 0-1 line). Following practice with aligned analogous problems, children's fraction estimation approached that of college students. This improvement occurred without any instruction or feedback on their estimates, highlighting the power of analogy to promote learning. Children in a no-alignment control group, who practiced the same problems in random order, did not improve. Alignment may prove even more effective if combined with explicit instruction; in other domains, people struggle to notice and use analogies when solving unfamiliar tasks without explicit instruction or cognitive supports (e.g., Duncker, 1945; Gick & Holyoak, 1980). Notably, Yu et al. (2022) neither compared analogy instruction to no instruction nor measured whether alignment training improved fraction magnitude knowledge outside of a number line context.

An analogical approach to fraction NLE may offer multiple advantages when compared to the more common partitioning approach. Most importantly, it does not require children to count parts or focus on the numerator and denominator separately, which can contribute to extending whole number strategies inappropriately to fractions, known as the whole number bias

(Matthews & Ellis, 2018; Ni & Zhou, 2005). Whole number bias often leads to errors in thinking about fraction magnitude (e.g., 4/8 is bigger than 1/2) and fraction arithmetic (e.g., adding across numerators and denominators such as 1/4 + 1/2 = 2/6) (e.g., Braithwaite & Siegler, 2018). Whereas the partitioning approach highlights each component of x/y separately, which potentially allows the child to continue to think about them as separate numbers, the analogy lesson focuses directly on the relation between x and y. For example, a child using an analogy approach to think about the magnitude of 3/8 is directly encouraged to think about approximately how big three is compared to eight. An analogical approach thus emphasizes multiplicative relationships and leverages children's intuitions and informal experiences with visuospatial proportions (e.g., Carraher, 1993; Jacob et al., 2012; Matthews & Chesney, 2015; Park et al., 2021). Because children are not required to draw partitions to use the analogy strategy, they may be empowered to focus on estimating approximate magnitudes of fractions, rather than focusing on steps of a procedure.

Skeptics of the analogical approach might argue that an estimated or "fuzzy" sense of fraction sizes is not precise enough to perform exact calculations with fractions. This may be true when thinking about supporting direct calculation, but strong estimation skills can be useful for error-checking, rejecting flawed procedures, and speeding up calculations when an imprecise answer is acceptable (Butterworth, 1999; Siegler et al., 2011). An analogical approach may be especially helpful for estimating the sizes of fractions with large denominators (e.g., 15/42), because children can use their approximate knowledge of proportions with whole numbers (e.g., "about how big is 15 compared to 42?") rather than visualizing a number line partitioned into many equal parts.

In sum, there are multiple theoretical and empirical reasons to suspect that an analogical approach that avoids partitioning may be even more effective than partitioning for improving fraction magnitude knowledge. We put these alternatives to the test. We developed a 15-minute lesson that combined alignment of whole number and fraction estimation problems used by Yu et al. (2022) with explicit instruction and other strategies that have been shown to improve learning through analogy in past research. Specifically, we used color-coding (Holyoak & Koh, 1987), linking gestures (Richland, 2015), and relational language (Loewenstein & Gentner, 2005), which have been all shown to support analogical transfer. In a preregistered [https://osf.io/3y5an] initial pilot study with 43 children ages seven to nine years old, we examined children's improvements in fraction estimation and comparison following our 15-minute analogy lesson compared to children's performance after a 15-minute control activity where they practiced the same estimation problems with no alignment or lesson. Results from the pilot study showed that the analogy lesson reduced children's fraction estimation error by almost 40%. This was not merely an effect of practice or rote memorization, because children in the analogy group outperformed the control group even on untrained fractions. However, the improvements in fraction estimation following the analogy lesson did not transfer to fraction comparison in the pilot study, so we adapted the analogy lesson slightly to scaffold fraction comparison more directly and to use language that emphasized fraction sizes.

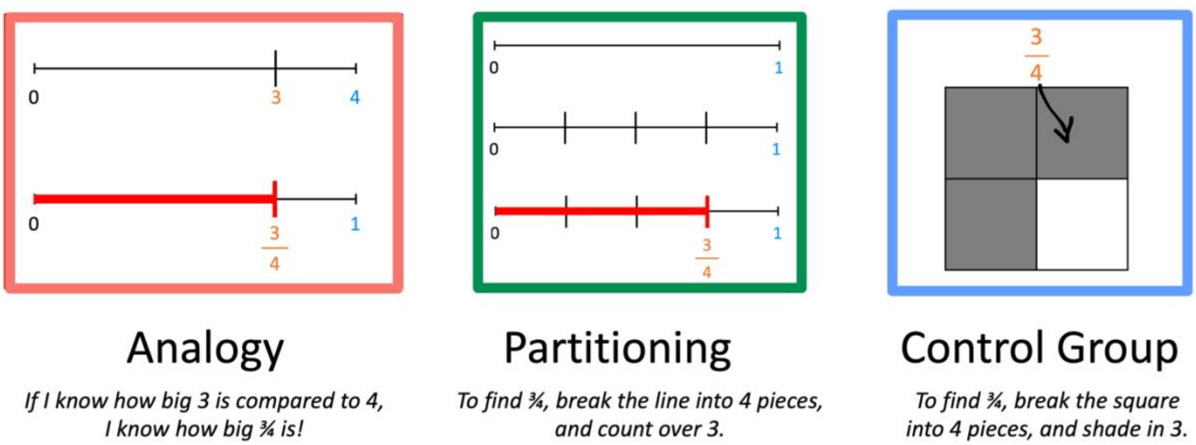
Current Study

The current study compares the effectiveness of three fraction magnitude lessons: a number line lesson focused on *Partitioning*, a number line lesson focused on *Analogy*, and a *Control* lesson focused on partitioning a square area model (Figure 2). We sought to conceptually replicate and extend prior work (i.e., Gunderson et al., 2019; Hamdan &

Gunderson, 2017; Yu et al., 2022) by more extensively testing the transfer, maintenance, and robustness of children's learning following the analogy and partitioning lessons. The study design included a one-week retention test and multiple transfer measures to support these aims. To our knowledge, this study is also the first to compare the effectiveness of the partitioning and analogy number line lessons for teaching children fraction magnitudes.

Figure 2

Examples of the three fraction magnitude lessons.



Preregistered Hypotheses. We preregistered the following hypotheses on OSF:

1. Both the analogy and partitioning groups would outperform the control group on fraction NLE (H1a) and fraction comparison (H1b) at posttest and retention, based on prior findings from our pilot study, Yu et al. (2022), and Gunderson et al. (2019).
2. Whole number estimation skill would interact with the effect of the analogy lesson, such that children with better whole number NLE skills would benefit more from the analogy lesson which builds on these skills (H2).

3. The *analogy* lesson would be more helpful than the other two for large-denominator fractions, as children could use an approximate sense of how whole numbers relate instead of breaking the models into many small pieces (H3).
4. The area model *control* lesson would be more helpful than the other two for children's area model estimation (H4).
5. We also preregistered a hypothesis that the *partitioning* lesson would be more helpful than the other two for children's understanding of *fraction schemes*, or their ability to use partitioning and iterating with unit and non-unit fractions through drawing. However, due to time constraints, few children fully (11.9%) or partially (44.0%) completed the drawing measure, and we did not give a pretest for this measure, which reduced our ability to detect group differences. Therefore, fraction schemes results are reported only in the online supplement.

Methods

Transparency and Openness

We preregistered this study's design, hypotheses, and analyses at [https://osf.io/6zvkd]. We report all data exclusions, manipulations, and measures in the study, and we follow Journal Article Reporting Standards (Kazak, 2018). All deidentified data, analysis code, and research materials are available on OSF at [https://osf.io/3e7q9]. Data were analyzed using R, version 4.1.3 (R Core Team, 2022) and the *tidyverse* (Wickham et al., 2019), *ggpubr* (Kassambara, 2023), *lm.beta* (Behrendt, 2023), and *emmeans* (Lenth, 2023) packages.

Participants

A preregistered power analysis using the *pwr* package (Champely, 2020) showed that a sample size of at least 25 students per group was sufficient to yield 80% power to detect a meaningful effect size of Cohen's $f^2 = .15$ with $\alpha = .05$ using multiple regression with three coefficients (i.e., pretest score, whole number NLE, and condition). To account for possible attrition, we therefore recruited 90 children to participate, following our preregistered plan. Six of the 90 students whose parent/guardian gave consent and/or confirmed interest never scheduled sessions or kept canceling scheduled sessions, leading to a final sample size of eighty-four children. A sensitivity power analysis showed that this slightly smaller sample was sufficient to detect an effect size of Cohen's $f^2 = .13$ at 80% power with $\alpha = .05$ using multiple regression with three coefficients.

Eighty-four children who had recently completed 2nd and 3rd grade (Mean age=8.83 years, 46 female and 38 male) participated during summer 2021 through videoconferencing on Zoom. All children participated in three sessions, although eleven children missed specific measure(s) in one or more sessions due to time constraints, technology issues, or opting to skip a particular task. More details about missing data are provided in the analysis plan section. Children were recruited from five states in the central time zone of the United States using digital flyers distributed to parents through PeachJar. According to parent/guardian report, the sample was 71% White, 23% Asian, 1% Black or African-American, 1% American Indian or Alaska Native, and 2% multiple races, with one student's race unspecified. Parents/guardians reported that 4% of the children were of Hispanic or Latino/Spanish Origin, 92% were not, and 4% did not specify. We did not collect socioeconomic information from individual families but can report on school-level SES: approximately 57% of the children attended Title I schools, which receive federal support due to the large percentage of students in poverty, and the median percentage of students eligible for free- and reduced-price lunch at children's schools was 30%.

Prior to the first experimental session, we randomly assigned participants to receive one of three lessons: Analogy on number lines (n = 29), Partitioning on number lines (n = 27), or Control using square area models (n = 28). In the control lesson, children received instruction about how to partition square area models to represent fraction sizes, adapted from Hamdan and Gunderson (2017) and Gunderson et al. (2019). The Control script was almost identical to the Partitioning script. This study was approved by the University of Wisconsin-Madison Institutional Review Board.

Measures

Children completed three measures of fraction knowledge at pretest, posttest, and retention: NLE, comparison, and area model estimation. They also completed additional measures of mathematical and cognitive ability at sessions 1 and 3. All task stimuli, including instructions and practice trials, are publicly available on OSF.

Fraction Knowledge

Number Line Estimation (NLE). Children completed fraction NLE on their own computers using the experiment hosting platform www.Gorilla.sc (Anwyl-Irvine et al., 2020), while sharing their screen with experimenters. One fraction at a time appeared above an unpartitioned line with endpoints labeled 0 and 1, and children estimated its position by clicking on the line. After seeing 1/2 placed on the number line as an example, children completed a subsequent practice trial with 1/2. There was no time limit, and children could use scratch paper. Children completed 14 trials with the following single-digit-denominator fractions: 1/2, 2/3, 1/4, 3/4, 1/5, 2/5, 3/5, 4/5, 1/6, 4/6, 5/6, 1/7, 7/8, and 3/9. The analogy lesson included seven of these fractions but only showed the correct estimates for three (1/2, 1/4, and 3/4). The partitioning and control lessons included and showed correct estimates for six of these fractions

(1/2, 1/4, 3/4, 1/5, 2/5, and 4/5). Therefore, six of these fraction NLE items were untrained (i.e., not included in any of the lessons: 1/7, 1/6, 3/9, 4/6, 5/6, and 7/8).

Children also completed seven NLE items with fractions containing larger denominators. Items were chosen to span the numerical distance from 0 to 1 and to approximately match the values of the untrained small-denominator fractions: 12/83, 5/30, 12/36, 7/18, 67/100, 10/12, and 88/100. If children followed the procedure they were taught, we expected that children in the partitioning group would have a more difficult time with these items than children in the analogy group, because mentally breaking a line into 30 or 18 equal pieces is more difficult than thinking about approximately how big 5 is compared to 30 or approximately how big 7 is compared to 18. Performance was calculated using Percent Absolute Error (PAE; |*actual magnitude* – *estimated magnitude* |/*range* *100; Siegler & Booth, 2004).

Fraction Comparison. In each trial, two fractions appeared on the screen surrounded by different colored boxes and children verbally answered, “Which fraction is bigger: the green or the blue?” The experimenter recorded the child’s response via click. We used verbal responses to increase engagement and comfort with the experimenter. There was no time limit, and children could use scratch paper.

To parallel the NLE task, children completed 20 trials with small-denominator fractions, and 10 with untrained large-denominator fractions. The fraction pairs for the 20 small-denominator trials were taken from Hamdan and Gunderson (2017), and fraction pairs for the 10 large-denominator trials were modified to approximately match the numerical values of the untrained small-denominator set. In fraction comparison tasks, trials can be consistent with a whole number strategy, such that the fraction with bigger components is larger (e.g., 3/4 vs. 1/2), inconsistent (e.g., 2/7 vs. 1/3), or ambiguous (e.g., 3/7 vs. 4/6). These stimuli were constructed to have some of each type of trial in each set. Numerical distances between the fraction magnitudes (i.e., larger magnitude – smaller magnitude) ranged from 0.05 to 0.55, with an average distance of 0.24. Half of the trials showed the larger fraction on the left, and half showed it on the right.

Area Model Estimation. To test whether the conditions were differentially effective helping children represent fraction sizes on visual models other than number lines, we asked children to represent symbolic fractions on square area models at pretest and at posttest. Because of time constraints, children did not complete this transfer task at the retention. Following Gunderson et al. (2019), children saw blank squares on the screen below a symbolic fraction, and they were asked to show the fraction on the square. Children responded by drawing on the screen using the Zoom "Annotate" tools. To help children feel comfortable using Annotate tools, they completed 4 practice drawing exercises unrelated to math (e.g., connect each animal to its habitat, trace the shape, etc.) before completing the area model task.

After viewing an example of how to show ½, children estimated 7 items with single-digit fractions, none of which were shown in the lessons. To score each item, we calculated the proportion of the overall square the child shaded, to arrive at a Percent Absolute Error (PAE=|*actual proportion of pixels shaded – correct proportion of pixels that should have been shaded*| *100) score for each item, following Gunderson et al (2019) (see online supplement for details). All items were coded by 2 researchers with an initial agreement rate of 94.1%. Discrepancies of more than 2.5% were then resolved by the first author (who was not one of the original coders) to achieve 100% agreement.

Other Cognitive Skills

Whole number estimation. Children completed whole number NLE to account for variability in children's prior knowledge of number lines and whole number magnitudes. The procedure was identical to that of fraction NLE, except that the lines used different end points. Each screen showed a number line from 0 to some whole number (i.e., 2, 3, 4, 5, 6, 7, 8, 9, 12, 18, 30, 36, 83, 100) and a target number, which children estimated by mouse click. Children completed 21 randomly-ordered trials, and stimuli were chosen to match the fraction NLE stimuli (e.g., because 3/5 was included in fraction NLE, children were asked to judge 3 on a 0-5 number line). Children were invited to use scratch paper if needed.

Multiplication and division fluency. Children's fluency with multiplication and division facts may influence their use of benchmarks during NLE (e.g., Siegler et al., 2011). To test whether arithmetic fluency differed across groups, children completed a 2-minute fluency test of multiplication and division facts within 100. Children were given 1 minute to complete as many multiplication facts as possible, and after a short break 1 additional minute to complete as many division facts as possible. The problems were shown vertically, and children responded verbally. Children were instructed to say "skip" or guess if they did not know an answer. Each child's fluency score was the total number of correctly answered items within the allotted time.

Working memory. To test whether groups differed in auditory working memory, all children completed the Backward Digit Span subtest of the WISC-IV standardized test. Participants listened to a series of numbers (e.g., 5–7–4) and then repeated them in reverse order (e.g., 4–7–5). Children began with two practice trials that contained two-number sequences and continued with increasingly long sequences until they answered two incorrectly in a row. After the sessions, we scored each item as correct or incorrect based on the recording and calculated each child's longest string of correctly repeated numbers.

Nonsymbolic ratio comparison. To test whether groups differed in nonsymbolic proportional reasoning, which can influence NLE, children completed a test of nonsymbolic ratio processing at the final session (Park et al., 2021). Using identical stimuli and procedures as Park et al. (2021), we showed children pairs of line ratios (e.g., ||) on the screen and asked them to choose the larger of the two ratios via mouse click. Children completed 40 trials.

Procedure

The study used a pretest-lesson-posttest-retention design over three one-on-one 60-minute sessions on Zoom. Before the sessions, parents completed consent and reported their child’s demographic characteristics. After each session, families received an e-gift card as compensation.

At session 1, the experimenter introduced the study and secured verbal assent from the child. Then children completed the Whole Number NLE task and were asked short survey items that are not analyzed in the current study. Children then completed pretest Fraction NLE, pretest Fraction Comparison, a brief non-math drawing activity to introduce them to Zoom’s Annotate tools, and pretest Area Model Estimation.

At session 2, children received the lesson that they were randomly assigned to: analogy, partitioning, or the control lesson without number lines. All lessons took about 15-20 minutes, and all lessons were presented via slideshows shared on the computer screen. A full script and materials are available on OSF.

In all three lessons, children were first shown an example of how to use the assigned strategy to estimate the fraction 1/2. Then, following Hamdan and Gunderson (2017), children in the Partitioning and Control lessons were asked to estimate the fractions 1/4, 2/4, 3/4, 1/5, 2/5, 3/5, and 4/5—in that order—using these steps to place x/y: *partition* the [line/square] into y equal parts, *shade* x equal parts, and *place* the fraction x/y on the line or square. After each step the child was shown the correct response for that step, and if the child’s work did not match the correct response they were told to correct it to make it match. Children in the Partitioning and Control lessons practiced placing each fraction twice, first on a pre-partitioned model and then on an unpartitioned model.

The Analogy lesson began by walking children through using the analogy strategy to estimate 1/2, 3/4, 1/4, and 2/4 (in that order) on a number line by using a “helper number line” with the analogous whole numbers, as shown in Figure 2. For these first 4 fractions, the correct estimates were shown on the screen. Then, the Analogy lesson switched to practice problems, where the child estimated the fractions 3/5, 5/7, 1/8, 1/5, 7/9, 3/8, 1/3, and 5/9 without receiving feedback on the correctness of their estimates. Finally, the analogy lesson ended by explicitly scaffolding fraction comparison, showing children that they could use two helper lines to place two fractions on the same number line: 3/7 vs. 1/8, 2/3 vs. 4/9, and 4/5 vs. 5/9. Children did not receive feedback on the accuracy of their whole number or fraction estimates during the analogy lesson, but they did receive reminders of how to use the analogy strategy if they made a mistake with any of the steps (e.g., put the numerator as the endpoint of the whole number helper line).

Immediately following the lesson, children completed posttest Fraction NLE, Fraction Comparison, and Area Model Estimation tasks. Children in all conditions were asked to “remember what we worked on in the lesson” and encouraged to “write down whatever you need to on scratch paper to help you figure it out.” At the end of session 2, students were also asked to complete the test of fraction drawing on the computer screen.

At session 3, after a 2-minute reminder of the lesson, children completed a retention test with Fraction NLE and Fraction Comparison. They also completed Multiplication and Division Fluency, Working Memory, and Nonsymbolic Line Ratio Comparison tasks.

Analysis Plan

We first examined basic descriptive statistics and used a series of one-way Analysis of Variance (ANOVA) tests to test whether our randomly-assigned groups differed on any pretest measure or covariate. Finally, as preregistered, we tested for differences between groups on each primary outcome measure (i.e., small- and large-denominator estimation and comparison at posttest and retention), using linear regression models predicting posttest performance by condition, controlling for pretest scores on the same measure. We also tested the interaction between pretest and condition. Specifically, we compared a series of nested models as shown in Table 1. In all models, Condition refers to two dummy variables (Analogy and Partitioning) relative to the reference control group. Because some models contain interactions with pretest and/or whole number estimation, we centered these variables at the sample mean. This allowed us to interpret any effects of condition as average main effects (i.e., effects of analogy (vs. control) or partitioning (vs. control) when pretest and whole number estimation scores are held constant at the sample mean). We then used the BIC, which accounts for variance explained and for parsimony, to select the best model for each outcome. We report and interpret only the results of the selected model in detail. Results of the other models are reported in the online supplement. Lastly, to test whether there were differences between groups in area model estimation, which we measured only at pretest and posttest, we followed our preregistered plan to predict posttest error by condition, controlling for pretest scores.

For each analysis, we omitted any participants who had incomplete data on the measure being tested. Specifically, for large-denominator NLE, we excluded two children (one due to time constraints, one because the child opted out) from the analysis of the posttest and one child (due to time constraints) from the analysis of the retention. For small-denominator comparison, we excluded one child from the posttest analysis due to the participant opting out of the task but did not exclude any children from the retention analysis. For large-denominator comparison, we excluded three children from the posttest analysis due to time constraints (two children) and the participant opting out (one child), and we excluded two children from the retention analysis due to technology difficulties (one child) and experimenter error (one child). Lastly, we excluded nine children from the area model estimation analysis because they did not complete the measure at the posttest due to time constraints (six children), opting out of the task (two children), or not following instructions (one child).

Table 1
Nested Models to be Compared in All Regression Analyses

Model	Specification
Null	Post ~ Pre
1	Post ~ Pre + Condition
2	Post ~ Pre + Condition + Pre*Condition
3	Post ~ Pre + Condition + Pre*Condition + WNL
4	Post ~ Pre + Condition + Pre*Condition + WNL + WNL*Condition
5	Post ~ Pre + Condition + WNL + WNL*Condition
6	Post ~ Pre + Condition + WNL

Note. WNL=Whole Number NLE.

Results

Descriptive Statistics

Descriptive statistics for fraction measures at each session are shown in Table 2. As expected, at pretest students made more accurate estimates with small, single-digit denominators (M_{PAE} = 15.4%, SD = 10.8) than with large, two-digit and three-digit denominators (M_{PAE} = 18.3%, SD = 13.1), $t(83) = -2.89$, $p = .005$. Average pretest accuracy on fraction comparison was similar for small-denominator fractions (M_{ACC} = 66.6, SD = 26.1) and large-denominator fractions (M_{ACC} = 65.9, SD = 19.9), $t(83) = .33$, $p = .74$.

Next, we examined means and standard deviations of pretest and covariate measures by the three conditions, as shown in Table 3. One-way ANOVA tests showed that the groups were similar in age, pretest scores, covariate scores (i.e., digit span, multiplication and division fluency, nonsymbolic ratio comparison, and whole number NLE), and the duration of the delay from posttest to retention. A chi-squared test showed that the groups also had a similar proportion of second graders vs. third graders, $\chi^2(2) = 4.29$, $p = .117$.

Table 2

Means and SDs Averaged Across All Groups

Task	N	Pretest Mean (SD)	Posttest Mean (SD)	Retention Mean (SD)
Fraction NLE (PAE)				
Small Denominator	84	15.4 (10.8)	12.2 (9.74)	12.4 (9.82)
Large Denominator	81	18.3 (13.1)	15.5 (11.5)	14.5 (10.9)
Fraction Comparison (PA)				
Small Denominator	83	66.6 (26.1)	73.3 (25.3)	72.9 (24.9)
Large Denominator	79	65.9 (19.9)	67.5 (21.9)	68.4 (20.0)
Fraction Area Estimation (PAE)	75	10.7 (9.55)	6.58 (5.93)	-

Note. NLE=Number Line Estimation. PAE=Percent Absolute Error (|estimate – actual|/range*100). PA=Percent Accuracy. Sample size refers to the number of participants with complete data across all sessions for that measure.

Table 3
Means and SDs for Pretest and Covariate Measures by Condition

Measure	Analogy (n=29) Mean *(SD)*	Partitioning (n=27) Mean *(SD)*	Control (n=28) Mean *(SD)*	Test of condition difference
Days between Posttest and Retention	8.31 (2.88)	8.59 (2.02)	8.93 (3.92)	$F(2, 81)=.294$, $p = .747$
Grade (Proportion in 2nd grade)	0.65	0.41	0.43	$\chi^2(2) = 4.29$, $p = .117$
Age (Years)	8.71 (.68)	8.88 (.59)	8.90 (.61)	$F(2, 81)=.791$, $p = .457$
Digit Span (Digits Correct)	3.66 (1.11)	3.96 (1.16)	3.96 (1.29)	$F(2, 81)=.639$, $p = .530$
Arithmetic Fluency (Number Correct)	14.86 (11.20)	22.07 (15.14)	19.86 (12.0)	$F(2, 81)=2.34$, $p = .103$
Nonsymbolic Ratio Comparison (PA)	91.1 (6.18)	88.8 (18.8)	91.1 (13.7)	$F(2, 80)=.246$, $p = .782$
Whole Number NL (PAE)	10.28 (6.39)	9.45 (5.65)	8.59 (4.59)	$F(2, 81)=.644$, $p = .528$
		Pretest		
Number Line Estimation (PAE)				
Small-Denominator	15.90 (9.35)	15.53 (10.80)	14.70 (12.32)	$F(2, 81)=.090$, $p = .914$
Large-Denominator	19.02 (14.29)	17.51 (13.30)	18.34 (11.91)	$F(2, 81)=.092$, $p = .912$
Comparison (PA)	61.72 (27.03)	65.37 (25.07)	72.68 (25.77)	$F(2, 81)=1.31$, $p = .277$

Small-Denominator				
Large-Denominator	61.53 (19.25)	68.15 (16.18)	68.21 (23.58)	F(2, 81)=1.06, p = .352
Area Model Estimation (PAE)	12.48 (10.10)	7.77 (7.71)	11.88 (9.29)	F(2,78)=2.00, p = .142

Note. PAE=percent absolute error; PA=percent accuracy

Learning: Number Line Estimation

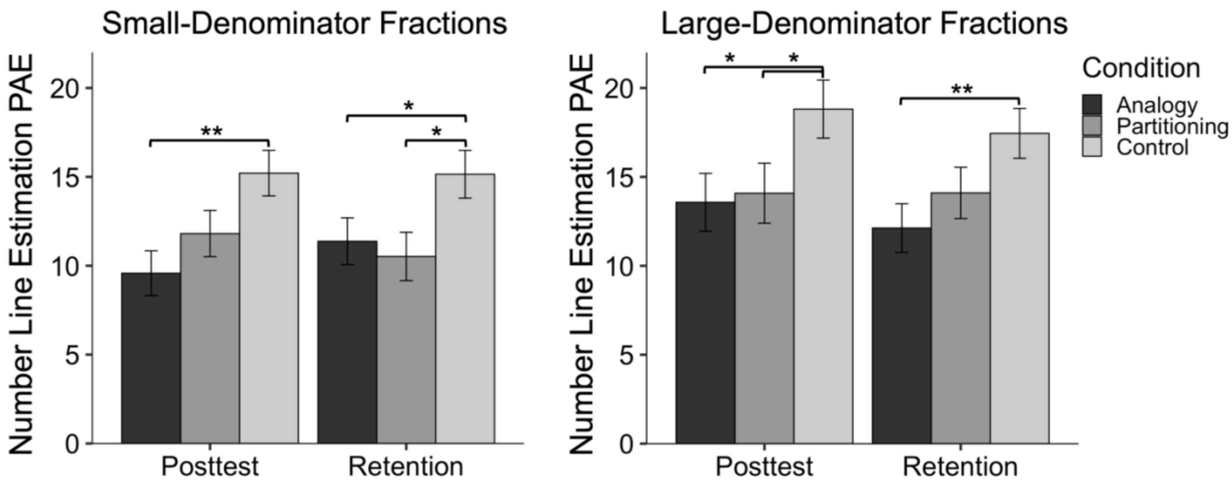
Improvements at Posttest

As preregistered, to test whether children learned the target skill, we used the linear models shown in Table 1 to predict the child's average PAE on the posttest small-denominator Fraction NLE by their condition, controlling for pretest scores. The BIC values indicated that Model 6 (Post ~ Pre + Condition + WhNumNLE) was preferred, when accounting both for variance explained and for parsimony of predictors, $R^2 = .54$, $F(4, 79) = 23.4$, $p < .001$. We report standardized regression coefficients here and for all subsequent analyses.

Children's posttest fraction NLE scores were predicted by their pretest fraction NLE scores ($\beta = 0.458$, $p < .001$) and whole number NLE scores ($\beta = 0.357$, $p < .001$). After controlling for these indicators of prior knowledge, children in the analogy group ($\beta = -0.276$, $p = .003$) made more accurate estimates than the control group at posttest (Figure 3). The partitioning group's performance was not significantly different than the control group's ($\beta = -0.164$, $p = .066$). This shows that the analogy lesson, but not the partitioning lesson, was more effective at teaching small-denominator fraction NLE compared to the control lesson. However, post-hoc contrasts using Tukey's correction for multiple comparisons ($t(79) = 1.23$, $p = .438$) indicated that the analogy and partitioning groups did not significantly differ from each other.

Figure 3

Fraction Number Line Estimation Performance



Note. PAE=Percent Absolute Error. Scores shown are adjusted means, controlling for pretest and whole number NLE using the *emmeans_test* function in R. Error bars show standard errors.

Sustained Learning at Retention?

To test whether these improvements were sustained, we tested the six nested models in Table 1 to predict participants' fraction NLE PAE at retention. Again, Model 6 was preferred by the BIC, $R^2 = .51$, $F(4, 79) = 20.34$, $p < .001$. As shown in Figure 3, after controlling for fraction NLE pretest score ($\beta = 0.391$, $p < .001$) and whole number estimation ability ($\beta = 0.403$, $p < .001$), both the analogy group ($\beta = -0.184$, $p = .049$) and partitioning group ($\beta = -0.221$, $p = .018$) outperformed the control group after a one-week delay by .18 and .22 standard deviations, respectively. A post-hoc Tukey test showed that the analogy group and the partitioning group did not perform statistically differently at retention ($t(79) = .451$, $p = .893$). The results support our hypotheses that both Analogy and Partitioning would lead to better fraction NLE than the area model Control. Exploratory analyses conducted solely on untrained fractions showed that the small-denominator NLE results also held for untrained fractions at posttest and retention (see online supplement).[1]

Is Analogy Especially Helpful for Estimating Fractions with Large Denominators?

We hypothesized that the analogy strategy might be especially helpful for thinking about the sizes of fractions with large denominators (e.g., 12/52 or 34/100). Following our preregistered plan, we estimated the same models selected for small-denominator NLE to test this hypothesis with large-denominator NLE. However, as shown in Figure 3, after controlling for pretest large-denominator fraction NLE (β = 0.603, p < .001) and whole number NLE (β = 0.077, p = .459), both number line groups outperformed the control group at posttest (analogy β = -0.216, p = .026; partitioning β = -0.192, p = .048), R^2=.47, F(4,77) = 17.29, p < .001. A post-hoc Tukey test showed that the analogy and partitioning groups did not perform statistically differently at posttest (t(77) = -0.218, p = .974).

After a one-week delay, results showed an advantage of the analogy lesson, but not the partitioning lesson, over the control lesson for large-denominator fraction NLE, partially supporting our hypothesis. After accounting for the significant effects of pretest score (β = 0.509, p < .001) and whole number estimation skill (β = 0.301, p = .003), the analogy group showed better PAE scores than the control group at retention (β = -0.235, p = .008), R^2=.57, F(4, 78) = 25.41, p < .001. As shown in Figure 3, the analogy group's average PAE with large-denominator fractions at retention was 4.1% lower than that of the control group. However, the partitioning group was no better at large-denominator NLE than the control group at retention (β = -0.143, p = .101). Although only the analogy group outperformed the control group, we cannot necessarily conclude that the analogy group performed better than the partitioning group: a post-

[1] Supplemental exploratory analyses also showed that almost all results for NLE and comparison were consistent when controlling for nonsymbolic ratio comparison and arithmetic fluency. See online supplement for details.

hoc Tukey test showed that the analogy and partitioning groups did not perform statistically differently at retention ($t(78) = -0.994$, $p = .583$).

Transfer: Comparison

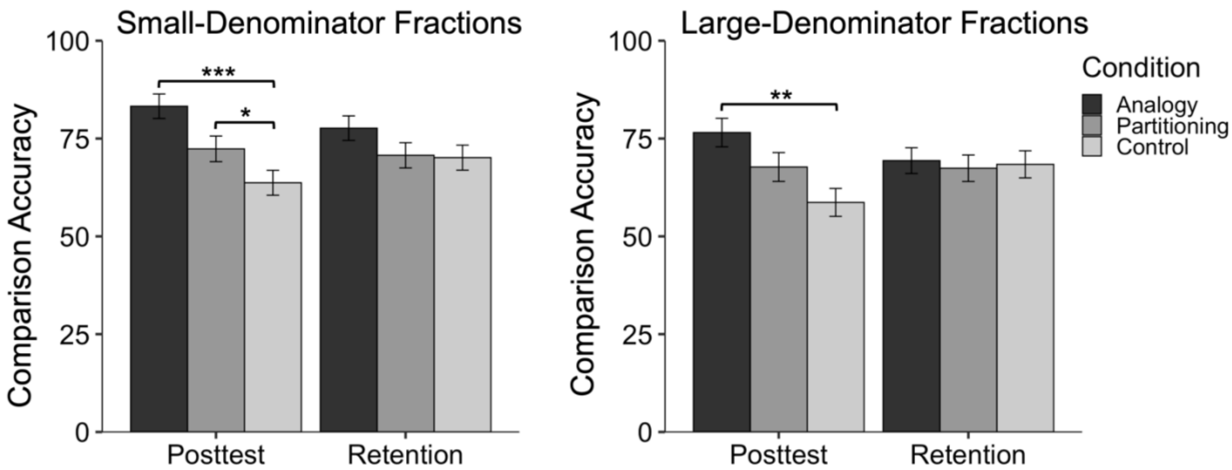
Improvements at Posttest.

To test whether children's learning about fraction estimation transferred to the fraction comparison task once the number line was removed, we compared the six nested models in Table 1 to predict participants' fraction comparison accuracy at posttest. Model 2 (Posttest ~ Pretest + Condition + Pretest × Condition) was preferred by the BIC, R^2=.66, $F(5,77) = 29.86$, $p < .001$. There were significant main effects of both pretest accuracy and condition, as well as a significant interaction. For clarity, we interpret the interaction effect first. There was a significant interaction between pretest accuracy and condition for the analogy group only ($\beta = -0.415$, $p < .001$). We can interpret this interaction to indicate that children with a *lower* accuracy at pretest benefited more from being placed in the analogy group than children with a higher accuracy at pretest.

Overall, unsurprisingly, fraction comparison pretest score ($\beta = 1.10$, $p < .001$) predicted fraction comparison accuracy at posttest. Note that, because of the interaction, this effect can be considered the effect of pretest within the control condition (i.e., when analogy and partitioning are set to 0). There was also a main effect of condition. As shown in Figure 4, children in both analogy ($\beta = 0.376$, $p < .001$) and partitioning groups ($\beta = 0.197$, $p = .013$) had higher posttest small-denominator comparison accuracy than children in the control group. That is, when pretest scores were held constant at the sample mean, the analogy group outperformed the control group by almost 0.4 standard deviations and the partitioning group by about 0.2 standard deviation. A post-hoc Tukey test showed that the analogy group and the partitioning group did not perform statistically differently at posttest ($t(77) = 2.20$, $p = .077$).

Figure 4

Fraction Comparison Accuracy



Note. Scores shown are adjusted means, controlling for pretest scores and whole number estimation. Error bars show standard errors.

Retention of Learning?

To test whether the number line groups continued to outperform the control on small-denominator comparison one week later, we compared the six models in Table 1 to predict accuracy at retention. Model 3 (Posttest ~ Pretest + Condition + Pre × Condition + Whole Number Line PAE) was preferred by the BIC, R^2=.62, $F(6,77) = 21.2$, $p < .001$. As with the posttest, retention was predicted by a significant interaction between pretest comparison accuracy and condition for the analogy group ($\beta = -0.325$, $p = .002$), indicating that children with lower pretest accuracy benefited more from the analogy lesson. After accounting for this interaction, there was no effect of condition, as shown in Figure 4. Children in the analogy group

(β = 0.144, p = .086) and partitioning group (β = 0.040, p = .631) were no longer more accurate than the control group at retention. A post-hoc Tukey test showed that the analogy group and the partitioning group did not perform statistically differently at retention ($t(77) = 1.26$, p = .421).

Again, pretest comparison accuracy predicted performance on the same task at retention (β = 0.878, $p < .001$). Whereas whole number estimation knowledge was not a significant predictor of comparison accuracy at posttest, it did predict accuracy at retention (β = -0.193, p = .020). Note that, because of the interaction, these effects can be considered the effects of pretest and whole number estimation within the control condition (i.e., when analogy and partitioning are set to 0). Children with higher PAE on whole number estimation (i.e., worse estimates) had lower accuracy on fraction comparison at retention.

Is Analogy Especially Helpful for Comparing Fractions with Large Denominators?

We also tested our hypothesis that the analogy lesson would be particularly useful for large-denominator fractions. Following our preregistered plan, we estimated the same models selected for small-denominator fractions. We estimated Model 2 (Posttest ~ Pretest + Condition + Pretest × Condition) to predict fraction comparison accuracy with large-denominator fractions at posttest and Model 3 (Posttest ~ Pretest + Condition + Pre × Condition + Whole Number Line PAE) to predict accuracy at retention.

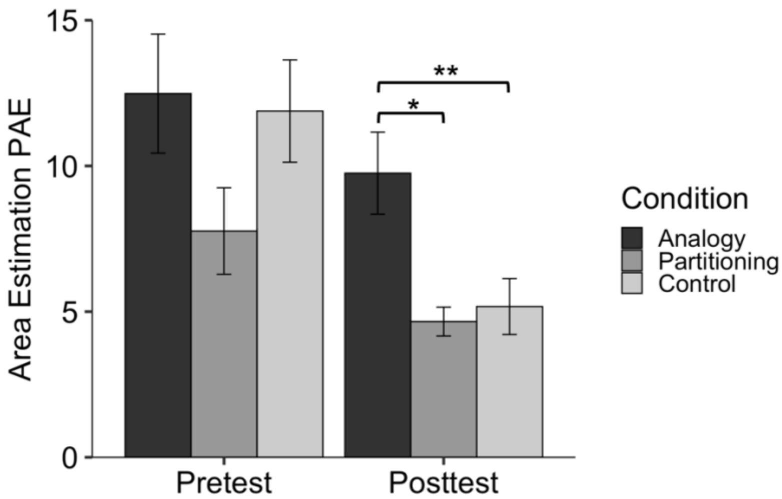
As hypothesized and as shown in Figure 4, the analogy group (β = 0.355, p =.002), but not the partitioning group (β = 0.192, p = .086), outperformed the control group at posttest on comparing large-denominator fractions, R^2=.31, $F(5,75) = 6.72$, $p < .001$. However, a post-hoc Tukey test showed that the analogy group and the partitioning group did not perform statistically differently from each other ($t(75) = 1.42$, p = .335). Pretest accuracy with large-denominator fractions also significantly predicted accuracy at posttest (β = 0.649, $p < .001$).

At retention, there was no longer a significant effect of the analogy lesson on comparison accuracy (β=.019, p=.872), R^2=.29, F(6,75), p<.001. Only pretest accuracy (β=.511, p=.001) and whole number estimation PAE (β=-.289, p=.006) were significant predictors of comparison accuracy with large-denominator fractions. The models predicting large-denominator fraction comparison explained only about 30% of the variance in children's accuracy, compared to about 60% of the variance in small-denominator fraction comparison.

Transfer: Area Estimation

Finally, to test whether the area model control condition successfully promoted its targeted skill, we analyzed children's estimations of fraction magnitudes on area models at posttest. As preregistered, we regressed PAE on area model estimation at posttest from condition and pretest PAE, with the control group as the reference group. Controlling for pretest score (β=.237, p=.015), the Analogy group performed .35 standard deviations worse than the control group at posttest (β=.353, p=.005), whereas the Partitioning group performed similarly to the control group at posttest (β=.003, p=.979). A post-hoc Tukey test showed that the Analogy group was also significantly worse at posttest than the partitioning group ($t(71)$=2.70, p=.023). The findings reported in Figure 5 suggest that the partitioning lesson on number lines was just as effective at teaching children to estimate fractions on area models as on number lines, whereas the analogy lesson was not as helpful for teaching area model estimation.

Figure 5. Improvements in Fraction Area Model Estimation



Note. Area estimation was not included at retention. Error bars show standard errors.

Table 4

Summary of Results

		Posttest		Retention	
	Hypothesis	Small	Large	Small	Large
H1a.	Analogy and Partitioning will outperform Control on fraction NLE.	**Only Analogy outperformed Control**	✓	✓	**Only Analogy outperformed Control**
H1b.	Analogy and Partitioning will outperform Control on fraction comparison.	✓	**Only Analogy outperformed Control**	✗	✗
H2.	Analogy will be more effective for children with strong whole number NLE skills.	✗	✗	✗	✗

H3.	Analogy will be best for large-denominator fractions.	**N/A**	**NLE: Not Supported** **Comparison: Weakly Supported**	**N/A**	**NLE: Weakly Supported** **Comparison: Not Supported**
H4.	Control will best support area model estimation.	**Control and Partitioning outperformed Analogy**	**N/A**	**N/A**	**N/A**

Note. Small = Small-Denominator Fractions, Large = Large-Denominator Fractions, NLE = Number Line Estimation. A checkmark indicates that the hypothesis was supported, an X indicates that the hypothesis was not supported, and N/A indicates that the hypothesis was not applicable for that outcome measure. Hypotheses which were partially supported are explained with additional text.

Discussion

The current study makes two important contributions to research on mathematical development and fraction cognition. First, fraction magnitude instruction often emphasizes partitioning, but this approach may not fully leverage children's approximate proportional reasoning. We showed that a brief 15-minute lesson that taught children an analogy using the proportional structure of number lines (e.g., 3:4::3/4:1) was more effective for promoting fraction magnitude understanding than a partitioning lesson using area models, and it was at least as effective as a partitioning lesson using number lines. As summarized in Table 4, we showed that the analogy group more consistently outperformed the control group, especially with large-denominator fractions, although there was not a significant difference between analogy and partitioning groups. One week after the lesson, only the analogy group sustained an advantage over the control group on estimation of large-denominator fractions, and only the analogy group generalized their fraction magnitude learning to comparison with large-denominator fractions at the immediate posttest. These results suggest that the analogical approach, which allows children to leverage their approximate sense of relative magnitudes of whole numbers to estimate fractions, may be an important route to promote fraction magnitude learning.

Second, our findings replicate previous studies showing that lessons using number lines support children's understanding of fraction magnitudes more effectively than lessons using area models (Gunderson et al., 2019; Hamdan & Gunderson, 2017) and extend these studies by showing this advantage is maintained for at least one week on estimation but not on comparison. Immediately after the 15-minute lesson, both analogy and partitioning groups outperformed the control group on symbolic fraction comparison, showing generalizable fraction magnitude learning. The two number line lessons yielded advantages over the control group with meaningful effect sizes: controlling for pretest scores and pretest-condition interaction effects, the analogy group and partitioning groups scored 0.38 and 0.20 standard deviations better than the control group, respectively, on fraction comparison at posttest. As summarized in Table 4, the number line partitioning group also matched the performance of the control group on area model estimation, demonstrating that fraction number line instruction with partitioning may transfer to other visual models more readily than area model instruction with partitioning.

Explaining the Advantages of Analogy

Unlike the other lessons, the analogy lesson directly capitalized on children's approximate proportional reasoning with space and whole numbers. Rather than representing fractions by discretizing a continuous number line and counting the resultant parts, as partitioning approaches do, the analogical approach allows children to estimate the position of a fraction using their prior knowledge of the relative magnitude of whole numbers. Number line estimation, even with whole numbers, involves mapping a symbolic proportion (e.g., 3 out of 10) to a visuospatial proportion (Matthews & Chesney, 2015; Matthews & Ellis, 2018; Sidney et al., 2017). Typical approaches to fraction magnitude instruction that emphasize partitioning may not fully take advantage of this proportional structure of number lines, but we explicitly taught children to notice and use this proportional analogy.

Moreover, the analogy approach represents a case in which children's perceptual capacities can be used to improve knowledge of number symbols. Humans can perceive proportions in nonsymbolic visuospatial formats as early as infancy (McCrink & Wynn, 2007), with perceptual acuity improving throughout childhood (Park et al., 2021). Some evidence suggests that fraction symbols may be grounded in these nonsymbolic representations (Jacob et al., 2012; Kalra et al., 2020), but attempts to leverage nonsymbolic proportional reasoning to support children's fractions knowledge have so far been met with limited success (Abreu-Mendoza & Rosenberg Lee, 2022). Unlike some other training studies in numerical cognition (e.g., Gouet et al., 2020; Hyde et al., 2014), we did not train children's perception alone with the goal of improving math performance. Instead, we sought to leverage children's perceptual proportion processing for didactic purposes, using analogy as a bridge.

We specifically designed the analogy lesson to leverage children's approximate, nonsymbolic proportional reasoning. Past research suggests that children are more likely to successfully use nonsymbolic proportional reasoning with *continuous* visual representations, such as the unpartitioned number lines used in the analogy lesson, than with *discrete* or *discretized* visual representations like those used in the partitioning and control lessons (Begolli et al., 2020; Boyer et al., 2008; Park et al., 2021). However, the partitioning and control conditions may also have engaged children's nonsymbolic proportional reasoning to a lesser extent, as several studies show that elementary-aged children can sometimes reason about nonsymbolic proportions even with discrete or two-dimensional representations (Abreu-Mendoza et al., 2020; Boyer & Levine, 2015; Kalra et al., 2020). It is also possible that some children in the analogy lesson may have used partitioning without being instructed to, as prior research shows that children sometimes spontaneously engage in partitioning strategies to estimate whole numbers on number lines (Barth & Paladino, 2011; Zax et al., 2019). Indeed, one advantage of the analogy lesson over partitioning lessons may be that it allows children to flexibly use a combination of strategies to identify the relation between the numerator and denominator (e.g., nonsymbolic proportional reasoning, estimating using a single reference point such as the midpoint, or explicit partitioning), depending on the fraction to be estimated.

Unlike the partitioning group, the analogy group retained an advantage over the control group after a one-week delay on the transfer measure of large-denominator estimation. The analogy lesson may have been more resistant to forgetting because it may have been more easily encoded both verbally and visually compared with the other two lessons. All three lessons featured both verbal scripts and visual models, but the analogy lesson restated at the end of every example "If *this* is how big X is compared to Y (point to whole number helper line), *this* is how big *x/y* is compared to 1." When children then had to estimate unfamiliar fractions after a delay, they could use both their memory of the visual analogy and the verbal analogy. Such multi-modal encoding tends to produce longer-lasting learning (Moreno & Mayer, 2007). It may have been more challenging to recall the verbal instruction in the partitioning and control lessons, where each step was verbally repeated but there was no concise overall verbal reminder after each example.

Partitioning Lesson Effectiveness Partially Replicated, but Not all Benefits were Sustained

Our results partially replicate previous findings that partitioning lessons on number lines are better for teaching fraction magnitudes than are partitioning lessons on area models (Gunderson et al., 2019; Hamdan & Gunderson, 2019). Surprisingly, unlike previous studies, we found no significant difference between the number line partitioning and area model control groups on small-denominator fraction estimation at posttest. One possible explanation for this may be that our sample had more prior knowledge of small-denominator fraction estimation, and therefore less room to improve, compared to samples in previous work (i.e., mean pretest PAE at pretest of 15.4% vs. 26-29% in Gunderson et al. (2019)). Supporting this hypothesis, the partitioning number line group *did* outperform the control group on estimation with the more difficult large-denominator fractions. Despite the somewhat confusing findings at posttest, the partitioning lesson outperformed the control group on estimation of both small- and large-denominator fractions at retention, which extends prior findings and suggests that number lines may be better than area models for retention of fraction magnitude learning.

Also replicating previous findings, we found that the partitioning number line lesson, even when delivered on Zoom with children drawing on the screen, led to better fraction comparison performance at posttest than the control area model lesson. However, this was only true with small-denominator fractions and only at the posttest. Our findings show that a single 15-minute online partitioning lesson is not enough to sustain generalized fraction magnitude understanding after a one-week delay, which is not surprising given that many children struggle to compare fractions even after months or years of instruction (Resnick et al., 2016; Rosli et al., 2020).

Limitations

Although we matched all three lessons for time on task, the partitioning and control lessons were more closely aligned than the analogy lesson. Using the exact lessons from prior studies allowed us to replicate and extend prior findings about partitioning on number lines versus area models in an online setting. Nevertheless, it is possible that features of the analogy lesson other than analogy, per se, may have contributed to its success. For example, the analogy lesson interspersed fractions of different denominators, whereas the other two lessons grouped by denominator. Future studies are needed to untangle which aspects of the analogy lesson may drive its effectiveness.

This study was conducted online during the COVID-19 pandemic. Because of the variability in U.S. schooling during the pandemic, children in our sample had a wide range of comfort using a computer and drawing on the screen. We attempted to mitigate any discomfort with computers by providing practice exercises and drawing warm-ups, but individual variability added noise to our data. Variability in computer experience and drawing skill could have especially impacted the area estimation results. Importantly, there were also benefits of using an online format for this study. We reduced barriers to participation due to geographic constraints or transportation issues, and we showed that children could benefit from fraction instruction delivered online, which is increasingly common in K-12 education. Conducting studies online, as we did, also opens the possibility for comparative studies using samples that typically are too dispersed for close comparison, up to and including international comparative studies.

We found compelling evidence that brief number line lessons using either analogy or partitioning can improve fraction magnitude knowledge in our ethnically homogenous, relatively high-achieving US sample. However, more research is needed to test the effectiveness of these fraction magnitude lessons for students with different geographic, socioeconomic, and academic backgrounds, given that mathematical inputs and knowledge vary greatly by geographic region and socioeconomic status (e.g., DeFlorio & Beliakoff, 2015; Torbeyns et al., 2015).

Future Directions

We showed initial evidence that a scripted one-on-one fraction number line lesson using analogy is at least as effective as a similar-length lesson focused on partitioning number lines, and more effective than a lesson about partitioning area models, at supporting children's fraction magnitude understanding. The benefits of both number line lessons faded somewhat after a one-week delay, with neither number line lesson showing sustained advantages on fraction comparison. Repeated, spaced practice with number lines may be necessary to support retention of fraction magnitude learning (e.g., Barbieri et al., 2020). Future work should also test lessons and practices that could be more directly applied in classrooms, including integrating other effective pedagogical methods, such as explicit attention to the whole (e.g., Lamon, 2020; Pedersen & Bjerre, 2021), self-explanations (e.g., Bisra et al., 2018), and peer-assisted reflection (Calkins et al., 2020; Rohrbeck et al., 2003).

Future work should continue to investigate when and for whom different fraction number line instruction is effective. Analogy, partitioning, and other approaches to number line instruction may emphasize or highlight different aspects of fractions. For example, our results suggest that the analogy approach may be helpful for thinking about the sizes of large-denominator fractions, whereas past work suggests that the partitioning approach may be especially useful for teaching struggling children about small-denominator fractions (Dyson et al., 2020). Future research should investigate how and when to deploy each approach most effectively to foster children's fraction magnitude learning and lay a foundation for future math achievement.

Conclusion

Effective teaching for school-age children builds on what students already have. This involves leveraging both children's existing knowledge and general learning mechanisms to support cognitive development. In the current study, we took advantage of children's prior knowledge of whole numbers using analogy. Our approach not only has implications for fraction magnitude instruction, but it illustrates one example of how cognitive developmental science may be used to improve instruction more generally. Future work should continue to design and evaluate instruction that builds on children's prior knowledge, aligns with children's intuitions and perceptual abilities, and leverages general learning principles.

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