An Analysis of Multiplicative Thinking Development in Years 3 to 6

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Research has shown that many primary students experience transition barriers between additive and multiplicative thinking. This paper analysed responses from 253 Years 3 to 6 students to a diagnostic assessment which consists of whole number multiplication and division problems involving equal groups, arrays, multiplicative comparison and Cartesian product situations. Based on the Rasch analysis, item responses were differentiated into five developmental Stages indicating a wide range of understanding and pointing to different transition barriers that students experience. The reasons for these are discussed in the paper and some advice is presented for teachers.

This paper intends to share some of the findings from a larger study aimed at identifying Years 3 to 6 students' transition barriers between additive and multiplicative thinking by analysing students' written responses from the *Multiplicative Thinking Diagnostic Assessment* through Rasch analysis. The study was stimulated by previous research findings indicating that students in the middle years of schooling experience transition barriers while moving from additive thinking to multiplicative thinking (e.g., Bao, 2023; Hurst & Hurrell, 2016).

Introduction

The transition from additive to multiplicative thinking is a slow climb, involving a conceptual leap which constitutes an obstacle for many students (Siemon et al., 2006). There is a considerable body of research pointing to the transition barriers students experience during their development. Early studies (e.g., Jacob & Willis, 2003; Mulligan & Mitchelmore, 1997) claimed that recognising the equal grouping structure (groups of equal size and the number of groups) is a barrier for many students. Others such as Sophian and Madrid (2003) and Downton et al. (2022) pointed out that students have difficulty in understanding the abstract concept of *many-to-one correspondence*. For example, to understand 5×3, students need to understand five units of one is distributed over the elements of one unit of three (Steffe, 1994). Studies such as Larsson (2016) suggested that repeated addition as a procedure to solve whole number multiplication problems can potentially hinder students' development. Larsson (2016) also reported another barrier for making inappropriate generalisations while solving two-digit by two-digit multiplication where students solve 19×26 as 10×20+9×6. Other studies by, for example, Hurst and Hurell (2016), added that procedural based learning could limit students' ability to recognise multiplicative relationships and apply their properties to solve problems. Research by Downton and Sullivan (2017) and Bao (2023) also noted that students have difficulties in solving multiplicative problems involving Cartesian product situation.

Transition from early notion to sophisticated thinking about a targeted concept is a process of cognitive change along a developmental stage where cognitive difficulties are encountered (Tzur, 2019). Cognitive difficulties are critical transition points throughout the stages of concept development (Griffin, 2020). Many researchers (e.g., Tzur, 2019; Griffin, 2020) have argued that these transition points appear to be boundaries between two stages of a developmental progression. Therefore, the construction and testing of a developmental progression for multiplicative thinking is a vital step to reveal students' transition barriers.

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The Research Project

The research reported here aimed at investigating Years 3 to 6 students' understanding of multiplicative thinking to reveal their transition barriers. This study is framed in terms of the need to identify students' transition barriers and build on what is known. It used a design-based research approach involving an iterative process of construction, evaluation, and refining of assessment (e.g., Plomp & Nieveen, 2013) and the use of Rasch modelling (Masters, 1982) for the pilot study in phase 1 and implementing the assessment in the current phase. This paper reports some key findings indicating what transition barriers students across Years 3 to 6 experience during their development of multiplicative thinking.

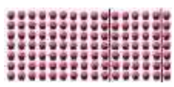
Method

The *Multiplicative Thinking Diagnostic Assessment* (Table 1) was tried out on a purposeful sample of 253 Years 3 to 6 students from six government schools in the Geelong region, Victoria, Australia. There were 42 Year 3 students, 71 Year 4 students, 68 Year 5 students and 72 Year 6 students. A sample of this size provided a substantive amount of data on students' responses to the test and allowed important generalisations to be made about students' performance (Rogers, 2014). Students were from a mixed of high, medium, and low social-demographic backgrounds which are a representation of the key characteristics of the population. This adds weight to the generalisability of the results (Rogers, 2014).

Between November 2022 and March 2023, teachers from research schools administered the assessment based on instructions provided by the researcher. Students completed the assessment within 45 minutes during class time. Their responses were collected, coded, and scored by the researcher to ensure the process was as systematic and objective as possible.

Table 1

Multiplication and Division Word Problems

Multiplicative thinking diagnostic assessment items	
1a. The value of five 20c coins is same as one $1 coin. How many 20c coins are the same as $4?	
1b. If you have 45 20c coins, how many $1 coins can you make?	
2a. Michelle bakes 40 biscuits. She puts them in rows of 8 biscuits on a baking tray. How many rows of biscuits does Michelle bake?	
2b. Michelle puts party pies on a baking tray like in the picture and fills the tray. How many party pies does Michelle bake?	
2c. Michelle bakes 18 pies. She also bakes 4 times as many sausage rolls as pies. How many sausage rolls does Michelle bake?	
2d. Michelle sold 15 pies on Friday and 60 pies on Saturday. How many times as many pies were sold on Saturday?	
2e. Michelle needs 25 boxes to pack 200 pies. How many pies are in each box?	
3a To work out the total number of cupcakes, Beth divided the cupcakes into 3 sections like in the picture. How did Beth work out the total number of cupcakes?	
3b. Beth baked 12 rows of cookies with 15 cookies in each row. To work out the total number of cookies, Sam did $12 \times 15 = 10 \times 17 = 170$. Tom did $12 \times 15 = 10 \times 10 + 2 \times 5 = 110$. Emily did $12 \times 15 = 6 \times 30 = 180$. Who do you think is correct? Why?	
4a. Sam has 4 jumpers and 3 shorts. If Sam chose a blue jumper, what might be Sam's choice of outfits?	
4b. Sam has 4 jumpers and 3 shorts. How many different outfits are there in total?	
4c. Sam has 5 jumpers and 30 outfits. How many shorts does Sam have?	

Multiplicative thinking diagnostic assessment items
4d. Sam's Dad has 18 jumpers and 13 shorts so he has 234 different outfits. Sam's younger brother has 13 jumpers and 18 shorts. How many different outfits does Sam's younger brother have?
4e. Sam's older brother has 13 jumpers and 19 shorts. How many different outfits does Sam's older brother have?

An earlier study Bao (2023) provided a detailed rationale for each of the above items showing how each is supported by relevant research literature. Following established procedures for using Rasch analysis (Bhatti, et al., 2023; Callingham & Watson, 2005; Siemon et al., 2021), each item has two scoring rubrics: one was designed to score dichotomous data where the item was scored 0 or 1(correct or incorrect) and another was to score polytomous data where the same item was scored 0, 1, 2 etc. for different levels and types of thinking. The numerical scoring code assigned to each response ranged from 0–1 to 0–4. The number of codes for each item depended on the complexity of responses to particular items which was determined by qualitative analysis of students' responses taken from previous studies, as well as Rasch analysis in the pilot study in Phase 1. Scoring rubrics designed to fit Rasch Partial Credit Model did not need to have the same number of categories (Callingham & Watson, 2005). Scoring rubrics aim not only to differentiate cognitive skills between answering the item correctly with a correct reasoning and giving a correct answer without explanation but are also able to acknowledge students' partially correct thinking (Arieli-Attali & Liu, 2016). Scoring rubrics for Task 2d are shown in Figure 1 below.

Figure 1

Task 2d and Scoring Rubrics in Michelle's Bakery

d. Michelle sold 15 party pies on Friday and 60 party pies on Saturday.

How many times as many party pies were sold on Saturday? [2d1]

Scoring rubrics for [2d1]

Scores	Description
0	No response or irrelevant response or incorrect answers
1	Correct answer (4)

You can write or draw to show how you worked out your answer.

Scoring rubrics for [2d2]

Scores	Description
0	No response or irrelevant response or no indication of strategies, e.g. an incorrect response resulting from adding or subtracting two given numbers.
1	A response reflects an error of overreliance of additive thinking, e.g., 4 -1 = 3
2	A response based on using additive strategies such as skip counting or repeated addition/subtraction.
3	A response based on a multiplicative approach.

Analysis

The data was calibrated using the Winsteps 5.3.3.1 (Linacre, 2022) to fit the Rasch Partial Credit Model (Masters, 1982) based on the scoring rubrics for each item. This allowed both students' performance and item responses' difficulties to be measured and placed on the same scale. The Winsteps program evaluates the fit of the data to the Rasch model, showing the mean Infit MNSQ for each item value is between 0.73 and 1.26 which are within the default acceptable value (between 0.7 and 1.3) and indicates that the data fit to the model. The values of the Separation Reliability for both items and persons were high, indicating consistent behaviours of both items and persons.

Figure 2

The Variable Map Based on Rasch-Thurstonian Thresholds

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The variable map based on Rasch-Thurstonian thresholds is shown in Figure 2. It produced an overall list of item thresholds which differentiated items based on students' responses. In Rasch analysis, students who appear on the scale at about the same logit value as an item response have a 50% chance of exhibiting that level of thinking. For an item response above the student's position on the scale there is less than 50% chance of demonstrating that level of thinking for that item and for item response below the student's position the chance is greater than 50%. The '#s' represent the distribution of the students according to their ability estimate. Each person is shown by '#' and the position on the scale indicates the ability measure. The right-hand side of the figure shows the distribution of the test items with responses on the same scale in reference to their difficulty estimates. Easier and more accessible items had relatively low item thresholds. For example, the item threshold (1a2.1) associated with a score 1 for part 2 of the item 1a of the *Australian Coin* Task (possible scores 0, 1, 2 or 3) was -3.16 where students used drawing and *one-to-one correspondence*. The item threshold (3b2.4) associated with a score of 4 on part 2 of the item 3b of the *Beth's Cupcakes* task (possible scores 0, 1, 2, 3 or 4) was 4.4 where students needed to demonstrate understanding of associative property.

The Rasch model used has the capacity to order assessment items from the least difficult to the most difficult with logit scores along the scale. According to Griffin (2020), locating substantive change in item difficulty based on logit scores can determine the transition points or "natural breaks" where a change in cognitive skills occurs. Griffin (2020) described this method as criterion-referenced interpretation. This is an empirically grounded method in many different mathematical research topics (e.g., Griffin, 2020; Rogers, 2014; Siemon et al., 2021). Using this method, the test data were used to derive the developmental progression for multiplicative thinking. Since Rasch analysis connects differences in the logit scores of related items to increased cognitive demands imposed on students (Griffin, 2020), transition barriers can be identified where Rasch analysis shows a substantive difference in the logit scores of related items between stages along a developmental progression.

Firstly, each item response was ordered from the least to the most difficult according to the Rasch data. It was classified according to the underpinning cognitive skills that the item exhibits (Griffin, 2020). Next, the difference between the difficulties of adjacent item response was calculated to determine the location of substantive changes in item difficulty as "natural break" where a change in the cognitive skills appears to be required to correctly answer items (Griffin, 2020). Based on logit difference, substantive changes between adjacent item response were identified: 4d2.3 and 4a2.2 (0.45); 4c2.3 and 1b2.4 (0.15); 4d2.2 and 4c2.1 (0.07); 3b2.2 and 4a2.1 (0.2); 2c2.2 and 3a2.1 (0.28); 2c2.1 and 1a1.1 (0.57). Locating substantive changes allows item responses to be clustered together in Stages (Griffin, 2020) which is shown in Figure 2. The horizontal double lines on the map indicate the points where there appeared to be some significant change in the cognitive demands of the item responses and the single horizontal lines indicate a less pronounced change (in the judgement of the researcher). These points were confirmed after considering the content of the items, the skills required for correctly answering the items, and apparent discontinuities in the difficulty levels. Then, item responses within each stage need to be analysed qualitatively to determine if they exhibit a common substantive interpretation (Griffin, 2020). According to Rogers (2014) the clustered item responses at each stage should differ distinctly from those at other stages so that a hierarchy of students' understanding can be implied and allow exploration of transition barriers between the developmental stages of multiplicative thinking.

Results

Identifying where students lie on the scale and interpreting the nature of the item responses around the same location, common skills and similar demands from the items can be identified (Griffin, 2020; Siemon et al., 2021). A detailed content analysis of item responses led to the identification of eight substantive changes consisting of several sub-stages which sit within five relatively discrete Stages. Table 2 provides a succinct summary of item responses and their underpinning multiplicative knowledge and thinking skills required for each Stage.

In Stage 1 students generally relied on drawing and *one-to-one correspondence* to solve problems involving familiar situations such as equal groups and arrays. Stage 1 includes two sub-stages: 1a and 1b. From sub-stage 1a to 1b, there is a change in size of numbers in item response 3a2.1 and 2c2.1 (Figure 2) which involve single and two-digit factors. Students classified in Stage 1 had difficulty in recognising the equal grouping structure. In sub-stage1b, students showed overreliance of additive thinking while dealing with multiplicative comparison situation involving single-digit by two-digit multiplication where students had difficulty in understanding the notion of "times as many". According to Table 2, item responses with thresholds ranging from -1 to 0 were grouped together to form a discrete Stage 2 where students used skip counting or repeated addition to solve single-digit by two-digit multiplication problems involving equal groups, arrays and multiplicative comparison situations, but switched to *one-to-one correspondence* for division problems with single and two-digit factors involving equal group situation. In Stage 2, even though students recognised groups of equal size and the number of groups, they still failed to recognise the abstract multiplicative relationship, *many-to-one correspondence*. Students still showed overreliance of additive thinking in item 2d2 (Figure 1) for multiplicative comparison situation involving division operation. Many responses in Stage 2 showed inappropriate generalisation of additive thinking (Squire et al., 2004; Larsson, 2016). For instance, in item response 3b2.1, students wrote $12 \times 15 = 10 \times 17$ by moving 2 to 15 or $12 \times 15 = 10 \times 10 + 2 \times 5$ by splitting 12 into 10 and 2, and splitting 15 into 10 and 5. The barrier for these students is that they apply place value partitioning incorrectly for multiplication problems. The Rasch analysis confirms that inappropriately using partitioning is a serious barrier to the development of multiplicative thinking.

Table 2

The Developmental Stages for Multiplicative Thinking

Stages	Thinking skills and knowledge
1a	Using drawing and one-to-one correspondence to solve single-digit by single-digit multiplication problems involving arrays and equal groups situations
1b	Using drawing and one-to-one correspondence to solve multiplication and division problems with single and two-digit factors involving arrays and equal groups but showing an error of overreliance on additive thinking while attempting multiplicative comparison situations problem involving multiplication operation
2	Using skip counting or repeated addition to solve single-digit by two-digit multiplication problems involving equal groups, arrays and multiplicative comparison situations, switching to one-to-one correspondence for division problem with single and two-digit factors involving equal groups situation; showing an error of overreliance of additive thinking for division problem with single and two-digit factors involving multiplicative comparison situation; showing an error of inappropriate generalisation of additive thinking for two-digit by two-digit multiplication problems involving arrays situation
3a	Using multiplicative strategies such as spitting or doubling, multiplication facts to solve single-digit by two-digit multiplication problems involving equal groups, arrays and multiplicative comparison situations; switching to one-to-one correspondence for single-digit by single-digit multiplication problem involving Cartesian product situation with a visual diagram
3b	Using multiplicative strategies to solve multiplication and division problems with single and two-digit factors involving equal groups, arrays and multiplicative comparison situations; switching to skip counting or repeated addition for two-digit by two-digit multiplication involving equal groups situation and one-to-one correspondence for multiplication and division problems with 2 single-digit factors involving Cartesian product situation; showing an error of inappropriate generalisation of additive thinking for two-digit by two-digit multiplication problems involving Cartesian product situation
4a	Using procedural based multiplicative approaches to solve multiplication and division problems with 2 two-digit factors involving equal groups and arrays situations; switching to additive strategies for Cartesian production situation. Showing a procedural understanding of commutativity
4b	Using procedural based multiplicative approaches to solve multiplication and division problems involving various multiplicative situations
5	Applying properties of multiplication such as commutativity, distributivity and associativity to solve multiplicative problems involving various multiplicative situations

According to Table 2, students experienced more difficulty in understanding multiplicative relationships involving division operations than multiplication operations; and identifying the equal grouping structure in division items appears to be a barrier for students. Hurst and Linsell (2020) also draw attention the fact that many students in the middle primary years fail to notice the inverse relationship between multiplication and division. In Stage 3 students also experienced more difficulty in understanding multiplicative comparison situation than equal groups and arrays as students had to switch multiplicative strategies to repeated addition or skip counting. The analysis shows that Cartesian product situations present a clear barrier for many students who could not access simple Cartesian product situation items until Stage 3a with item responses 4a1.1 and 4a2.1 where they relied on drawing and *one-to-one correspondence* strategy. Another key transition barrier between Stage 3 and 4 relates to students' ability to deal with two-digit by two-digit multiplication problems. This is evident in item response 4e2.1 when students wrote $13 \times 19 = 234 + 1 = 235$ because in item 4d $13 \times 18 = 234$ and

19 is 1 more than 18. In Stage 4, even though students were fluent with procedural methods to solve multiplication and division problems, many failed to understand the relationship between 13×18 and 13×19 which requires adding another group of 13. Instead, these students used vertical multiplication to solve 13×19 as a new problem. Similarly in item 3b2, further barrier was evident when students did not see the relationship between12×15 and 6×30, instead relying procedural methods to solve the problem. The Rasch analysis shows that Stage 5 requires an understanding of properties of multiplication such as distributivity and associativity.

Conclusion

Understanding areas where students encounter difficulties should inform ways and means to overcome these barriers (Hurst & Hurell, 2016). Siemon et al. (2021) argue that identifying students' transition barriers and building on their prior knowledge is the key to improve learning outcomes. Based on the Rasch analysis, the detailed item response analysis reported in this paper identified eight substantive changes, comprising five developmental Stages of multiplicative thinking which provides a more detailed and richer description of transition points between Stages along the developmental progression of multiplicative thinking.

The Rasch analysis also shows the importance of recognising the equal grouping structure and understanding of the concept of *many-to-one correspondence* at the early Stages of the development. Also important are size of numbers and multiplicative contexts such as two-digit by two-digit multiplication problems and Cartesian product situation during students' development for multiplicative thinking. It is possible that some students had limited encounters with Cartesian product situations which involve repeated equal sets, unlike equal groups and arrays. This may create a barrier for these students. The Rasch analysis clearly confirms students' lack of knowledge in the properties of multiplication. Understanding of the distributive and associative properties of multiplication clearly remains a barrier for middle and upper primary students. This study shows that reliance on procedural based methods may provide correct answers but limit students' ability to see underlying multiplicative relationships, including properties of multiplication between the pairs of quantities involved in the operation. It is important to assist students to see the structural properties of multiplication and to be able to identify and explain correct and incorrect solutions to multiplicative situations. Equally important is that for students being able to use the known results to arrive at a new result without having to do a full calculation, which remains a challenge for these students. The *Multiplicative Thinking Diagnostic Assessment* test was designed to explore these barriers and transition points. The Rasch analysis has confirmed the presence of these barriers and draws attention of teachers and educators to key transition points in the development of multiplicative thinking.

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