# UNDERSTANDING ONE CALCULUS INSTUCTOR'S CLASS PRACTICES USING A POSSIBLE STUDENTS' COGNITIVE MODEL

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*In this study, we examined one experienced mathematician's class practices, with particular attention to cognitive model described in genetic decomposition. Our findings indicate that students only had limited opportunities to be familiar with the first three steps in genetic decomposition, which may potentially lead students to answer limit tasks correctly, but not necessarily having a deep conceptual understanding behind those tasks. With limited opportunities, it will be challenging for students to overcome well-known learning challenges in limit.*

Keywords: cognitive model, genetic decomposition, limit concepts, mathematician's class practices

## Introduction

Among several topics in calculus, the concept of limit is considered fundamental to the study of advanced mathematics, including derivatives, integrals, and series (Bezuidenhout, 2001; Cornu, 1991). In recent years, attempts have been made to understand how students come to understand the concepts of limit by exploring possible cognitive model (Larsen et al., 2016). Using genetic decomposition, a construct of APOS theory (Action, Process, Object and Scheme), some students were able to overcome challenges of understanding infinite process of finding limit (Swinyard, 2011; Swinyard & Larsen, 2012). This suggests that genetic decomposition can be a useful tool for calculus instructors in designing limit tasks and lessons to support students (Arnon et al., 2014a; Trigueros, 2022). In this study, we examined one mathematician's teaching practices in presenting the concept of limit as a springboard to further discussions about what and how we teach foundational topics in calculus as what students experience in their classes is an important influencing factor that allow students to complete the calculus sequence successfully (Hagman, Johnson, & Fosdick, 2017). The following research questions guided our present study:

- How does an experienced mathematician introduce the concept of limit?
- How are students' learning challenges and cognitive model treated in limit lessons?

## Relevant literature

### Students' understanding of the limit concept

Students often think the limit of a function cannot be equal to a number, the limit is unreachable (Güçler, 2013; Szydlik, 2000), only continuous functions have limits (Bezuidenhout, 2001), limit is an approximation so it will not be equal to a number (Nagle, Tracy, Adams, & Scutella, 2017; Roh, 2008), and the limit is the same as the value of the function at the point (Nagle et al., 2017). To overcome these difficulties, students need to be able to coordinate how two varying quantities, x and y, behave (Jones, 2015; Thompson & Carlson, 2017). However, the mathematical notation, limit $\lim_{x \to c} f(x) = L$, itself seems to emphasize how $x$ approaches a number, domain process, without clearly describing how $y$ behaves, range process.

Studies show that students are very familiar with plugging in x values to find the limit, but unable to or do not describe how y values change (Cottrill et al., 1996; Keene et al., 2014).

**Developing a cognitive model for students' understanding**

Recent studies have moved towards to understanding and describing how students come to understand limit (Cottrill et al., 1996). One such attempt used APOS theory that describes models of what might be going on in the mind of a student when he or she is trying to learn a mathematical concept and uses these models to design instructional materials (Arnon et al., 2014b). In the process of understanding students' constructions of mathematical concepts, genetic decomposition plays an important part in APOS theory as genetic decomposition describes the mental structures and mechanisms that a student might need to construct in order to learn a specific mathematical concept (Arnon et al., 2014a). Although genetic decomposition is not unique, when it is empirically tested, it can serve as a guide for instruction of particular mathematical concept (Trigueros, 2022).

**Genetic decomposition of limit**

Genetic decomposition for the limit describes a sequence of mental constructions students may make in the process of coming to understand the limit concept informally and formally (Figure 1).

1. The action of evaluating $f$ at a single point $x$ that is considered to be close to, or even equal to, $a$.
2. The action of evaluating the function $f$ at a few points, each successive point closer to $a$ than was the previous point.
3. Construction of a coordinated schema as follows.
a. Interiorization of the action of Step 2 to construct a domain process in which $x$ approaches $a$.
b. Construction of a range process in which $y$ approaches $L$.
c. Coordination of (a), (b) via $f$. That is, the function $f$ is applied to the process of $x$ approaching $a$ to obtain the process of $f(x)$ approaching $L$.
4. Perform actions on the limit concept by talking about, for example, limits of combinations of functions. In this way, the schema of Step 3 is encapsulated to become an object.
5. Reconstruct the processes of Step 3(c) in terms of intervals and inequalities. This is done by introducing numerical estimates of the closeness of approach, in symbols, $0 < |x - a| < \delta$ and $|f(x) - L| < \varepsilon$.
6. Apply a quantification schema to connect the reconstructed process of the previous step to obtain the formal definition of limit.
7. A completed $\varepsilon - \delta$ conception applied to a specific situation

**Figure. 1: Refined genetic decomposition of limit (Cottrill et al. 1996, p. 177–178).**

Figure 1 illustrates genetic decomposition used by Cottrill et al. (1996). Genetic decomposition can be revised when needed. For example, step 1 in Figure 2 was added to the previous preliminary decomposition because some students evaluated function at a single point (Arnon et al., 2014a; Cottrill et al., 1996). Also, the coordinated process (step 3) replaced the single process as well because students tended to construct a separate Process for $x$ approaching a number apart from application of the function $f$ (Arnon et al., 2014c; Cottrill et al., 1996). Since process of finding limit is not restricted to a finite computation, students who are only able to evaluate functions with a finite number of values are at the Action stage. At this stage, they are only able to evaluate a function at a few finite points without considering how *y* values behave, so it is possible for them to think that the limit and the value of the function at a point are equal to each other (Cottrill et al., 1996; Nagle et al., 2017). In order to move to the Process stage (interiorizing the action is needed), students need to be able to know what will happen to the function when evaluating it at infinitely many points as they construct both domain and range processes (Cottrill et al., 1996; Dubinsky, 1991). However, interiorizing this process to think about the infinite process of domain and range is difficult for students (Swinyard, 2011; Swinyard & Larsen, 2012). Often, *close as we want, close enough,* or *sufficiently close to* are used describe the infinite process, but they are vague and it is often very challenging for students to conceptualize the meaning of “close enough” (Keene et al., 2014). When they are able to coordinate and interiorize both processes, students are able to think that the limit can be reached, but it may not necessarily be equal to the value of the function. With empirical evidence for the first four steps (Cottrill et al., 1996), those four steps can be considered as a useful model of students’ cognition that we may see in most students’ work (Arnon et al., 2014a). Thus, it may be beneficial for calculus instructors to use the first three steps in genetic decomposition to design limit lessons to provide learning opportunities for students to possibly overcome learning challenges, as others have suggested (Cottrill et al., 1996; Nagle et al., 2017).

## Methods

### Data sources

A Midwestern research university in the United States was the setting for this study. The calculus instructor, Dr A (a pseudonym), has taught Calculus1 several times and has a PhD in mathematics with a specialization in topology. Each of Dr A’s classes was 50 minutes long. The source of data came from class video and audio recordings. All calculus classes taught by Dr A were videotaped. For this study, we examined four video and audio recordings during the initial introduction of limit concept since Dr. A spent four lessons to cover limit topics In all, we examined discussions of 32 implemented tasks. In addition to videotaped lessons, we interviewed Dr. A to see if she had any particular reasons to make her instructional decisions.

### Analysis of Implemented Tasks

For our analysis of Dr. A’s lessons, we used the first three steps in Cottrill et al’s genetic decompostion.

**Table 1: Analytic Framework**

| Genetic Decomposition | |
|---|---|
| Cognitive model | Step 1: evaluating function at just one point.<br>Step 2: evaluating function at several points<br>Step 3: coordinating and interiorizing both domain and range processes |

Table 1 describes our analytic framework. For each of these areas, we examined Dr. A’s lessons and task implementations to see if her students had opportunities to experience and discuss first three steps of cognitive model. We carefully examined each task discussion to determine if students had opportunities to use a cognitive model described in genetic decomposition. If students are just familiar with step 1 of genetic decomposition, it is possible for students to think that the limit and value of a function are equal to each other (Cottrill et al., 1996). Even when students are able to be at step 2, without being able to coordinate both domain and range processes, they could still think that the limit and value of a function are equal to each other (Cottrill et al., 1996). When students are at step 3, being able to interiorize both domain and range processes, they might be able to see the infinite process of finding the limit and conceptualize "close enough" because you can't plug in numbers infinitely many times, calculations need to be contemplated (Cottrill et al., 1996).

**Examples of analysis**

The unit of analysis is discussion of one task. For each unit of analysis, the analytic framework was employed.

Dr. A: Very good. So what you do is your look when x goes very very close to 1 what does it happen to y and I've attempted to indicate the y approaches 2. Now I am going to change the graph, what is the limit of f of x as x approaches 2?
S: 4?
Dr. A: I have four answer,
Dr. A: I have three answer,
Dr. A: I have does not exist answer … Alright the key is to remember that you do not care what happens at 1 you just care what happen near 1. What happens near 1? what does f of, I mean, near 2. What does f of x approach x is getting close to do?
Dr. A: Three. The value of f of three of f of 2 is four that actually completely does not matter as far as limit is concerned. So what happen happens at the number is irrelevant. All you care about is what happen when you get close to the number. You can also get asked about limits at the infinity. For example, what is the limit of f of x as x goes to the infinity. So you should thinking about x gets very very large. What does x, what does f of x approaches when x gets very very large?
S: Zero
Dr. A: Excellent.

**Figure 2: A sample discussion of a task**

Figure 2 shows implementation of a task. Dr. A drew a graph to demonstrate how to find the limit when a graph is given. We observed several different findings here. First, she described both domain and range processes, "f(x) approaches L as x approaches a" to describe limit. She used the term "very, very close" (or "very, very large") to describe closeness. She also mentioned that the limit and the value of the function are not necessarily equal to each other. Dr. A provided learning opportunities to address important issues in learning the concept of limit. However, we reviewed the discussion more in depth to see more precise learning opportunities that Dr. A's students have. In the discussion, some students mentioned limit is 3, 4 and does not exist. Then, here is what she said to describe the limit of this function.

The value of f of three of f of 2 is four that actually completely does not matter as far as limit is concerned. So what happens at the number is irrelevant. All you care about is what happen when you get close to the number.

She mentioned that "what happens at the number is irrelevant," and "all you care about is what happen when you get close to the number" to describe the limit is not always equal to the value of the function at that point, but describing what it precisely means to be close enough and how x and y behave near 2 using table of values would be more helpful. Students can be asked to confirm that they can always do better as they coordinate domain and range processes to describe the process if limit exists. Such discussion gives them opportunities to think about the difference between what happens to the function at 2 and what happens when x is getting closer and closer to 2; ultimately, why limit exists while function value is not 3. In previous studies, using the zoom feature helped students interiorize and conceptualize the processes (Swinyard, 2011; Swinyard & Larsen, 2012). We are not suggesting that the formal $\varepsilon$ and $\delta$ definitions need to be mentioned here, but with how this task was presented and discussed, they may not be able to conceptualize closeness (it is not clear how close you have to be to have limit), coordinate both domain and range processes, or understand why the function value does not matter when finding limit. Lastly, students were able to find the limit at 2 with a graph. However, this can leave students at Step 1 of genetic decomposition, evaluating functions at one point.

**Coding Reliability**

We reviewed previous studies carefully to develop our codes (Hong, 2023; Nagle et al., 2017; Swinyard & Larsen, 2012). Once they were developed, two independent readers watched and read the lessons and transcripts several times and compared findings to ensure reliability of results. The initial agreement rate for task discussions was 93.5% (30 out of 32). Discussion was done until there was 100% agreement.

## Results

In this section, we describe our findings from our analysis of Dr. A's four limit lesson tasks. We describe learning opportunities that students have to understand and use research – based students' thinking. Table 2 shows the results of our analysis. One clear finding is many tasks were discussed at Step 1 of genetic decomposition. We will discuss these in more detail.

**Table 2: Percent distribution of class discussions.**

| Cognitive model | |
|---|---|
| Genetic Decomposition: Step 1 | Genetic Decomposition: Step 2 |
| 26 (81%) | 3 (9.4%) |

**Genetic decomposition**

Dr. A: So here's an example. Alright, [pauses] I'm going to go through this one example using the rules. So you have a bunch of functions added and subtracted from each other. So the first rule told you that the limit of two functions added is the sum of the limits. So this splits up into limit of $2x^2$ minus limit of $3x$ plus limit of 4, this is all x goes to 5 . So this is by the first two rules that told you that you can subtract and add the limits. Alright, what rule do we apply to this kind then? You can think of it as multiplying or you can think of it as a constant times the function. And you know that constants can come out – this was like, rule 3 or so… And we can apply the same rule to the next kind?

Dr. A: So here's an example. Alright, [pauses] I'm going to go through this one example using the rules. So you have a bunch of functions added and subtracted from each other. So the first

rule told you that the limit of two functions added is the sum of the limits. So this splits up into limit of $\boldsymbol{2x^2}$ minus limit of $\boldsymbol{3x}$ plus limit of 4, this is all x goes to $\mathbf{5}$. So this is by the first two rules that told you that you can subtract and add the limits. Alright, what rule do we apply to this kind then? You can think of it as multiplying or you can think of it as a constant times the function. And you know that constants can come out – this was like, rule 3 or so… And we can apply the same rule to the next kind?

Dr. A: Alright, and now I think we're done because we have the rule that says that the limit is $\boldsymbol{x = a}$ of $\boldsymbol{x^2}$ is $\boldsymbol{a^2}$. We have it for general n so the limit of x is $\boldsymbol{x^2 = 5}$ is what?
S: 25
Dr. A: And what is the limit of x is x=5?
S: 5
Dr. A: And what is the limit of the constant? The constant itself.

**Figure 3: Dr. A's discussion of a task**

Figure 3 describes discussion of the task $\lim_{x\to 5} 2x^2 - 3x + 4$. Dr. A said the following, "Alright, and now I think we're done because we have the rule that says that the limit is $x = a$ of $x^2$ is $a^2$. We have it for general n so the limit of x is $x^2 = 5$ is what?" This task was discussed right after discussing several limit laws. This discussion is mostly about Step 1, evaluating a function at one point, of genetic decomposition. There was no consideration of how x and y behave, what it means to having x closer and closer to 5 or interiorizing the process. Among the 32 tasks that we examined, 26 task discussions were very similar to this. Students are mostly becoming familiar with plugging in numbers to evaluate functions, Step 1 of genetic decomposition. We do acknowledge that once students see some limit tasks and limit laws, it is realistic to think that calculus instructors mostly discuss plugging in a number, and other procedures such as factoring and dividing by a term; however, at this point in the lesson, students have a very limited number of learning opportunities to think about coordinating domain and range processes and conceptualizing closeness.

**Coordinating domain and range processes**

The following shows the typical way Dr. A introduces and describes how x and y behave.

Dr. A: Okay, so today, we'll start calculating limits. We'll actually do this algebraically without, um, without needing to graph things.

Dr. A: So if I do want the limit $\boldsymbol{x = 1}$ of $\boldsymbol{x^2 + 1}$, the way we did this – like, two seconds ago – was we graphed the function which was a continuous function which means that you can draw it the graph without picking up your pencil – for now – and, um, we saw that the x=1 the y value was 2 and because this is all nice and there's no holes or breaks in the graph, this told us that limit is actually 2 because when x gets close to 1, y got close to 2. Yes?

Dr. A: So when things are nice, you can actually just plug in a number into the equation and that will tell you what the limit is. And that's basically what we're going to do today.

**Figure 4: Dr. A's discussion of another task**

Figure 4 describes Dr. A's class discussion of $\lim_{x\to 1} x^2 + 1$. Dr. A mentioned how x and y behave, "x gets close to 1, y got close to 2" (two other tasks are also discussed this way). However, discussion of this task did not describe both processes in more detail. Instead, she mentioned, “So when things are nice, you can actually just plug in a number into the equation and that will tell you what the limit is.” This may give students the idea that limit is simply plugging in numbers into x and the limit is the same as the value of the function at that point. Coordinating domain and range processes was mentioned briefly, and the main focus was more about plugging in numbers rather than describing how to coordinate x and y values – step 1 of genetic decomposition.

**Conceptualizing closeness**

Defining “close” in an iteratively restrictive manner is one way to help students operationalize the notion of infinite closeness. This leads to the idea that every possible requirement for closeness must be met for the limit to exist.

Dr. A: Let’s see how far we get. F of x, as x goes to negative 1 from the negatives. Is this big enough can you guys see this or show they are omitted?
Dr. A: alright. x is going to -1 from the negatives
S: 2
Dr. A: -1 from the negatives. I love it. -1 from the positives.
S: 2
Dr. A: Fantastic. Grey person. -1.
Dr. A: Yes.
Dr. A: -1 would be 2 I guess?
Dr. A: Sounds great.
Dr. A: the limit for -4 does not exist?
Dr. A: Very nice. The limit to -4 does not exist because from the left it’s -2 from the right it’s -3.
Dr. A: does not exist.
T: What I said a second ago was incorrect, -4 from one side is 3 from one side is 4 and four also does not exist and that one from one side is -3 and from the other side

**Figure 5: Dr. A’s discussion of several limit tasks**

Figure 5 shows the discussion of several tasks that students need to find the limit from a graph. Dr. A mentioned “going to -1 from negatives” or similar terms. There were 23 tasks that were discussed this way, where students needed to find the limit using graphs. Although they correctly found the limit using graphs, without more clearly describing infinite process, as the value of the function can be easily found from the graph, it is possible for students to think that limit is the same as the value of the function or limit is about finding the value at one point.

**Interview with Dr. A**

With our findings, we were very interested in knowing what influenced her instructional decisions. We asked Dr. A about what shapes her instructional practices. Here is what she said about what her students usually want in calculus class.

> They do not want concepts. They want to follow procedures. They often want to know what is going to be on the exam.

This is very similar to what other researchers found out about how students forced instructors to alter their instructional decisions (Hemmi, 2010; Kontorovich, 2016). We also asked her whether she is familiar with APOS theory and cognitive model proposed in genetic decomposition. She said although she is not familiar with APOS theory, she thinks that using students’ thinking might be beneficial in teaching calculus.

> I definitely think that building understanding using their own thinking might be only valuable way to teach calculus. If I do not have pressure to how much material that I have to cover, I will definitely figure out what students understand and start from that point to entirely build everything on understanding.

Her comments show that she has similar issues as other studies pointed out and her class practices might be dictated by other factors that force her to alter her pedagogies.

## Summary and Discussion

Our study examined how one mathematician presents the concepts of limit. Our results are very similar to previous studies that describe what mathematicians do in calculus classes (Güçler, 2013; Park, 2015; Wagner, 2018). Our results show her students focused on finding the value of the function at one point (step 1 of genetic decomposition – action stage) or performing procedures. We do not claim that finding limit by plugging in numbers or procedures of factoring and dividing by a term are not necessary; however, they were discussed with limited discussions of a possible research-based findings of how students may come and understand limit concept. Students only accustomed to plugging in values to find the limit will find it difficult to coordinate both domain and range processes, conceptualize infinite process of closeness, and overcome the challenge that limit and the value of the function may not be equal to each other. Although calculus students may successfully answer many limit tasks on assessments, they most likely lack conceptual understanding. What we observed from Dr. A's four lessons was limited opportunities for students to be familiar with three steps in genetic decomposition. It would be useful for students to conceptualize and interiorize these concepts with tables of values to more precisely see what happens to each function if x values get closer and closer to a number. In previous studies, using the zoom feature was helpful for students to interiorize and conceptualize the processes (Swinyard, 2011; Swinyard & Larsen, 2012). Although using $\varepsilon$ and $\delta$ may not be necessary, it would be useful to have tasks or activities that allow students to see those processes and have opportunities to use three steps in genetic decomposition to possibly avoid students' learning challenges in limit.

We also need to consider what shapes calculus instructors' class practices. The natural questions are "Is that what happens in most calculus classes?" and "Are there any pedagogical reasons that shape Dr. A's class practices?" Other studies often indicated that college mathematics instructors often altered their class practices because of their students' needs or limited content knowledge (Hong & Choi, 2019; Hong & Lee, 2023). Moreover, other researchers also indicated that it is natural to think that when important conceptual ideas are not being assessed, calculus instructors and students focus on what will be assessed on exams (Tallman et al., 2021). For limit, when simply plugging in numbers, factoring or dividing provides correct answers to assessment items, they will be less likely to feel the need to consider other methods, which can greatly influence what and how calculus instructors plan and implement their lessons. Dr. A's students lack prerequisite content knowledge and they pay more attention to what will be on exams than the concepts behind limit, which all contributed to her class practices. As we mentioned, Dr. A and other mathematicians have other issues to consider when plan their lessons. Then, is it fair or realistic to expect mathematicians like Dr. A to implement teaching practices that build on findings from limit research? However, one thing that is worthwhile to mention is Dr. A also acknowledges that using students' own thinking will be beneficial in teaching calculus. Although this study only examined one mathematician in one country, Dr. A's teaching practices can be quite widespread in other calculus classes, as Wagner (2018) suggested. Finally, we acknowledge that the results of this study are just limited to practices of one mathematician and we examined one genetic decomposition. Although other mathematicians may have similar issues that shape their instructional practices, it will be interesting to examine other mathematicians' practices as well. As Dr. A mentioned, more mathematicians might feel the need of using students' thinking in developing their lessons.

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