“$\frac{\pi}{5}$... WASN’T ON THE UNIT CIRCLE ”: A PRESERVICE MATHEMATICS TEACHER’S MEANINGS FOR RADIAN

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Researchers have reported that preservice mathematics teachers’ (PMTs’) conceptions for radian angle measure are dominated by conceptions of degrees, memorization, and calculational strategies, and are always expressed in terms on π. In this report, I unpack the mathematical meanings of a PMT, Henry (pseudonym), as he engaged in three task-based interviews involving radian angle measure. In the first interview, Henry described radians procedurally, emphasizing memorized special angles written in terms of π. By the end of the final interview, Henry described radians conceptually, including generalizing special angles. I conclude by highlighting the importance of tasks designed to build and support conceptual understanding.

Keywords: measurement, preservice mathematics teacher knowledge, proportional reasoning.

While a coherent understanding of radian angle measure is essential in higher mathematics such as trigonometry, precalculus, and calculus (Thompson et al., 2007), researchers have reported that learners struggle to conceptualize radian angle measure and how it relates to trigonometry (Akkoc, 2008; Çekmez, 2020; Fi, 2003; Moore, 2013, 2014; Moore et al., 2016; Tallman & Frank, 2020; Thompson, 2013; Topcu et al., 2006). Radian angle measure can be described by building on the conventional approach to measuring angles. Specifically, angles are measured relative to a benchmark associated with a circle centered at the angle’s vertex. The angle’s subtended arc is measured relative to a specific, yet arbitrary, fractional amount of the circle’s circumference. In degrees, the subtended arc is measured relative to $1/360^{\text{th}}$ of the circumference, while in radians, the subtended arc is measured relative to $1/(2\pi)^{\text{th}}$ of the circumference. This convention makes angle measure a proportional relationship since the measure involves a fractional amount of a circumference. Moore’s (2013) teaching experiment provided precalculus students the opportunity to use ideas of proportionality to conceptualize angle measure with this arc approach. However, in-service teachers questioned the practicality of using ideas of proportionality and the arc approach to measure angles for precalculus and calculus understanding (Thompson et al., 2007). Additionally, while preservice mathematics teachers (PMTs) procedurally converted between degrees and radians, they struggled to describe the results of their conversions as involving a proportional relationship (Akkoc, 2008; Çekmez, 2020; Fi, 2003; Topcu et al., 2006). However, when asked to use different diagrams to describe radian angle measure, some PMTs attended to proportionality (Alyami, 2022b). This finding suggests that PMTs’ procedural conversion between units of angle measure might connect to conceptual understanding (Baroody et al., 2007; Maciejewski & Star, 2019; Nilsson, 2020; Nordlander, 2021; Star, 2005, 2007), a connection that is under-explored in the context of angle measure. In this report, I unpack one PMT’s *mathematical meanings* for radian angle measure, a construct I will elaborate on in the next section. The research aim is to describe one PMT’s meanings for radian angle measure through a series of three task-based interviews to unpack the connections between procedural and conceptual meanings. The research question guiding this report is “What mathematical meanings does a PMT demonstrate upon engaging with a series of tasks that involve radian angle measure?”

Theoretical Framing

Mathematical Meanings

I take a constructivist perspective and build on the work of Thompson et al. (2014), where mathematical meaning refers to assimilating to schemes associated with an understanding, including “the space of implications that the current understanding mobilizes” (Thompson et al., 2014, p. 13). This description of mathematical meanings builds on the duality between thinking and understanding, which are rooted in mental actions that are demonstrated through common cognitive characteristics given a mathematical situation (Harel, 2008). From Harel’s (2008) and Thompson et al.’s (2014) descriptions, I interpret mathematical meaning as the understanding that enables a learner to reason about a particular concept, including implications brought to bear from active reasoning about the mathematical concept. I also consider ways of thinking as the patterns a learner develops to reason about a particular concept, given a particular situation that evokes such reasoning. In this report, the concept is radian angle measure, and the active reasoning is expected to be brought forth by various mathematical tasks. For example, PMTs described radian angle measure with an emphasis on special angles written in terms of π (Akkoc, 2008; Fi, 2003; Topcu et al., 2006). Additionally, Akkoc (2008), Fi (2003), and Moore et al. (2016) reported that PMTs incorporated procedural calculations with radian angle measure using the unit circle (Figure 1), a circle diagram typically labeled with special angles in radians written as integer multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$ radian. Another version of the unit circle depicts special angles in radians along with the equivalent measure in degrees.

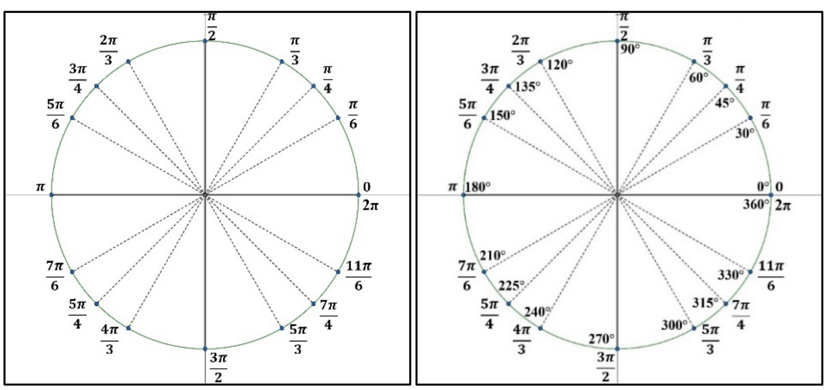


Figure 1. Typical Representations of the Unit Circle with Special Angles

The previous findings suggest that the PMTs’ previous experiences could have led to mathematical meanings and ways of thinking that incorporate procedures (e.g., conversion, calculations with the unit circle, memorizing special radian angles in terms of π, etc.) to reason about angle measure. Yet the National Council of Teachers of Mathematics (NCTM) calls for learners to move beyond procedural use of memorized facts, to flexibly applying mathematical problem-solving through a foundation of conceptual understanding (NCTM, 2014). Specifically, if applying procedures is expected to build on a foundation of conceptual understanding (NCTM, 2014), then attention to the amalgamation of procedural and conceptual meanings is needed.

Proportional Reasoning and Angle Measure

Considering the convention for measuring angles described earlier, attending to ideas of proportionality is appropriate for this report. Thompson (2011) described proportional reasoning as involving multiplicative comparisons between two quantities. Measuring angles involves a multiplicative comparison between the angle's subtended arc and the circumference of the circle containing the arc, making angle measure a proportional relationship. The arc approach to angle measure demonstrates learners' use of fractions and ratios to reason about angle measure (Moore, 2013). Angle measure can also be described using other proportional reasoning concepts. For example, ninth-graders described angle measure using partitioning of familiar angles to create a 1° angle (Hardison, 2020). The ninth-graders attended to the measure of a 1° angle relative to familiar angles (i.e., 90° as the right angle), demonstrating equipartitioning as a strategy for describing angle measure.

Based on the literature discussed above, I anticipate a PMT's thinking about radians would involve special angles written in terms of π (Akkoc, 2008), with attention to proportionality (Alyami, 2022b; Moore, 2013), and partitioning (Hardison, 2020).

Methods

Research Design

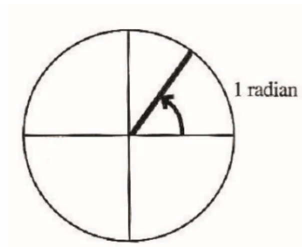
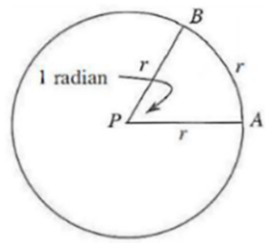
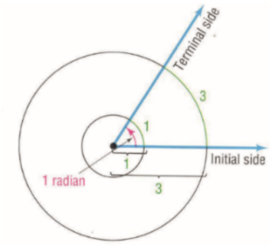
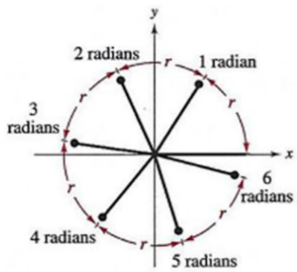
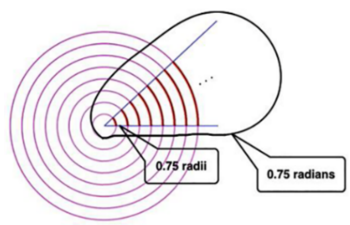
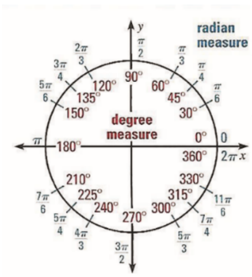
To examine the PMT's mathematical meanings about radian angle measure, I employed a qualitative case study design (Flyvbjerg, 2011) during participation in a series of task-based interviews involving radian angle measure (described below).

Participant and Tasks

The participant in this report was Henry (pseudonym), a PMT enrolled in a mathematics teacher preparation program at a large Midwestern university. Henry volunteered to participate in three separate task-based interviews and was compensated for his time. I did not offer any learning sessions about radians prior to this research; however, Henry has likely learned about radian angle measure during his K-16 schooling. In the following sections, I describe each task and the timeline for Henry's participation.

Task 1. The first task-based interview took place in spring of 2019. Henry was asked to describe radian angle measure through examining a series of radian diagrams (Alyami, 2022b). The diagrams were handed to Henry one at a time, in the order listed in Table 6.

Table 6: Radian Angle Measure Representations Used for Task 1

Diagram 1	Diagram 2	Diagram 3
 (Kysh et al., 2009, p. 47)	 (Kysh et al., 2009, p. 46)	 (Sullivan & Sullivan, 2009, p. 355)
Diagram 4	**Diagram 5**	**Diagram 6**
 (Larson et al., 2008, p. 259)	 (P. W. Thompson, personal communication, 23 February 2018)	 (Larson et al., 2010, p. 861)

Task 2. The second task-based interview took place in spring of 2021. Henry was asked to describe what it would mean for an angle to have the measure of 1 radian. Henry also described how he would determine the measure of angles in radians, given specific measures in degrees (i.e., 360°, 90°, 72°, 36°, 112°, x°).

Task 3. The third and final task-based interview took place in fall of 2021. Henry engaged with a digital activity that involves radian angle measure and light reflection (Alyami, 2022a). Henry was asked to input angles measured in radians to situate a laser and one or two mirrors so the laser beam would pass through three stationary targets. A benefit of the *Radian Lasers* task is that the angles needed to situate the mirror are not limited to common special angles (Figure 2).

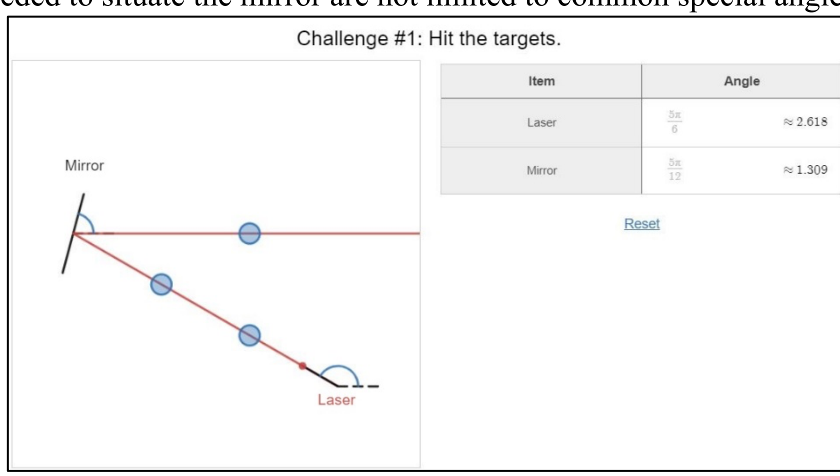


Figure 2: A Radian Lasers Challenge, With the Mirror Angle not a Common Special Angle

Data and Analysis

Each interview followed the guidelines of structured, task-based interviews (Clement, 2000; Goldin, 2000), lasted approximately one hour, and was audio- and video-recorded. Each interview was transcribed, and the transcripts comprise the data for this report. To unpack Henry's meanings for radian angle measure, I attend to the strategies he used when addressing each task. By *strategies*, I refer to utterances and observable actions (including writing and sketching) made to address the various tasks. By characterizing Henry's strategies, I describe a collection of reasoning actions that were brought to bear through engagement with the tasks, representing the space of implication that resulted from assimilating to a scheme (i.e., Henry's meanings for angle measure). I used thematic analysis (Saldaña, 2013) to categorize Henry's mathematical meanings with attention to conceptual and procedural strategies. To answer the research question, the analysis focused on unpacking Henry's strategies throughout the three interviews "so that more can be gleaned from the data than would be available from merely reading, viewing, or listening carefully to the data multiple times" (Simon, 2019, p. 112).

Findings

I organize this section in the chronological order of the interviews to illustrate the development in Henry's meanings for radian angle measure during the interviews. I then provide a summary of the findings in relation to the relevant literature in the discussion.

First Interview (Task 1)

While Diagram 1 does not depict radian angle measure in terms of π, Henry described radian angle measure that involves π. He stated that "the unit of 1 radian doesn't really mean a whole lot...since [radian] is always measured in π." Henry continued referring to π when describing radians from Diagrams 1 through 5 (none of which depicted radian angle measure in terms of π). This suggests that Henry has developed thought patterns to reason about radian angle measure that involve angles measured in terms of π.

While Henry emphasized radian angle measure in terms of π, he identified the arc approach to radian angle measure when he was given Diagrams 2 through 5. For example, when looking at Diagram 2, Henry indicated that "there is the relationship between the lengths of the sides [points to the radius] and the arc length." When looking at Diagram 3, he elaborated that "the three to one ratio of the radii is the same as the three to one ratio [of] the arc lengths, and for me I'm just looking at this as a special case of that property in which the arc length just happens to be the same as the radius," which highlights his attention to the special case of equivalence. He gave the same response when describing radians in Diagram 4 with "each of these arc lengths is the same length as the radius," and in Diagram 5 as "each arc length is 0.75 of, the length of each arc is 0.75 times the radius." The previous statements suggest Henry's awareness of the multiplicative relationship between the arc length and the radius when measuring angles in radians.

Despite recognizing the proportionality involved when measuring angles in radians, Henry still preferred to think of radian angle measure in terms of π and questioned the practicality of such approach. This is evident from Henry's description of radians when looking at Diagram 6:

> this is the most satisfying representation of radian, when you multiply it by π and you get easy fractions of a circle... you just have to immediately go to halfway around the circle is π … you kind of just have to immediately identify π with the fraction of the circle and not spend time thinking about it's actually the physical length of the arc.

The previous excerpt demonstrates Henry's preference for the unit circle representation, along with special angles written in terms of π. While Henry's reference to fractions of a circle could be interpreted as partitioning, he emphasized the symbolic multiplication of π with symbolic association to the circle. By the end of this interview, Henry's meanings for radian can be described as procedural, with awareness of conceptual meanings demonstrated by his attention to proportionality. Figure 3 represents Henry's emphasis on procedural meanings for radian angle measure without connecting to the conceptual meanings he demonstrated.

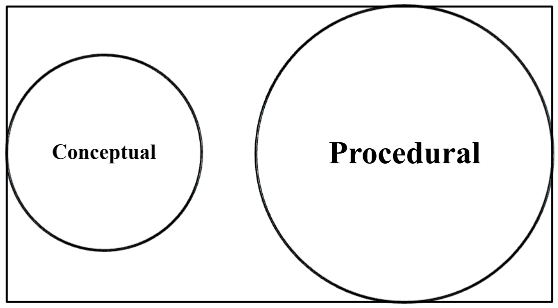


Figure 3. A Model of Henry's Meanings for Radian in the First Interview

Second Interview (Task 2)

When Henry was asked to describe what it would mean for an angle to have the measure of one radian, he stated "I remember this from the last study, but I'm going to take it a little bit to how I came to think of it after that study, how I came to think of it on my own." This suggests that Henry's description (below) resulted from reflecting after the interview for the first task:

> we learn the formula for circumference before we learn anything about angles... the circumference is 2π times the radius... this really didn't hit me until... we were asked... what's the length of this thing [points to arc in Figure 4]... it turns out that whatever that angle is, if you measure that angle in radians, then that's just the length of the arc. I mean, still times r... that was the part that kind of like, blew my mind... because... this circumference formula... [points to $C = 2\pi r$] is just a special case of the more general formula $S = \theta r$... this is what really made this significant for me, what happens if you just make this 1 [points to θ in $S = \theta r$]?... that arc length equals the length of the radius (Figure 4)... it's just one radian.

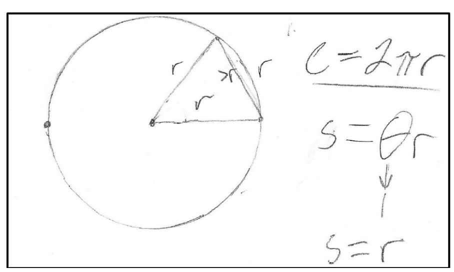


Figure 4. Henry Describing the Measure of 1 Radian Angle in Relation to the Arc Formula

While the previous excerpt contains references to formulas and calculations, Henry's description involves connections between the radian angle measure in relation to the arc length and circumference formulas. He recognized the circumference formula as a special case of the arc length formula, where the circumference is the arc length associated with a full rotation angle. Henry also uses the arc length formula to describe the measure of one radian as the situation when the arc length subtending a one radian angle equals the length of the radius.

While Henry was able to describe radian angle measure by describing the arc length in relation to the radius, he still referred to "easy fractions of π" when given angle measure in degrees and was asked to describe the measure in radians. For example, when describing 72° and 36° in radians, Henry's strategy involved using the measure relative to special angles that he memorized, such as $\frac{\pi}{6}$, $\frac{\pi}{3}$, and 2π (Figure 5). However, Henry was surprised by $\frac{\pi}{5}$ radians, which satisfied the "easy fractions of π" description, but he was not familiar with it. He wondered "why wasn't I able to look at 36 and immediately say, 'Oh, that's $\frac{\pi}{5}$?" A justification he provided right away was "because it wasn't on the unit circle."

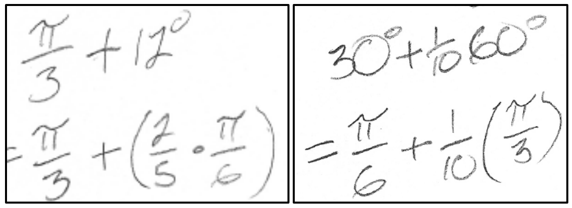


Figure 5. Henry's Description of 72° and 36° as $\frac{2}{5}\pi$ and $\frac{\pi}{5}$ Radian Relative to Special Angles

The previous examples demonstrate connections between Henry's meanings for radian. Henry's description of radian angle measure involved special angles written in terms of π along with awareness of proportionality. However, by the end of the second interview, Henry demonstrated various connections between ideas that involve radian angle measure. Figure 6 represents Henry's meanings for radian angle measure after the second interview as involving both procedural and conceptual meanings with a connection between the two. Henry recognized the circumference formula as a special case of the arc length formula, and also recognized the limitation of describing radian angle measure as "easy fractions of π" when he was only thinking about the few special angles typically depicted on the unit circle.

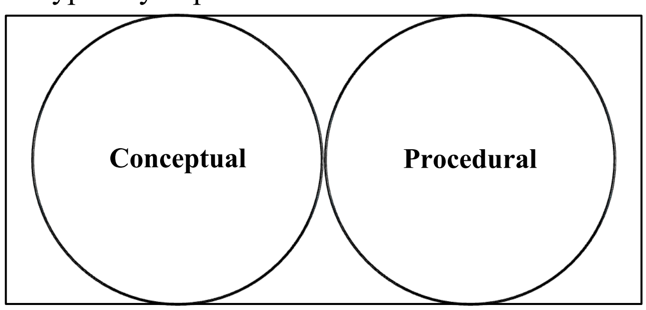


Figure 6. A Model of Henry's Meanings for Radian in the Second Interview

Third Interview (Task 3)

The *Radian Lasers* activity challenged Henry's dependence on the special angles typically depicted on the unit circle diagram. For example, when working on Challenge 1 (Figure 2), Henry noticed that positioning the mirror at "$\frac{\pi}{3}$ gives me too small of an angle," while positioning the mirror at "$\frac{\pi}{2}$ gives me too big of an angle, but each of those are off by the same amount, just in opposite directions. Okay, so I must just need the value in between those two." After seeing that "the answer was $\frac{5\pi}{12}$," Henry explicitly stated,

> once I saw that this is a problem about radians, I had already decided that the answer had to be on the unit circle, and so I looked for every one of those [special angles] before I even thought about, well, maybe I actually need to think about this problem, instead of just looking for the most applicable unit circle value.

Henry's statements illustrate both dependence on unit circle values and his acknowledgment of the limitations of such an approach. However, since the angles needed were not limited to common special angles, Henry reflected on the limitation of depending on unit circle values:

> The knowledge that we're trying to give to students when we introduce the unit circle … exercises like this will still get you to that same place … but they won't lock you into like, the diagram. Like everyone knows the diagram, it's got all the multiples of $\frac{\pi}{4}$ and all the multiples of $\frac{\pi}{6}$. That's the diagram … but this develops a better intuition of actually knowing that … $\frac{\pi}{n}$ radians is one nth of the way around the semicircle.

Henry's statement suggests that even if students are introduced to the unit circle diagram, it can be done in a way that promotes conceptual meanings for radian, where students consider radian angle measure beyond the few special angles depicted on a typical diagram of the unit circle. While Henry initially used the special angles from the unit circle, he was able to flexibly think about radian angle measure beyond those special angles using conceptual meanings. Figure 7 represents Henry's meanings for radian angle measure after the third interview as involving overlapping procedural and conceptual meanings. Henry generalized radian angle measure as "easy fractions of π" by describing that any fraction of π radians represents an angle that is "one n^{th} of the way around the semicircle."

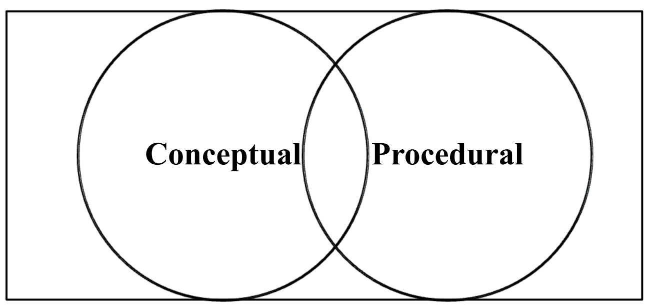


Figure 7. A Model of Henry's Meanings for Radian in the Third Interview

Discussion

This report aims to extend previous research exploring PMTs' conceptions of radian angle measure (Akkoc, 2008; Çekmez, 2020; Fi, 2003; Moore et al., 2016; Topcu et al., 2006) by tracking one PMT's mathematical meanings during engagement with three tasks involving radian angle measure. Henry's initial description of radians demonstrates the influence of common representations of radians in terms of π and in relation to the unit circle (Akkoc, 2008; Fi, 2003; Moore, 2013; Topcu et al., 2006). Additionally, during the first interview, Henry attended to the role of proportionality between the angle's subtended arc and radius (Alyami, 2022b). However, similar to in-service teachers in Thompson et al. (2007), Henry questioned the practicality of using ideas of proportionality and the arc approach to measure angles.

While not his preferred strategy, Henry elaborated on the relationship between the arc length and radius at the beginning of the second interview by relating the formulas for measuring arc length and circumference. Using these relationships to describe one radian as the special case when the angle's subtended arc and the radius are equal reflects Henry's multiplicative comparison of the two lengths (Thompson, 2011). While Henry demonstrated this comparison procedurally using the formulas, his description incorporated conceptual strategies that focused on *how* and *why* these formulas are related (Nordlander, 2021).

During the second interview, Henry continued to describe radians as "easy fractions of π," with an emphasis on special angles depicted on a typical unit circle diagram. However, Henry acknowledged the limitation of this strategy upon encountering $\frac{\pi}{5}$, which he was not familiar with because "it was not on the unit circle." This situation relates to how procedural strategies can be beneficial in limited situations (Thompson & Carlson, 2017). Eventually, Henry generalized his description of radians from "easy fractions of π" to fractions that are "one n^{th} of the way around the semicircle," suggesting a partitioning approach (Hardison, 2020). When reasoning about radians, procedural strategies might not be devoid of conceptual understanding (Baroody et al., 2007; Maciejewski & Star, 2019; Nilsson, 2020; Nordlander, 2021; Star, 2005, 2007). While Henry used conceptual strategies from the beginning, his procedural explanations outweighed his conceptual ones. Through exposure to situations with radians in contexts that challenged procedural strategies, such as non-familiar diagrams and measurement (e.g., $\frac{\pi}{5}, \frac{5\pi}{12}$), Henry connected his procedural and conceptual meanings for radian angle measure.

The main implication from this report is the recommendation to capitalize on tasks that are designed to build and support conceptual understanding. Instead of tasks that emphasize procedures, using tasks that encourage learners to unpack the mathematical foundation behind the procedures potentially contributes to developing conceptual and productive meanings for radian angle measure.

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