PROOF WITHOUT CLAIM: A NOVEL TOOL FOR EXPLORING STUDENTS' CONCEPTIONS OF PROOFS

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In order to learn more about student understanding of the structure of proofs, we generated a novel genre of tasks called "Proof Without Claim" (PWC). Our work can be viewed as an extension of Selden and Selden's (1995) construct of "proof framework"; while Selden and Selden discuss how the structure of a proof can be discerned by the claim it proves, we leverage the consideration that one can discern a claim by the structure of its proof. In these PWC tasks, the student is presented with a proof with any explicit mention of its claim being proven removed and is instructed to discern the proof's claim. We introduce the construct of Personal Proof Framework (PPF) to describe the varied ways in which students relate the claim being proven to the content of the proof.

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Understanding how to comprehend and write proofs is an essential skill for transitioning to advanced mathematics (Stylianides et al., 2017). One tool aimed at helping students transition to being effective proof writers is based on Selden and Selden's (1995) construct of *proof frameworks.* This construct accounts for the fact that the logical structure of a theorem (a claim) can influence the structure of its proof. For example, if we are proving an implication of the form "for all x, P(x) implies Q(x)", we might start our proof with the assumption that P(x) is true for an arbitrary x in the universe of discourse, and end with the conclusion that Q(x) is true. Similarly, we might begin writing a proof by contradiction of a statement X by starting with something akin to "Toward a contradiction, assume not-X". As discussed by Selden et al. (2018), this structure provides a valuable way for students to start their proofs via outlining. Some authors have indeed observed some improvements in student proving when these proof frameworks are explicitly taught (Gaber et al., in press).

Selden and Selden (2015) describe distinctions between "proof validation", "proof comprehension", and "proof construction". Proof validation is the process by which one reads a proof and reaffirms its soundness, proof comprehension is the process by which one "understands" (p.10) a proof, and proof construction is the process of writing a proof. Selden and Selden (ibid.) call for more research on the interrelationships between these aspects of proof. Our study looks at the relationships between students' proof comprehension, proof construction, and proof validation. Specifically, we address the questions: how do students interact with and conceptualize common proof frameworks? How do students relate the claim being proven to the content of a valid proof?

We do this by leveraging Selden and Selden's (1995) construct of *proof frameworks* to study how students conceptualize the connections between a claim being proven and its proof. While Selden and Selden (2018) explore how students can start with a claim and use its logical structure to outline a proof that coheres with that structure, our study does the reverse. Specifically, we examine how students discern the claim being proven when given only the content of a proof. For example, if a proof begins with "toward a contradiction, suppose not X" and ends with "therefore, we have found a contradiction", we can discern that the claim being proven is “X”. In other words, the logical structure of the claim appears in *both* the claim and in the body of the proof.

In order to facilitate our investigation, we generated a novel genre of task we call Proof Without Claim (PWC). These tasks begin with a correct proof of some claim. Guiding sentences which sometimes bookend a proof and allude to what is being proven by stating “we will prove that X” and “thus we have shown that X” are removed. Students are given the remainder of the proof and tasked with figuring out and then writing down what claim the original proof proved. Observing what students pay attention to when figuring out these PWC tasks thus provides a lens into their conceptions of how proofs prove, and particularly how a proof is related to the claim it is proving. Instead of prescribing what connections students should ideally see between a claim and its proof, we study what connections students do see. This necessitates a shift from thinking about proof frameworks in the normative sense to what we call Personal Proof Frameworks (PPFs). By investigating what students write down as the proof’s claim, we are investigating an aspect of *proof construction.* By investigating how students read the provided valid proof and relate it to the claim, we are investigating an aspect of students’ *proof comprehension* and *proof validation.* Hence, our work can be viewed as looking at the relationships between proof comprehension, validation, and construction (Selden & Selden, 2015).

Literature

One important aspect of student conceptions of proving is the role equations and computations play within proofs. Several studies have identified differences between expert and novice provers’ reading behaviors that suggest that experts and novices might understand the role of equations and computations differently. This research includes eye-tracking studies that revealed differences in where readers’ attention is focused; experts tended to spend less time looking at equations and calculations than novice readers (Ingles & Alcock, 2012). While experts tended to read nonlinearly and attend to the structure of a proof, students tended to read linearly (ibid.). Read aloud studies have revealed that novices tend to sound out equations by vocalizing the individual symbols, while experts tend to verbalize the purpose or meaning of the equations/calculation (Shepherd & van de Sande, 2014).

Recent work has posited that some students’ prior experiences with calculation-centered mathematics have significant influence on their interpretations of the purpose of equations within proofs. For example, in a number theory context, an equation might serve no calculational purpose and instead be used to infer divisibility properties of variables in that equation. Dawkins and Zazkis (2021), conducted a study of students’ in-the-moment understanding of proofs. They displayed proofs line-by-line with the previously read lines visible but subsequent lines hidden. This allowed the researchers to observe that some students assumed equations in a proof would be used for future calculation, despite the non-calculational purpose regarding divisibility being explicitly stated and accomplished within the proof. This result suggests that these students conceptualize equations within proofs as being limited to a purely calculational role.

Other research gives insight into what students might focus on when reading proofs and when identifying a proof’s central argument. For example, in Zazkis and Villanueva (2016), students often did not connect an informal argument of a result to a more formalized version of that same argument (the proof). Specifically, students were given an informal argument for a theorem and two correct formal proofs of that theorem. Only one of these formal proofs mirrored the informal argument. However, students generally did not correctly identify which of the two proofs was a formalization of the argument. Students tended to focus on (what we consider to be) a superficial detail of the proof to make their assessments instead of its core argument. In our study, we seek to better understand how students interpret another aspect of proof: the connection between the content of a proof and the claim that it proves.

Framework and Methods

In this study we are in a sense doing the reverse of what Selden et al. (2018) did with their *proof frameworks* construct (described in more detail above). While they suggest that students start writing the structure of a proof by using the structure of its claim, we have students discern the claim from the content of the proof. So, we use their construct as a starting point for our work. The proof framework construct described in Selden and Selden (1995) can be conceptualized as a normative expert view on the connection between the claim and set up of a proof based on the proof technique being implemented. Since we are instead interested in student conceptions of the connections between a proof and the claim it proves, we propose an anti-deficit construct that we call *personal proof framework (PPF)*. This is a student's construal of the connection between the proof and the claim being proven, which is reflected in their approaches to PWC tasks.

A recruitment email was sent to all instructors of Anonymous State University's (ASU) introduction to proof course (David & Zazkis, 2020). The email requested that the instructors inform their students of the opportunity to take part in the study. All interested students who responded were interviewed – consequently, six students participated. All participants were compensated for their time in accordance with our Institutional Review Board (IRB) approval. Each interview consisted of an orientation task in which the student was shown a complete proof and asked to identify what the proof proves: i.e., what claim or theorem the proof validates. Further, the student was asked to identify the location(s) in the proof where the claim being proven was stated. The student was then shown a version of that same proof with the first and last line removed. These removed sentences stated what would be proven and what had just been proven, respectively. The student was then told that all remaining proofs in the interview would be of the second type with the removed lines, and that their task was to figure out what the proof was proving in order to fill in these missing sentences (PWC tasks). It was explained to the participant that all the proofs presented are error-free and do indeed successfully prove a claim. Interviews were video, screen, and audio recorded. The first two authors collaboratively analyzed the interviews in a way that is consistent with Simon's (2019) three layers of analysis. This involved iteratively documenting and interpreting interviewees' processes for determining what a proof in a PWC tasks was proving and comparing these individual processes to each other to identify similarities and differences. The collection of PWC tasks implemented a variety of proof techniques, including induction, contradiction, direct proof, and disproof by counterexample. However, due to space constraints, we will primarily focus on the task in which students were presented with a standard proof by contradiction that the square root of two is irrational.

Results

We propose three categories of Personal Proof Frameworks (PPFs) as explanations for how students interact with PWC tasks. Although we describe these frameworks within the context of our PWC tasks, we hypothesize that these conceptions extend to how students view proofs more generally. Our hypothesis is somewhat confirmed by the fact that students responded similarly to the standard proof (without its claim removed) in the orientation task (discussed more below).

1) Calculational Personal Proof Framework (Calculational PPF): This category is reminiscent of results from the aforementioned reading studies wherein students primarily focus on the calculations within proofs, treating them as the primary aspect of a proof. In our study, this

framework specifically refers to a conception that proofs are calculations, and that what is being proven by a proof thus must be what is calculated by the equations and computations it contains. Two participants demonstrated Calculational PPFs (Maria and Guillermo).

2) Cohesive Personal Proof Framework (Cohesive PPF): With this conception of proof, the proof technique used and the details of how the proof is proven are both integral to understanding what is being proven. Two students demonstrated Cohesive PPFs (Javon and June).
3) Claim-Setup Personal Proof Framework (Claim-Setup PPF): With this conception, the setup and conclusion of a proof are all that is needed to discern what is being proven. The details in the middle of a proof may be needed for the proof to be complete and correct, but they are not needed for determining the claim being proven. Two participants demonstrated Claim-Setup PPFs (Thomas and Manuel).

Although at face value it might appear that these categories are mutually exclusive, it is important to note that novice provers often have varying familiarity with different proof techniques and mathematical objects. So, it might be perfectly reasonable for a student to demonstrate one conception when working on some proof and then another when they are working within a different context or technique. Nonetheless, within the confines of the tasks in this study, participants each fit into one of the three categories.

Calculational Personal Proof Framework

Undergraduate mathematics students beginning to take proof-centered courses have typically had years of experience interacting with equations in calculational contexts. Thus, it would make sense for some students to focus on equations while overlooking what may be inferred from the sentences around them. Consider the following excerpt from Guillermo's interaction with Task A, which presented the student with the proof that $\sqrt{\mathbf{2}}$ is irrational.

Guillermo: So we are pretending that root 2 is rational.
Interviewer: What do you mean by pretending?
Guillermo: "Suppose to the contrary," for the sake of the proof we are pretending. Because it isn't normally rational. Since they are saying "suppose to the contrary" that means that in this world. Rad 2 is normally irrational
Interviewer: Square root of 2 is irrational.
Guillermo: Okay, I think this is saying that this is proving algebraically that if you have the quotient of two squared numbers that you can take the square root of both of those numbers. And if you have it set equal, you can take the square root of that particular number. I believe that's what it's trying to prove. We are pretending that root 2 is rational (highlights the centered equations $\sqrt{2} = \frac{p}{q}$ and $2 = \frac{p^2}{q^2}$).
Interviewer: So, what would you write in the first box?
Guillermo: We will prove that r equals p over q implies that r squared equals p squared over q squared (see Figure 1 below for Guillermo's inscriptions).

In the excerpt above, the interviewee notices the contradiction setup at the beginning of the proof when he refers to "pretending that root 2 is rational." However, the square root of two is not part of what he concludes the proof is proving. He instead shows evidence of a Calculational PPF when he focuses on the two equations present in the proof and infers what the proof proves is instantiated in those equations. Focusing on this computation, Guillermo concludes that the fact that squaring both sides of an equation yields a new valid equation is what is being proven, instead of the validity of this operation being utilized as part of a larger proving process.

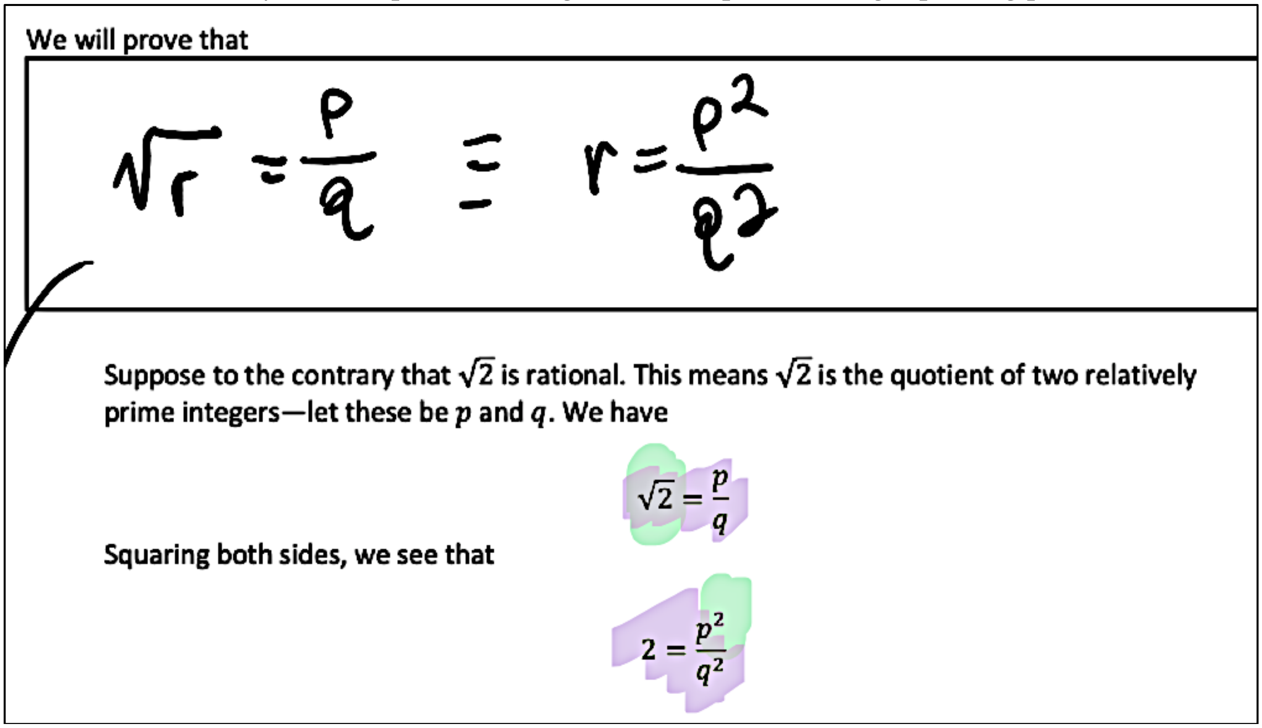

Figure 1: Guillermo's work on Task A.

Here, we can see that Guillermo focused on the equation and ignored much of the rest of the proof when deciding what the proof is proving. In so doing, he demonstrated a Calculational PPF. As a consequence of this PPF, he did not see the argument in the proof as being about the infeasibility of those quantities being relatively prime. For him, the supposition that $\sqrt{2}$ is rational seemed not to be relevant for determining the claim being proven – instead, the claim being proven was simply being done in the counterfactual "world" in which $\sqrt{2}$ is rational.

Cohesive Personal Proof Framework

A Cohesive PPF involves conceptualizing a proof as a whole, in which every part of the proof is relevant to the claim being proven. During the orientation task, Javon demonstrated this outlook when he stated that, "You kind of have to look through the whole proof, because we are trying to get to just one center goal." In fact, Javon highlighted both the beginning, the end, and parts of the middle in the orientation task when asked where he saw what parts of the proof indicate what was being proven.

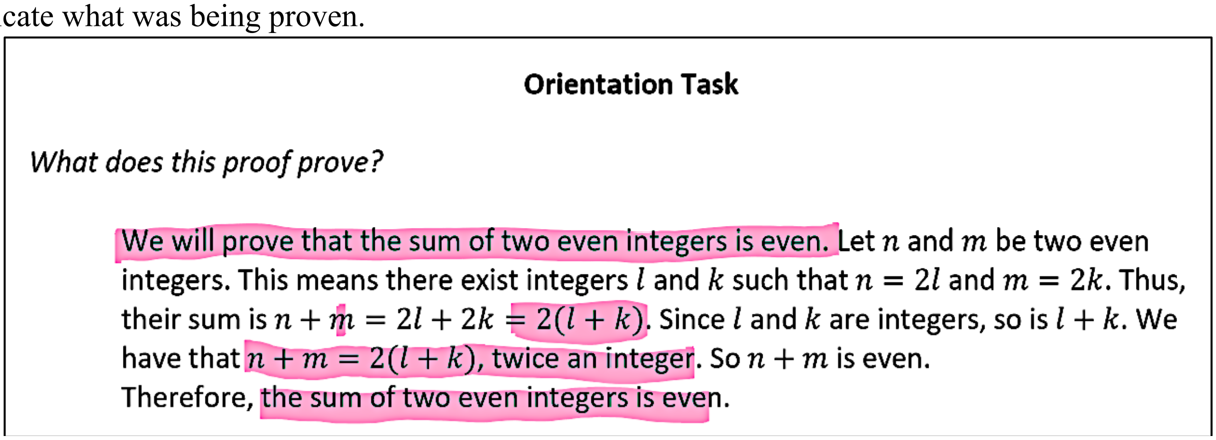

Figure 2: Javon's work on the orientation task.

Excerpts from later in the interview help illustrate his cohesive proof framework in other tasks. When presented with Task A, Javon initially used the first line of the proof to hypothesize that the claim being proven is that the square root of 2 is irrational. However, he explained that he was "more comfortable" with his answer if he reads through the entire proof:

> Javon : I'm going to look through the question just a little bit more. Um… I can also see that, um, we, uh, can't have a ratio in a, uh, base form, so we can see p and q are relatively prime so we can't simplify the function any more, which of course means that square root of two is not rational.

Consistent with his prior statement regarding the importance of the whole proof, he continued to read the proof to glean how it achieves the contradiction. It is important to note that, when writing down his answer, he included aspects of the middle of his proof in the second claim box. Specifically, he wrote " We have proven that $\sqrt{2}$ cannot be represented by a simplified ratio of two integers, thus $\sqrt{2}$ is not rational." In doing so, he documented not just what was proven, but also *how* it was proven. He went beyond simply formulating the claim to be the last sentence in the proof by including this extra information about how the claim was justified. We hypothesize that he did this because of past experiences with proof validation and proof comprehension. While the PWC are a type of task that he has not experienced prior to this interview, he had significant experience with proof comprehension and validation tasks from his coursework. Proof comprehension and validation involves parsing the details of a proof, so it seems reasonable that he applied strategies from past experiences with proof comprehension to PWC tasks.

Claim-Setup Personal Proof Framework:

The Claim-Setup PPF closely mirrors what Selden and Selden (1995) call a "proof framework". That is, there is a structural relationship between the claim being proven, the proof technique, and the beginning and ending portions of the proof. In other words, students with this framework readily see the beginning and ending of a proof as mirroring the structure of the claim being proven. Thus, if the task is simply to glean what is being proven in a PWC task, there is little to be gained from looking at the middle of the proof.

In Task A, Manuel immediately underlined the first ("suppose to the contrary that $\sqrt{2}$ is irrational") and last ("contradicting our supposition") line of the proof and did not annotate anywhere in the middle. He then explained:

> Manuel: We have what I would call a contradiction. Proof by contradiction. And so we assume that somewhat contrary, something is not true. So, we assume that radical two *is* rational, but then we proved something, and it ended up being a contradiction. And therefore, that's not gonna work. Therefore, proof by contradiction proves that the hypothesis is true. Meaning, we have proven that radical 2 is irrational.

Having a Claim-Setup PPF does not guarantee that a student can easily, confidently, and accurately determine what a proof is proving. Although students with this framework understand the structure of the proof as mirroring the claim, there are a multitude of ways that students might interpret the structure of the proof. For example, while Thomas also demonstrated a Claim-Setup PPF, his response to Task A diverged greatly from Manuel's, even though Thomas also focused on the beginning and ending portion of the proof. While thinking about the task, Thomas explained how he was using the beginning and ending portion:

> Interviewer: What's standing out?
>
> Thomas: So a lot of what I'm looking for is things that I can use in the beginning in the end.

Although he focused on the beginning and ending sentence, he felt conflicted about how to relate them. This appeared to arise from his own interpretation of the word "supposition" at the end of the proof ("contradicting our supposition that p and q are relatively prime"). He interpreted this use of the word "supposition" to indicate that "p and q are relatively prime" was the hypothesis of a conditional being proven. He identified the claim as a conditional that was being proven by contradiction: "From the concluding sentence, we kind of contradicted one of our givens". However, he had trouble reconciling the generality of p and q with the specificity of $\sqrt{2}$:

> Thomas: I'm having trouble figuring out what the *then* portion of the proof is, in the sense that it seems like I would assume from the first sentence, where it says "suppose to the contrary that the square root of 2 is rational", that I would believe is kind of the, I don't know, I don't remember the exact terminology for it, it's the *then* portion of the conditional statement. But, it says that the square root of 2 is rational, and that seems more like, specific than if we start with a proof and we have p and q where we prove that the square root of *some* number is rational, rather than square root of 2 is rational."

We can see that Thomas's PPF was a Claim-Setup Proof Framework. He focused on the word "suppose" and used the beginning and end of the proof to suggest that the claim being proven was "if p and q are relatively prime, then the square root of 2 is irrational (or perhaps rational)."

Interpretation of Claims

Additionally, it bears mentioning that determining the immediate claim being proven does not necessitate a particular interpretation of this claim. It seems reasonable to believe that a student might identify a conditional (for example) being proven but still interpret its meaning or applicability differently than a mathematician would. Indeed, this phenomenon occurred with Task D, a PWC that proved that a particular linear function is injective. The word "injective" was not used – it was proven that the implication "if $x_1=x_2$, then $f(x_1)=f(x_2)$" holds. For example, Manuel, who demonstrated a Claim-Setup PPF, used the beginning and ending and identified the proof as proving the implication, but then explained "there is a function f(x) where any unique input x gives a unique output f(x)". In other words, despite readily identifying the implication being proven, Manuel interpreted it to be an existence statement that described meeting the criteria for being a function. Here, we see an analogue to the Shepherd and van de Sande (2014) work by noting that interpreting the literal statement being proven does not necessarily determine an interpretation of that statement.

Future Directions and Discussion

From our data, it can be seen that students vary in their conception of how a proof relates to the claim being proven. Students with a Claim-Setup PPF focused on the beginning and ending portions of a proof to discern the logical structure and content of the claim. This tended to involve paying attention to the language being used in the proof (e.g., words like "suppose"). We can contrast this approach markedly with students who used the Calculational Proof Framework. These students often ignored the verbiage of the proof and focused on the symbolism and equations. Like the students in the eye-tracking studies (e.g., Ingles & Alcock, 2012), they honed in on the equations and computations and appeared to view them as constituting the proof. Such students seemed not to connect the calculations within a proof to the overall structure of the proof and the verbiage surrounding the equations. Finally, students with a Cohesive PPF often needed to make sure they understood every or several of the sentences in the proof before feeling comfortable enough to confidently state what the proof was proving. It appears that their sensemaking of the middle portion of the proof (including the computations and equations) aided them in constructing the claim. Unlike students with a Claim-Setup PPF, these students tended not to stop and take note of structure-indicating words such as "suppose". Future work can investigate how such students interpret such words.

We saw evidence that students were being influenced by their prior experience with mathematics. For students with the Calculational PPF, prior experiences in calculation-based courses appeared to be playing a role. This observation aligns with Dawkins and Zazkis' (2021) claim that students' prior calculational experiences were influencing the fact that students tended to view equations within proofs as existing solely for computational purposes. Similarly, we noticed that students with the Claim-Setup PPF appeared to be influenced by their experiences with textbooks or instruction that prompted them to identify "givens" and "goals" when producing a proof. We note that, although the students with the Claim-Setup PPF appeared to exclusively use the beginning and end of the proof to write their claim, they still sometimes read portions of the middle of the proof. We hypothesize that such students are influenced by the fact that much of their past experiences with reading proofs involved proof validation and proof comprehension.

Importantly, even though for mathematicians the structure of the claim mirrors the structure of a proof that proves it, students do not readily make the same connections. While Selden and Selden (2018) posit that it can be useful to teach students to start their proofs using proof frameworks, we believe our PWC tasks can be similarly useful. PWC tasks have the utility of being includable on assessments as short-answer or multiple-choice questions. Future research can study the how PWC tasks can be implemented for instructional or assessment purposes.

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