STUDENTS' UNDERSTANDINGS OF THE DEFINITION OF A FUNCTION

Camille S. Burnett
Prairie View A&M University
csburnett@pvamu.edu

The study discussed here aims to describe students' understandings of the definition of a mathematical function, which was achieved through a pilot case study of clinical interviews with four participants – two ninth graders and two twelfth graders. The participants were recruited from the same urban public high school in the northeast of the United States, which serves a diverse racial, ethnic, and cultural community. All four participants were selected by their mathematics teachers because of their high grades and skill level. The participants were interviewed individually about questions pertaining to the definition of a mathematical function. Analyses of the interview responses revealed that the twelfth graders understand the definition of a mathematical function differently than do ninth graders.

Keywords: Algebra and Algebraic Thinking

Purpose of Study

Functions are one of the fundamental objects of mathematics (Doorman & Drijvers, 2011; Dreyfus & Eisenberg, 1982; Eisenberg, 1992; Gagatsis & Shiakalli, 2004; Hitt, 1998; Schwartz, 1999; Schwartz & Yerushalmy, 1992; Zandieh et al., 2017). They are present in (1) elementary school mathematics courses, where students are required to identify patterns or interpret data tables and graphs; (2) middle and high school mathematics courses, ranging from Algebra 1 to Pre-Calculus; and (3) college or university mathematics courses, including Calculus at all levels and Real Analysis. Therefore, functions are a core concept in mathematics, appearing at all mathematics curriculum levels. There is much research within the field of functions. An example of one of the most highly cited pieces of literature regarding functions is Dubinsky and Harel (1992), whose work examines: the development of the function concept; teaching experiments; students' and teachers' conceptions of functions; and the use of technology. The research presented here is a pilot case study that focuses on students' conceptions of functions and aims to further contribute to this line of research, particularly students' *understandings* of the *definition* of a *mathematical function*, given its presence throughout mathematics. Previous studies have shown that (a) junior high school teachers were better able than college students to determine if a given relation was a function based on their concept image of a function (Dreyfus & Vinner, 1982); (b) most prospective teachers knew of the univalence requirement for a relation to be a function, but many did not know why this was a requirement (Even, 1993); (c) there is initial evidence to suggest that secondary students' conceptualization of functions may be connected to the curriculum to which they were exposed (Ayalon & Wilkie, 2019); and (d) many undergraduate mathematics students are unable to define a function or determine if a given graph or rule represents a function (Bardini et al., 2014). The research presented here will highlight differences in ninth and twelfth graders' understanding of a function's definition.

Theoretical Framework

Definition of a Function

Several definitions of a function exist, each with a slight variation depending on the audience (e.g., high school vs. college student) receiving the definition. A quick search on the internet suggests that one possible explanation of a function is "A special relationship where each input has a single output" (Math is Fun, 2018). A high school mathematics textbook suggests the following definition: "Suppose A and B are two sets of objects. A function from A to B is a pairing between A and B such that each element in A pairs with exactly one element of B" (CME Project Development Team, 2009, p. 104). A college-level textbook suggests that the definition of a function is, "A function f from a set A to a set B is a relation that assigns to each element x in the set A exactly one element y in the set B. The set A is the domain (or set of inputs) of the function f, and the set B contains the range (the set of outputs)" (Larson, 2014, p. 173). The latter two definitions of a function are known as the Dirichlet-Bourbaki definition of a function. They include the set-theoretic notions of Bourbaki and the rule-based notions of Dirichlet. They are also considered a *modern* definition of function for these same reasons. The former definition of a function does not explicitly mention the set-theoretic notions of Bourbaki. Thus, for this paper, the latter two definitions of a function will be the accepted definitions.

Characteristics of a Function

In order to get to this modern definition of a function, the concept of function has undergone a curious evolutionary process due to the change in knowledge of mathematics over time (Kleiner, 1989; Malik, 1980; Markovits et al., 1986; O'Connor & Robertson, 2005; Sfard, 1992; Sierpinska, 1992). Functions may have first appeared in tabular representational form and as trigonometric functions (2000 B.C.E. – 1299 C.E.), then successively as a relationship of dependence (1300 C.E. – 1499 C.E.), a relationship between varying quantities (1500 C.E. – 1599 C.E.), and in the algebraic and graphical representational form (1600 C.E. – 1699 C.E.). Next, functions were defined from an algebraic perspective by Euler (1700 C.E. – 1799 C.E.), then based on an arbitrary correspondence by Dirichlet (1800 C.E. – 1899 C.E.), and finally considered as an arbitrary correspondence between two sets, which followed the emergence of set theory (1900 C.E. – present). Thus, this *modern* definition of a function has two distinct characteristics – *arbitrariness* and *univalence* (Freudenthal, 1983) – which are used in this paper to form the framework for analyzing the data in this study.

Arbitrariness. The arbitrariness of a function refers to "both the relationship between the two sets on which the function is defined and the sets themselves" (Even, 1990, p. 528; 1993, p. 96). In terms of the relationship between the two sets, there does not need to be any specific rule of correspondence, i.e., there does not need to be a specific algebraic expression, a set pattern in a table of values, or a graph with a specific shape. In terms of the sets themselves, the sets do not need to be defined on any specific set of objects, i.e., the sets do not necessarily contain numbers. In other words, the sets (which are referred to as variables by Freudenthal [1983]) can consist of "numbers, number tuples, points, curves, functions, permutands, elements of arbitrary sets" (Freudenthal, 1983, p. 528).

Univalence. The univalence characteristic of a function refers to the part of the definition that states that for each element in the domain, there is only one element (image) in the range (Even, 1990, 1993). Thus, in terms of a relation between two sets, a relation in which every single element in the domain is mapped to its own single element in the range (i.e., a one-to-one relation) or a relation in which more than one element in the domain is mapped to the same single element in the range (i.e., a many-to-one relation) could be a function. While a relation in which every single element in the domain is mapped to more than one element in the range (i.e., a one-to-many relation) or a relation in which more than one element in the domain is mapped to more than one element in the range (i.e., a many-to-many relation) could not be a function.

Methods

Participants

The participants included two ninth graders (Student 9-1 and Student 9-2) and two twelfth graders (Student 12-1 and Student 12-2). These students were from the same urban public high school in the northeast of the United States of America, which serves a diverse racial, ethnic, and cultural community. The ninth graders were learning basic topics in algebra (from an equational perspective), geometry, and probability and statistics in their current mathematics class, and the twelfth graders were learning a combination of topics from pre-calculus and calculus with a focus on various types of functions, as reported by their mathematics teachers. The teachers at the school designed the curriculum for both grades, and it was not supplemented with any specific textbook. Their mathematics teachers selected the participating students to participate in the study based on their performance (grades and skill level) in their current mathematics class.

Individual Interviews

The interviews consisted of seven questions focused on the definition of function and the transformations and comparisons of functions in various representations. This paper will focus only on the responses to the two questions regarding the definition of function (see Figure 1).

Question 1

(a) Give me an example of a function. Explain.
(b) Give me an example of something that is not a function. Explain.
(c) What is the definition of a function? Explain.

Question 7

(a) Which of the following seems to you to be the best definition of a function?

i. A function is an algebraic expression in which you can substitute various values for an unknown.
ii. A function is a computational process that produces an output (y) from an input (x).
iii. A function consists of two sets S and T together with a "rule" that assigns to each element of S a specific element of T.

(b) The following table of values were computed from a function, but the person who made the table forgot to label one column x and the other $f(x)$.

1	9
4	2
4	-2
5	25
9	1
16	-4

Which of the following labeling is correct? How do you know?

x	$f(x)$
1	9
4	2
4	-2
5	25
9	1
16	-4

x	$f(x)$
9	1
2	4
-2	4
25	5
1	9
-4	16

Figure 1: Interview Questions

Question 1 was open-ended and consisted of three parts. The participants were asked to give an example of a function, an example of a non-function, and the definition of a function, with an accompanying explanation for each. Question 7 consisted of two parts. The participants were asked to determine which of three options defined a function correctly. They were also asked to determine which of two given relations (presented in the tabular representational form) were functions. The order of the format of the two questions was intentional. Participants could provide an initial spontaneous and untainted response to the definition of a function in Question 1 before being presented with possible meanings of a function in Question 7. The clinical interviews (Ginsburg, 1997; Piaget, 1929/1976) were videotaped and lasted between 15 and 60 minutes, depending on the participants' length of time to process and answer each question. All interviews were conducted within a month to reduce the likelihood of any student being exposed to more mathematics instruction than their grade-level counterparts.

Analysis

The participants' interview responses to Questions 1 and 7 were transcribed. Their written responses and oral responses were combined and summarized using direct quotes and paraphrasing their answers. The general idea of their responses was noted from this summary (see Tables 1, 2, 3, and 4) by identifying mathematical terms related to functions. The participants' responses to Questions 1 and 7 were then coded based on the essential characteristics of a function – *arbitrariness* and *univalence* (see Table 5). Each explanation was scored with a "1" or a "0" depending on whether it met the following criteria:

- Arbitrariness (A): The *relationship* between the two sets comprising the function is arbitrary, and/or the members of the *sets* themselves are arbitrary.
- Univalence (U): The relation between the two sets comprising the function is either one-to-one or many-to-one.

Results

For each participant, the results are presented here; they include a narrative summary and a table of their responses to Questions 1 and 7 (see Tables 1, 2, 3, and 4). Brief comparisons of grade-level counterparts and a coded summary of their responses (see Table 5) are also presented. Note that even though the emphasis is placed on the participants being able to define a function, it is even more critical for them to understand the concept of function. Thus, it is more important for a participant to suggest an example of a function or a non-function and determine if a given relationship is a function, indicating that they understand the concept of function. It is also hoped that if the participants can this, then stating the definition of a function will be less challenging for them and will become less critical in describing their understanding of function.

Student 9-1

Student 9-1 was consistent in their responses to Questions 1 and 7 (see Table 1). This student believed that a function is an equation such that there would be a specific rule and that there should be unknowns instead of variables involved in a function. This understanding is consistently evidenced in the responses given to the questions posed. It should be noted that this student had never been formally exposed to the *modern* definition of a function per their mathematics teacher and that the exposure to examples of functions was limited to linear functions. Therefore, it is understandable that the student was not fully aware of the need for two arbitrary sets, an arbitrary rule of correspondence, and univalence in a function. It also indicates that instruction within this area is necessary for understanding the definition of a function and understanding various examples of functions, even linear functions.

Table 1: Student 9-1 Responses

Question	Response	Explanation	General Idea
1(a) Function	$a^2 + b^2 = c^2$	"it is an equation, a way to find something out"	equation, unknown
1(b) Non-function	PEMDAS[11]	"an order in which you do something, and it's not a function, so it doesn't give you a way to find something out"	unknown
1(c) Definition	"Something that is used to get an answer; a function has a purpose, and the purpose is the answer."	"a function usually like has a purpose, so the purpose would be the answer. So the function is like an equation that you can ultimately get the answer of"	equation, unknown
7(a) Definition	i	"because like what we were doing before, we were substituting variables, and we were trying to find out what the unknown was."	variables and unknowns
7(b) Function?	neither	no consistent pattern in either of them	rule

Student 9-2

Student 9-2 consistently stated or implied that a function is an equation in their responses to Questions 1 and 7 (see Table 2). This implies a belief in the need for a specific rule of correspondence. In addition, since there was constant emphasis on using a function to find something out or solve a problem, the student likely believed that there are unknowns, not variables, in a function. However, the response to Question 7(b) was surprising because the student thought that the lack of a rule of correspondence in the relations made them functions, which, unbeknown to the student, contradicted the definition repeatedly given by the student. The question probably caused a state of disequilibrium (Beth & Piaget, 1966) in the student's understanding of the definition of a function, as evidenced by the discrepancy in the student's responses to the questions about the definition of a function. It is possible the student was never asked questions of this nature before, which caused the student to respond with such contradictory answers.

Finally, it should be noted that this student, much like the first ninth grade student, had never been formally exposed to the *modern* definition of a function per their mathematics teacher. The exposure to examples of functions was limited to linear functions, as is expected for ninth grade

[11] PEMDAS – Parentheses, Exponents, Multiplication, Division, Addition, Subtraction, also known as Please Excuse My Dear Aunt Sally, is the acronym used to help students remember the order in which mathematical operations should be applied to an algebraic expression.

students. Therefore, it is understandable why the student was not fully aware of the need for two arbitrary sets, an arbitrary rule of correspondence, and univalence in a function. Again, it seems that instruction within this area and exposure to various functions is necessary for understanding the definition of a function and understanding of multiple examples of functions, even linear functions, that are relatively simple.

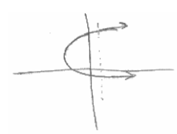
Table 2: Student 9-2 Responses

<table>
<tr><th>Question</th><th>Response</th><th>Explanation</th><th>General Idea</th></tr>
<tr><td>1(a) Function</td><td>y = mx + b</td><td>“it’s an equation to help you figure out something else”</td><td>equation, unknown</td></tr>
<tr><td>1(b) Non-function</td><td><table><tr><td>People</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td></tr><tr><td>Donuts</td><td>2</td><td>4</td><td>6</td><td>8</td><td>10</td></tr></table></td><td>does not match definition of a function, which is “a function is like an equation to help you solve…to help you solve a problem”</td><td>equation, unknown</td></tr>
<tr><td>1(c) Definition</td><td>“An equation or problem used to solve something.”</td><td>functions are like equations that are used to find something out</td><td>equation, unknown</td></tr>
<tr><td>7(a) Definition</td><td>i, ii, iii</td><td>all three options are indicative of different types of functions</td><td>types of functions</td></tr>
<tr><td>7(b) Function?</td><td>either</td><td>no consistent pattern in either of them</td><td>rule</td></tr>
</table>

Student 12-1

In response to every question posed regarding the definition of a function, Student 12-1 identified the two characteristics of a function (see Table 3). In fact, the student emphasized the need for univalence more than the need for arbitrariness, but the variability in emphasis was consistent throughout the interview. The student also showed a solid awareness of the need for two arbitrary sets but was inconsistent in indicating that the rule of correspondence could be arbitrary.

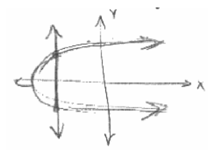
Table 3: Student 12-1 Responses

Question	Response	Explanation	General Idea
1(a) Function	$f(x) = x^3 + 3x^2 - 4x + 3$	there is an input, being referred to as x, and an output, and the input "will give you one and only one outcome in the output which makes it a function"	input/output, mapping
1(b) Non-functiorn		"Something that's not a function is, basically if you put in something for the input and you get two different answers or multiple answers for the output" … fails the vertical line test	input/output, mapping, vertical line test
1(c) Definition	"A function is an equation that will give only one output for one specific input."	if there is more than one output for each input, then the graph would be inaccurate	input/output, mapping
7(a) Definition	iii	"two sets … S and T are … combined with … with a rule, that basically says that for each assignment of S you can only get one specific … element of T, or like one specific output which would be T"	arbitrariness, univalence
7(b) Function?	second table	"you are giving one input, and it's giving two different outputs, so this [the second table] would be the correct one"	equation, univalence

Student 12-2

For Student 12-2, there is continuous awareness of the need for two arbitrary sets to define a function (see Table 4). The student also continually expected that the rule of correspondence should be specific. Regarding the characteristic of univalence, the student could only identify and explain it through the vertical line test, which is a graphical test. This suggests that the representation of the function may limit the student's understanding of this characteristic.

Table 4: Student 12-2 Responses

Question	Response	Explanation	General Idea
1(a) Function	$f(x) = x^2 + 9x + 3$	it has roots, factors, and it is a parabola … passes the vertical line test	vertical line test
1(b) Non-function		has an x-axis and y-axis …, and "fails the vertical line test"	input/output, vertical line test
1(c) Definition	"If you have two different inputs and have two similar outputs, then it is not a function. If you have two inputs and get two different outputs then it is a function."	it does not fail the vertical line test	vertical line test
7(a) Definition	i and ii	one needs to plug in an input in order to obtain an output	input/output, rule
7(b) Function?	second table	created graph, then noted that it failed the vertical line test	vertical line test

Comparison of Student 9-1 and Student 9-2

In comparing Student 9-1 and Student 9-2, both students have a relatively unsophisticated understanding of function. This is evidenced in their examples and their definitions. There is a consistent emphasis on the presence of unknowns and solving, as opposed to the presence of a variable. This unsophisticated understanding is further evidenced in Table 5, where we can see that both students have a similar non-understanding of a function's characteristics. Both students could not fully articulate the need for arbitrariness in a function.

Additionally, both students are similarly unaware of the idea of univalence. These results parallel the work of Ayalon and Wilkie (2019), Bardini et al. (2014), Dreyfus and Vinner (1982), Even (1993), Vinner (1983), and Vinner and Dreyfus (1989), whose research shows that some students are less aware of the univalence characteristic of function than other students. These ninth-grade students' responses also clearly revealed the limited variety of functions they might have been exposed to in their mathematics education, further reinforcing their understanding of function yet indicating the potential for a shift in their understanding.

Comparison of Student 12-1 and Student 12-2

In comparing the responses of Student 12-1 and Student 12-2, the first of the two students has a slightly more sophisticated understanding of the concept of function than the second twelfth grade student. This is evidenced in their examples being reasonably similar but somewhat different explanations. In their answers, Student 12-1 consistently refers to the arbitrary nature of a function, and Student 12-2 does not. Their responses to Question 7a are also different as Student 12-1 chose the option in which the univalent nature of function was prominent, and Student 12-2 did not. This difference in understanding is further emphasized in Table 5, where the slight difference in their understandings is more clearly seen. Again, these results parallel the work of Ayalon and Wilkie (2019), Bardini et al. (2014), Dreyfus and Vinner (1982), Even (1993), Vinner (1983), and Vinner and Dreyfus (1989), whose research shows that some students are more aware of the univalence characteristic of function than other students.

Table 5: Coded Summary of Responses to Questions 1 and 7

	1(a) Function		1(b) Non-function		1(c) Definition		7(a) Definition		7(b) Function?	
Participant	A	U	A	U	A	U	A	U	A	U
Student 9-1	1	0	1	0	1	0	1	0	1	0
Student 9-2	1	0	1	0	1	0	1	0	1	0
Student 12-1	1	1	1	1	1	1	1	1	1	1
Student 12-2	0	1	1	1	1	1	1	0	1	1

Discussion

Overall Findings

This paper aimed to describe students' understandings of the definition of function through a case study analysis. The results and discussion lead to three main findings about the participants:

- the ninth-grade students are slightly more aware of the arbitrariness than of the univalence of functions, while the twelfth grade students are more mindful of both the arbitrariness and univalence of functions;
- the differences in understandings of the definition of a function coincide with a student's grade level and, therefore, a student's exposure to mathematics, which implies that progression from one level of understanding to the next level of understanding can only be achieved through direct learning experiences of the concept (Duckworth, 1973, 1996; Piaget 1975/1985; Vygotsky, 1978); and
- the differences in understanding a function's definition parallel the historical development of the concept of function (Piaget & Garcia, 1983/1989).

Implications for Teachers and Researchers

This study is essential to both mathematics teachers and mathematics education researchers. For mathematics teachers, it describes students' responses to various questions about functions and highlights their misconceptions. For mathematics education researchers, it implies possible levels of understandings of the concept of function. These descriptions and implications are necessary, given that function is a core concept in the teaching and learning of mathematics. The more we know about how a student understands a concept, the better we will teach it and develop appropriate curricula, as evidenced in the work of Dubinsky and Wilson (2013) and Sherman et al. (2019), and therefore, engage all students in learning.

Future Research

The research presented here is a pilot case study and is merely descriptive. Thus, more research needs to be done with a larger sample size to make truly conclusive statements regarding students' understanding of functions. Despite these limitations, the research discussed can form the basis of such a research endeavor, for example, a study focused on one particular grade level. Also, the types of questions asked should be modified based on the results presented here. For instance, the responses to questions about examples of functions were quite varied, implying that a question asking one to determine whether a given relation is a function might provide more streamlined and valuable data and, therefore, more substantiated conclusions.

References

Ayalon, M., & Wilkie, K. J. (2019). Exploring secondary students' conceptualization of functions in three curriculum contexts. Journal of Mathematical Behavior, 56, 1-17. https://doi-org.pvamu.idm.oclc.org/10.1016/j.jmathb.2019.100718

Beth, E. W. & Piaget, J. (1966). Mathematical Epistemology and Psychology (W. Mays, Trans.). Dordrecht-Holland: D. Reidel Publishing Company.

Bardini, C., Pierce, R., Vincent, J., & King, D. (2014). Undergraduate mathematics students' understanding of the concept of function. Indonesian Mathematical Society Journal on Mathematics Education, 5(2), 85-107.

CME Project Development Team. (2009). Algebra 2: Common Core. Pearson.

Doorman, M., & Drijvers, P. (2011). Algebra in function. In P. Drijvers (Ed.), Secondary Algebra Education (pp. 119-136). Sense Publishers.

Dreyfus, T. & Eisenberg, T. (1982). Intuitive function concepts: A baseline study on intuitions. Journal for Research in Mathematics Education, 13(5), 360-380.

Dreyfus, T. & Vinner, S. (1982). Some aspects of the function concept in college students and junior high school teachers. Proceedings of the Sixth International Conference for the Psychology of Mathematics Education, Antwerp, Belgium.

Dubinsky, E. & Harel, G. (Eds.). (1992). The concept of function: Aspects of epistemology and pedagogy. Mathematical Association of America.

Dubinsky, E., & Wilson, R. T. (2013). High school students' understanding of the function concept. The Journal of Mathematical Behavior, 32(1), 83-101. https://doi.org/10.1016/j.jmathb.2012.12.001

Duckworth, E. (1973, October). Piaget takes a teacher's look. Learning, 22-27.

Duckworth, E. (1996). Either we're too early and they can't learn it, or we're too late and they know it already: The dilemma of "applying Piaget." In E. Duckworth, The having of wonderful ideas and other essays on teaching and learning (2nd ed.. pp. 31-49). Teachers College Press.

Eisenberg, T. (1992). On the development of a sense for functions. In E. Dubinsky & G. Harel (Eds.), The concept of function: Aspects of epistemology and pedagogy (pp. 153-174). Mathematical Association of America.

Even, R. (1990). Subject matter knowledge for teaching and the case of functions. Educational Studies in Mathematics, 21(6), 521-544.

Even, R. (1993). Subject-matter knowledge and pedagogical content knowledge: prospective secondary teachers and the function concept. Journal for Research in Mathematics Education, 24(2), 91-116.

Freudenthal, H. (1983). Didactical phenomenology of mathematical structures. D. Reidel Publishing Company.

Gagatsis, A., & Shiakalli, M. (2004). Ability to translate from one representation of the concept of function to another and mathematical problem solving. Educational Psychology, 24(5), 645-657.

Ginsburg, H. P. (1997). Entering the child's mind: The clinical interview in psychological research and practice. Cambridge, UK: Cambridge University Press.

Hitt, F. (1998). Difficulties in the articulation of different representations linked to the concept of function. Journal of Mathematical Behavior, 17(1), 123-134.

Kleiner, I. (1989). Evolution of the function concept: A brief survey. The College Mathematics Journal, 20(4), 282-300.

Larson, R. (2014). College Algebra (9th ed.). Cengage Learning.

Malik, M. A. (1980). Historical and pedagogical aspects of the definition of function. International Journal of Mathematics Education in Science and Technology, 11(4), 489-492.

Markovits, Z., Eylon, B., & Bruckheimer, M. (1986). Functions today and yesterday. For the Learning of Mathematics, 6(2), 18-24.

Math is Fun. (2018). Definition of a function. https://www.mathsisfun.com/definitions/function.html

O'Connor, J. J., & Robertson, E. F. (2005). The Function Concept. MacTutor History of Mathematics. http://mathshistory.st-andrews.ac.uk/HistTopics/Functions.html

Piaget, J. (1976). A child's conception of the world (J. and A. Tomlinson, Trans.). Maryland: Littlefield Adams Quality Paperbacks. (Original work published in 1929)

Piaget, J. (1985). The equilibration of cognitive structures (T. Brown and K. J. Thampy, Trans.). The University of Chicago Press. (Original work published in 1975).
Piaget, J. & Garcia, R. (1989). Psychogenesis and the history of science (H. Feider, Trans.). Columbia University Press. (Original work published in 1983).
Schwartz, J. L. (1999). Can technology help us make the mathematics curriculum intellectually stimulating and socially responsible? International Journal of Computers for Mathematical Learning, 4(2-3), 99-119.
Schwartz, J. & Yerushalmy, M. (1992). Getting students to function on and with algebra. In E. Dubinsky & G. Harel (Eds.), The concept of function: Aspects of epistemology and pedagogy (pp. 261-289). Mathematical Association of America.
Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification – the case of function. In E. Dubinsky & G. Harel (Eds.), The concept of function: Aspects of epistemology and pedagogy (pp. 59-84). Mathematical Association of America.
Sherman, M. F., Meagher, M. S., Lovett, J., & McCulloch, A. (2019). Transforming pre-service teachers definition of function. North American Chapter of the International Group for the Psychology of Mathematics Education.
Sierpinska, A. (1992). On understanding the notion of function. In E. Dubinsky & G. Harel (Eds.), The concept of function: Aspects of epistemology and pedagogy (pp. 25-58).
Vinner, S. (1983). Concept definition, concept image and the notion of function. International Journal of Mathematics Education in Science and Technology, 14(3), 293-305.
Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. Journal for Research in Mathematics Education, 20(4), 356-366.
Vygotsky, L. (1978). Mind In Society. Harvard University Press.
Zandieh, M., Ellis, J., & Rasmussen, C. (2017). A characterization of a unified notion of mathematical function: The case of high school function and linear transformation. Educational Studies in Mathematics 95(1), 21-38. https//doi.org10.1007/s10649-016-9737-0