# WHAT IS A UNIT? BROADENING UNITS COORDINATION

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*Units coordination, defined by Steffe (1992) as the mental distribution of one composite unit (i.e., a unit of units) "over the elements of another composite unit" (p. 264) is a powerful tool for modeling students' mathematical thinking in the context of whole number and fractional reasoning. This paper proposes extending the idea of a numerical unit to an algebraic unit and a covariational unit. Evidence to support this extension is taken from two qualitative studies conducted with middle-grades students. Results suggest the coordination of two unknowns in algebraic contexts is limited for students who assimilate with composite units. Additionally, in the context of covariational reasoning, results describe a new partitioned unit that students who assimilate with composite units may apply. Implications for engaging all learners are discussed.*

Keywords: Algebra and Algebraic Thinking; Learning Theory; Number Concepts and Operations; Middle School Education

## What is a Unit?

Researchers over the past several decades have found children's construction of whole number is characterized by the types of units they construct and the operations or unit transformations they enact. Several different classifications of student's unit structures emerged from the literature including number sequences (Olive, 2001; Ulrich, 2016a), Units Coordination (UC) stages (Boyce & Norton, 2016; Hackenberg, 2007; Steffe, 1992), and multiplicative concepts (MC) (Hackenberg & Lee, 2015; Hackenberg & Tillema, 2009). Central to each of these is the level of units a student assimilates with, as well as characterized by different types of operations and transformation of those units (Ulrich, 2015, 2016a).

Steffe (1992) imagined UC as a cognitive foundation to numerical reasoning, where *UC* describes the mental distribution of one composite unit (i.e., unit of units) "over the elements of another composite unit" (p. 264). For instance, to reason multiplicatively about finding the number of plants in four rows with seven plants in each row, one must mentally join, or construct, a row as a unit containing seven units. This constitutes a composite unit of seven. Then, one can mentally distribute, or think about inserting, the unit of seven plants into each of the four rows. The garden as a unit of four rows constitutes the second composite unit in this example. This typifies one possible way that UC can apply to students' whole number reasoning, and how it forms a foundation for multiplicative reasoning.

In the UC paradigm, when a student has assimilated one level of units, their foundational unit are units of 1. This aligns with the Tacitly Nested Number Sequence (TNS) (Olive, 2001) and MC1 (Hackenberg & Tillema, 2009). With this unit of 1, stage 1 students, in activity, construct other units through iterating their unit of 1 $n$ times to make a unit of $n$ units of 1. This unit of units is a composite unit (Steffe, 1992). For example, a stage 1 student can construct 4 in activity by iterating their unit of 1 four times.

Stage 2 students, have composite units as their foundational unit because they have assimilated the process of constructed composite units into an object they can act upon. Thus, a stage 2 student

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can engage with multiple composite units to solve multiplication tasks multiplicatively rater than additively, as described above in the garden problem. In the garden

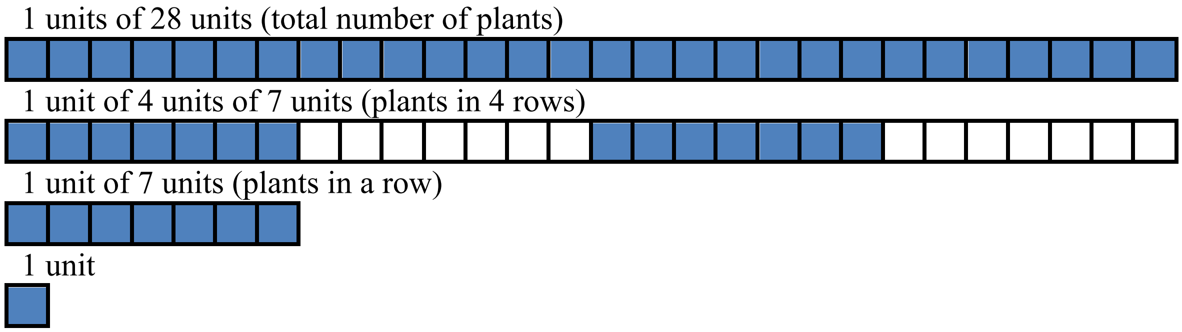



**Figure 1: Example of UC through solving 4 × 7 garden problem**

problem example, a student reasoning multiplicatively constructs a third level of units (1 unit of 4 units of 7 units) in activity to reach an answer of 28. Because a stage 2 student's foundational unit is a two-level unit, composite unit, when they arrive at the answer 28, it gets reduced back to a composite unit and the third level decays. This differs from a stage 3 student who assimilates with three levels of units. These students can assimilate the three levels of units needed to reason multiplicatively to get the answer of 28 while maintaining all three levels. Stage 3 students can also construct a fourth level in activity.

Overall, UC has proved to be a powerful tool in modeling students' mathematical thinking in situations of counting (Steffe, 1992; Steffe & Cobb, 1988), and additive and multiplicative reasoning (Hackenberg & Tillema, 2009; Steffe, 1992) but also in contexts of fractional reasoning (Hackenberg, 2007; Steffe, 2002; Steffe & Olive, 2010), combinatorial reasoning (Tillema, 2013), proportional reasoning (Steffe et al., 2014) and measurement (Hackenberg et al., 2021; Zwanch et al., 2023) These are a few representative examples of the immense body of UC research literature that has begun to explain students' mathematical behaviors by considering their mental coordination of numerical units.

More recent literature, however, begins to extend UC to a more diverse set of mathematical concepts including algebra (Hackenberg, 2013; Hackenberg & Lee, 2015; Hackenberg et al., 2017; Hackenberg et al., 2021; Zwanch, 2022a, 2022b) and covariational reasoning (Kerrigan, 2023). As seen in the example of UC above, the core of numerical reasoning with whole numbers the iterable units. This is also true of students' construction of fractions as measures. However, as researchers have explored connections with students' UC in other mathematical domains, the question arose of if there is an different foundational unit than an iterable units (Castillo-Garsow, 2014). Therefore, the purpose of this paper is to report two exemplars that broaden Steffe's (1992) unit to algebraic and covariational reasoning.

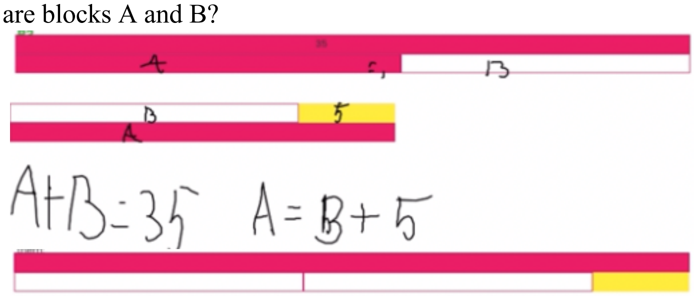
## Exemplar 1: Algebraic Reasoning

### Methods

This exemplar is a case of algebraic reasoning demonstrated by a sixth-grade student, Candace, participating in a constructivist teaching experiment (Steffe & Thompson, 2000). The student participated in eleven, one hour teaching episodes across one year during which she worked with a teacher-researcher to test the boundaries of her algebraic reasoning in a variety of contexts. In the majority of teaching episodes, Candace worked with a partner, but in the episode presented here, her partner, Liam, only joined them for the last 15 minutes. Prior to the teaching episodes, Candace and Liam were found to iterate units of 1 in mental activity and assimilate numerical situations with two levels of units (i.e., Candace and Liam had each constructed an advanced tacitly nested number sequence (aTNS). See Ulrich, 2016b). Their numerical reasoning was measured using a paper and pencil assessment written and validated by Ulrich and Wilkins (2017) and by a follow up clinical interview with the first author.

The task presented in this exemplar is a word problem that could be modeled by a system of two linear equations (Figure 2). The teacher-researcher had a goal of testing the extent to which Candace and Liam could write and solve a system of two linear equations in two variables to model and algebraically solve the task. In this portion of the teaching experiment, Candace is using virtual Cuisenaire rods. In previous teaching episodes, Candace and Liam worked with physical Cuisenaire rods, which were selected to make visible the students' operations with composite units (e.g., a yellow five block reflects a composite unit of five). Virtual Cuisenaire rods were selected for this teaching episode because they contain a variable pink and variable white block that can be adjusted to any integral length greater than or equal to one unit. Therefore, the students could use the variable blocks to represent values greater than 10 and could adjust the lengths of the blocks to represent unknown quantities.

Systems of Equations Task: Find two blocks, A and B, that together are 35 units long. Block A is 5 units longer than block B. How long are blocks A and B?




Note that the teacher-research created and locked the "35" blocks shown above. All other blocks were introduced, adjusted, and labeled by Candace.

**Figure 2: Candace's Blocks and Labels**

## Results

Initially when asked to solve a word problem that could be modeled by a system of linear equations in two variables, Candace used guess and check strategies. The teacher-researcher set a goal of introducing algebraic notation and determining the extent to which Candace could reason algebraically about the equations to solve the system. The teacher-researcher's hypothesis was that utilizing Cuisenaire rods to represent the numerical values in the problem might support equation writing and subsequent algebraic reasoning, however, Candace quickly memorized the lengths of each Cuisenaire rod based on their color (e.g., the yellow block is five units long) and she continued to solve the word problems using numerical reasoning.

**Candace's reasoning with numbers greater than 10.** The researcher next hypothesized that Candace might operate with unknowns more meaningfully if she were given tasks that she could not solve by relying on the lengths of the Cuisenaire rods that she had memorized. Accordingly, Candace was given the task in Figure 2. Larger numbers were selected to encourage Candace to work with the variable-length Cuisenaire rods, thereby limiting her ability to solve the task by recalling the length of each color of Cuisenaire rod. Working with the variable-length Cuisenaire rods was also designed to support Candace's thinking about unknowns.

To begin the problem, Candace added a variable pink (block A), variable white (block B), and yellow block (5) to the screen. She adjusted A longer than B, then aligned B and yellow end to end and placed A underneath (See Figure 2, Block Structure #2). Then, she made copies of A and B, and aligned them end to end under the existing 35 block. She said, "Too long!" and adjusted A and B to be shorter. The shorter A and B blocks did not hold the difference of five, so she adjusted their lengths a second time and said, "This might take me all day." After the third adjustment, Candace compared A and B to the 35 block, and the researcher observed that A and B together were 33 units long.

Teacher-research (T): How do you think you'll have to adjust them?
Candace (C): I think that I'm going to have to make each of them one longer.
T: Okay. Let's try it.
C: [Adjusted A and B to be one unit longer. Compared them to 35 (Block Structure #1). Compared them to each other (Block Structure #2).] Yay!
T: Awesome! And can you write two equations to represent your block structures?
C: [Writes A+B=35 and A=B+5]
….[Interruption as Liam joins and Candace explains her progress on the problem.]
T: So using only As, Bs, and fives, how else could we make 35? Is there another way?
C: I think I have an idea. [Makes 2 copies of B and adds a yellow (5) to the screen. Aligns the two Bs and yellow end to end under a 35 block (Block Structure #3).]
T: Ah, now you did that really quick. How'd you know that that was going to fit?
C: Because I figured that if you take two of those [B], put it down there, and then add a five that it would work.
T: Okay, that's good. [To Liam] Do you see how she figured that out?
Liam: Not really.
T: Yeah, let's see if we can get her to explain a little more. [To Candace] How'd you know it was going to be two Bs and a five?
C: Well, because, so when I was looking at that [points to Block Structure #2], I realized that it was 35 and kind of figured out what these [Bs] are. I figured out that they're 15 I'm pretty sure because 15 plus 15 equals 30 plus 5 equals 35.
T: Ah, so you figured out the numbers in your head?
C: Yeah.

The purpose of introducing this task was to motivate Candace's use of the equations to solve for A and B. From the teacher-researcher's perspective, Candace created equations and block structures consistent with inventing the algebraic substitution method for solving a system of linear equations when she created Block Structure #3 because the A in Block Structure #1 was replaced by a B and a yellow in Block Structure #3. However, this was not Candace's perception of the situation. Rather, Candace identified a numerical combination that summed 35 without consideration of the relationship between A and B. Furthermore, in the remaining three teaching episodes, Candace successfully wrote equations to model systems of linear equations but did not move from numerical to algebraic reasoning.

**Analysis of Candace's reasoning with numbers greater than 10.** The teacher-researcher's goal in this teaching episode was to pose situations that might motivate Candace's algebraic reasoning, however, that goal was not met. Instead, Candace reasoned strategically that she needed to "make each of them [variable blocks] one longer," which aTNS students may do by leveraging their operations on embedded composite units (Ulrich, 2016b). This suggests that Candace's foundational composite unit did not support her algebraic reasoning in this context. This is consistent with Hackenberg et al.'s (2017) finding that students who assimilate numerical situations with two levels of units tend to reason about numerical values rather than unknowns to reduce the complexity of the UCs in a task. The need for students, such as Candace, to reduce the complexity of the UCs is due to the nature of an unknown. To operate on an unknown quantity requires operations on two levels of units because an unknown is a unit that contains an unknown number of units of one (Hackenberg et al., 2017). Although aTNS students can operate on two levels of units (Ulrich, 2016b), the systems of equations task demonstrates that Candace's foundational units did not support reasoning about two unknowns (i.e., two two-level unit structures), nor her reasoning about the relationship between the two unknowns.

**What is a Unit in Algebraic Reasoning?: Implications for Algebra**

The exemplar presented here demonstrates one way that students with foundational composite units might be supported to solve problems that can be modeled by systems of linear equations in two variables, although not in the way that we, as mathematics educators, might expect. Candace leveraged an assimilatory composite unit to strategically guess and check values for the lengths of the variable Cuisenaire rods and to determine the numerical solutions to the problem. Candace did not leverage the operations of her aTNS, however, to coordinate two unknowns. These findings continue to broaden the applicability of UC to different forms of algebraic reasoning, as a way of supporting mathematics educators in better understand how students construct algebraic ideas. This finding also demonstrates how students' numerical UCs may relate to their operations in the context of systems of linear equations in two variables.

Additionally, classroom instruction is often limited to teaching algebraic methods to solve systems of equations (Oktaç, 2018; Proulx et al., 2009), which may limit the opportunity for students like Candace to engage meaningfully with the mathematics she is being taught in school. To engage all students, mathematics educators must continue to develop instructional strategies that differentiate in a manner that is sensitive to each student's operations with numerical and algebraic units.

## Exemplar 2: Covariational Reasoning

**Methods**

This exemplar is a case of covariational reasoning of an eighth-grade student who was part of a larger study that examined connections between middle-grades students' covariational reasoning, UC, and working memory. Through semi-structured clinical interviews and written assessments six students were selected from a pool of eight to achieve cognitive diversity in both UC and working memory. This report focuses on Daniel's, an advanced stage 2, work on one of the 12 covariation tasks given throughout the study. Each task consisted of its own Zoom semi-structured clinical interview (Clement, 2000) lasting 30-45minutes. The tasks were GeoGebra applets consisting of different dynamic situations the students engaged with on an iPad.

Each task protocol consisted of asking students general questions about what quantities they noticed and how those changed. After general questions about quantities, the students were asked about a specific relation between two quantities in the animation. For example, the fifth task in the task sequence, presented in this exemplar (Figure 4), consisted of a shape tripling in size starting with a square, then jumping right to a rectangle, then up to a square again. It repeated this process for several changes before restarting. In the second part of the interview students were asked to relate how the area of the shape changed with respect to time. For these two parts of the interviews, general and specific quantity relations, students were not allowed to write anything down to capture students' work relying solely on their working memory for another part of the study not reported here.

**Figure 4: First three stages of Task 5**

**Results**

A portion of Daniel's reasoning about how the size of the shape changes as time passes is reported here. Throughout the task, Daniel reasoned about how the shape changed through multiplication by three. This report includes responses from two of the task questions that highlight how he used his units coordinating structures to reasoning about how the quantities changed. The first of them is from the first part of the protocol and asked Daniel whether any of the quantities he identified as changing, changed in the same way. The second question was from the second part of the interview and asked Daniel how the area changed as time passed.

**Daniel's initial reasoning about how the area changed.** Prior to answering how the changing quantities changed, the interviewer asked what quantities in the animation changed. Daniel identified the size of the shape saying, "how big it's getting," and then specified length. The next question in the task protocol asked Daniel how any of the changing quantities he previously identified in the animation were changing in the same way. Daniel's initial response to this question was that the quantities changed in the same way. He described the change as being the same, "percentage-wise," before specifying they were changing by two-thirds. After running through the first few steps of the animation verbalizing his two-thirds idea while making corresponding sweeps with his hands, Daniel restated his claim to include a multiplicative change, "I think it's going up by two-thirds or tripling the whole thing."

Daniel was then asked a follow-up question how he concluded the shape tripled and not some other multiple. His initial response was that he "just estimated that it was going up by three or you know tripling itself." Here is the exchanging following his initial estimation explanation:

I: Okay, and so, how did you come to that estimate of what, why do you think it was tripling instead of doubling?

D: Well, a double would be shorter, like if it doubled itself it wouldn't work as long. I think I looked at how many spaces it would take up [makes length measure with thumb and index finger and makes a kind of iterating motion] if there, there were three of them, and if there were two of them.

I: Okay, and so, when you say like taking up space, what do you mean by that?

D: Or, okay I looked at how many how big like the square was by itself [points to bottom left corner of animation, starting square position]. And then I looked at um how many [makes bouncing along a line motion with stylus] if like, if you added on two more of those same squares [makes length form both index fingers and iterates it twice] how big would it be.

I: Okay, so you're sort of imagining copies of it?

D: Yeah.
I: And it looked like there would be three copies for each step?
D: [nods] mmhmm.

**Analysis of Daniel's initial reasoning about how the area changed.** In this first question, Daniel shows evidence of several different unit structures to capture how the shape's size changed. He first focused on the shape's form (square to rectangle to square) and spatial direction of change (right and up) in connection with the change of size. Then when he answered the question about changing similarly, Daniel transitions from using gross quantities to measured quantities when he introduces numerical transformations of two-thirds and tripling. His two-thirds answer suggests he had a partitioned unit and thought additively but then transitioned or encapsulated that into a multiplicative structure to get his tripling action. This description of the change as repeated multiplication by three is found in Ellis and colleague's (2016) learning trajectory for exponential growth.

In the follow-up question asking about how he generated his tripling idea we gain further evidence of Daniel's use of a unit to measure the size of the shape and construction of his tripling action. Daniel used the starting square as a measuring unit and mentally partitioned the rectangle into three and knew that could be represented by iterating his unit two more times. This led him to generate a partitioned unit of 3 as 1 and 2 (1,2). He then repeated that process with the rectangle to square change only he re-unitized his unit of measure to be the rectangle. Thus, he re-unitized and lost his composite unit structure going through the next shape change. However, he did maintain his partitioned unit structure of 3 as (1,2). The partitioned unit is a new structure identified by Kerrigan, (2023) and might have structural connections to higher dimensional units (Tillema, 2014).

**Daniel's shift in reasoning about how the area changed.** Later in the task, Daniel was asked, "As time passes, how is the shape getting bigger?" To which he responded:

> Um…Okay, so as time goes on, it's increasing it's like tripling itself or actually I think it's… I think it's like…Since it's tripling itself to the right and then doing it three more times, it's technically adding on just nine, nine of that unit, [slight pause] or eight more of that unit until like the one square in this corner. So, it's going to make 1,2,3 [moves finger over to the left with each number spoken] and then six up here [points above previous 1,2,3]. So, it's doing that three times, but it stops when it does look another 1,2,3 like this because but it's a very big square so.

The researcher asked more follow-up questions to confirm the hypothesis that Daniel now maintained his initial unit of the starting square and composite unit when constructing the larger square by two successive multiplications by three. In the follow-up questions, Daniel gave a more detailed description of how he got from square to square as a transformation of multiplication by three twice or by adding eight. Both of these resulted in Daniel measuring the second square as 9 units.

**Analysis of Daniel's shift in reasoning about how the area changed.** From this question, Daniel shifted from re-unitizing after each multiplication by three to maintaining the rectangle as a composite unit and constructed a third level in activity to make the square nine units. He also categorized the transition from square to square through addition by saying his multiplication by three twice was equivalent to adding eight of his starting unit square. Thus, Daniel was able to think about the resulting square as nine in two ways, additively as 1+8 and multiplicatively as 3x3.

The additive decomposition also shows that Daniel did not maintain the three-level structure of nine outside of his multiplication scheme which aligns with his stage 2 assessment.

Note, in his previous tripling scheme, Daniel also leveraged his partitioned unit of (1,2) to describe change from step to step. Here, he only had one utterance that suggested he also maintained the partition unit structure in this 3x3 scheme when he said, “So, it's going to make 1,2,3 and then six up here.” The six indicates that Daniel had two copies of the three units he just constructed in his 1,2,3 utterance (the rectangle). In the follow-up questions to confirm how he constructed his 1+8 and 3x3 models, Daniel did not mention 6 or any kind of (1,2) partition.

**What is a Unit in Covariational Reasoning?: Implications for Covariational Reasoning**

This exemplar shows one way in which units construction and transformations were leveraged by a stage 2 student in reasoning about how area changed. Daniel’s generation of a model of exponential growth as repeated multiplication by three for each time step is not surprising based on previous findings on exponential growth (Ellis et al., 2016). This analysis extends this to focus on the specific use of the unit structures to capture the change.

Daniel generated several unit structures as his reasoning evolved throughout the task sequence. He first generated a partitioned unit (1,2) to describe how he saw the quantities as changing the same “by two-thirds.” This transitioned to represent a multiplicative relationship of multiplication by three. This structure was a reconciliation of his additive and multiplicative worlds reflective of Confrey and Smith’s (1995) approach to covariational reasoning. The partition unit also might be fundamental for determining how amounts of change, change, a component of Carlson and colleagues’ (2002) covariational reasoning framework. Daniel’s generation of the six added units to the 3 rectangle units to get the 9 units of the square shows he constructed three levels in activity and represented and a second level of amount of change from his (1,2) structure applied to his composite unit rectangle.

An important part of Daniel’s covariational reasoning depended on his construction of a measuring unit. Many covariation tasks are designed for graphical settings and posed without any measuring units. Similarly, this task was designed without any given measuring units or tools. However, Daniel displayed his most sophisticated covariational reasoning when he generated is own measuring units and was thus able to leverage his units coordinating structures. This suggests an added importance to incorporating a measuring component to covariational task designs to increase accessibility of the tasks and for researchers to consider how students are using their unit structures when reasoning covariationally.

## Engaging All Learners

Engaging all students calls mathematics educators to consider and respond to the strengths and needs of individual students in our research and teaching. Les Steffe’s units coordination research programs have accomplished just that by positioning children’s mathematics in the forefront of research and teaching (Tillema & Hackenberg, 2017). We seek to extend this line of research to algebraic and covariational reasoning as a means of positioning these learners’ mathematics as legitimate and as building on the students’ mathematics as a foundation upon which mathematics educators can develop responsive curricula and instruction.

Better understanding the mental structures and activities needed for algebraic and covariational reasoning will allow researchers and educators to design activities and curricula that supports mathematical development in these areas. In particular, better ways to add supports that will make tasks accessible to students with different cognitive diversity in terms of units construction.

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