RELATIVE REASONING AND THE TRANSITION FROM ADDITIVE TO MULTIPLICATIVE THINKING IN PROPORTIONALITY

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Abstract. Research studies are abundant in pointing at how the transition from additive to multiplicative thinking acts as a core challenge for students' understanding of proportionality. This said, we have yet to understand how this transition can be supported, and there remains significant questions to address about how students experience it. Recent work on proportional reasoning has pointed to a type of strategy, called "relative", that appears to be lodged right between additive and multiplicative ways of thinking. This sort of "in-between" strategy raises significant interest and motivates further analysis. In this paper, I explore several of these relative strategies engaged in by a 13-year-old student, Marie, during a series of individual interviews. The analysis outlines several dimensions that can inform as much the transition from additive to multiplicative thinking than proportional reasoning itself.

Keywords: Proportionnality; Relative reasoning; Additive thinking; Multiplicative thinking

Proportionality is as much a fundamental topic in school mathematics as it is an enduring challenge for students of various grades. This situation continues to raise the need to investigate ways in which proportionality can be made sense of and explored in the classroom. As such, research studies have for a long-time documented difficulties experienced by students when attempting to solve proportionality problems (e.g. Behr et al., 1992; Lamon, 2007). One main issue is that proportional problems require a shift from additive to multiplicative thinking (Van Dooren et al., 2010). Whereas most problems students face in their initial school years focus on additive situations, proportionality confronts them with multiplicative ones, and there remains significant questions as to how this transition is experienced and how it can be supported.

Work by Copur-Gencturk et al.'s (2022) has recently drawn attention to a type of strategy, called "relative", that appears to sit right in between additive and multiplicative thinking. For example, they gave to teachers the following rectangle problem:

> The Science Club has four separate rectangular plots for experiments with plans. Which rectangle(s) looks more like a square? Explain your answer.
>
> a) 1 foot by 4 feet b) 17 feet by 20 feet
> c) 7 feet by 10 feet d) 27 feet by 30 feet

Usual type of strategies, ranging from incorrect (additive or else) to multiplicative, were used by the teachers. However, 17% of them engaged in another type of strategy, one that used the additive attributes of the problem, the difference of 3 feet, but went beyond it: the difference of 3 feet between both sides of the rectangle was assessed precisely in relation to the sides of each rectangle, when evaluating which one would look more like a square. Examples of answers were: "Because, since the side lengths are longer than all of the other rectangles, the 3 feet difference is less noticeable." (p. 8); "I think [option] d is most square because although the difference in feet of the dimensions is the same, the larger number will make it harder to tell that there is 3 feet difference." (p. 11). Copur-Gencturk et al. position these strategies as lying between additive and multiplicative ones, without necessary detailing more on them (their scientific objectives being different). Hence, their work raises interest in better understanding these relative strategies and what they entail. Taking the form of a case study, this paper reports on the analysis of these kind of relative strategies, here engaged by a 13-year-old student, Marie, during a series of individual interviews focused on solving proportional problems. The consideration and investigation of her relative strategies offers enriched understandings of the mathematical strengths lying at the core of these sorts of "in-between" strategies, informing in turn matters concerning the additive and multiplicative transition in the unfolding of proportional reasoning in students.

Transition from Additive to Multiplicative Thinking in Proportionality

The transition from additive to multiplicative thinking in proportionality is mostly known through the difficulties students experience: e.g., students use additive procedures in problems where multiplicative ones would be needed (Van Dooren et al, 2010). Solving problems in a proportional way requires the consideration of the multiplicative structure of the problem (Vergnaud, 1988), something that additive procedures do not necessarily succeed in doing. One main difficulty can be defined as the "additive invariant" conception, where students focus on the constant differences between quantities. Brousseau's (1998) puzzle situation is a good example of such (Figure 1), where students are required to enlarge a square-shape puzzle, where a segment measuring 4 units in the initial puzzle has to measure 7 units in the new one. To do so, many students engage in additive thinking, adding 3 units to all other sides of the puzzle. This ends up modifying the initial square format, and the pieces no longer fit together.

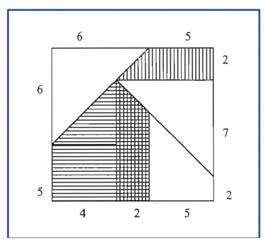


Figure 1: Illustration of Brousseau's (1998) puzzle situation

To inform the transition from additive to multiplicative thinking in proportions, numerous researchers have brought forth development models to help appreciate the many "levels" students are found to be working in concerning proportionality. Based on a variety of studies, Steinthorsdottir and Sriraman (2009) have evoked "three levels of strategies that students use as they grow in understanding proportional relationships" (p. 8). The first level is of qualitative nature, the second is additive, and the third multiplicative. To elaborate more precisely on each, consider the following "cat food problem" taken from their work:

> It is lunchtime at the Humane Society. The staff has found that 8 cats eat 5 large cans of cat food. How many large cans of cat food would the staff members need to feed 48 cats? (p. 8)

The kind of qualitative strategy that some students could display would be centered on an appreciation of the relation existing between numbers without using direct calculations. For example, one student could say that "they would need a lot more cans because 48 cats is a lot more than 8 cats" (p. 8). Without being false, it might be said that it lacks precision as to how many cans of food will be needed. The second type of strategy is based on additive thinking, which has often been termed *building up* strategies (Hart, 1981; Pulos & Tourniaire, 1985). In these kinds of strategies, students could, for example, add up values by jumps of 8 cats and 5 cans to arrive at the aimed-for number of cans for 48 cats: 8 for 5, 16 for 10, 24 for 15, 32 for 20, 40 for 25, and 48 for 30 (they could also combine these, like 8 for 5, 16 for 10, and directly 32 for 20, and another for 48 for 30). Although additive, these building up strategies are most often successful, especially when the number of jumps is described by a natural number (here 5). Finally, there is the level of multiplicative strategies, often represented by the following algebraic equation: 8/5=48/x. In using this equation, students can focus on the ratio existing between 8 and 48 (6 times more) and adjust for 5 and 30. They can also focus on the ratio existing between 8 and 5 (5/8 of) and adjust for 48 and 30. This said, those are only examples, as other strategies like the unit-rate or the cross-product could be engaged with.

Carpenter et al.'s (1999) model raise other distinctions for the additive dimensions. Whereas the first level concerns students' random calculations or focus on additive differences, the second level also concerns building up strategies; where students see the ratio of 8 for 5 as a single unit and can repeat it until the desired value is found (or can use the multiplication by a whole number, where 8 times 6 gives 48 and so 5 times 6 gives 30). These strategies are successful and intertwine with some multiplicative dimensions. However, in this second level, students are not able to work with fractional multiples, that is, when a problem does not imply the multiplication by a whole number. This ability would represent the third level, where a ratio is seen as a single unit but can also be repeated a noninteger number of times (or when fractioning the ratio is needed). E.g., if the problem called for 44 cats instead of 48 cats, students at level three could add 8 for 5 up until 40 for 25, and then split in two the ratio to get a 4 for 2.5, leading to 44 cats for 27.5 cans of food. Finally, at the fourth level, the focus goes beyond the ratio as a single unit and into a consideration of the multiplicative relations between the numbers in the problem. Hence, the whole-number multiplicative relation from 8 to 48 or the fractional one from 8 to 5 can both be considered in solving the problem for finding the desired value.

Another developmental model frequently referred to is the Piaget sequences on proportional reasoning (Inhelder & Piaget, 1958; Piaget et al., 1967). Slightly different, Piaget's model also has four steps ending in a mastery of the multiplicative relations in the problem. Piaget mentions as well that the first steps taken in solving a proportional problem are often additive, focusing on respecting the differences between the numbers in the problem. Then students migrate to what he calls pre-proportionality, which still uses additive, and incorrect, thinking, but adapts the numbers in relation to their size: the additive difference need not be constant and varies with the size of the numbers. To understand the difference between these two steps, imagine that in the catfood problem there is a third house with 24 cats, and one wonders how many cans of foods would be needed. Students in the first step would focus on the additive difference of 3 between 5 and 8, and would reproduce it between 24 and 21 cans, as well as between 48 and 45. Students in the pre-proportionality mode could, however, focus on the fact that the third house seems somewhat "in the middle" of the first one of 8 cats and of the second one of 48; hence the number of cans would have to represent that middleness. Whereas the first house of 8 cats needs 5 cans of food, the second one could use 15 and the third 25: differences change and follow the size of the numbers associated with it. This pre-proportionality step is followed with the logical proportions step. This time, students consider the set of (multiplicative) relations existing between the four numbers in a proportion. Thus, they understand that increases or decreases in any number cause corresponding increases or decreases in the other. Students in the logical proportions step often succeed with simple problems (ratios of 1 for 2 or matching numbers like 2 and 8), and mostly develop local understandings of the problem at hand: the functioning of these relations are not extended to other problems, nor conceived as a general law of proportions. This abstract level (e.g. 8/5=48/x) would constitute the final and fourth step. As Pulos and Tourniaire (1985, p. 187) explain: "According to Piaget, adolescents' proportional reasoning develops from a global compensatory strategy, often additive in nature, to an organized proportional strategy without generalization to all cases, to finally the formulation of the law".

Other models and theories have been proposed over the years, refining or offering other distinctions between their levels (e.g., Resnick & Singer's, 1988, protoquantitative reasoning). These help flag the significance of mastering the multiplicative relations between the numbers in proportionality problems and raise the fact that there exists important steps to get from an additive take on the problem to a consideration of its multiplicative structure, and then to be able to reason proportionally on it. As this transition continues to need to be better understood, efforts being made toward investigating it have led to important advances (e.g., Singh, 2000, study using Steffe's coordination of units; Van Dooren et al., 2010, study of students' strategies in proportional and non-proportional problems). These studies raise recognitions of how additive and multiplicative strategies differ, how students' strategies can oscillate between the two, as well as how specific characteristics of the problems can create difficulties for students (e.g., numbers, context). These studies also show how hard it is to pinpoint at specific elements that can play a role in (understanding) this transition, and how there continues to be much to be understood about it. Can the investigation of relative strategies contribute to understandings about this transition? This is the declared intention of this paper, through exploring examples of strategies that one student, Marie, engaged in when solving proportional problems.

Methodological Considerations

The data analyzed in this paper comes from a larger study on proportionality, consisting of two phases: a first one about conducting individual interviews and a second one consisting of classroom-wide experiments. Because the objective of this paper is to scrutinize the use of relative strategies for solving proportional problems, it only reports on selected strategies that Marie (13-year-old student, Grade-8), engaged in during the first phase of individual interviews. Over a period of three months, interviews were conducted with Marie, who had recently been taught proportions during her school year. The interviews were conducted by the PI, using an online environment for recording purposes (except for one face-to-face interview). The interviews consisted of presenting a problem to the student, who had to solve it and then explain her answer (orally, using paper, or drawing on the computer). Each interview lasted between 20 to 30 minutes, depending on the students' availability. In total, Marie solved 16 problems. These were chosen or inspired from a review of the current literature, which abounds in problems on proportionality, and mostly ranged from missing-value problems, comparison problems, and transformation problems (as classified in Lesh et al., 1988). This paper reports on some of Marie's strategies with comparison problems, in which she engaged in relative strategies.

The data reported on are circumscribed and only concern strategies engaged in by Marie that align with Copur-Gencturk et al.'s synthetic description: engaging with additive attributes while considering the relative value of these same additive attributes in relation to other quantities in the problem. As such, the data is analyzed through elements outlined about the transition from additive to multiplicative thinking. Note that the goal here is not to report on all of Marie's strategies, nor to discuss the evolution that might have happened or the long-term outcomes of this work with her. Instead, the scientific intention is to gain a better understanding and make sense of relative strategies when solving proportionality problems, as some sort of case study. As

Lamon (1994) and Singh (2000) argue, the detailed analysis and precise identification of students' ideas and thinking processes can enhance understandings of how to work with specific mathematical content. This is particularly so, Lamon flags, if these processes and ideas can act as connectors for relating different mathematical dimensions (here, for additive and multiplicative thinking). In other words, the intention of the data analysis is to directly address and explore relative strategies, and investigate how these can inform understandings of proportionality, ways of working with it, and the transition from additive to multiplicative thinking.

Investigation of Marie's Relative Strategies

In the following, three examples of relative strategies engaged by Marie are presented. These are discussed and analyzed through aspects of the additive and multiplicative transition.

The Rectangle Problem

The rectangle problem was given to Marie during her interviews (using meters instead of feet). Marie's response is that the 27x30 rectangle is the one closer to a square. She explains that even though all rectangles have a difference of 3 meters between their sides, these 3 meters would appear particularly small for the 27x30 rectangle, and big for the 1x4 rectangle. Using her fingers (Figure 2a), she shows what the 27x30 rectangle would look like and how, because each side is almost the same, the 3 meters would appear small. She then contrasts this for the 1x4 rectangle. Keeping one finger up from her 27x30 representation, she elongated from the tip of that finger to simulate how far apart the 4-meter side is from the 1-meter one (Figure 2b).

Figure 2: Marie's illustration of (a) the 27x30 rectangle, (b) the 1x4 rectangle

This was Marie's way to show how "not-square" the 1x4 rectangle is, and how the difference of 3 would be greater in this case, making it obvious that it is a rectangle, because it is "4 times more" as a length. She continues by saying that it would be the same for the 17x20, meaning that the 3 meters would also make it more apparent *than the* 27x30 that it is a rectangle, and again for the 7x10 where the 3 meters would make "even more" apparent (than this latter 17x20) that it is a rectangle. Her explanations offer a sort of gradient of rectangles, from being more rectangle-like with 1x4, toward a lesser one with 27x30. She concluded by insisting that the sides of the 27x30 rectangle are almost the same, but with a difference of "merely 3 small meters", adding that with the side of 27 meters, the 3 meters is "almost nothing, it is a mini-fraction of it".

Marie's strategy appears to go beyond a quantification of the difference between the ratios forming the sides of the four rectangles (for then comparing with a 1 for 1 ratio for a square). Her strategy is focused on a relative quality of the constant difference of 3 between the sides of the rectangle. This translates by a movement toward getting closer and closer to a square, the more the dimensions of the rectangle enlarge: the "same" difference of 3 is at times small, at times big, bigger, "merely 3 small meters", or "almost nothing", in relation to the dimensions of the rectangle. The consideration of this additive invariance of 3 meters is something that can lead students into difficulties, where they could answer that all rectangles are as “square-ish”. But Marie’s take on this additive difference of 3, although invariant, is not conceived in absolute terms: it is considered in relation to other quantities that are themselves varying (i.e. each rectangle’s side dimensions), which in turn make this 3 vary *qualitatively*. It thus makes these 3 meters a kind of variable entity in relation to each rectangle to which it gets associated. But this 3 meters does not vary in terms of quantity since it is the same numerical 3 everywhere. It varies in terms of its relative nature, its quality. It somehow stops being a plain constant 3 meters and merges into a qualitatively variable 3 meters, relative to its referent, that is, the dimensions of the rectangle from which it comes. Marie’s relative strategy raises a qualitative-quantitative interplay, leading her beyond an additive strategy in considering values as relative in the problem. She also engaged in similar relative strategies when faced with a purely numerical problem and even one without numbers. The following two are examples of such.

The Group Increment Problem

The following problem was given to Marie (inspired by MEO, 2012):

> Which one of the two groups of persons experienced the most important increase?
> The Frimousses were 3 and are now 9
> The Grippettes were 100 and are now 150.

A focus on the numerical differences, before and after the transformation, would lead one to assert that the increase of 50 for the Grippettes is a more significant one than the one of 6 for the Frimousses. Marie did not opt for this additive route and engaged as much relatively as multiplicatively in it. She expressed that it is the Frimousses who have a more important increase, since 9 is 3 times 3 and 150 is only 1.5 times 100. She was then questioned on the fact that from 100 to 150 there is a difference of 50, and that from 3 to 9 there is only a difference of 6. She responded by saying that it is the increase one needs to focus on, and not the 50 nor the 6.

> Marie: it is not the number that counts, it is the biggest increase that we look for. In the 1st group there is 3 times more people than before, and in the 2nd group there is to the half more.

Her explanations reveal two types of multiplicative considerations: there is the “3 times more”, and there is a blend of an additive and a multiplicative entry with her “to the half more”. In the first case, the difference of 6 is not explicitly considered, where in the second case the difference of 50 is. This shows how she here oscillates between an additive and a multiplicative mode. Her “to the half more” concerns the 50 and is as much a multiplicative relation (50 is half of 100), as it is an additive relation (50, this half, is added to 100). Questioned about what she meant by “looking for the increase”, she added how considering the increase leads one to go beyond the numbers and consider them in relation to another number (i.e. the initial value):

> Marie: well, it is not linked to the number itself [the numerical value of the increase], it is related to how much times more it is [than the initial value].

Marie’s strategy highlights again the difference between the numbers themselves, a quantity in the absolute, and that same number “in relation” to another one, which gives this number its true value in the problem. This points anew to the varying quality that a number can take in these situations, and how its value is relative. Counterintuitively, an increase of 6 is now somehow becoming bigger than one of 50, when seen in relation to where it came from; in the same way that getting to 9 is more impressive than getting to 150, since what counts is relative to where it started. Here, this 50 becomes a relative number, a “relative 50”, in relation to 100.

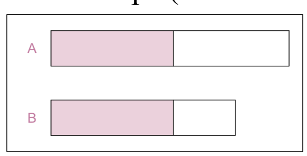
These explanations illustrate at least a part of what underlies Marie's relative consideration of the numbers at stake, and how it goes beyond an additive consideration of them: she considers not only the number itself, but the number in its relation to other numbers. There is more: considering the relation between quantities is not enough, since an additive relation is still a relation! Marie's take on the relativity focuses on the *multiplicative relation* that exists between these numbers: the 3 times 3, the half of it, etc. Hence, the relativity of the numbers in this proportional problem is of a multiplicative nature. More than seeing 6 as big because of 3, and 50 as small because of 100, the relativity for each number emerges from a multiplicative connection: where 9 is three times as 3, and 50 is only half of 100. This leads her to bridge additive and multiplicative issues as well as to establish differences between them. She can see how the additive differences can be considered, and also sees them in a multiplicative way in relation to, relative to, another number. This became salient when, at the end of the interview, she was challenged once more on these same matters. The following discussion took place:

PI: So, if you had 3 pairs of earrings, you would prefer that your dad buys you 6 more pairs than if you have 100 pair of earrings and your dad buys you 50 more?
Marie: Oh no! The thing is not that he buys me 6 more, but that he buys me 3 times more!

Marie is here making quite plain how her relative connotation of the additive difference in the problem is grounded in multiplicative matters. This shows how her consideration of quantities as relative is a move toward multiplicative thinking itself, about considering the multiplicative relations that exist within the quantities in the problem; and how these relations transform the qualitative value, thence the quantitative value, of the numbers.

The Color Comparison Problem

Marie engaged in similar relative proportional reasoning in the following problem without numbers, where she had to compare colored strips (from MEO, 2012; inspired by Small, 2008).



Which figure contains more color?

Marie asserted that the B strip had more color, relative to its length: "they both have the same amount, but B is smaller". She explained that both colored parts were the same, but that she considered them in relation to their respective strip, which made them different. The amount of color was seen as constant, but became relative to its entire strip, hence part of a different whole (and not considered in the absolute). She explained this by referring to surfaces and volumes:

Marie: they have the same surface of color. But the cylinder A is bigger so the fraction of color will be smaller whereas in B it will be bigger.

Her reference to the fractioning of the strip illustrates again how the notion of relativity is connected to a multiplicative consideration of the situation. The length of color is not viewed as absolute, but is in relation to another length, of which it is a part, and that modulates its value. Her reference to "a fraction of" points to how the same length of color becomes smaller when taken in relation to another length, which makes it a fraction of it.

Relative Proportional Reasoning and Multiplicative Thinking

Not only are the quantities relative in the examples of Marie's relative strategies, but these quantities become grounded in multiplicative considerations. Numbers, amounts, differences, etc., are not considered in absolute terms, but are constantly placed in relation to other ones that modulate their value in the problem: it is a part of, it is in relation to, etc. This relativity appears to act as a core element for going beyond a unique additive consideration of numbers for embarking in multiplicative thinking. These quantities are not, again, taken as absolutes, but become relative through their multiplicative link with another quantity. This multiplicative link transforms their worth in the problem. To better appreciate this difference, consider the following problems from Van de Walle et al. (2008, p. 168; Figure 3).

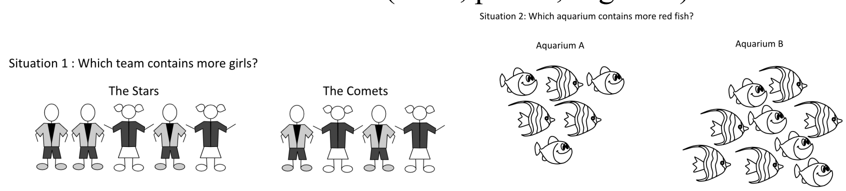


Figure 3: Reproduction of comparison problems from Van de Walle et al. (2008)

To engage additively in these problems, one needs to consider the number of girls or of fish, and then compare them. The answer to "which has more?" becomes one of comparing the numbers obtained and asserting which is the biggest. However, engaging proportionally in these problems means considering the number of girls or of fishes in relation to another amount, which in turn modulates its value. For the fish problem, this other amount can be the number of white fish or the total number of fish in each aquarium. An additive answer leads one to say that there are 4 fish in B and 3 in A; hence aquarium B has more red fish, because 4 and 3 are considered in absolute terms. A proportional answer, however, would lead one to say that e.g. aquarium A has 3 red fish for 3 white fish, whereas aquarium B has 4 red fish for 5 white fish; hence B has more white fish than red, and thus aquarium A has more fish when the other number of fish is considered. The answer to these problems becomes relative to these other amounts in it.

It is this relativity that not only leads beyond an additive take on the problem, but also to the consideration of the relation between the quantities in the problem: here a 1 to 1 or a 3 to 3, for example, in the fish situation, or what Marie has called "a fraction of" in the color problem. Being "relative to" leads to consider, literally, the relation that exist between the quantities. Marie's relative strategies thus taps on some important reflections on the transition from additive to multiplicative thinking in proportionality. Proportional reasoning is not only about quantities, it is about quantities in relation to other ones, hence relative, in their multiplicative relation. This is what happens in Marie's relative strategies: problems can be solved additively, but it is when they are considered under their multiplicative nature that they become proportional.

Concluding Remarks

That students experience difficulties in proportional reasoning is not new. One main obstacle has often been seen as a matter of transitioning from additive to multiplicative thinking, and how there seem to be a wide gap lying between these. The notion of relative strategies raised by Copur-Gencturk et al. seems to have some potential for contributing to our understanding of this difficult transition, being situated right in between both. As shown, Marie's relative strategies call upon the additive and multiplicative structure of the problem, while controlling the quantities or magnitudes of the problem proportionally. The “answers” given were not obtained uniquely through calculations but understood in relation to other quantities and magnitudes in the problems to solve. This might offer conceptual keys for informing the difficult transition from additive to multiplicative thinking. Marie’s relative strategies point to how the passage from additive to multiplicative called for a qualitative shift, and how it connected the two. This qualitative shift on numbers might be an element to reflect on concerning the multiplicative structure that proportional problem requires: 3 is not merely a 3, it is a 3 in relation to a 27, to a 9, etc.; hence it is multiplicatively connected to another quantity in the problem, related to it. Obviously representing an avenue requiring more study, the consideration of aspects of relative strategies has potential to push further our current understandings of the transition from additive to multiplicative thinking in proportionality.

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