INTUITION AND FORMALIZATION IN THE UNDERSTANDING OF THE MATHEMATICAL INFINITE: THE CASE OF OMAR

INTUICIÓN Y SIMBOLIZACIÓN EN LA COMPRENSIÓN DEL INFINITO MATEMÁTICO: EL CASO DE OMAR

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The development of the mathematical concept of the infinite, through the reflections that arise from personal notions and perceptions and the analysis of some ideas of Galileo and Cantor, invites us to investigate the relationship between intuition and formalization for the understanding of the said concept. This paper aims to observe and describe the shift from intuition to formalization that Omar, a first-semester undergraduate student in applied mathematics, goes through when accounting for the number of objects between two equal mirrors, one in front of the other. The results indicate that the interaction between intuition and formalism is necessary for Omar to conceive of mathematical infinity.

Keywords: advanced mathematical thinking, cognition, learning theory, mathematical knowledge for teaching.

Mathematical infinity is a concept that has had a diversity of meanings throughout the history of mathematics (Kidron and Tall, 2015). Across the times, it has been observed that the efforts of thinkers to make sense of this concept were characterized by a permanent intertwining between symbolism and intuition. The following mental exercise represents how this intertwining can occur. If you could have a bag full of 5,000,000 candies and you decide to give one of them away, you would surely feel that the bag is still full; it would be said that still the same amount of sweets remains. If we instead have a bag with 50 candies, and we decide to give one away, perhaps we could still affirm that the bag is full, but our intuition suggests that the number of candies is a bit less than 50. In number symbols, we rather doubt that 49 is 50. Finally, if we have a bag with 5 candies, and we give one away, we could no longer say that we continue to have the same number of candies. Our intuition cannot allow us to say that 4 is almost 5; neither could a symbolic justification allow it. From this mental exercise we can affirm that as a set has increasingly more elements, our intuition exhibits a certain insensitivity to the loss of a few elements, that is, a small number with respect to the original size of the set. However, if the original set is small, losing one is a sensitive loss. Reading the thought experiment, we felt a certain familiarity with its plot line. We have internalized our experiences with numbers and have transformed them into a symbolic, mental model that we can manipulate as if it were something material.

Donald (2001), in his book, A mind so rare: The evolution of human consciousness, explains how human cognition, shared with other species, in addition to its analogical or holistic nature, has evolutionarily acquired a capacity for symbolic representation. That duplicates, in a certain sense, the world of our experiences from the material world. Of course, this symbolic ability gradually acquires a (relative) autonomy that is reflected both in the world of art and in the world of science.

And, of course, in mathematics. The mental exercise that we previously described is related to the encounter that we have, at an early age, with the sequence of natural numbers. When counting 1,2, 3..., at some point we feel that we are not going to finish. No matter how hard we try. Here we have an encounter between the meaning of the endless and the infinite —as an action without end. Our symbolic abilities, apart from allowing us to recode the world of our experiences, enable an access to a level of reality that no other species experiences: the world of the abstract.

Our cognitive ability is hybrid: we have a direct contact to the material world around us and we can symbolize (at least partially) this world transforming symbolically those raw materials. As if the symbolic structures were an external version of a world of experiences that lives within us. There is a permanent swinging between these two worlds every moment we begin exploring a piece of mathematical knowledge previously unknown to us. In general, the material world and the symbolic world are intertwined to generate a swing between our intuitions and the corresponding mathematical formalism. The infinite is right there: in the coming and going of intuitions and symbols as we induce, generalize and deduce. Like an endless dance, so is the cognitive act: a constant intertwining between the material world and the symbolic world. In mathematics education, the concept of mathematical infinity has been taught precisely from a symbolic perspective: infinity as the distant, the unattainable, the imperceptible. And, at the same time, as the uncountable, the inexpressible.

Let us now continue with our introductory discussion of infinity. To do this, let's go with our students to read a short passage from Galileo's book, Dialogue on two new sciences (Galilei, 1638, pp. 31-33). There Galileo's characters (Salviati, Sagredo, and Simplicius) discuss how each natural number can be associated with its square. Then a list of the squares can be made: 1, 4, 9, 16...and so on. Galileo concludes that such a list is endless. But then you must accept that the list is never-ending, since you can order the members of that list: first, second, third, fourth,... and this requires the use of all natural numbers.

Galileo is facing a paradox: All natural numbers are needed to count the squares. However, it is clear to him that there are considerably more naturals than there are squares. The squares constitute a fragment, so to speak, of the (set of) natural numbers. Then he concludes that, under these circumstances, it does not follow that the whole is greater than a proper part, as common-sense dictates. Thus, he manages to avoid the paradoxical situation. Our feelings are that Galileo found a dead-end as he followed this argument.

We consider that Galileo was quite close to opening a new perspective for mathematics at that time. He failed, although he advanced as far as the episteme of his time allowed. One lives in a cultural atmosphere, surrounded by cultural ways of understanding and ways of thinking that open up conceptual perspectives on some ideas and make it practically impossible to consider others. Though at some future moment, a visionary (individual or collective) opens what we could name a zone of potential development within this culture and slowly emerges a new conceptual reality.

Galileo looked at the correspondence of a natural m to its square m2 not as a way of comparing number collections, but as a way of counting how many squares he could produce: one, two, three...and so on. It is incorrect he was defining a bijection. Regarding the tension between our intuitive, sensory-motor knowledge, and the formalization of mathematical knowledge, the American mathematician Pierpont (1899, p.
406) wrote:

> The analysis of to-day is indeed a transparent science. Built up on the simple notion of number, its truths are the most solidly established in the whole range of human knowledge. It is, however, not to be overlooked that the price paid for this clearness is appalling, it is total separation from the world of our senses.

After the passage from Galileo´s book discussed earlier with our students, we asked them the question: What happens if we delete all the odd numbers from the list of natural numbers?As Galileo explain, we can count the even numbers and, for this, we will have to use all the natural numbers. So far, students were following Galileo's reasoning for counting squares. But now a mathematical notion that Galileo lacked was present: the notion of set, which opens a conceptual space so that the list of even numbers could be conceived holistically; that is, to conceive it, as a complete infinity, to distinguish it from infinite processes when trying to count the list of even numbers. There was an obstacle that Galileo was not able to surmount: while we perceive one set as a proper subset of another, it becomes almost unacceptable to see the subset as having the same number of elements than the larger set.

We cannot say that the students´ thinking and the paradoxical situation experienced by Galileo coincide at the cognitive level, nevertheless we can affirm that there is a common root between them explained by embodied theories of human intelligence.

M. Donald (2001, p. 155) explains, that:Basic animal awareness intuits the mysteries of the world directly, allowing the universe to carve its own image in the mind. This is a largely receptive mode of knowing, and we share it with our animal cousins. In contrast, the symbolizing side of our mind is more aggressive in its approach. It creates a sharply defined, abstract universe that is largely of its own invention.

Our symbolic universe permanently establishes its own rules trying to give meaning to what it perceives. These two ways of thinking and conceiving, the holistic one and the symbolic, continue to revolve around each other. It is in this sense that we affirm that our cognitive capacity is hybrid. The symbolic universe develops in a cultural atmosphere that gradually recodes, culturally, the natural world. Including, of course, our intuitions.

We agree in this sense with Pierpont´s words according to which the cognitive price we will pay is too high if we try to erect an insurmountable wall between our intuition and the formalized versions of mathematics, particularly those referring to infinity.

From a Cognitive Consideration to the Didactical Laboratory

According to Georg Cantor, an actual or completed infinite set is characterized by the existence of a one-to-one correspondence between the set and a proper subset of it. The concept of one-to-one correspondence was within Cantor's reach but was not available to Galileo. This is an important conceptual difference. Galileo was upset when he was not able to resolve the paradox of infinite sets. His mathematical (and cultural) ways of understanding made it impossible for him to conceive of infinity as a property of sets. The existence of one-to-one correspondence came to answer, with Cantor, the unanswered questions Galileo himself had formulated. Nevertheless, we can find an analogy involving finite collections. For instance, if everyone is sitting in a room except one person, we know that the number of persons exceeds by one the number of chairs. However, this commonsense observation or a similar one, was not enough for Galileo to lead him to conceive of infinite collections as Cantor did. A key conceptual tool was missing… This discussion generated a hot debate among the students interviewed in our small laboratory. Their previous knowledge and reflections had not been aimed at becoming aware of how an idea as logically as simple as the one-to-one correspondence between two sets, contained the key to access to the actual infinity that exists in that virtual reality called mathematics. Then, it was due time to reflect on how the original intuitions they had about the number of elements in a set, (obtained by usual counting), should be "forgotten" when dealing with infinite sets.

In a well-known letter to Dedekind, regarding this phenomenon of infinite sets Cantor wrote: "I see it, but I don't believe it," (Zimmerman, 2013, p. 452). Students, as had happened to Cantor, saw it, but they felt forced to accept it —even if they did not believe it!

Intuitions are the unarticulated consequences of embodiment, which we acquire through our experiences in the human world, even if we cannot make them explicit. We cannot change them at will. This is because our intuitions, when trying to explain a more or less known fact, do not obey a strict logical-deductive control and, therefore, their level of coherence is, at most, local. If we say that the whole is greater than each of its parts, this seems acceptable to us. But this acceptance ends when we remember (or learn) that we can establish a one-to-one correspondence between the set of natural numbers and the odd numbers. So, we must set limits to that popular sentence which seemed so natural as to be taken as universal.

Through school education, the moment arrives when the logical part of our rationality prevails temporarily, so to speak, over our intuitions. However, it is necessary to remember that this logical part of our thinking always hides deep down, an intuition that is fragmented or separated from our thinking. Our classroom experiences made tangible the belief that, for students, to learn the basics of infinity was such as walking through an abrupt territory. It was risky to try guessing the answers. They felt they were missing the necessary tools to address the problems emerging from the new cognitive and mathematical demands. We asked ourselves: how can we face the cognitive problems coming from the study of infinity, considering that the main problem is the development of meaning? The students were facing a formalized terrain.

Feeling the underlying tension we saw in their faces; a transition was a must. The transition from instrumental fluency to cognitive fluency is an educational problem of global concern. Indeed, what is simpler from the logical, formal point of view, does not necessarily coincide with what is simpler from the intuitive point of view. Both aspects are constitutive parts of human cognition.

Experimentation and Search

The activity El infinito en el espejo, is part of a sequence of activities whose purpose is to recognize, in freshman students of mathematics in a Mexican university, the swing between the intuitive conception of the concept of infinity and its formalization that generates the need to specify a mathematical, formal answer. The device designed for the activity consisted of two mirrors of the same size, placed one in front of the other. Among them were 9 plasticine balls with the same volume and the same color. The students had to look at their reflection in the mirrors and determine how many balls they could count. After allowing the students to observe the device, the teacher (the first author of this article) would ask the group the number of balls they saw. The answers should be spoken aloud and had the objective that all the students could reflect on what their classmates answered. One of them, Omar, reflecting on how the reflections of the balls would be in the mirror if they were removed one by one, makes a direct comparison between the amounts of balls that are reflected. He comments:

> infinities come in different sizes; and that depends, if it is one [he refers to a ball] then it will be smaller than an infinity of nine [ball], but at the same time it cannot be bigger, because being natural numbers they still belong to the same infinity.

In this statement, Omar clearly specifies that his intuition makes him think that there would be two infinities of different sizes: the one that is formed with the reflections of a single ball and that of the reflection of nine balls. For Omar the first infinity is formed by multiples of one; the second, by multiples of nine. It seems intuitive to him that these two infinities identified by him will have different sizes ("it will be smaller than an infinity of nine"). Recognizing that in both cases the number of balls is related to the natural numbers, his intuition is held back by his formal mathematical knowledge. This leads him to conclude that they would be infinities of the same size ("but at the same time it cannot be larger because, being natural numbers, they still belong to the same infinity"). Omar continues with his reflection aloud, establishing a difference between intuitive knowledge and formal knowledge:

In other words, he explains that intuitively it seems that it is bigger, but if we put mathematics into it, no. It is still the same infinity of natural numbers and there cannot be different infinities of natural numbers because it is the same infinity.

When Omar says, "but if we put mathematics in it", he reveals how he seeks justification from the logic that his formal mathematical training provides him. This search allows him to make sense of his answer.

Omar, motivated by the researcher's questions, talks about the existence of sizes of infinities different from the natural numbers:

He had said that there are larger infinities such as the infinity of the real numbers. And this is because there are irrational numbers like π and fractions like 1/4 that each "skyrocket to infinity". That is why it is a greater infinity. Because they are like many infinities combined into one; then it is bigger than a single infinity of a single thing: the natural numbers.

Omar intuitively expresses what real numbers are for him. Briefly explain that real numbers include irrational numbers, such as π, an rational numbers, such as 1/4, and clarifies that infinity is embedded in each number. By expressing "there are irrational numbers", Omar implies that he knows several of them, although he only gives π as an example, he can then make sure that he knows more, like √2 or e and adds them to the set of rational numbers (he says "fractions"). With this, Omar makes tangible that he knows how the set of real numbers is formed.

Now, by giving π and saying that "each one fires to infinity", he refers to the fact that infinity is embedded in each of the irrational numbers. This is so because, for Omar, the characteristic of the decimal, nonperiodic expansion of every irrational number shows the existence of an infinity for each number. But Omar does not end his reflection on infinity here. Thus, as each irrational number has its own infinity, the set of irrational numbers has an infinity greater than the infinity of the natural numbers. Omar's reasoning is based on his intuition that the non-ending fractional part of an irrational numbers means that they are not identifiable with the same clarity as natural numbers. In other words, the irrational numbers are not as accessible as the natural ones, since each of these does not have an associated infinity, since they do not have an unlimited decimal expansion. For example, for Omar, the number 3 does not have an associated infinity, since it does not have non-periodic non-zero digits; however, π does have an associated infinity, since the decimal part has non-periodic digits. In this way, Omar's intuition causes the natural numbers to be associated with a single infinity, quantitatively smaller than the infinity of the irrational numbers.

The colloquial way in which Omar expresses himself to describe the cardinality of the set of real numbers ("they are like many infinities combined into one") leads us to think again about how he relies on his intuition to make himself understood verbally. Omar reinforces his intuition considering the amount of numbers that make up the real ones. His sensorimotor intuition, when expressing words like "many" or "combined", seeks to give an explanation close to the experiences of daily life, close to that idea of larger infinities. The swing from intuition to the formal can be observed in the arguments that Omar uses to explain the existence of different sizes of infinities, in the go-between intuitive ideas, which are developed through experiences, and the certainty he obtain from the formal definitions, like the one that the irrational ones are immersed in the infinite.

Now, this argument, based on intuition, is also based on formal knowledge, acquired by Omar during his school years. The definitions of rational and irrational number are part of Omar's mathematical language. Given the observation of the reflections of the balls, Omar must, then, make use of his intuition and, at the same time, he finds support in his academic knowledge.

Thus, intuitively, if you put more balls it is a bigger infinity, but if you put mathematics on them, well no. But after what we're discussing right now, well no, I'm still in conflict thinking that the theory can work, if it's theoretical it's possible.

In this intervention, the swing between the intuitive and the symbolic is observed more clearly. Indeed, Omar bases his answer on intuition ("if you put more balls it is a bigger infinity"), and also on the formal ("but if you put mathematics on them, well no"). This going and coming between the intuitive and the theoretical is difficult for him to assimilate ("I am still in conflict thinking that the theory can work"). Even so, he seeks to separate what he visually identifies in the reflections of the balls from what he accepts in theory (abstract concepts). By making this distinction, Omar implies that the mathematical existence of objects is dependent on their material existence.

To conclude his reflection, Omar comments the following:

> Look, it is not feasible that there is a material infinity, not even the space [he is referring to the Universe] that we know is expanding, it may grow at a certain rate, but we do not know if it is infinite or not, because if it continues growing means that it has not stopped and, if it stops, we will never know because we would not notice. So, that is why it is not feasible to measure an infinity because we do not have a material infinity.

In the expression "we do not have a material infinity", Omar condenses the inevitable amalgamation of intuition and the formal, produced by trying to explain what is in front of his eyes: the reflection of nine balls produced by the arrangement of mirrors. Your intuition tells you that infinity is characterized by the unattainable and the immeasurable. The example that he himself gives in his argument helps to give credibility to this intuition.

Even if the Universe does not expand, there is no certainty of its dimensions; intuitively speaking it would be impossible to know its dimensions; that is, it would be impossible to know the infinite. It is this non-finiteness of space that makes it impossible for Omar to conceive of a "material infinity". In other words, for Omar infinity is insensitive, and that is why mathematics must be used (as he did in his second reflection) to explain it.

Conclusion and Perspectives

In this article we report how Omar, a first-semester mathematics student, intertwines his intuition and formal knowledge to account for the infinite reflection of a configuration of plasticine balls, placed in a straight line between two mirrors. In the analysis, Omar bases his answers both on the use of intuition and on the application of mathematical concepts. The constant tension between the intuitions and the formal approach is what allows Omar not to assign a cardinality to the set of plasticine balls, but to conceive this set as an infinite set. Our work takes advantage of the field of embodied cognition, as developed by Donald (2001) and Lakoff and Núñez (2000).

The notion of infinity lives, so to speak, between the sensory-motor experience and the symbolic formulations typical of mathematics. But the premature replacement of Galilean insights (which are actually inarticulate consequences of embodiment) with Cantorian formalization creates cognitive obstructions that are difficult for students to overcome. A hypothesis derived from this research consists in proposing that the conception of mathematical infinity may require not the substitution of intuition for symbolism, but the intertwining between them. Each time Omar needed to justify the infinity of plasticine balls, he needed to blend his mathematical background with his intuitions.

There emerges, as a didactic necessity, the notion of conceptual metaphor (Lakoff-Núñez) linking intuitions with their eventual formalization. All cases of infinity in mathematics, according to Lakoff-Núñez (2000, p. 158) (for example, limits, infinite series, infinite sets, etc.), correspond to processes that do not end but that we conceptualize as if they did really have an end.Human beings can imagine the result of a process that does not end. This is the case of the existence of irrational numbers defined by an infinite collection of digits that we cannot even fully know, except in that other dimension of our cognitive experiences: in the virtual reality of mathematics and other symbolic worlds.

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