IMPLICATIONS OF FASTER/SLOWER LANGUAGE ON UNDERGRADUATE PRECALCULUS STUDENTS' GRAPHING

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Researchers have recommended using tasks that support students in reasoning covariationally to build productive meanings for graphs, rates of change, exponential growth, and more. However, not many recent studies have been done to identify how students reason when engaging in covariational reasoning tasks in undergraduate precalculus courses. In this study, I analyze submitted classwork, including video submissions of that work, in an applied precalculus undergraduate course. In comparing current literature on students' covariational reasoning with these students' responses, there is some overlap that this study provides additional insights to, and there are also unique ways of reasoning these students exhibited tied to understanding the steepness of slope as being associated with ideas of speed. This study contributes to knowledge about how students develop covariational reasoning.

Keywords: Cognition, Undergraduate Education, Precalculus, Advanced Mathematical Thinking

Introduction

Attempts to study students' covariational reasoning—students' reasoning about how quantities change together—has been going on for decades. Back in 2002, Carlson et al. (2002) proposed a framework for covariational reasoning. This framework was built after interviewing precalculus students on what has been aptly referred to as the Bottle Problem. In this problem, students graph volume vs. height of a given image of a slice of a bottle. Variations of the Bottle Problem have been used with a variety of populations (with K-12 students, undergraduate mathematics students, preservice teachers, etc.) and contexts (in the US and internationally). The framework developed from Carlson et al.'s analysis of students' responses to the Bottle Problem has served several researchers for the past few decades in understanding how students reason covariationally. Moreover, in 2015, Thompson & Carlson (2015) proposed a new framework for (co)variational reasoning that attends to more nuanced mental actions—such as the construction of multiplicative objects and distinguishing between variation and covariation.

The study described in this report builds on the work of these and other researchers' understanding of covariational reasoning by using these newer ideas on covariational reasoning. In particular, the newer ideas are used to explain a common way of reasoning seen in the Bottle Problem with students in an undergraduate precalculus classroom—one of which was a way of reasoning that was offered by a student in the initial Carlson et al. (2002) paper to introduce the framework. Below is the excerpt from that paper (Carlson et al, 2002, p. 366) in which the student describes their reasoning for constructing a graph with a concave down curve.

B: OK, the more water, the higher the height would be [MA2]. In terms of height of the water, that is what we are talking about. If you are talking about the height left over, that is basically decreasing. Right here the height will be zero and the volume is zero.

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As you go up, a little more height increases and the volume increases quite a bit [MA3], so the amount by which the height goes up is not as fast [MA3]. Once you get there [pointing to halfway up the spherical part of the bottle], the height increases even slower [MA3]. I guess from here to there height increases the same as the volume increases, and once you

get here it increases slower [MA3]. No, I am wrong. So, every time you have to put more and more volume in to get a greater height towards the middle of the bottle and once you get here, it would be linear, probably [pointing to the top of the spherical portion]. So, it's always going up [tracing his finger along the concave-down graph], then it would be a line.

Int.: So, what does the graph look like?

B: Like this [pointing to the concave-down graph he has constructed], but it has a straight line at the end.

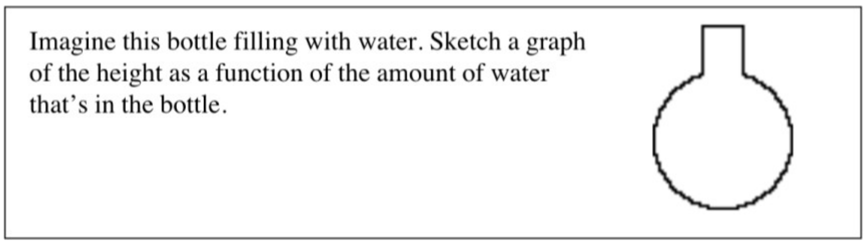

Figure 1: The Bottle Problem from Carlson et al. (2002, p. 360).

The goal of this study was to collect students' responses to the Bottle Problem from precalculus students and identify the covariational reasoning the students exhibited. In doing so, the following research question emerged and will be addressed in this paper: How do students who draw graphs for a situation involving two different rates reason covariationally? To answer this question, I will report on what literature on students' quantitative and covariational reasoning has said about this situation in the past, including ideas of iconic translations, shape thinking, and thematic reasoning. I will then describe the methods of how I collected data on this problem and coded it to pull out examples similar to the ones described in the research question and identify a unique but related way of reasoning the students exhibited. I will then report on the students who exhibited this reasoning and conclude by describing how these students justified their graphs and relate it to the existing research on covariational reasoning. I conclude by introducing a new way of understanding students' reasoning about concavity and linearity in graphs that has implications for future research on students' meanings for graphing.

Theoretical Framework

In 2002, Carlson et al. (2002) proposed a framework for covariational reasoning that included five mental actions: "coordinating the value of one variable with changes in another" (MA1), "coordinating the direction of change of one variable with changes in another variable" (MA2), "coordinating the amount of change of one variable with changes in the other variable" (MA3), "coordinating the average rate-of-change of the function with uniform increments of change in the input variable" (MA4), and "coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the

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function" (MA5) (p. 357). In classifying students based on this framework, they recognized that students can exhibit behaviors indicating particular mental actions without providing adequate evidence they "possessed an understanding that supported the behavior" (p. 358), citing Vinner's (1997) notion of pseudo-analytical behavior. They thought that the student in the excerpt above may fit this category of reasoning.

In moving forward with research on student's covariational reasoning, Moore & Thompson began to distinguish between evidence of understanding vs. pseudo-analytical behavior in students' graphing activities. In 2015, they introduced the notion of static and emergent shape thinking. Static shape thinking entails thinking about a graph "as an object in and of itself", possessing properties associated with learned facts. An example in the paper was that of a student referencing "the sine graph" as a "graph everyone knows about" and that has a particular shape and orientation. Emergent thinking emphasizes the creation of a trace constructed via representing a covariational relationship between quantities. Students' reasoning in the sine example might include highlighting magnitudes and constructing a point on the graph simultaneously representing both quantities' magnitudes (i.e., a multiplicative object (Piaget, 1970)), and then varying the quantities' magnitudes to construct a trace of a line with that point.

Their distinction was one contribution of many to research interested in making sense of graphical reasoning. For example, Stevens et al. (2016) looked at students' justifications of curvature, Frank (2017) looked at the process of constructing a multiplicative object, Paoletti (2020) looked at the impact of inverse functions on students' representational activity, Johnson (2022) looked at the impact of varying representations, and Ellis et al. (2013) focused on students' understanding particular kinds of relationships being represented (e.g., exponential). Moore and colleagues also continued to learn about students' construction of relationships between quantities via figurative and operative thought (Liang & Moore, 2020; Moore et al., 2019). This research included attending to iconic translations and thematic associations as evidence of figurative thought, thought focused on the "figure to ground" (Thompson, 1985).

Beyond understanding students' representational activities, colleagues also worked to develop the construct of covariational reasoning. For example, Castillo-Garsow, Johnson, & Moore (2013) introduced the notion of chunky and smooth shape thinking, attending to the discrete vs. continuous ways of thinking involved in reasoning about varying objects. Ely & Ellis (2018) also considered this in Calculus contexts in terms of a "zooming in" on functions' graphs. Meanwhile, work continued to develop the initial framework posed by Carlson et al. (2002), and in 2017, Thompson & Carlson (2017) proposed a new framework for covariational reasoning,
revised by attending to "students' variational reasoning separately from covariational reasoning" and to "how students coordinate their images of quantities' values varying" by considering the variational reasoning and their construction of a multiplicative object (Thompson & Carlson, 2017; p. 440). The resulting levels of covariational reasoning include: no coordination, precoordination of values, gross coordination of values, coordination of values, chunky continuous covariation, and smooth continuous covariation (Thompson & Carlson, 2017, p. 441).

The aforementioned body of literature has influenced this study. First, the Carlson et. al (2002) covariational reasoning provided an initial framework to analyze students' responses. The work on students' static and emergent shape thinking and figurative and operative thought provided a lens to tease apart students who focused on building quantitative relationships between quantities versus representing learned properties. The work on

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understanding the impact and importance of gaining evidence for students' reasoning via changing the representation systems helped make sense of the severity of students' association to particular shapes. The work on understanding various relationships helped to unpack the role in which relationship was being described was impacting the resulting graphical representation. Lastly, the updated Thompson & Carlson (2017) framework provided a way to make sense of students' actions in ways that did not rely solely on directional reasoning and amounts of change reasoning, but rather their coordination of quantities and their imagery of variation.

Methods

The study was conducted with 39 students from an undergraduate applied precalculus course at a medium-sized public university in the northeastern U.S. The course is coordinated and the instructor of the course is the researcher. The students in this course are not intended Mathematics or Engineering students, but rather the students with a variety of other majors (e.g., Biology, Conservation, Pharmaceutical Sciences, Marine Affairs, Journalism, Communication Studies, Psychology). The course consisted of students self-reported as White (27), Black (2), Hispanic (1), 2 or more races (8), and not specified (1). Throughout the semester, coursework was collected from the students, and the focus of this study is on the third class assignment of four total class assignments (see task description in the following section). All assignments had two parts: one done in class through groupwork and one outside of class time. Both parts were submitted individually online. The last two assignments included a video submission in which the students talked through their solutions on Part II of the assignment.

In analyzing students' covariational reasoning for this study, the researcher analyzed the 33 submitted written work with accompanying video submissions of Assignment 3. The work was analyzed using Carlson et al.'s (2002) framework. Additionally, using thematic analysis (Corbin & Strauss, 2009), notes were taken on similar work within and across those categories (e.g., mention of speed) and compared to literature on covariational reasoning. The researcher then identified that several of the students reasoned similar to the student mentioned in Carlson et al. (2002). The researcher then used the ideas of shape thinking, co(variational) reasoning, and
slope/rate of change to characterize students' reasonings in ways that are viable based on their responses.

Task Descriptions

In this section, I will summarize the goals of the first three assignments to provide a better understanding of the assignments collectively, and then I will focus on the Bottle Problem assignment, indicating adaptations that were made to target the precalculus concepts of polynomial and exponential growth.

Overview of Class Assignments

There were four class assignments throughout the course of the semester. Part I questions primarily focused on exploring a dynamic context, identifying quantities, and representing relationships. Part II primarily asked for more pointed questions about the relationships between quantities. In Assignment 3, the students received various cross-sections of bottles, descriptions of bottles, or graphs relating the height and volume of the bottle and asked to

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either draw the corresponding volume-height graph or the corresponding bottle. In Part II of the assignment, the students were asked to submit a 3-5 minute video in which they talked through their various graph and bottle constructions for the Part II bottles.

Class Assignment 3: The Bottle Problem

In the Bottle Problem, the students were given cross-sections of bottles, descriptions of bottles, or graphs of the volume-height relationship for bottles. Part I adopted materials used by Moore and colleagues from the NSF funded *Advancing Reasoning Project*. In Part II, the prompts included asking for specific reference to Direction and Amounts of Change talk. The following narrative was provided to introduce this talk to the students:

> **We** create graphs to represent quantities' measures and how these measures change together. So, when we talk about graphs, we should talk about quantities' measures and relationships. We call this **covariation.** There are numerous ways of talking about covariation including: **Correlation** As quantity A changes, quantity B also changes. **Directional** As quantity A increases (or decreases), quantity B increases (or decreases). **Amounts of Change** As quantity A *increases by equal increments*, the *amount of increase (or decrease*) in quantity B *decreases* (or increases). **Rates of Change** As quantity A increases, quantity B increases (or decreases) at an increasing (or decreasing) rate.

Part II of the assignment had three problems. In the first problem, the students were asked to sketch the volume-height graph and draw the cross-section of a bottle (a) from a volume-height graph of their own choosing, (b) a bottle in which the bottle doubles in volume for every additional inch of height, (c) a bottle in which the for each inch of height the bottle increases, the volume of the bottle increases two more inches cubed in volume than the previous increase. In the second problem, the students drew their own cross-section of a bottle and created the corresponding height-volume graph. In all these problems, the students were also asked to include Direction and Amounts of Change talk using color-coding. In the third problem, the students were asked to create a 3–5-minute video talking through their work in Part II using Direction and Amounts of Change talk. Only responses to Part II 1a and 2 are reported here.

Results

Although all the student work was analyzed using the Carlson et al. (2002) framework for covariational reasoning, the results reported here are two common themes of work in which students used some sort of covariational reasoning in their argument but also offered different justifications for the curvature of their graphs for at least some regions of their graphs. In particular, these categories are justifications based on faster/slower language (thematic associations) and justifications based on the shape of the bottle (iconic) (related to the aforementioned description of the student from Carlson et al. (2002).

Thematic Justification: Faster/Slower

Ten of the 33 students mentioned some of descriptions that associated the words *faster* and *slower* with the steepness of slope. For four of these students, these descriptions were associated with graphs that were sectioned into linear components (with, from the researcher's perspective, some moderate curvature between the sections that was unaddressed in any students' descriptions). Julia, for instance, gave the following verbal description associated with the work she produced in Figure 3.

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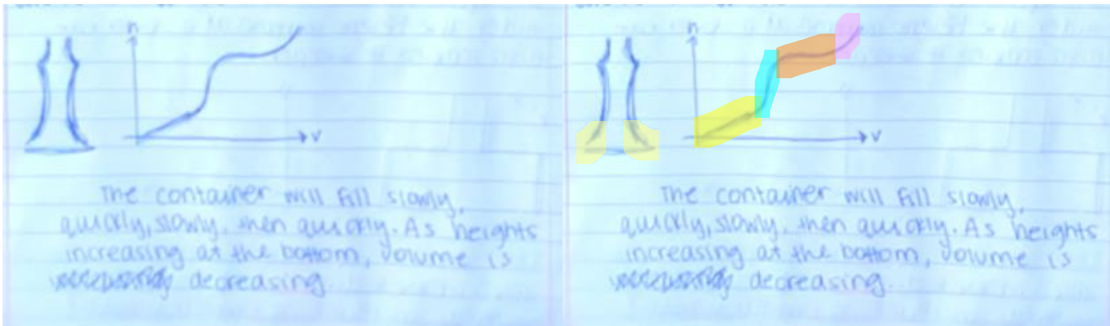

Figure 3: (Left) Julia's Bottle and Graph and (Right) Additional Highlights Added by Author to Represent the Sections of the Graph Referenced by Julia.

[Julia's Written Response in Figure 3: "The container will fill slowly, quickly, slowly, then quickly. As the height's increasing at the bottom, volume is decreasing."]

Julia: And as for this one, the container will fill slowly, then quickly, then slowly, and then quickly again. And you can see this because at the base [*points to the bottom of the cross section*], even though it's wider [*makes angled in motion with hands*], it doesn't allow as much water in 'cause it narrows and it becomes slimmer [*motioning along themiddle of the bottle*]. So this would fill in very slowly [*starts moving fingers alongside and up the cross section of the bottle*] but not-but at a more constant rate [*pointing to the first linear region on the graph*], and then when it changes directions again [*motions up the remainder of the bottle*], that's when this changes directions [*motions up the remainder of the graph*]. Yep. Then as height's increasing at the bottom, volume is decreasing.

Lorella and Leonhard produced similar bottles and graphs and gave similar justifications. When describing the middle section of her cross section and associated slanted line on her graph, Lorella stated that the volume will "start to fill up quicker and then as shown in the graph right here, is the portion that kinda closes in", pointing to the middle region of her graph. Leonhard offered the following description of the vase and his graph seen in Figure 4.

Leonhard: And as for this one, the container will fill slowly, then quickly, then slowly, and then quickly again The vase fills slowly, then quickly, then slowly. With the curve starting out less steep as we can see here [*pointing to the first section of the curve*], then steeper [*pointing to the middle section of the curve*]-steeper, very steep, then less steep [*pointing to the third section of the curve*]. Which is the same as it is here [*pointing to the first section of the curve*]. Which is why it will be the vase right here [*pointing to diagram of cross section of vase*].

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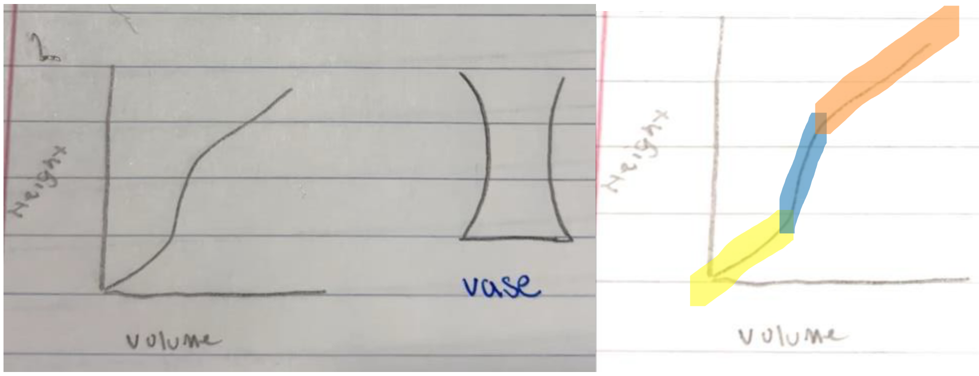

Figure 4: (Left) Leonhard's Bottle and Graph with (Right) Additional Highlights Added by Author to Represent the Three Sections of the Graph Referenced by Leonhard

These students all exhibited reasoning which involved an imagery of pouring in water that filled the glasses "slowly" or "quickly". Also, although volume was the quantity they were referencing as filling "quickly", a *higher* volume to height rate with the provided axes would imply a *less* steep slope based on their description. Thus, all these students associated speed with the steepness of a slope (without attention to the axes labels)-faster speed results in a steeper slope. Moreover, the "steeper" or "less steep" regions were referencing *regions* of the bottle (not *points*), and thus occurring over intervals in the corresponding graph.

Iconic Translations: Decreasing then Increasing

The results in this section correspond to the result that Carlson et al. (2002) reported for Student B discussed in the introduction. Although the bottles are different, like Student B, these five students all used a single concave up or down curve to represent the height and volume relationship of a spherical and an inward curved portion of a bottle (see Figure 5).

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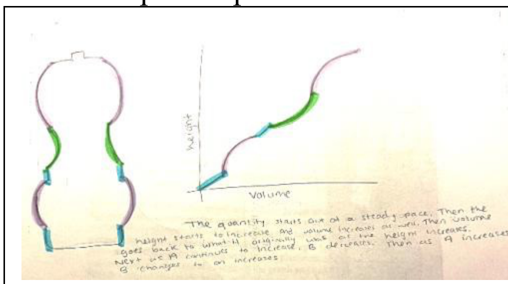
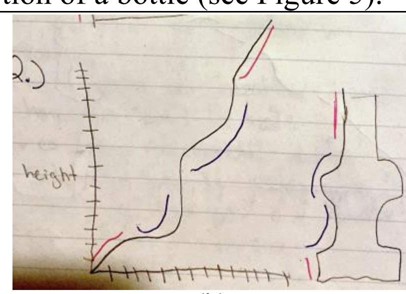
<tr><td>

(a)</td><td>

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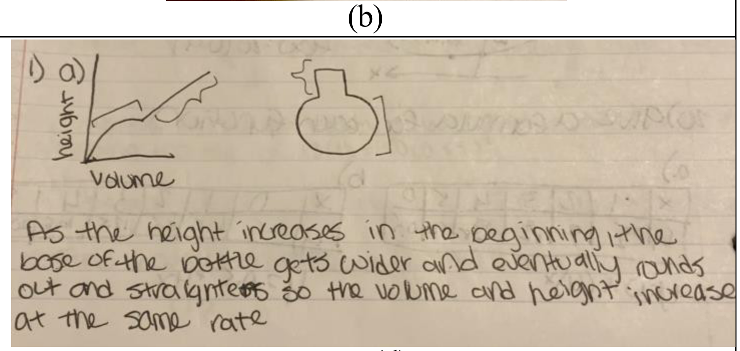
<tr><td></td><td>

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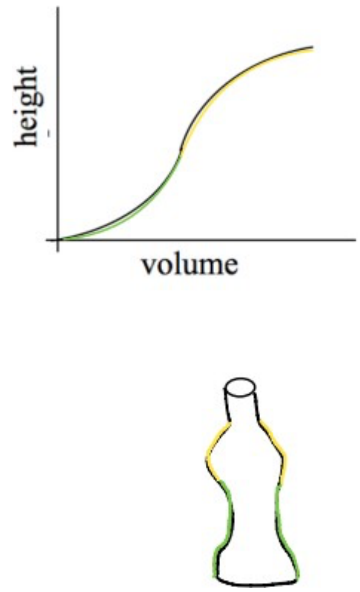

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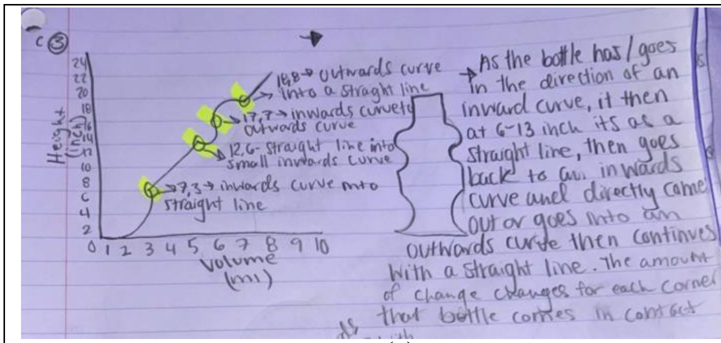

(e)

Figure 5: Bottles and Graphs (a) Louise, (b) Elizabeth, (c) Sandra, (d) Hertha, and (e) Alan.

In all the bottles in Figure 5, students associate concave down and up curves with regions that normatively have changes from increasing to decreasing (or vice versa) rates of height with respect to volume. Below are some excerpts from the various students describing their work:

Louise (5a): [*Referencing the green section on her bottle*] The volume now starts to decrease, so rate of change quantity, everything, the volume is now decreasing as the height is increasing. [*Referencing the purple section above the green section on her bottle*] But then as the height increases, the volume then starts to increase and then increases.

Elizabeth (5b): And then for question 2, I did this demonstrated with the pink being the straight line [*pointing at leftmost pink region on graph*] because it goes straight out, and then when there's a rate of change in the curve [*pointing at the left purple region on the graph*], it is demonstrated by the purple [*pointing to the bottom purple curve on the cross section*]. And this side it changed again [*pointing to the right purple region on the graph*], so it's purple here [*pointing to the top purple curve on the cross section*]. And then straight up [*pointing at the right pink region on the graph*] so it's pink [*pointing at the top pink region on the cross section*]. And that is what that bottle looks like.

Sandra (5c): The graph, it doesn't have any straight lines in this, so I knew it was gonna be a curvier bottle. So the green is going inwards so you see that inwards bottle there, and then it goes outwards, making it up to the cap of the bottle.

Hertha (5d): So the bottle does round out and at the bottom and it's straight at the top and you can see that in the graph. So at the top, it's increasing at more of the same rate, and at the bottom, the volume is increasing significantly faster than the height.

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From all these various pieces of work, we see evidence of iconic translations, static shape thinking associating a curve in the bottle with a similarly represented curve (albeit diagonally placed) on a graph. However, when analyzing their descriptions, as well as the description B gave from the Carlson et al., (2002) article, these students are referencing quantities varying. Specifically, they talk about height and/or volume increasing. Instead of the focus being on speed, the focus here is on the shape of the bottle as height (or volume) increases. There are references to the bottle getting "wider" or have an "inwards curve" vs. an "outwards curve" as that quantity varies. Thus, the iconic translations the students are making are occurring in sections that they consider as one quantity is varying. For covariational reasoning to occur, two quantities need to be related, but here instead the students associate features (not quantities) as one quantity increases. So referencing features such as having "more volume than height" or "volume increasing significantly faster than the height" are not quantitative comparisons of quantities, but rather descriptions of characteristics associated with particular shapes in a graph. What makes it more sophisticated than some other iconic translations described in the literature review (with purely figurative reasoning) is twofold: (i) all the graphs are monotonically increasing indicating some awareness of the varying (particularly the increase) in one quantity, and (ii) like the previous examples, a bulging out seems to always be represented by a concave down shape to be associated with a decreasing, in a way similar to the speed decreasing was getting represented by less steep slopes.

Discussion

Altogether, as seen in the results, the discussed corpus of literature enabled the identification of nuances in students' construction of a static object with apparent covariational reasoning (based on the Carlson et al. (2002) framework). Namely, rather than only characterizing the work of iconic translations or thematic associations, or simply as pseudo-analytical, the additional insights offered by the Thompson & Carlson (2015) frameworks enabled me to distinguish variational reasoning from covariational reasoning in ways that still attend to the students' descriptions of variation in their justifications, distinguishing figurative reasoning from operative reasoning. These nuances brought to light connections students were making about how to represent differing speeds graphically (e.g., faster region implies steeper line segment, changing from faster to slower implies a concave down curve). The contribution of this study is noting that the students' reasoning, although seemingly thematic, involves imagery of a changing speed occurring in a chunky matter—that is, with intervals of linear segments (thematic examples) or regions of steepness (iconic examples) whose steepness can be compared to one another to make claims about how changing quantities are related. The faster and slower language supported them to an extent, even the students making seemingly iconic translations based on shape of an object. I encourage continued research on how the development of chunky and smooth thinking might support student learning of various function types, especially during the transition from precalculus to calculus, when they will be introduced to tangent lines and instantaneous rates of change as resources to thinking about how quantities change together.

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