EXPLORING THE ASSOCIATION BETWEEN UPPER ELEMENTARY SCHOOL STUDENTS' MATURE NUMBER SENSE AND GRADE-LEVEL MATHEMATICS ACHIEVEMENT

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Students with mature number sense make sense of numbers and operations, use reasoning to notice patterns, and flexibly select the most effective and efficient problem-solving strategies (McIntosh et al., 1997; Yang, 2005). Despite being highlighted in national standards and policy documents (CCSS, 2010; NCTM, 2000), the association between students' mature number sense and other important outcomes is not well specified. For example, how does students' mature number sense relate to their grade-level mathematics achievement? We analyzed 153 upper elementary school students' scores on measures of mature number sense, fraction and decimal knowledge, multiplication fluency, and grade-level mathematics achievement. We found mature number sense to be measurably distinct from their fraction and decimal knowledge and uniquely associated with students' grade-level mathematics achievement.

Keywords: Number Concepts and Operations; Cognition; Assessment; Rational Numbers

In the late 1980's, students' "number sense" emerged as a principal goal of mathematics education. While many high-quality mathematics teachers were certainly already pushing students to make sense of numbers and operations before then, "number sense" was not yet a commonly used expression. Specifically in the United States at this time, national curriculum frameworks (e.g., National Council of Teachers of Mathematics (NCTM), 1989) and government commissioned reports (e.g., National Research Council, 1989) emphasized "number sense" as a core objective of K-12 mathematics education. Students with mature number sense exhibit the disposition to *make sense of numerical situations* and use a rich conceptual understanding of number and operations to flexibly solve problems (McIntosh et al., 1997).

Despite this emphasis in the field, there is little evidence in the literature on how students' mature number sense develops over time or relates to central goals in mathematics education. For example, how do students' levels of mature number sense relate to their grade-level mathematics achievement? In prior work, we began to address this gap by examining how middle school students' mature number sense related to other theoretically related constructs (Kirkland et al., 2022). We aim to extend that work here to upper elementary school students, testing the hypothesis that upper elementary school students' mature number sense is both distinct from their fraction and decimal knowledge and mathematics achievement and uniquely associated with grade-level mathematics achievement.

Background and Theoretical Framework

How number sense has been defined, operationalized, and measured looks very different across the disciplines of cognitive and developmental psychology, mathematics education, neuroscience, and special education (summarized in Whitacre et al., 2020). In this project, we want to clarify that our construct of interest is what Whitacre et al. (2020) have termed *mature number sense*, which we define in line with McIntosh et al. (1992) as "a person's general understanding of number and operations along with the ability and inclination to use this understanding in flexible ways." This focus is distinct from research on approximate number sense (e.g., Dehaene, 2001) or early number sense or numeracy (e.g., Jordan et al., 2009).

As a construct, mature number sense has often been subdivided into components (McIntosh et al., 1997; NCTM, 1989; Reys et al., 1999; Yang & Lin, 2015). Building from this work and in consultation with expert mathematics teachers, teacher educators, and mathematics education researchers, we specified a four-component framework of mature number sense for middle school as well as upper elementary students (Kirkland et al., *in press)* as follows:

3. Understanding basic number concepts and number magnitude: Strong mature number sense is characterized by a rich conceptual understanding of fractions, decimals, and whole numbers. Students use their understanding of place value and rational number magnitude to efficiently estimate results using concepts such as unit fractions.
4. Using multiple representations of a number: Strong mature number sense is characterized by proficiency in translating among multiple representations of rational numbers efficiently and flexibly to solve problems. Students use this understanding to translate between representations such as Arabic numerals and the number line.
5. Understanding the effect of arithmetic operations on numbers: Strong mature number sense is characterized by recognizing how the four core arithmetic operations affect whole numbers as well as fractions and decimals. Students demonstrate understanding that patterns observed with operations with whole numbers may not hold true for numbers between 0 and 1. Students use this understanding to efficiently estimate computational results and ensure the results make sense given the relationship between operations and rational numbers.
6. Understanding mathematical equivalence: Strong mature number sense is characterized by understanding the equal sign as a relational symbol, reflecting that the two sides of an equation are equal. Rather than approaching equations with an "operational" approach, students recognize patterns across the equal sign and use this relational thinking to flexibly solve problems (c.f. Jacobs et al., 2007).

We hypothesized, similar to Yang (2019), that mature number sense would be an overarching hierarchical latent construct, with the four components each theoretically related to it. That is, a student's understanding of mathematical equivalence reflects their mature number sense as well as a more specific relational understanding of the equal sign. In our initial development of a brief assessment of mature number sense for middle school students (Kirkland et al., *accepted)* the structure of the response data reflected this hypothesized framework. A bifactor model with each component as a specific factor best fit the response data over theoretically related models. We then created an upper elementary form of the brief assessment, and the response data in our elementary validation study reflected the same structure. Here we used this upper elementary form of the assessment to further study mature number sense in 3^{rd}-5^{th} grade students.

Rationale for Current Study

Mature number sense has been measured and studied internationally in the last three decades (summarized in Whitacre et al., 2020), most frequently in a single time point and with upper elementary and middle school students. Very few of these studies have analyzed any potential overlap with potentially related psychological constructs. For reference, Yang et al. (2008) is the only prior study we are aware of that examined mature number sense's association with grade-level mathematics achievement, and this was with 5th grade Taiwanese students. As a first step toward addressing this gap, we analyzed middle school students' (*N* = 129) scores on measures of mature number sense, fraction and decimal knowledge, grade-level mathematics achievement, and addition fluency (Kirkland et al., 2022). We hypothesized that students with strong mature number sense are flexible problem solvers with a deep understanding of number and operation and, thus, that they would do well on more "traditional" school assessments. In contrast, throughout the literature, mature number sense is often juxtaposed with standardized school mathematics achievement and the algorithmic, overly procedural mathematics instruction common in schools (Whitacre et al., 2020). We found mature number sense to be measurably distinct from their fraction and decimal knowledge and uniquely associated with students' grade-level mathematics achievement. However, one could argue that this observed pattern may be drastically different in upper elementary (e.g., 3rd-5th grade) than in middle school (e.g., 6th-8th grade). In upper elementary, students are first introduced to multiplication and division as well arithmetic operations as fractions and decimals (CCSS, 2010). In addition, by the end of grade 3, students are expected to recall fluently all single-digit multiplication facts. One therefore might expect 3rd-5th grade students' multiplication fluency or basic fraction and decimal computation to be more aligned with grade-level mathematics achievement than with mature number sense. However, we are not aware of a study with upper elementary school students that examined the association between their mature number sense and either their fraction computation or their fluency with single-digit multiplication facts or how these related constructs may be associated with their overall grade-level mathematics achievement.

In this study, we examined how US upper elementary school students' mature number sense related to their grade-level mathematics achievement, controlling for a variety of other potentially related factors, including fraction knowledge and multiplication fluency. Our research questions were as follows: 1) Is the mature number sense of upper elementary school students measurably distinct from their fraction knowledge as well as their grade-level mathematics achievement? and 2) Is mature number sense in upper elementary school uniquely associated with students' standardized grade-level mathematics achievement, even after controlling for their fraction knowledge and multiplication fluency?

Methods

Participants

One hundred fifty-three upper elementary school students (*N* = 47 in 3rd grade, *N* = 54 in 4th grade, *N* = 52 in 5th grade) from schools surrounding a university in the midwestern United States participated (48% identified as female; 81% identified as White, 12% Multiracial, 8% Hispanic, and 2% Black). Sessions took place after school hours in university building. Students were participants in timepoint 1 of a larger (still ongoing) longitudinal study. They completed eight measures over the course of two 45-minute sessions, scheduled about a week apart (median of 7 days between sessions). For this analysis, we focused on the four measures described below, but note that the results reported hold up to robustness checks that include all measures as well.

Measures and Procedure

Brief Assessment of Mature Number Sense: Elementary Form (Kirkland et al., *accepted*). This is an electronic, 24 item multiple-choice test of students' mature number sense, aligned with our theoretical framework detailed above. Items differ from a traditional curriculum and are designed to specially assess students' number sense. Each item has a time limit of 60 seconds and students are not allowed to use paper and pencil to discourage the use of traditional algorithms. Student's total sum score is used in the analyses. In this study, students completed the upper elementary form of the brief assessment. This form includes 8 common linking items with the middle school form used in our prior work (Kirkland et al., 2022) This allows us to track students' number sense development over time. Figure 1 includes an example item on the elementary form for each of the four components in our mature number sense framework. In the validation analysis, student scores on the elementary form brief assessment were reliable over time ($r = 0.84$) and had evidence from student think-alouds, expert reviews, factor analyses, and item response theory analyses to support our validity argument.

Understanding the Effect of Operations on Numbers	**Number Concepts and Magnitude**
What is true about the answer for 23 ÷ 4 (23 divided by 4)? ○ The answer is in between 2 and 3 ○ The answer is in between 3 and 4 ○ The answer is in between 4 and 5 ○ The answer is in between 5 and 6	Which fraction is equivalent to (has the same value as) $\frac{9}{3}$? ○ $\frac{1}{3}$ ○ $\frac{10}{4}$ ○ $\frac{6}{2}$ ○ $\frac{4}{3}$
Multiple Representations of a Number	**Mathematical Equivalence**
Which of the following numbers is the red arrow most likely pointing to on the number line below?  ○ 100 ○ 250 ○ 500 ○ 900	34 + 19 = ___ + 18 ○ 33 ○ 35 ○ 53 ○ 71

Figure 1: Sample Items Assessing Mature Number Sense Organized by Component

Massachusetts Comprehensive Assessment System (MCAS) Grade-Level Mathematics Test (2019). Students completed the released 2019 MCAS paper test appropriate for their grade level. This is a freely available standardized test designed to assess student proficiency with grade-level mathematics standards. Student scores are converted to percent correct because the maximum possible correct differs by grade level. Students had no time limit to complete each section of the test and could use scratch paper but not calculators.

Fraction and Decimals Assessment (adapted from the Rational Numbers Measure [Powell, 2014]). This is a 28-item paper and pencil test of students' knowledge of fractions, decimals, and percent. We adapted the Rational Numbers Measure (Powell, 2014) used in our prior work with middle school students to make it grade-level appropriate for 3rd-5th grade. To do so, we updated denominators to be those recommended by the Common Core State Standards (CCSS, 2010) for 4th grade, removed any percent or percentages question given that this content is not mentioned until 6th grade in CCSS, and reworded some word problems based on literacy levels for 3rd-5th grade. We chose to keep the bulk of the measure the same to allow for a more consistent comparison with our study of middle school students' number sense. Students are asked to compare fractions, perform the four operations with both fractions and decimals, find common denominators of fractions, and generate equivalent fractions. For example, one item was "$1\frac{2}{8} + 3\frac{3}{8} =$ _____" and another was "Write $\frac{14}{100}$ as a decimal." Students worked for 20 minutes or until they finished. They could use scratch paper but not calculators. Students received 1 point for each correct response.

Multiplication Fluency Task (Burns et al., 2015; Nelson et al., 2013). This measure includes all combinations of randomly presented single digit multiplication facts with the numbers 1-9. Students were tasked with solving as many correctly as they could in 1 minute. The order of the facts was predetermined randomly and then kept standard for all participants. Students received 1 point for each correct response.

Data Analyses

To address the research question on whether mature number sense is measurably distinct from students' general mathematics achievement, we first calculated the zero-order correlations between the constructs. We predicted mature number sense to be most closely related to students' grade-level math achievement on the MCAS and scores on the Fraction and Decimal Assessment. We expected mature number sense to be least closely related to multiplication fluency. We then ran a series of partial correlation analyses. We analyzed the correlation between mature number sense and both other constructs after we control for students' multiplication fluency scores. We then analyzed the associations between students' grade-level math achievement, fraction and decimal computation, and mature number sense, controlling for each in turn. In each case, we predict the relationship between mature number sense and the other construct to remain significant, even after controlling for a strongly related third variable.

For additional evidence that mature number sense and fraction and decimal computation are distinct, we then conducted a series of factor analyses (Shaffer et al., 2016). We used a common method where we tested a constrained model with 2 latent factors set with a covariance equal to 1 and an unconstrainted model where the 2 latent factors are allowed to freely covary with each other. If the unconstrained model provides a better fit through a RMSEA test that is significant, then the two measures of interest can be said to be distinct. We tested these two models using a chi-squared, $\chi 2$, difference test. Due to each grade's different test for the MCAS, we are unable to perform this same analysis on grade-level mathematics achievement.

To address research question 2, we ran a partial correlation test on the association between grade-level mathematics achievement and fraction and decimal computation, controlling for mature number sense. We then ran a linear regression model, regressing mature number sense and other predictors measured in the study on students' grade-level mathematics achievement.

Findings

We first begin with the overall student performance on the measures in the study. Students solved on average 11.21 (*47%)* items correctly on our brief assessment of mature number sense (SD = 5.09). Across the three grade levels, students solved on average 39% of the items correctly on the MCAS (SD = 0.22). This differed dramatically by grade level, with 3rd graders outperforming (M = 50% of items correct) both 4th graders (M = 35%) and 5th graders (M = 32%) on their respective grade level standards. Out of a maximum possible of 28 correct, students answered on average 4.43 items correct on the Fractions and Decimals measure (SD = 4.38). While this number may appear very low as a percentage correct (*16%*), this measure was adapted from a measure designed to include many problems to help differentiate students' fraction and decimal knowledge up to the college level. Finally, on average, students answered correctly 9.08 (*SD* = 6.36) single-digit multiplication problems on the multiplication fluency task.

To begin to address research question 1, the zero-order correlations between the measured constructs are summarized in Table 1 (* p <0.01, **p <0.001). As predicted, students' mature number sense scores correlated very highly with their fraction and decimal computation skills (r = 0.71). However, the correlation with grade-level mathematics achievement (r = 0.46) was much lower than expected given the observed correlation in middle school students (r = 0.76). It also was lower than the correlation with multiplication fluency (r = 0.67) and this difference was significant ($t(152) = 2.81$, p <.01). Given the disparity observed between grade levels on the MCAS, we considered the correlation between the constructs for 4^{th} and 5^{th} grade students only. Here, the patterns were more in line with our predictions. Mature number sense was still strongly correlated with fraction and decimal computation (r = 0.70) and multiplication fluency (r = 0.58), but the correlation with grade-level achievement was stronger (r = 0.65).

Table 1: Correlation Between Measures in Study

Construct	*M*	*SD*	1	2	3	4
1. Mature Number Sense	11.21	5.09	-			
2. MCAS – Grade-Level Achievement	39%	22%	0.46**	-		
3. Fraction and Decimal Computation	4.43	4.38	0.71**	0.30**	-	
4. Multiplication Fluency	9.08	6.36	0.67**	0.20*	0.63**	-

With all three grades included and controlling for students' multiplication fluency scores, students' mature number sense still correlated highly with their grade-level mathematics achievement (pr =0.45, $t(150) = 6.15$, p < .001) and their fraction and decimal computation (pr =0.50, $t(150) = 7.14$, p < .001). Relatedly, students' mature number still correlated highly with their multiplication fluency when controlling for either their grade-level mathematics achievement (pr =0.66, $t(150) = 10.83$, p < .001) or their fraction and decimal computation (pr =0.40, $t(150) = 5.33$, p < .001).

We then further examined the association between mature number and grade-level achievement. Controlling for students' fraction and decimal computation, their mature number sense scores still were significantly positively related with their grade-level achievement (r =0.37, $t(150) = 4.83$, p < .001). The same was true when controlling for grade-level achievement and examining mature number sense's relationship with fraction and decimal computation (r =0.68, $t(150) = 11.32$, p < .001). We would predict that if mature number sense and grade-level mathematics achievement are redundant constructs, this relationship would no longer be significant. Thus, mature number sense appears to be distinct from grade level achievement.

As discussed above, we then ran a series of confirmatory factor analyses for evidence on if mature number sense and fraction and decimal knowledge are distinct (Shaffer et al., 2016). These analyses provided evidence that the Fractions and Decimals Assessment and the Brief Assessment of Mature Number Sense measure distinct constructs, χ^2 difference = 67.03, p < 0.001; *CIF* difference = 0.02 compared to benchmark of 0.002. This suggests the constructs, as measured in this study, are highly related, but not redundant.

To further examine the unique importance of mature number sense, we first tested the partial correlation between fraction and decimal computation and grade-level achievement.

Interestingly, when controlling for students' mature number sense, there is no longer a significant correlation between the two other constructs (r =-0.04, $t(150)$ = -0.50, p =0.62). This is the same pattern we observed with middle school students as well, even with a weaker observed correlation between grade-level mathematics achievement and the other constructs with upper elementary school students.

We then ran a linear regression on students' grade-level mathematics achievement (Model $R^2 = 0.23$). Results from the model are summarized in Table 2. Students' mature number sense was significantly positively related ($B_{NS} = 2.51$, $p < .001$) to students' grade-level mathematics achievement. However, students' fraction and decimal computation ($B_{Fract} = 0.05$, p =0.92) and multiplication fluency ($B_{Multi} = -0.66$, p =0.06) were not significantly related to achievement. To compare the relative importance of each regressor, we examined the semipartial correlation of each predictor in the model (Darlington & Hayes, 2016; Hayes & Rockwood, 2017). The semipartial correlation coefficient for mature number sense was significant (sr = 0.38, $t(149)$ = 4.95, p <.001). However, the semipartial correlation for neither fraction and decimal computation (sr = 0.01, $t(149)$ = 0.09, p = 0.93) nor multiplication fluency (sr = -0.14, $t(149)$ = -1.70, p = 0.09) were significant. We can interpret the ratio of the two semipartial correlations (0.38/0.01 = 38; 0.38/-0.14 = 2.7) as a measure of the relative importance of each in explaining students' grade-level achievement. As another way to quantify mature number sense's importance, we ran an additional linear regression model with mature number sense removed from the model. The model R^2 dropped from 0.23 to 0.09 (the same as sr_{ns}^2 = 0.14) and this was a significant decrease in model fit ($F(1, 149) = 27.46$, $p < .001$).

Table 2: Summary of Regression Model Predicting Grade-Level Mathematics Achievement

Variable	*B*	*SE*	*t*	*p*
(Intercept)	0.16	0.040	4.02	<0.001
Mature Number Sense	0.03	0.005	5.24	<0.001
Fraction & Decimal Computation	0.001	0.005	0.10	0.92
Multiplication Fluency	-0.007	0.003	-1.93	0.06

Discussion and Implications

From the evidence presented here, we see the same overall pattern of results as observed with middle school students and their mature number sense. Upper elementary students' mature number sense is distinct from their general grade-level mathematics achievement and fraction and decimal computation. Being identified as "strong with fractions" is not the same as displaying strong number sense. We also found students' mature number sense to be predictive within a single timepoint of their mathematics achievement, above and beyond their fraction and decimal computation and their single-digit multiplication fluency. When mature number sense was removed the model, there was a substantial and significant decrease in the explained amount of variance in students' grade-level achievement. Together, this suggests, that just as observed in middle school, students' mature number sense in upper elementary is uniquely predictive of their grade-level achievement, despite the significantly lower observed correlation between the two constructs.

The observed correlations between students' grade-level achievement and the other constructs in the study were much lower than expected or than observed with middle school students. As we noted above, students in 3rd grade were much more successful on their grade-level mathematics achievement test (M = 50% correct) than the 4th (M = 35%) and 5th graders (M = 32%). This was especially surprising to us as much of the 3rd grade content on the MCAS includes content that, based on the Common Core State Standards, would be brand new to students in 3rd grade: fraction magnitude, formal multiplication, division, and area. However, there were several items on multidigit addition and subtraction as well as interpreting bar graphs that students did very well on overall (M = 67% correct). In addition, students did quite well on fraction items related to verbal situations (e.g., "There are 6 children on a bus. Each child is wearing a hat. What fraction of the children on the bus are wearing a hat?") or basic area models (M = 75% correct). Therefore, even though "fraction" does not formally appear in the Common Core State Standards until 3rd grade, the equal "partitioning" or "shares" of shapes in Grade 2 may prepare students well enough to solve initial fraction concept items. Further research into the strategies students used to solve these items would be necessary to better understand the mechanisms behind their strong performance.

Because students did better by grade on the brief assessment of mature number sense (M = 8.62 in 3rd, 11.93 in 4th, 12.98 in 5th) and the 3rd grade MCAS was the easiest of the grade levels, the observed correlation between mature number sense and math achievement is much lower across 3rd-5th than within each grade level. Within 3rd grade for example (r = 0.75), the correlation is in line with our initial expectations based on middle school students (r = 0.76) and 4th and 5th graders (r = 0.65). A deeper analysis of the 3rd grade MCAS performance in comparison to 4th and 5th grade MCAS performance is needed to shed light on the grade-level differences observed here.

Given the importance of fraction skill (e.g., Barbieri et al., 2021; Booth et al., 2014; Booth & Newton, 2012; Siegler et al., 2013) and multiplication fluency (e.g., Nelson et al., 2016) for students' mathematics achievement, it is perhaps unanticipated that mature number sense was the strongest predictor of grade level mathematics achievement. Perhaps students' ability to make sense of numbers and operations and use that knowledge flexibly to solve problems is a better indicator of future mathematics achievement than fraction or decimal computation or simple fact retrieval. Further research is needed to understand these relationships more fully, especially in a longitudinal setting where any potential mediating roles can be accurately tested.

Overall, we have initial evidence extending our understanding of mature number sense and its nomological network with other mathematical constructs across upper elementary and middle school. This is an important step forward to future research more fully examining how mature number sense might lay a foundation for future mathematics achievement or whether improving mature number sense might be a more important target for instruction than, say, procedural problem-solving skills with fractions. This is, however, only one initial study with a sample of students that may not be wholly representative of upper elementary school students more broadly. By continuing to characterize mature number sense's relationship with other constructs, researchers can more fully explain how students develop a key proficiency in mathematics education and provide rigorous evidence to educators on how to help ensure all students have the disposition to *make sense of numerical situations*.

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