INSERVICE SECONDARY TEACHERS' BELIEFS ABOUT DEDUCTIVE DISCOURSE FOR EQUATION SOLVING

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We investigate teacher beliefs about discourses for equation solving and the challenges these beliefs might pose for the implementation of instructional practices that promote deductive reasoning in algebra. To uncover these beliefs, we recorded three video explanations of solutions to the same linear equation with distinct discursive characteristics and analyzed seven secondary mathematics teachers' small-group critical discussions of these explanations. Three prevalent themes surfaced in our thematic analysis. Teacher beliefs about discourse for equation solving specified different roles and potential benefits of deductive explanations, estimated students' capacity to understand deductive explanations, and hypothesized differences between teachers' and students' potential to understand deductive reasoning. We discuss implications of these beliefs for opportunities to engage all learners in conceptual thinking about equations.

Keywords: Algebra and Algebraic Thinking, Teacher Beliefs, Classroom Discourse, Reasoning and Proof

The study of algebra serves as an entry point into postsecondary studies and careers in science, technology, engineering, and mathematics. Yet success in algebra remains elusive for many students at the secondary and college levels. Reasons for this are varied and include a lack of equitable access to well-prepared mathematics teachers (Lee, 2012; Sutcher et al., 2019), teaching practices that do not build upon students' knowledge assets (including knowledge from their communities as well as foundational understandings of number and operations; see for example Civil, 2016), and a focus on symbolic manipulation at the expense of opportunities for sensemaking (Chazan, 1996). A key concept within algebra is equation solving, which may be introduced to students using various approaches and choices of language. Our study investigates the language that teachers consider effective in teaching students to solve equations. The Common Core State Standards call for students to "Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution" (NGA & CCSSO, 2010, HSA-REI.A.1). This standard aligns with a view of equation solving as a deductive process: the steps of a solution process can be viewed as steps in an argument which assumes the equality of the values of two expressions and makes successive inferences about the value of a variable. In classroom settings, this deductive process may be modeled using concrete models such as a balance scale (Vlassis, 2002). On the other hand, evidence suggests that in some algebra courses, textbooks and teachers describe novel problem-solving procedures in terms of actions on symbols without attending to underlying algebraic objects and their properties (Patterson & Farmer, 2018).

Although there has been some research conducted to assess how language specific to the algebra of equations is used in mathematics classrooms (e.g., Planas, 2021), work is still needed to advance our understanding of how language may be leveraged to further students' deductive understanding of equation solving. Our research team considers teacher beliefs a driving force behind teaching practices and choices in how mathematical content is communicated. Therefore, we suggest that a productive first step in understanding how discourse considerations shape teaching and learning about equations is to gain insight into teacher beliefs.

Theoretical Framework

Our work draws from the commognitive perspective, in which thinking is viewed as a process of communication, and to learn mathematics is to undergo a change in one's participation in a discourse community (Sfard, 2007, 2020). A foundational assumption of our work is that the ways in which teachers communicate when explaining algebraic concepts, and the discourse in which they invite students to participate as they grapple with these concepts, are consequential for students' opportunities to learn to reason flexibly and fluently about algebra. This view is supported by empirical research on connections between language, conceptual understanding, and student achievement (e.g., Bills, 2002; Huntley et al., 2007; Knuth et al., 2006; Van Amerom, 2003). In our study of discourses associated with equation solving in algebra, we draw from the *arithmetical discourse profile* of Ben-Yehuda, Lavy, Linchevsky, and Sfard (2005), which analyzed learners' discourse about concepts and problems in arithmetic along several key dimensions: their uses of *words* and the extent to which these explicitly describe mathematical objects, their uses of *mediators* (symbols and visuals that represent mathematical objects), their *endorsed narratives* and apparent meta-rules for accepting and rejecting narratives, and their uses of routines. In framing our work, we condense the *words* and *mediators* dimensions into a single dimension and use the resulting three dimensions (*words/mediators*, *endorsed narratives*, and *routines*) as a framework for algebraic discourse.

Guided by this discourse framework, we have developed a survey with fourteen open-ended items related to algebraic expressions, equations, functions, and modeling. Each item asks teachers to analyze a hypothetical student solution to an algebra problem, resolve a potentially ambiguous situation (such as what happens when the process of solving a system of linear equations culminates in an equation of the form $c = c$ where c is a constant), or explain the conceptual underpinnings of an idea often taken for granted in algebra (such as "combining like terms" or "keeping the sides balanced" when solving an equation). In-service teacher responses to this survey have supported a preliminary finding that teachers' talk about algebraic concepts varies significantly along all three dimensions (Patterson et al., 2021), suggesting that the arithmetical discourse profile can be extended to study the discourse practices of secondary algebra classrooms.

We define a *deductive discourse for equation solving* to be one in which words and mediators frequently serve to make the objects of the discourse (*e.g.*, values of expressions, operations, equality) and their properties explicit, in which narratives about equations and unknowns are endorsed or rejected by deduction from assumptions and other endorsed narratives, rather than by appeal to authority or other communicative rituals lacking an explicit deductive basis; and in which routines are used as flexible tools for generating new narratives about mathematical objects. Our larger study investigates the extent to which teachers engage in deductive discourse when explaining processes for solving equations and how teacher beliefs might support or constrain students' opportunities to engage in deductive discourse for equation solving. This research report focuses on our investigation of the second question.

Our analysis of teachers' beliefs about equation solving is informed by Leatham's *sensible systems* theory, which suggests that rather than focusing on apparent contradictions among beliefs held by an individual teacher, we should view beliefs as occupying an interconnected network in which some beliefs may take precedence over others at specific times (2006).

Leatham encourages mathematics teacher educators to present opportunities for teachers to explore their beliefs in practical contexts. While teacher beliefs may pose challenges for instructional change, beliefs can both inform the design of professional learning experiences and be shaped by collaboration with teachers and teacher educators (Goldsmith et al., 2014).

Guided by this framework, we address the following research questions:

1. What beliefs do teachers have about teaching students a deductive discourse for solving linear equations?
2. What challenges might teacher beliefs pose for teaching students how to reason deductively about equation solving?

Method

Informed by teachers' responses to survey items dealing with equations and solution processes, we developed an activity titled *Linear Equation Talk-Throughs* that we implemented as part of an 80-hour content-based professional development workshop for seven middle and high school algebra teachers in 2022. In the first stage of this activity, teachers privately recorded "talk-throughs" – video explanations of solution processes – in which they solved the linear equation $7x - 20 = 3x$, as if teaching students "who are just learning how to solve this type of equation" (per the written activity instructions). They also privately watched three different video talk-throughs recorded by the second author. These three researcher talk-throughs were designed to exemplify different possible discourse features of explanations of solutions to linear equations. Video 1 exemplified an approach focused on mediators, their spatial arrangement (*e.g.*, which "side" of the equation terms are on), and strategic actions-on-mediators needed to solve the equation. Video 2 exemplified an approach that interprets the given equation as a statement that the values of two expressions are equal and identifies the solution set through a sequence of deductive steps using properties of equality. Video 3 exemplified an approach that the teachers had come to know from a previous workshop activity as "solving by inspection": using the structure of the equation to make successive inferences about values of various terms and factors using number sense. Because the steps taken in Videos 1 and 2 are equivalent in their symbolic representation, these two talk-throughs served to illustrate a contrast between explanations that use words, mediators, and narratives differently. Because the approach taken in Video 3 is noticeably different, assigning values at intermediate steps to terms and factors in the equation, we see this talk-through as illustrating a more flexible approach to the use of narratives and routines during the equation-solving process. Table 1 provides a description and a representative transcript excerpt for each of the three researcher talk-throughs.

After the seven teacher-participants independently watched the researcher's talk-throughs, they divided into small groups (See Table 2) to compare the three talk-throughs and discuss the affordances and drawbacks of each. These discussions were video/audio recorded and transcripts were electronically generated and verified for accuracy. Each author independently coded the discussion transcripts; for each talk turn, we indicated any implicit beliefs about teaching and learning of equation solving that were evident in the teachers' analysis. Subsequently, we came together to discuss the independent themes to arrive at agreement on broad themes related to teacher beliefs. This research report presents the findings from this analysis and discusses possible implications of the teacher beliefs that surfaced.

Table 1: Researcher Talk-Through Video Samples

Video	Description	Sample from Researcher Talk-Through (Emphasis ours)
Video 1	Focus on actions-on-symbols Duration: 1:18	"My approach here is I'm going to try to **get all my x's on one side** of the equation and **put all the constants on the other side** of the equation. Because there's already a $3x$ on the right side of the equation, **I think I want to move the term $7x$, so it's over on the right side**, and I'll leave the -20 on the left. I'm going to take this $7x$ here, and I'm going to **change sides and change signs**, so I'm going to move it over to the right and **put a negative sign on it**."
Video 2	Focus on deductions about equal values Duration: 2:14	"So, I'm going to start by saying **because $7x$ minus 20 has the same value as $3x$, if I add $-7x$ to both of those values, I should get the same result**. So, in other words, $-7x$ plus $7x$ minus 20, that should be equal to $-7x$ plus $3x$, **'cause I took two equal values, $7x$ minus 20 and $3x$, and I added the same thing to each**. Now, if I look at the left side of the equation, I have $-7x$ plus $7x$ minus 20. $-7x$ plus $7x$, those are additive inverses of each other, so they add to zero. That means I'm left with 0 minus 20 equals $-7x$ plus $3x$."
Video 3	Using structure and number sense to solve by inspection Duration: 0:58	"**Well, one thing I notice about this equation** is I'm starting with $7x$ and I'm subtracting 20, and that leaves me with $3x$. One thing that I know is that if I start with $7x$ and subtract $4x$, that leaves $3x$. So that means that if I'm subtracting $7x$ minus 20 and getting $3x$, **that means that 20 has to be equal to $4x$**. And so now I have an equation that says that 20 is equal to 4 times x, 4 times my number x. **So I think what number multiplied by 4 gives me 20? Well, I know that 4 times 5 is 20**, so that indicates that x is equal to 5."

Table 2: Group Composition of Teacher Participants

Group	Participant	Level(s) Taught
Green	Danielle	High school
	Pablo	High school
	Frances	High school
Pink	Benjamin	High school
	Viola	Middle school (K–8 academy)
Yellow	Denise	High school
	Felipe	Middle school (K–8 academy)

Summary of Findings: Teacher Beliefs

Three prominent themes surfaced in our analysis of teachers' discussion of the researcher talk-throughs. Each of the three groups discussed the role or importance of understanding that solving linear equations is a deductive process. Notably, there seemed to be conflicting perspectives on when in the learning process the role of mathematical properties in the equation-solving process should be made explicit to students. A second common theme was estimation of students' capacity to understand solving linear equations as a deductive process. Two of the groups (Green and Yellow), comprising of five teacher participants, suggested that a deductive approach to solving equations would not be suitable for *all* students. Some teachers drew a distinction between those students who would be confused by "too many steps" and students who would benefit from an explicit development of the deductive reasoning behind the problem-solving process. The third salient theme was the perception of a difference between those more experienced in algebraic reasoning (teachers) and novices to algebraic reasoning (students) in terms of the potential to understand and engage in deductive algebraic discourse. We present some excerpts from the teachers' discussions to illustrate the major themes and our interpretations.

Role or Importance of Understanding Solving as a Deductive Process

A common thread among the three group discussions was beliefs related to the role or significance of a deductive discourse for solving linear equations, or of specific features of this discourse. Some teachers believed that it is productive to highlight algebraic properties upon introducing linear equations, thus providing students with the rationale supporting the steps of the problem-solving process, as was the case with Viola and Benjamin in the Pink Group in their exchange about Video 1, which focused on actions-on-symbols rather than deduction.

> Viola: It's that part that if I was new to algebra, I would not understand, "Why am I changing sides?" I'm assuming that a student who'd do this is well-versed in why I'm changing sides and why I'm changing signs. That statement assumes understanding is what I'm just saying.
>
> Benjamin: Especially with negative numbers, and that's where they get confused. My experience, they get confused a lot.
>
> Viola:…I'm going to tell you straight up; sixth grade is where it's introduced. If it's not introduced with concrete [models], they will struggle for a long, long time. Otherwise, you're going to have to rely on rules and they don't know why it works. So, this is key, right? …So, the question, "Why does it work?" needs to be happening way down before you... Yeah, because you're too far. You're advanced.

The teachers emphasize the importance of illuminating the "why" behind each step of the solution process, which directly aligns with the call-in standards documents to equip students with the deductive tools necessary to explain and justify each step. As Viola suggests, the language employed within Video 1, "changing sides" and "changing signs," phrases often used in describing the steps of solving an equation, "assumes understanding" that students may not yet possess.

The Yellow Group made a contrast between the explanations in Video 1 and Video 2 that suggested that Video 2, the deductive explanation, would be appropriate for introducing equation solving, whereas Video 1 is the conventional method to describe the steps of solving equations and would be deemed the "easier" approach.

> Felipe: I was going to say, I think using that method [from Video 2] would be a good way to introduce it, which sounds counterintuitive, but I feel like you show it to them, and they're like, "Okay." They can kind of see it, and then you show them the way we usually do it [referring to Video 1's method], and then to them, that seems easier, so they're like, "I like that a lot more."
>
> Denise: Yeah. Okay. I very much would tend to do that with my students. The first time we do it, I make them do it that way, so then when I show them an easier way, it makes sense, and they prefer it, and they're going to do that.

Counter to the belief that students *should* be exposed to the deductive reasoning behind the algebraic steps in equation solving as they learn the steps to solving, Danielle posited that introducing properties too early in the learning process may confuse learners.

Danielle: But that's after they already have learned to solve equations in ninth grade, in algebra one. Then we're doing it in geometry, we're saying, "Okay, these are what these properties are called now to practice those justifications." So, from that standpoint, but again, doing that not on the first time they're learning this. It's like, the second time. So, I love the use of properties, but I agree, I think it would be confusing to the people learning for the first time, and that's what I thought, too.

From Danielle's comment about Video 2, it would seem that she views the introduction of properties as a stepping-stone for inducting students into formal mathematics; she later clarified that she considered this appropriate only for students in advanced-track courses.

Our findings suggest that most teachers recognize benefits of exploring the justifications for steps of the problem-solving process. However, teachers also exhibited beliefs about students that stood in apparent tension with their view of the benefits.

Students' Capacity to Understand Solving as a Deductive Process

Another common theme that arose from our analysis relates to the teachers' beliefs about students' capacity to understand solving as a deductive process. For example, two of the teachers in the Green Group agreed that their students would be confused by the number of steps in Video 2, which explains the algebraic properties underpinning the deductive view of equation solving.

Frances: It's too many steps. And then, I would have simplified the right side instead of taking it to the next step. I would have simplified as I went to the next step on both sides. And he would simplify one side, then bring down to another one, another step, and then simplify on the first side. He wouldn't simplify it as he would go along; he would wait, go to the next step, next step, next step.

Danielle: Yeah.

Frances: Like, step one, step two, step three. My kids would get confused. Danielle: Yeah.

Frances: Yeah, my kids would get confused; too many steps. I already know that, too many steps. Now, the ones that are real bright, they would catch on real easily. But you have to realize you have to accommodate everybody in the class…

Additionally, Frances suggests Video 2, with the explicit reasoning steps, would be appropriate for her "real bright" students, implying that deductive reasoning is for more advanced students. Guided by an imperative to accommodate all students, Frances seemed to consider it important to rely on the equation-solving explanation that she believed would be most accessible.

Felipe and Denise in the Yellow Group reacted similarly to the explanation provided by Video 2, particularly the step where the researcher talk-through included a step to illustrate the reasoning behind combining like terms, which included factoring out the variable x to first add the coefficients, then redistributing x.

Felipe: I think right here they were getting confused.

Denise: Yes. They were not really understanding what you were doing there. Felipe: Yeah. This one right there, they'd be like, "What did you do?"

Denise: Yeah. Well, depending on what this is, combining like terms which is something you're going to do before I think you start doing solving, you need you look at that. So, they would see that combining like term, but doing like that, they would wonder where you got that.

The teacher participants implied that the conceptual reasoning behind "combining like terms" would be something introduced before equation solving, and that if it were integrated into the steps of equation solving, it could be a source of confusion for their students.

Differences Between Teachers' and Students' Potential to Understand Deductive Reasoning

The final notable theme that surfaced through our data analysis is related to teacher beliefs about the difference in potential to understand deductive reasoning between those deemed more knowledgeable about the equation-solving process, and those who are less experienced, as stated by two teacher participants. Regarding the explanation in Video 3, which relies on structure and number sense, Benjamin proclaimed, "For us [teachers] it's no big deal. It's trivial, we understand it," implying that it would be challenging for students to understand. Similarly, Felipe suggested a discrepancy between his view of Video 3 and the view his students would likely take:

> Felipe: That one [Video 3], I think is the more complex of them all. Well, no, not for us. For them to rationalize and understand because to them, when they see 5x, they generally, I think would see it as two units, 5 and x. Whereas we can see it as one thing that we can manipulate.

Discussion

We wish to acknowledge some limitations of the present study. Most notably, each researcher talk-through, by necessity, contained idiosyncratic features that may have diverted teachers' evaluations from the key discourse features we intended to embed in each video. For example, Video 2 contained steps justifying the process of combining $3x + -7x$ to obtain $–4x$; while linking this step to the distributive property may be edifying for students, we find that it is typically assumed that students are fluent in combining like terms before they learn to solve linear equations. Several of our teacher-participants, therefore, found it peculiar that the researcher justified this step in such detail. Because teachers understandably focused on critiquing specific choices that the researcher made in each explanation, they did not always discuss broader characteristics of each video, such as the commitment in Video 2 to reasoning deductively from assumptions. (Danielle was a notable exception: she aptly summarized Video 1 as "What do we do to isolate *x*?", Video 2 as "What keeps both sides equal at all times?", and Video 3 as "What makes that true?") We conjecture that adding prompts calling teachers' attention to some of these features in future iterations of the activity might enrich our understanding of teachers' beliefs about the feasibility and benefits of a deductive discourse for equation solving.

Participants' analyses of the researcher talk-throughs suggested that they saw potential benefits in the deductive explanation for the standard solution process given in Video 2 and the structure-oriented approach described in Video 3, though participants did see the role of these alternative explanations differently. For example, Felipe and Denise suggested that they would use an explanation like that in Video 2 to introduce students to the solution process before showing them an "easier" approach, while Danielle suggested that she would defer the in-depth explanation in Video 2 until her students began grappling with deductive reasoning and formal proof in geometry. While Viola and Benjamin stated that they found the "solving by inspection" approach in Video 3 to be "a fabulous tool," Denise and Felipe hypothesized that this method would be harder for students to understand and suggested offering it to students only as a "fun challenge."

Directions for Future Research

If there is strong consensus that deductive explanations and structure-oriented approaches for solving equations are potentially useful for students, why are actions-on-symbols explanations of solution processes so prevalent in teaching, as evidenced by reviews of curricular materials and our own teachers' recorded talk-throughs? In keeping with a sensible systems view of teacher beliefs, we seek to understand beliefs about instruction and about students that might mediate between teachers' generally favorable views of deductive discourse for equation solving and their likelihood of modeling this discourse in classroom practice. In this study we have discovered two such families of beliefs: (1) that a deductive perspective on equation solving is likely to prove difficult for students (especially those who have been the target of deficit attributions, such as students in an intervention course), and (2) that explanations that teachers find approachable (and in fact elegant or efficient) might nevertheless be beyond students' reach. Given that many teachers feel a strong sense of commitment to engaging all learners in successful mathematical practice, it is understandable that an explanation or approach that appears likely to confuse or frustrate learners might be disfavored in instruction. One goal of our project is to persuade teachers that it is feasible and worthwhile to engage all learners in deep and conceptually coherent algebraic reasoning.

Given that beliefs are deeply held and often resistant to change (Conner & Gomez, 2019; Philipp, 2007), we as mathematics teacher educators aim to design and provide professional learning experiences that allow teachers to reflect on, explore, and challenge their own beliefs about algebra teaching, while also helping to lower some of the perceived barriers that might impede students' access to deductive reasoning. Our teachers' analyses of the researcher talk-throughs offer some initial suggestions that we plan to incorporate into future iterations of the activity. At one point Frances noted that one factor that contributed to a general sense of "too many steps" in Video 2 was that the researcher rewrote the entire equation each time he wanted to simplify part of an expression; Frances stated that she would instead carry out detailed simplification steps in the margin and incorporate these changes into the solution process once done simplifying. We see this as entirely compatible with a deductive approach to equation solving: a sequence of simplification steps can be viewed as a sub-argument that generates an endorsed narrative about equivalent expressions; this sub-argument can be made separately from the main argument associated with the solution process. Viola pointed out that while she found the structural approach in Video 3 useful, she found it even more important to teach her middle school students to solve equations using concrete models first. Given that concrete models such as algebra tiles can act as mediators for unknown values and can encourage the kind of structure thinking embodied by Video 3, we see this suggestion as a potential bridge between the use of concrete models (which we have found that many teachers embrace enthusiastically) and deductive reasoning about equations. We have attempted this bridging with systems of linear equations with some success in our workshop and look forward to incorporating an explanation involving concrete models into the next iteration of the *Linear Equation Talk-Throughs* activity.

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