EFFECTIVE INSTRUCTIONAL PRACTICES THAT FOSTER THE DEVELOPMENT OF STUDENTS' EARLY ALGEBRAIC THINKING

Boram Lee
University of Texas at Austin
boram_lee@utexas.edu

Ingrid Ristroph
University of Texas at Austin
Ingrid.ristroph@utexas.edu

Despina Stylianou
City College of New York
dstylianou@ccny.cuny.edu

Eric Knuth
National Science Foundation
eknuth@nsf.gov

Maria Blanton
TERC
maria_blanton@terc.edu

Angela M. Gardiner
TERC
angela_gardiner@terc.edu

Ana Stephens
University of Wisconsin
Madison
acstephens@wisc.edu

Rena Stroud
Merrimack College
stroudr@merrimack.edu

Abstract

Is it possible to identify instructional practices that have an impact on student learning in mathematics? The study described here is part of ongoing efforts to understand and characterize effective instruction. We drew on the work of several recently developed frameworks for understanding teaching effectiveness to develop a protocol for studying effective instruction that both coordinates and extends existing research in the context of early algebra. Using a large-scale study, we characterized effective instruction in this context and documented the impact of such instruction on students' performance using both qualitative and quantitative analyses. Findings suggest that teachers' abilities to take up curriculum openings are important aspects of teaching. Furthermore, the manner with which teachers react to these moments strongly correlates with gains in student performance.

Keywords: Early Algebra, Effective Instruction, Teaching and Classroom Practice

Introduction

There is a convergence of belief in the field of mathematics education that the nature of classroom instruction significantly affects the nature and level of student learning. Ball and Forzani (2011) assert that "student learning depends fundamentally on what happens inside the classroom as teachers and learners interact over the curriculum" (p. 45). Scholars suggest that the nature of classroom teaching is by far the most significant factor in learning, surpassing all other aspects of schooling (e.g., Chetty et al., 2014; Opper, 2019). Policy recommendations echo these positions, presenting several principles describing what constitutes effective instruction that improves mathematics learning for all students (e.g., NCTM, 2000; NGA & CCSO, 2010; RAND, 2003). Yet, research evidence that links particular teaching practices with student outcomes is somewhat lacking. As Blazar (2015) contends, "the nature of effective teaching still largely remains a black box. Given that the effect of teachers on achievement must occur at least in part through instruction, it is critical that researchers identify the types of classroom practices that matter most to student outcomes" (p. 16).

In a rich body of work, several scholars have already *proposed* critical instructional practices that appear to contribute to student learning in mathematics. These practices include selecting rich, cognitively demanding mathematical tasks, and maintaining students' engagement with these tasks at a high level (Stein et al., 2000), building on students' mathematical thinking (e.g., Cengiz et al., 2011; Fennema et al., 1996), and facilitating discussions that support students in connecting mathematical ideas within the curriculum (Smith & Sherin, 2020).

We are interested in understanding the teacher actions within these practices that are associated with student learning. Hence, we examined this literature on teachers' instructional actions, particularly the frameworks that have been proposed as potentially fruitful for understanding teaching effectiveness, and then drawing upon this literature, we proposed a framework for understanding instructional effectiveness in the context of early algebra. We sought to understand the ways in which teachers take up "curriculum openings" (Remillard & Geist, 2002), that is, opportunities that arise in the moment in response to student reasoning and contributions to classroom mathematical work and how teachers engage with such opportunities. Finally, we used this framework to explore the relationships that exist between teachers' practices around these curricular openings and students' early algebra learning.

Literature Review

Teacher Moves and Frameworks

Critical moments within mathematics lessons that afford teachers opportunities to notice and act to further student reasoning and understanding have been well documented (e.g., Leatham et al., 2015; Stockero & Van Zoest, 2013). Ball and Cohen (1999) introduced the notion of teachers ''siz[ing] up a situation from moment to moment'' (p. 11) and using what they learn to improve their practice. Remillard and Geist (2002) referred to these moment-to-moment situations as "*openings in the curriculum*" (p.13, emphasis in original). These moments might include student questions or solutions that afford teachers the opportunity to engage students more deeply in mathematics. Similarly, van Es and Sherin (2002) used the term "noticing" to describe the ways in which teachers may take up unexpected moments and incorporate them into the lesson. Specifically in the context of algebra, Walkoe et al. (2022) referred to moments that arise as extensions of the curriculum that teachers can take up as "moments of algebraic potential." Additionally, Stockero and van Zoest (2013) defined "pivotal teaching moments" (PTMs) as "instance(s) in a classroom lesson in which an interruption in the flow of the lesson provides the teacher an opportunity to modify instruction in order to extend or change the nature of students' mathematical understanding" (p. 127). Other researchers have described such moments as "teachable moments" (Stockero & van Zoest, 2013), "mathematically significant pedagogical opportunities" (Leatham et al., 2015), "significant mathematical instances" (Davies & Walker, 2005), and "crucial mathematics hinge moments" (Thames & Ball, 2013). Unfortunately, little of this research has been done in the context of early algebra – with the work of Blanton and Kaput (2005), who documented the development of teachers' "algebra eyes and ears"—as one exception.

We chose to examine the specific teaching moves that occur during these opportunities, particularly as they occur during classroom discussions. What actions do teachers take to specifically foster mathematical development and extend students' engagement with rich mathematics in early algebra? To assist our examination, we turned to work on facilitating mathematical discourse.

Classroom Discourse

Classroom discourse is closely related to responsiveness. As Bakhtin (1986) explains, "every utterance must be regarded as a response to preceding utterances" (p. 91), and, as such, classroom discourse is a responsive act. Studies addressing discourse define responsiveness to students' mathematical thinking as "a characteristic of discourse that reflects the extent to which students' mathematical ideas are present, valued, attended to, and taken up as the basis of instruction" (Bishop et al., 2022, p. 11).

Several studies have characterized discourse as a responsive act and have examined its function and patterns. Prominent among these are patterns that view discourse as productive versus unproductive. For example, Knuth and Peressini (2001) discussed univocal versus dialogic discourse. Univocal discourse is characterized by an effort to convey information from the speaker (teacher) to the student (listener). The teacher responds to an incorrect student comment with the goal to correct it. Dialogic discourse, on the other hand, is characterized by a dialogue, a back-and-forth where both teacher and students participate equally within which new meaning is created. Wood (1998) described a funneling versus a focusing pattern in classroom discourse. The former involves the teacher leading a series of explicit questions or comments that aim to lead students to a certain correct response while the latter aims to co-develop a mathematical idea while authority is dispersed among all participants (students and a teacher).

Using the concepts of univocal versus dialogic, several studies have identified instructional "moves" that determine the degree of responsiveness or moves that determine the direction of the responsiveness in a univocal-dialogic continuum. At a more fundamental level are those studies that distinguished between the action of acknowledging and taking up a student comment or question, and the action of not taking or setting aside a student contribution (e.g., Ellis et al., 2019). Once a student contribution is taken up, several categories of moves may occur. Truxaw and DeFranco (2008) in particular, detailed moves such as acknowledging student responses (or ideas), revoicing them, responding to them by evaluating them (validating or correcting), eliciting a desired response (asking questions in a funneling pattern to draw a particular response), and extending student thinking (deepening student understanding by inviting generalizations and connections). Extending student thinking often involves moves that invite the participation of several students (e.g., Franke et al., 2015).

Situating the Study

The study draws from a larger longitudinal study that examined the impact and nature of an early algebra intervention in Grades 3–5. A curricular sequence based on an early algebra learning progression was developed that consisted of 18 lessons per year (Blanton et al., 2018). The lessons aimed to develop students' abilities to generalize, represent, justify, and reason with mathematical structure and focused on big ideas of early algebra: (a) mathematical equivalence; (b) generalized arithmetic; (c) functional thinking; and (d) variable (Kaput, 2008). For a detailed discussion of the larger study see Blanton et al. (2018, 2019).

Our Framework – Conceptualizing Early Algebra Moments

In this study, we take up the challenge to examine what constitutes effective instruction of early algebra, using the early algebra data corpus from the aforementioned larger study. We hope to add to the body of early algebra research by increasing our understanding of moments that afford teachers the opportunities to foster students' early algebraic thinking. In particular, we use videotaped lessons across the grades to address the following questions: (1) What types of teacher moves occur spontaneously (outside the curricular materials) in an inquiry-oriented early algebra setting? (2) How can we characterize the instructional moves within these moments? (3) To what extent do these moments impact student learning?

Methods

Data Corpus

The data corpus for this work is drawn from a large-scale, randomized, longitudinal study of the effectiveness of a Grades 3–5 early algebra intervention. The study involved approximately

3,200 students and 100 teachers at each grade level. It occurred in 46 schools (23 experimental, 23 control) in diverse settings. For this study, we focus on one lesson at each grade level, and the videotaped lessons of 12 teachers at each grade level implementing the respective lessons. Students completed an assessment addressing the intervention's big algebraic ideas that was administered as a pre-test at the beginning of Grade 3 and then at the end of each academic year in Grades 3, 4, and 5.

Data Coding and Analysis

The videotaped lessons were transcribed and coded independently by two team members, and any differences were reconciled through discussion. We used an open coding system (Saldana, 2013) to first identify *anticipated moments* (AMs) - expected instructional actions that might occur based on each lesson and on any accompanying teacher guidance for the lesson. For example, in a task that required students to create a graph, an AM might consist of a teacher inviting students to consider what type of graph would be appropriate or how to label the axes.

We next identified spontaneous early algebra moments (SMs) - unanticipated early algebra moments that arise during the enactment of a lesson. SMs often occur when students introduce unexpected, yet potentially fruitful, mathematical ideas that prompt teachers to take advantage of opportunities to advance students' algebraic thinking, even if this means deviating from the planned lesson. Using the earlier example, a student's suggestion to write an equation for the situation described by the graph, or to find how aspects of that equation relate to the graph were considered SMs.

When these moments were taken up, we subsequently coded for actions within these moments. Earlier work that characterized discourse as univocal versus dialogic (e.g., Knuth & Peressini, 2001) as well as categorizations of moves within responses (e.g., Truxaw & DeFranco, 2008; Ellis et al., 2019) framed our coding: setting aside, acknowledging, responding, eliciting, facilitating, and extending students' algebraic reasoning. These prior frameworks served as the basis for our coding (Syed & Nelson, 2015).

To address our research questions, AM and SM frequencies were noted, and SMs were each assigned one of six hierarchical response categories (i.e., setting aside, acknowledging, responding, eliciting, facilitating, and extending). Each teacher received an SM score by averaging the SM scores across that lesson.

We examined the correlation between the use of AMs and SMs and student overall performance using a simple linear regression (AM or SM frequencies and ratios being the independent variable, and student performance in each teacher's class being the dependent variable). Student performance was calculated by using the proportional difference between pre- and post-test growth on the assessment administered in each grade as part of the broader study. We also examined the correlation between the level of SM scores and student performance.

Results

We first present the frequency of AMs and SMs identified and taken up by teachers in the 36 analyzed videotaped lessons. We then take a closer look at SMs by presenting examples of such moments as well as the teacher actions associated with these SMs. Finally, as the goal of the work is to investigate instructional practices that are effective in fostering the development of early algebra learning, we examine the impact of teachers' interactions with AMs and SMs on students' early algebra learning. We do so by examining both the frequencies of these AMs and SMs as well as the characteristics of SMs and teachers' SM scores.

Algebra Moment Counts and Relationship with Student Performance

Based on the project team's identification of anticipated moments in each of the three lessons, there were 19 AMs in Grade 3 (lesson 15), 12 AMs in Grade 4 (lesson 5), and 9 AMs in Grade 5 (lesson 12). However, in the videotaped lessons teachers varied in taking up these opportunities. Table 1 shows the variation in the degree to which teachers took up AMs. Overall, Grade 3 teachers took up an average of 7.58 AMs (min 2, max 16) out of 19 potential AMs, 7.62 were taken up in Grade 4 (min 3, max 12) out of 12 potential AMs, and 6.8 in Grade 5 (min 5, max 8) out of 9 potential AMs. Viewed as a ratio of AMs taken up compared to the total AMs available to each teacher (in other words, the ratio of AMs one could possibly implement within the time that was available to them), Grade 3 teachers took up 86% of AMs (min 60%, max 100%), 88% in Grade 4 (min 55%, max 100%), and 86% in Grade 5 (min 55%, max 100%). (For a more detailed look at Grade 5 AMs, please see Ristroph et al., (2022))

SMs (spontaneous moments) also varied across the implemented lessons we analyzed. Overall, in Grade 3 teachers took up an average of 2.58 SMs (min 1, max 6), 2.00 in Grade 4 (min 1, max 5), and 5.76 in Grade 5 (min 1, max 11). By definition, there was not an expected number of spontaneous moments for each lesson.

Table 1: AM and SM Frequencies in Grades 3, 4, and 5

		Minimum	Maximum	Mean	r^2
AMs Taken	Grade 3	2	16	7.58	0.088
	Grade 4	3	12	7.58	0.134
	Grade 5	5	8	6.8	0.083
Proportion of AMs taken	Grade 3	0.6	1	0.86	0.001
	Grade 4	0.55	1	0.88	0.035
	Grade 5	0.55	1	0.86	0.185
SMs Taken	Grade 3	1	6	2.58	0
	Grade 4	1	5	2.00	0.098
	Grade 5	1	11	5.76	0.43

A linear regression of the relationship between the frequency of both AMs and SMs in lessons, and student growth as measured by the early algebra assessment (Blanton et al., 2019) did not detect any significant correlations (r^2 = 0.088 and 0.134 and 0.083 for AMs taken, 0.001 and 0.035 and 0.185 for AM ratio, and 0 and 0.098 and 0.43 for SMs taken for Grades 3, 4 and 5 respectively – see Table 1). In other words, students whose teachers took up more AMs or SMs during these lessons experienced just as much growth in the early algebra assessment as their counterparts whose teachers took fewer AMs/SMs.

Characterizing SMs

Each SM was assigned one of six "response categories" (i.e., setting aside, acknowledging, responding, eliciting, facilitating, and extending). With the exception of *setting aside*, an indication of teachers not engaging with student reasoning, the remaining five categories occurred as a continuum with an increasing degree of responsiveness and patterns of dialogic discourse. Hence, we considered these categories as "levels", starting with *setting aside* as Level 0 to "extending" as Level 5.

Table 2: Categories of Teachers' Responsiveness to Spontaneous Moments

Response Categories & Teacher Moves	Classroom Examples
L1: Acknowledging Acknowledges a response but does not act on it	Task: Fill in the blank: $m = \_ \times m$ St: Or we could put in "0" T: Hmm.. Interesting. Alright, next problem…
L2: Responding Validates correct answers or Corrects errors Revoices or highlights a response or contribution	Task: "John is thinking of a number. If he multiplies it by 1 and adds 0, what does he get? What kind of number was he thinking of?" S: Any number. T: Marco says "any number." Yes.
L3: Eliciting Elicits procedures, answers & facts, solutions of problems. Bounded by the need to produce an answer to the task	T: What's something you might need to do? S: An equation... we can use variables. T: Okay, what variables would you use? S: t T: Okay but for what? What is our equation? S: $t = 2s$ T: And, what would t stand for? S: Chocolate.
L4: Facilitating Facilitating understanding (asking questions) without validating, correcting, or cueing. Facilitating the building of conceptual ideas underlying the task	T: I want you to share your thoughts. S: We have strawberry and chocolate. T: Say more. What other pieces of evidence are there? S: Twice as many strawberry as chocolate. T: What does that mean? S: times 2 T: How are we going to find out how many strawberry? What else can you say? S: We can write two times as many strawberry more than chocolate.
L5: Extending Encouraging other students to add to the response	T: Let's start with Anna. S1: You don't know how much so you can use a variable. T: Oh, you can use a variable to represent what you don't know. Anymore or do you want to pass it on to someone else?

Pressing for Justification Pressing for Generalization Facilitating connections among mathematical ideas	S2: I want to add. I agree with Anna. So, like, you can do *v* times 2 equals something. Ss: You can use *v* times 2 equals *s* because that’s how many strawberry sandwiches. T: What does *v* mean, Sean? S: It means that number of chocolates. You don't know how many. T: So we don't know how many, but this is how many chocolate. S: You can use *c*. T: Can we use *c*? And, wait, what does this mean? S: Sandwiches. T: Do these numbers make sense in these problems? S: Because *s* is the amount of strawberry and c is chocolate. T: How do you know that these numbers (pointing to function table and then to the equation) work?

In this continuum, we identified instructional moves that gradually came to define these categories. As the categories changed from acknowledging, to responding, eliciting, facilitating, and extending, we noted an increase dialogic discourse. To that end, teachers gradually reduced the validation or correction of student responses/reasoning and increased the involvement of students in the discussion and in sharing the mathematical authority. We observed teachers gradually shifting the classroom discourse from focusing on a correct response or procedure to allowing students to negotiate a statement and, ultimately, inviting students to reason algebraically by generalizing, justifying, and connecting algebraic ideas.

As shown in Table 2, instructional engagement and responsiveness increased in each level. In *acknowledging* and *responding* (Levels 1 and 2 respectively), teachers did not build on student responses at all. *Eliciting* (Level 3), represents a relatively dramatic change in which teachers begin to build on student responses by drawing out (cueing, funneling) the desired response. Teachers elicit student participation, but at a very procedural level. The discourse is bounded by the need to produce a (predetermined) answer/solution/strategy/response to a given task. *Facilitating*, Level 4, is an extension of *eliciting* in that, once again, the teacher is drawing responses from students by inviting them to engage in more conceptual reasoning. Teachers orchestrate discourse around conceptual understanding but remain the main authority in the room as they guide the direction of the lesson. The difference between *eliciting* and *facilitating* may be subtle, unless one pays attention to the teacher’s cues− “What is something you might *need to do?*” (the teacher’s invitation to students in the illustrative eliciting example), as opposed to “I want you to *share your thoughts.*” (the teacher’s invitation in the illustrative facilitating example). The latter is more open-ended than the former in that it welcomes general student observations and questions. Finally, *extending*, once again, elevates discourse to a new level. The teacher openly shares authority and while pressing students for justification, generalization and for connections among ideas and also invites students to bring their curiosities and understandings to the floor and embraces these new mathematical directions.

Relationships Among SMs and Student Performance

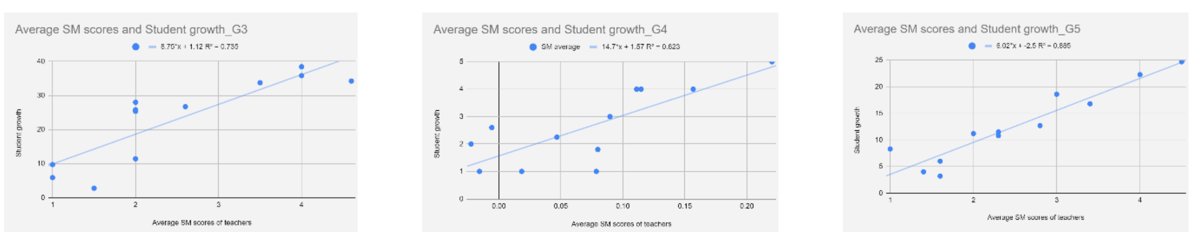
As the last part of our analysis, we examined students’ overall performance in early algebra and its relationship with the algebra moments during instruction. To this end, we conducted a linear regression of the relationship between the frequency of both AMs and SMs in lessons and student growth as measured by the early algebra assessment. As we did not detect any significant correlation between frequencies of either AMs or SMs and student performance, we proceeded to examine the SMs more closely.

Each SM was again assigned one of six characterizations (i.e., setting aside, acknowledging, responding, eliciting, facilitating, and extending). Each teacher received an SM score by averaging the SM levels across that lesson. Table 3 shows the variation in teachers' SM scores. Overall, in Grade 3 teachers had an average of score of 2.48 (min 1, max 4.6), 2.67 in Grade 4 (min 1, max 5), and 2.47 in Grade 5 (min 1, max 4.5). A simple linear regression between SM score and early algebra growth (r^2 = 0.735, 0.623, and 0.885 for Grades 3, 4 and 5 respectively as shown in Table 3) suggests a strong correlation overall between SM-scores and student growth.

Table 3:

		Minimum	Maximum	Mean	r^2
SMs	Grade 3	1	4.6	2.48	0.735
Score	Grade 4	1	5	2.67	0.623
	Grade 5	1	4.5	2.47	0.885

In contrast to the findings regarding *frequency* of either AMs or SMs, the level, or *quality* of SM implementation has a strong correlation with student growth. Students who were in classrooms in which teachers not only chose to take up the algebraic moments that arose spontaneously during classroom discourse, but also chose to respond to these moments by engaging in dialogic discourse in a manner that advances student algebraic reasoning (i.e., justification, generalization and encouraging connections), tended to have higher gains in the algebra assessment. The engagement in dialogic discourse is not a dichotomy, but a continuum of teachers' moves that increase student engagement with these algebraic practices, and with each other's reasoning. Figure 1 shows a graphic display of this relationship.



1a. Grade 3
$r^2 = 0.735$

1b. Grade 4
$r^2 = 0.623$

1c. Grade 5
$r^2 = 0.885$

Figure 1: Student Growth as it Relates to Average SM Teacher Level

Discussion and Conclusion

We began this study by asking whether it is possible to identify instructional practices that have an impact on student learning in mathematics. We examined teachers' implementation of an early algebra intervention and looked closely at curriculum openings – anticipated and spontaneous moments in instruction. Previous work suggests that teachers' responsiveness to these spontaneous moments are an important characteristic of "good" instruction. Teachers' abilities to take up openings in the curriculum and to identify important mathematical moments that arise outside of the curriculum have been identified as a *potentially* important aspects of teaching and important to student learning.

Our work corroborates these earlier findings and, in fact, finds a substantial positive relation between teachers' manner of taking up curriculum openings and actual student performance. It also brings us one step closer to understanding aspects of effective instruction regarding early algebra. As discussed earlier, attention to early algebra instructional effectiveness has been sparse, but our work gleans the untapped potential of this area of study. Our examination of several lessons across grades shows a clear pattern that levels of implementation of these spontaneous moments have a strong correlation to gains in student performance. As teachers attended to students' reasoning, be it correct or incorrect, and invited discussions on these issues as they organically arose, student performance on these concepts improved. These results offer promise that this is a fruitful area of research that we can continue to explore, and provide directions for improving teacher preparation to promote more rich early algebra teaching and learning.

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