EPISTEMOLOGICAL OBSTACLES RELATED TO TREATING LOGICAL IMPLICATIONS AS ACTIONS: THE CASE OF MARY

Anderson Norton
Virginia Tech
norton3@vt.edu

Rachel Arnold
Virginia Tech
rlongley@vt.edu

Joseph Antonides
Virginia Tech
jantonides@vt.edu

Vladislav Kokushkin
Virginia Tech
vladkok@vt.edu

Understanding how students reason with logical implication is essential for supporting students' construction of increasingly powerful ways of reasoning in proofs-based mathematics courses. We report on the results of an NSF-funded case study with a mathematics major enrolled in an introductory proofs course. We investigate the epistemological obstacles that she experienced and how they might relate to her treatment of logical implications as actions. Evidence shows that an action conception may pose challenges when students transform or quantify implications and may contribute to erroneous assumptions of biconditionality. Our report on available ways of operating with logical implications as actions is a first step in designing instructional tasks that leverage students' existing reasoning skills to support their continued development.

Keywords: Cognition, Learning Theory, Reasoning and Proof, Undergraduate Education

From a constructivist perspective, engaging all learners requires teachers and researchers to understand students' available ways of operating and invite students to bring them forth to make meaning in new mathematical contexts. At the same time, teachers and researchers must recognize their role in supporting students' constructions of increasingly powerful ways of operating. For example, Dubinsky (1986) identified two distinct ways that students operate with logical implications (LIs): as actions or as objects. Instruction on LI must therefore be sensitive to students who reason in each of these ways.

As an action, an LI ($P \rightarrow Q$) involves three components: a predicate, P; a conclusion, Q; and a transformation between them. Students who treat LI as an action can reason by *modus ponens*: the truth of P transfers, by implication, to the truth of Q. However, in treating an LI as an action across three components, rather than as a single object, students might experience persistent challenges in transforming and quantifying them. In particular, they might not reason by *modus tollen*s: Q is false implies P must also be false ($\sim Q \rightarrow \sim P$). They might also not attend to the quantification of an LI when determining its negation.

Here, we report on results from a case study with Mary–a mathematics major enrolled in an introductory proofs course in Fall 2022–who consistently treated LIs as actions. Data comes from three clinical interviews, as part of our larger study called The Proofs Project. Within our project, we frame epistemological obstacles (EOs) as persistent challenges experienced within mathematical interactions between the teacher and the student (cf. Brousseau, 2002). The purpose of this report is to document EOs related to LIs, their transformations, and their quantification, especially related to the treatment of LIs as actions. We address the following two research questions:

1. What specific EOs arise in mathematical interactions with Mary?

2. How might these EOs relate to Mary's treatment of LIs as actions?

Theoretical Framework

We adopt an overarching Piagetian framework in which students' logical-mathematical reasoning is understood in terms of their available ways of operating. In this section, we elaborate on what that means in terms of reasoning with LIs. Then, we interpret existing research on students' logical reasoning within that framework.

Action-Object Theory Applied to Logical Implications

We conceptualize student reasoning about LI using an action-object perspective, grounded in Piaget's (1970) genetic epistemology. According to this view, mathematical concepts arise as objects first through the coordination of mental actions (a process called *reflective abstraction*). APOS theory (Arnon et al., 2014; Dubinsky, 1991) is one Piagetian-based perspective that adopts an action-object view of mathematical thinking. Within APOS, an *action* is an explicit transformation that must be carried out step-by-step. Through interiorization, sequences of actions become coordinated into *processes* that may be carried out in thought. Note, however, that in our analyses, we do not distinguish between an action and a process; so for us, an "action" understanding may involve processes. Processes may be subsequently encapsulated into *objects* on which actions may be applied. When a student assimilates a task situation, they call to mind a *schema* of associated actions, processes, and objects.

Dubinsky (1991) hypothesized that for students to develop powerful ways of reasoning about logic and proof (e.g., to make sense of mathematical induction), they need to hold LI as an object. For this to occur, LI needs to be encapsulated (see Figure 1). Characteristically, a student with an action conception of LI might think about the statement "If P, then Q" as a command to operate: verify that P is true, and if so, then Q is true. A student with an object conception of LI has interiorized this way of operating, and they can meaningfully transform an LI, such as by taking its converse, contrapositive, or negation.

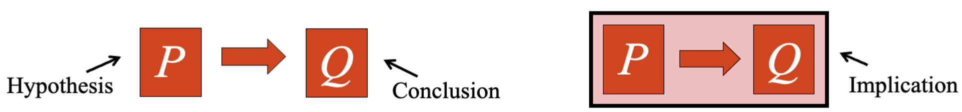

Figure 1: Logical Implication as Action or Object

Epistemological Obstacles Related to Logical Implications

The concept of an EO originates from the work of Sierpińska (1987) and Brousseau (1997, 2002), in which EOs were conceptualized as the necessary challenges in students' mathematical development. We frame EOs as cognitive challenges experienced by both teachers and students during instructional interactions. EOs persist over time, even in research-based instruction.

Prior literature has identified multiple challenges related to student reasoning with LIs. For example, students tend to interpret a conditional statement (*if P, then Q*) as a biconditional one (*P if and only if Q*) (e.g., Girotto, 1990; Wilkins, 1928). Although such interpretations are not valid from the standpoint of formal logic, they are deeply rooted in everyday language and can be explained from a pragmatic point of view (Epp, 1999; Geis & Zwicky, 1971; Rumain et al., 1983; Wagner-Egger, 2007). Transforming LIs presents a related challenge. Empirical studies have shown that undergraduate students often conflate an LI ($P \rightarrow Q$) with its converse ($Q \rightarrow P$) (Durand-Guerrier, 2003; O'Brien et al., 1971) and with its inverse ($\sim P \rightarrow \sim Q$) (Goetting, 1995; Knuth, 1999, 2002). Other scholars reported on students' struggles to understand the equivalence between an LI and its contrapositive ($\sim Q \rightarrow \sim P$) (Dawkins & Hub, 2017; Stylianides et al., 2004) or between an LI and a disjunction ($\sim P \vee Q$) (Hawthorne & Rasmussen, 2015).

Quantification presents another challenge, particularly when quantifiers are hidden. Shipman (2016) observed that the LI $P \rightarrow Q$ in fact has the form $P(x) \rightarrow Q(x)$. The latter contains a hidden universal quantifier and is conventionally interpreted as "$\forall\, x,\ P(x) \rightarrow Q(x)$." Overlooking hidden quantifiers can result in logical fallacies, which are especially evident when students attempt to transform LIs (e.g., negate them). In a similar vein, Durand-Guerrier (2003) reported on the challenge of dealing with LIs when the quantifiers are implicit.

Logical statements containing multiple quantifiers (i.e., multiply quantified statements) exacerbate students' struggles with quantification. Prior studies have documented naïve readings of such statements (Dubinsky & Yiparaki, 2000; Epp, 2003). Dawkins and Roh (2020) found that undergraduate students tend to read multiply quantified statements semantically, i.e., applying their previous knowledge to make sense of the formal statements, thereby neglecting the syntax of the statement and the order of the quantifiers in particular (Piatek-Jimenez, 2010). In our framing, such EOs should be deliberately evoked and carefully addressed over time.

Methods

Our study took place in the context of an introductory proofs course in the Fall of 2022. Using initial data from the classroom, we applied the technique of purposeful sampling (Maxwell, 2013) to select students who we thought would provide diverse ways of thinking and who seemed to articulate their reasoning. In total, four students agreed to participate in three clinical interviews. Each interview was between 30-60 minutes and was video recorded to capture students' interactions with the interviewer. The first interview focused on students' reasoning with LIs. The second interview focused on quantification of variables, statements, and LIs. The third interview employed stimulated recall, using video clips from the first two interviews to question students about their prior responses and underlying reasoning. Tasks for the first two interviews were given separately on slips of paper, and students were provided with a Livescribe pen to record what they wrote and drew.

We used the constant comparative method (Corbin & Strauss, 2014) to build and iteratively refine a system of codes for categorizing interview data. Two researchers conducted a first round of coding of each student's first two interviews, first analyzing all students' first interviews, then all students' second interviews. This initial analysis allowed us to (a) build a draft codebook drawn from literature reviewed within our theoretical framework; and (b) select important moments in each student's first two interviews to be presented to students during the stimulated recall portion of their third interview.

In a second round of coding, the researchers applied the draft codebook to analyze video data student-by-student. During this process, we clarified the definitions and indicators of codes in the initial draft codebook, defined new and emergent codes as needed, and resolved disagreements by discussing our interpretations of the data with the research team until reaching consensus. As we coded, we also wrote analytic memos of our hypotheses and interpretations of the data, and we noted quotes that seemed to capture important moments in students' reasoning.

Results

Within our framework, we report results in two sections: one assessing Mary's treatment of LIs as actions, versus objects; the other identifying EOs experienced during the interviews.

Mary's Treatment of Logical Implications as Actions

As indicated in Table 1, we coded 13 segments indicating how Mary operated with LIs: nine segments from the first interview and four segments from the second interview. Each segment provided evidence that Mary treated LIs as actions. The strongest evidence came from the middle of the first interview, as Mary responded to the first question (a) shown in the following Probability Task:

> Let *P* and *Q* be events that have some nonzero probability of occurring, and suppose the following two implications are true:
>
> - If *P* and *Q* are mutually exclusive, their probabilities are not independent.
> - If the probabilities of *P* and *Q* are independent, the probability of *P* and *Q* is the product of the probability of *P* and the probability of *Q*
>
> (a) What can you conclude if *P* and *Q* are independent?
> (b) What can you conclude if the probability of *P* and *Q* is not the product of the probability of *P* and the probability of *Q*
> (c) What can you conclude if *P* and *Q* are mutually exclusive?

Mary: It's an if statement, so it's either, that happens, or it doesn't happen.
R1: Oh, I see. So, you are talking about this part happening or not [pointing to the left side of the second LI (second bullet point)].
Mary: So, we already know that this part happens because P and Q are independent, and it's asking if they are independent [pointing to the LI]. So, this is true; if is a true statement in this case [points thumb up]… If happens. But in this one [pointing to the first LI], we don't necessarily know if if happens because… In my understanding of how if-statements kind of work, being dependent has to depend on being mutually exclusive, but being mutually exclusive does not necessarily mean it's dependent. So, if P and Q are mutually exclusive… Actually, if they are mutually exclusive… [long pause] No, I'm going to stick with that. If P and Q are mutually exclusive, they could still be independent… So, we don't know if P and Q are mutually exclusive, but we do know what their combined probability is, but I can't say anything else because I don't have…
R1: Okay, so in addition to knowing that P and Q are independent, you can also tell me what other fact?
Mary: For "a", the only thing else we know for a fact is that the probability of P and Q is the product of the probability of P and the probability of Q.

Table 1: Common Codes

Code	Description	Occurrences
AO	evidence of student treating LI as an action or object	9, 4
LIff	interpreting LIs as biconditional	6, 0
LIt	transforming LIs into their converses, contrapositives and negations	5, 1
Qh	hidden quantification, especially leading to ambiguity in meaning	2, 6
Qord	reasoning about order of quantification in the meaning of a statement	0, 7

Mary's response at the start of the transcript indicates that she thought about LIs as on/off switches: when the "if" part of an LI switches on, the "then" part switches on. This way of operating fits an action conception of LI: the action of turning on the first part of the LI (the hypothesis) is followed by the action of turning on the second part of the LI (the conclusion). This way of operating worked well for Mary when the hypothesis was given, as it was for the case of the second bulleted LI. However, when the hypothesis was not given, Mary was prone to reversing the LI (as she did for the first bulleted LI) and did not attempt to reason with *modus tollens* via its contrapositive.

Mary's reference to LIs as on/off switches persisted across all three interviews (including the SRI) and, at times, led her to think of LIs as biconditional (if and only if): either the hypothesis and conclusion of an LI were both true or both false. We will see evidence of this reasoning in the next section, specifically as we discuss the LIff code.

Epistemological Obstacles Experienced during Instructional Interactions with Mary

Among the potential EOs identified in prior literature, four appeared frequently (more than three times each) in our coding of Mary's first two interviews: LIff, LIt, Qh, and Qord (see Table 1). In this section we provide examples for each code and characterize the EOs in relation to Mary's ways of operating with LIs.

LIff. We have seen that Mary could apply an LI to a given hypothesis to draw the pertinent conclusion (modus ponens), using her on/off switch. When representing an LI abstractly (outside of directly applying it to a given hypothesis) Mary relied on a subset relationship and generated concrete examples. For instance, during the first interview, in considering $P \rightarrow Q$, Mary introduced a diagram showing that horses, dogs, and porcupines form subsets of the set of mammals. She explained that if we knew something were a horse, we would already know it was a mammal; but there are other subsets of mammals, so an animal being a mammal does not necessarily imply it is a horse (i.e., the converse $Q \rightarrow P$ does not necessarily follow). Elsewhere in the first interview, she explained that if-and-only-if relations occur when there is only one possible subset: "A is the only subset of B."

After reviewing the first interview, the researchers identified a potential conflict between Mary's two ways of operating with LIs: the on/off switch and the subset relationship. We found initial evidence as Mary responded to question (c) in the Probability Task. Mary read the question aloud and proceeded as follows.

Mary:	So if they are not mutually exclusive, then they're independent, and then this is also true [pointing to second bulleted LI].
R1:	How did you know that?
Mary:	Because, if they are mutually exclusive then their probabilities are not independent [pointing to first bulleted LI], so that means they are dependent. So, if they are mutually exclusive, they are dependent. But we are saying that they are not mutually exclusive, so they can't be dependent, so they have to be independent. Because dependent and independent is just like an on and off switch [waving hands back and forth]. It's either one or the other. It's binary. And since we know that it's independent, we know… [points to second bulleted LI]

We inferred from Mary's explanation that she had applied her binary on/off switch to the first LI, taking the negation of the hypothesis to infer the negation of the conclusion: because the "if" was off, it followed that the "then" was off. Note that this way of operating equates the LI $P \rightarrow Q$ with its inverse $\sim P \rightarrow \sim Q$ (the converse of the contrapositive) rendering the relation biconditional. This way of operating served Mary well in reasoning with the contrapositive of an LI, but it conflicted with her subset meaning for LIs whereby she had explicitly denied that the converse must follow. We investigated this potential cognitive conflict further during the third interview, which we discuss in the next subsection.

LIt. Early in the first interview, Mary drew an "umbrella" set with two subsets under it. She argued that if something did not exist in the umbrella set, it could not exist in either of the subsets under it (an argument for the contrapositive). However, shortly after, she opened up the possibility that there could be a second umbrella under which the subset could fall so that something could exist in the subset without existing in the first umbrella. Thus, it seems that Mary was not certain about the logical equivalence of an LI and its contrapositive. Further evidence for this claim comes from Mary's responses to the questions in the Probability Task. For example, in response to question (a), Mary drew the direct conclusion using the second bulleted LI, summarizing, "that's the only thing I can say with 100% certainty." She did not apply the given condition to the negation of the conclusion in the first bulleted LI to infer the negation of its hypothesis.

On the other hand, when Mary treated LIs as biconditional, using her on/off switch, both the contrapositive and converse logically followed. There is no need to transform a biconditional statement because the hypothesis and conclusion are paired with the same truth value: they are both on or both off. Thus, Mary's treatment of LIs as actions resolved any ambiguity she might have experienced in attempting to transform an LI into its contrapositive. At the same time, it equated an LI with its converse, which Mary knew, via her subset representations, did not always follow. The researcher attempted to induce this potential cognitive conflict in the third interview by showing Mary a video clip of her responses (from the first interview) to the Probability Task.

Mary:	[having just reviewed an audio-video clip of her saying, "they could still be mutually exclusive without being dependent"] I would like to amend that statement.
R1:	How would you amend it?
Mary:	If-then, um, if P and Q are mutually exclusive, their probabilities are not independent. Since we know P and Q *are* independent, they are *not* mutually exclusive.
R1:	Oh! OK. That *is* something new. So, what do you think changed from that Mary to this Mary, to where you had that realization?
Mary:	I think we went through the if-then unit [in class]. Before that I always used if-then as, like, it *could* work, but kind of like two inputs could converge and get one outcome, but now I know that's not true, I guess.
R1:	What do you mean "two inputs could converge and give you one output."
Mary:	Like, before I thought, with an if-then statement, the then could be true but the if not necessarily, but if the if is true, then the then also has to be true. But on its own the then could be true. I thought that before, but now I do not think that.
R1:	Oh, you don't think the then can be true unless the if is true.

After Mary affirmed the researcher's interpretation, the interview continued with the researcher showing Mary a video clip of her explaining (to the contrary) that "being dependent has to depend on being mutually exclusive, but being mutually exclusive does not necessarily mean it's dependent." Just then, Mary interjected, "yes!" and explained as follows.

Mary:	It's like, if W implies X, just because you have X doesn't mean W is true.

R1: But I thought you said just a little while ago that the only way the then happens is if the if happens.
Mary: [Long pause] Mm-hmm. I did say that.
R1: But what you just said now…
Mary: Contradicts that [laughing]? It does. Can I draw this out?

Mary then drew an Euler diagram in which the circle representing cases in which W is true was contained within the circle representing cases in which X was true. Then she exclaimed, "Oh dear! That just contradicts directly with what I was saying before. I take back what I said before. I'm going back to my original."

In treating LI as an action, Mary had accommodated her on/off switch to account for the logical equivalence of the contrapositive. However, this action of switching hypotheses and conclusions on and off together introduced the logical equivalence of the converse as well. Relying on a subset representation, such as an Euler diagram, Mary recognized the contradiction, but by the end of the third interview it remained unclear how she might reliably transform LIs into their contrapositives without also equating them with their converses.

Qh. Mary attended to the language of quantification within mathematical statements, including LIs, but she sometimes did so in unconventional ways. For instance, at the start of the first interview, she was considering an LI whose conclusion specified that all objects x in set S satisfy a particular property, P. Then, when considering a statement that "not all" objects satisfy that property, rather than taking this statement as the negation of the conclusion (thus, negating the hypothesis as well, via the contrapositive), she indicated that "not all" meant the conclusion might or might not happen. Rather than treating the conclusion as an invariant condition (that all x in S satisfy P), she had broken it into two cases: the case where the condition was met and the case where it was not. The quantification "not all" meant for her that both cases could exist, and so the conclusion was true for one case and not the other. In the second interview, we find stronger evidence for how Mary quantified LIs themselves.

At the start of the second interview, the researcher asked Mary to respond to the following task from Shipman (2016), which contains an LI within an LI: "$If\ A\ and\ B\ are\ sets, then\ (x \in A \Longrightarrow x \in B) \Longrightarrow A \subseteq B$." The following exchange began as the researcher asked Mary whether she could quantify the LI shown in parentheses.

R1: Does this mean there exists an x such that x of A implies x of B, or does it mean for all x, x of A implies x is in B? Yeah, how would you quantify this implication, just this part [pointing to the LI in parentheses]
Mary: Oh, I would say there is an x within A that implies… There exists an x such that an x within A implies that x is also in B.
R1: What if I changed the quantification and said for all x, x is in A implies x is in B. Would that change your answer?
Mary: Yes. For all values of x, x of A implies x of B. Actually, it's false, because there could be a number outside of x, like a y outside of x, that doesn't make the implication true.
R1: Alright, I'm going to quantify it one more time: for all x in A, x being an element of A implies x is also an element of B. So, if this implication is true for all x in A…
Mary: Then it is a subset, yes.

R1: What changed?

Mary: Because you told me that x is A. You basically told me that x is within A, and for any x in A, it's also going to be in B

Mary seemed more confident about the truth of the conclusion (that A is a subset of B) when the universal quantification of the LI was restricted to A. This would seem illogical because the universal quantification of the LI within the entire universe of numbers would include its quantification within A. However, Mary's reasoning makes more sense when we consider her tendency to move the quantification of an LI within the LI, applying it to the hypothesis that x is in A, instead: When x is in A, the first switch is on. when x is not in A, it's off. Thus, we conclude that Mary noticed the role of quantifications of statements within implications, but she did not seem to quantify LIs themselves.

Qord. The second interview included two sets of tasks designed specifically to assess whether students attended to the order of quantification when a statement included multiple quantifiers. All of the Qord codes occurred in our analysis of Mary's responses to those tasks. One such task asked Mary to evaluate the following statement: "For all positive real numbers k, there is a natural number M such that $1/k < M$" (Piatek-Jiminez, 2010). Mary responded as follows: "k could be like 1 over a billion, and therefore this $1/k$ would be a billion. And there technically is an M–a billion and 1–that is bigger than $1/k$, so I say that, yes, it's true. You could literally make M the inverse of k, plus 1."

From this response and similar responses to the other tasks, we inferred that Mary did not seem to attend to the order of quantifiers and how this order might alter the meaning and validity of statements. However, because these quantifiers occurred within statements and not across LIs, we do not currently have a hypothesis about how Mary's treatment of LIs as actions might influence her treatment of multiple quantifiers and their order. We conjecture that students might need to objectify sequences of quantifiers in the same way that they objectify LIs.

Conclusions

Mary had developed powerful ways of reasoning with LIs (e.g., reasoning by *modus ponens*). However, in treating LIs as actions (not objects), Mary seemed to experience challenges anticipated within our action-object framework (Dubinsky, 1991; Piaget, 1970), including EOs identified in prior research (e.g., Epp, 1999; Wagner-Egger, 2007). Specifically, she experienced persistent challenges in transforming and quantifying LIs because they were not, for her, objects to be transformed or quantified. In attempting to transform LIs into their contrapositives, Mary accommodated her on-off scheme for LIs to include biconditionality, rendering the LI logically equivalent to its converse (Goetting, 1995). In attending to quantification of an LI, Mary moved the quantification into the LI, applying it to the hypothesis instead (Norton et al., 2022). Thus, we find that many EOs identified in prior research might relate to treatment of LI as an action.

Mary's reasoning relied on particular ways of operating with LIs, such as her on/off switch and her use of subset relationships. Our understanding of those ways of operating is a necessary starting point for engaging students like Mary and for supporting their continued development. It enables us to "design learning environments that take student engagement and learning into account." We are particularly interested in designing tasks that might support Mary's objectification of LIs, as a potential means of meaningfully addressing the EOs she experienced with regard to transforming and quantifying LIs. We recognize that as much as we have learned about logical ways of operating from Mary, we have more to learn from other students.

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