

A STUDY OF WHAT STUDENTS FOCUS ON AND NOTICE ABOUT QUADRATIC FUNCTIONS REPRESENTATIONS DURING INSTRUCTION

Charles Hohensee
University of Delaware
hohensee@udel.edu

Sara Gartland
National University of Ireland Galway
sara.gartland@nuigalway.ie

Yue Ma
University of Delaware
yuemajoy@udel.edu

Srujana Acharya
University of Delaware
srujana@udel.edu

Student focusing and noticing, which drive reasoning, are important but under researched aspects of student learning. Quadratic functions representations are perceptually and conceptually complex and thus, offer much for students to focus on and notice. Our study compared a teacher's goals for student focusing and noticing during quadratic functions instruction with what students actually focused on and noticed. Qualitative analysis revealed some alignment but also informative ways that the teacher's goals and student outcomes for focusing and noticing were misaligned. These results will further the field's understanding of how students learn about quadratic functions and may have implications for student focusing and noticing of other mathematics topics as well.

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This study is about what students focus on and notice during quadratic functions instruction. The topic of student focusing and noticing is a vitally important consideration for the field of mathematics education because (a) focusing and noticing are crucial processes for any and all reasoning (i.e., what remains unnoticed cannot be reasoned with or about; Bransford & Schwartz, 1999), and (b) what students focus on and notice might not always align with what their teachers want students to focus on and notice (Mason, 1999). Although teacher noticing has become a thriving area of research among teacher education researchers, student focusing and noticing has yet to garner the same kind of attention among student learning researchers. Our research sets out to help address that imbalance. Foremost, our research sets out to produce insights about how students learn about quadratic functions. However, we also view our findings as having potential relevance for the learning of other topics as well.

Theoretical Orientation Towards Student Noticing and the Learning of Quadratic Functions

In this section, we first present our theoretical orientation toward student focusing and noticing. We then present our theoretical orientation toward the learning of quadratic functions.

Student Focusing and Noticing

We conceive of student focusing and noticing as the last two parts of a three-part process involving *attending*, *focusing*, and *noticing*. All three are needed for student noticing to occur (i.e., *attending* and *focusing* are precursors to noticing). To conceptualize these processes, we turn to von Glasersfeld (1995). We conceptualize attending as “pick[ing] a chunk of experience, isolate[ing] it from what came before and from what follows, and treat[ing] it as a closed entity,” (p. 91). In other words, to attend to an object in a complex perceptual or conceptual context requires that the object be mentally isolated or foregrounded. While driving, for example, someone could foreground traffic signals against a background of other aspects of the

environment.

Next, we conceptualize focusing on an object as “the mind, then, ‘to posit it as object against itself’ . . . to re-present it,” (von Glasersfeld, 1995, p. 91). In other words, focusing on an object in a complex perceptual and/or conceptual field goes beyond simply attending to the object, and requires the creation of a mental record of the object, and the re-presentation of the object to oneself. For example, while driving, someone may focus on the license plate of the car in front of them by creating a mental record of the license plate and re-presenting that mental record to themselves in a way that goes beyond the way they simply attend to the traffic signals.

Finally, we conceptualize noticing as about “(‘establish[ing] regularities in the flow of experience” (von Glasersfeld, 1995, p. 144). In other words, noticing requires more than just focusing on an object in a complex perceptual and/or conceptual field; it requires mentally identifying particular regularities, features, properties, etc. of the object. For example, someone who is focusing on the licence plate in front of them while driving may notice that the plate is from out of state, that the numbers on the plate form a pattern, etc.

Although noticing can be viewed as the culmination of the three-part process, our study examined both student focusing *and* student noticing because what one focuses on does not predetermine what one notices (Goodwin, 1994). So, for instance, two students could focus on the same perceptual or conceptual object yet notice different features. We did not consider attending for this study because, of the three processes, attending is more difficult to track (e.g., can occur at a subconscious level), and in our view would not add significantly to what a study of focusing and noticing would reveal.

Quadratic Functions

As Lobato et al. explained, quadratic functions represent a “complex mathematical domain” (2012, p. 85). We add that quadratic functions are complex, both perceptually and conceptually. Quadratic functions are a particularly important topic for which to study student focusing and noticing because there are many aspects of quadratic functions teachers could want students to focus on and notice and many aspects students could actually focus on and notice. An overarching goal teachers may have with respect to quadratic functions is to help students focus on and notice those features of quadratic functions that put students in position to understand what makes a function quadratic. This was the instructional goal for our study as well.

Although there are a number of ways to determine a function is quadratic (e.g., the distance from vertex of a parabola to a point on the parabola being equal to the shortest distance from the point on the parabola to a directrix of the parabola, second order function, etc.), the way quadratic functions were promoted in our study were as that “the rate of change of the rate of change . . . of a quadratic function is constant” (Cooney et al., 2010, p. 9). Thus, our study emphasized focusing on and noticing features of quadratic functions that put students in position to understand that functions with a constant rate of change of the rate of change are quadratic.

An instructional approach to teaching functions that aligns with the instructional goal described above is a *covariational reasoning approach*. This approach aligns with our goal because covariational reasoning is needed for understanding and reasoning about rates of change (Carlson et al., 2002), and hence also for understanding and reasoning about rates of change of rates of change. Confrey and Smith (1994) described a covariational reasoning approach as “being able to move operationally from y_m to y_{m+1} coordinating with movement from x_m to x_{m+1} ” (p. 137). Moreover, Confrey and Smith (1995) found that “creating tables of data can often serve as a point of entry for students” (p. 78) for covariational reasoning. Additionally, Carlson et al. explained that covariational reasoning can be fostered by creating dynamic situations.

For our instructional approach, we fostered covariational reasoning about quadratic functions by using data tables as the primary representation. Additionally, we created dynamic quadratic functions on SimCalc Mathworlds™ software. Specifically, we created distance-time quadratic function (DTQF) animations (see Figure 1). For those unfamiliar with SimCalc, this computer software can be programmed with a great variety of distance-time functions, including all manner of DTQFs. When played, animation characters walk across the screen according to the pre-programmed function and tools exist that enable students to collect precise measurements of the accumulating distance and time that they can then explore for patterns and relationships.

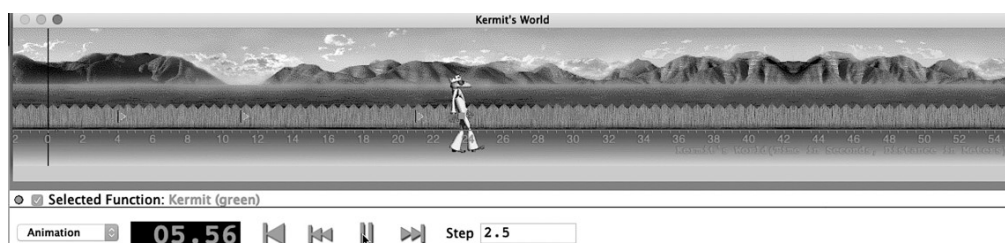


Figure 1. SimCalc Mathworlds™ DTQF animation

Research Question

The research question that guided our study was the following: During quadratic functions instruction that emphasizes covariational reasoning with dynamic situations, how does what students focus on and notice about quadratic functions during instruction align with the teacher’s goals for student focusing and noticing? As stated above, in addressing this question, we hoped to generate insights about the learning of quadratic functions, as well as more general insights that potentially could also apply to the learning of other mathematics topics.

Methods

Participants and Context

This study took place during a summer mathematics program for 9th and 10th grade students held at a university in the Mid-Atlantic region of the United States. Participants included 18 students ($N = 18$), and one teacher, who was also the first author, and thus a researcher- participant. The remaining four members of the research team, consisted of the second author, another researcher, and two public high school mathematics teachers from the Mid-Atlantic region.

The summer program was designed to focus on quadratic functions and to promote students’ covariational reasoning. During the two-week program, the students met each day for two 1-hour instructional sessions, separated by a short break. Each 1-hour instructional session typically involved a single instructional activity. Students manipulated and explored DTQF SimCalc animations on laptops in small groups during these instructional activities. Then, students engaged in whole-group discussions about the quantities and relationships associated with each DTQF.

Data Collection and Analysis

Data were also collected during the summer program. In particular, the instructional sessions were recorded. These recordings were of whole-group and small-group activities, although only whole-group recordings were reported on for this paper. Specifically, recordings and the subsequent transcripts of these whole-group discussions were analyzed qualitatively.

As stated above, data analysis focused on the whole-group discussion that occurred at the end of each instructional activity (we called each discussion an episode). Analysis was used to develop inductive codes for the teacher’s goals for student focusing and noticing and the student outcomes for focusing and noticing (Strauss & Corbin, 1998). Analytic memos were written to develop

themes. This report focuses on themes emerging from four early episodes, namely Episode 2B (E2B), E3A, E3B1, and E3B2.

Results

We now present four themes that emerged during data analysis with respect to the teacher’s goals and students’ outcomes for focusing on and noticing features of DTQFs, a general theme and three quantity-specific themes. The general theme was about the association between the teacher’s goals and student outcomes for focusing and noticing and the amount of structure in the instructional activities. The three quantity-specific themes involved *distance* (3-part theme), *time* (2-part theme), and *speed* (2-part theme). Table 1 presents a comparison of teacher goals and student outcomes for focusing on DTQFs. Table 2 presents a comparison of teacher goals and student outcomes for noticing features of DTQFs. We say more about the tables below.

Table 1: Comparison of Teacher Goals and Student Outcomes for Focusing on DTQFs

Episode	Teacher Goals for Student Focusing	Student Focusing Outcomes
E2B	Focus on individual pairs of accum. time & distance	Focus on individual pairs of accum. time & distance, speed*
E3A	Focus on multiple accum. times	Focus on multiple accum. times, total time* , speed*
E3B1	Focus on multiple accum. distances	Focus on multiple accum. distances, total distance* , displacement* , speed*
E3B2	Focus simultaneously on multiple accum. times & distances	Focus simultaneously on multiple accum. times & distances

*Aspects of teachers’ goals and student outcomes for focusing and noticing not aligned.

Table 2: Comparison of Teacher Goals and Student Outcomes for Noticing DTQFs

Episode	Teacher Goals for Student Noticing	Student Noticing Outcomes
E2B	Noticing features of accum. time and distance pairs that indicate which DTQF is faster	Noticing features of accum. time and distance pairs that indicate which DTQF is faster, speed constant* , speed changing*
E3A	Noticing accum. times at multiple notable points in DTQF, how accum. time changing*	Noticing accum. times at multiple notable points in DTQF, total time* , speed constant* , speed changing*
E3B1	Noticing accum. distances at multiple notable points in DTQF, how accum. distance changing*	Noticing accum. distances at multiple notable points in DTQF, sign of the distances (pos/ neg)* , total distance* , displace. distance* , speed changing*
E3B2	Noticing multiple associated accum. times and distances	Noticing multiple associated accum. times and distances

*Aspects of teachers' goals and student outcomes for focusing and noticing not aligned.

Association Between Focusing and Noticing and the Amount of Structure in the Instructional Activities

The first theme we present is about the association between how aligned the teacher's goals and students' outcomes for focusing on and noticing features of DTQFs were and the amount of structure in the instructional activities. This theme was more general than the other themes. What we specifically observed was that more structure in the DTQF instructional activities was associated with more alignment between the teacher's goals and students' outcomes for focusing and noticing. We explain the kinds of structure we observed and describe how more versus less structure was associated with alignment between the teacher's goals and student outcomes for focusing and noticing.

For the episodes reported on in this paper, there appeared to be two kinds of structure in the DTQF instructional activities that guided student focusing and noticing. One type of structure was when the instructional activity had a clearly defined goal. E2B had a clearly defined goal because it involved students in deciding which of two DTQF animations, which could not be played simultaneously, would win a race that went beyond the length of either animation (i.e., each animation ended before 150 m and students were to decide which animation would win a 150 m race). E3A, E3B1, and E3B2 did not have as clearly defined a goal.

The second type of structure was when the activity involved clear patterns. E3B2 involved focusing on and noticing patterns when students used patterns to focus on and notice associated accumulating distances and times. Some students focused on and noticed accumulating times associated with a pattern of accumulating distance (e.g., every 10 m). Other students focused on and noticed accumulating distances associated with a pattern of accumulating times (e.g., every 0.8 sec). E2B, E3A, and E3B1 did not involve clear patterns.

Comparisons between the teacher's goal column and the student outcome column for Table 1 and 2, indicate greater alignment between the teacher's goals for focusing on and noticing features of DTQFs in E2B and E3B2 than in E3A and E3B1. In E2B and E3B2, all of the teacher's goals for noticing were evident in the student outcomes but not in E3A and E3B1.

Moreover, in E2B and E3B1, fewer aspects of DTQFs not part of the teacher's goals were focused on and noticed by students (e.g., for E3A, total time and speed were aspects students focused on that were not part of the teacher's goals, whereas for E3B2 no aspects we observed students focus on were not part of the teacher's goals). Thus, our conclusion was that EB2 and E3B2, which involved activities that had more structure in the form of having a more clearly defined goal or that involved focusing on and noticing patterns, also involved greater alignment between the teacher's goals and the student outcomes for focusing on and noticing DTQFs.

Focusing On and Noticing Distance

The second theme that emerged with respect to the teacher's goals and students' outcomes for focusing on and noticing features of DTQFs was quantity-specific and involved *distance* (i.e., the dependent variable of a DTQF). The theme about focusing on and noticing distance had three parts, (a) focusing on accumulating distances, (b) noticing that the accumulating distance changed direction, and (c) noticing the speed that accompanies the accumulating distance.

Focusing on accumulating distance. A teacher's goal in these episodes for student focusing in the context of DTQFs was for students to focus on the accumulating distance. Our use of the term accumulating is consistent with how it is used in Carlson et al. (2002), in particular that "the accumulating quantity can be imagined to be made of infinitesimal accruals in the quantities" (p. 165). Our analysis revealed that, although students' outcomes for focusing with respect to the accumulating distances sometimes matched the teacher's goals, there were two additional aspects of the distance of DTQFs not part of the teacher's goals on which students sometimes focused.

One aspect of distance of DTQFs on which students sometimes focused, in addition to the accumulating distance, was the *displacement distance*. For example, students were considering a DTQF animation in which a dog on the negative side of a distance scale, walked toward and into the positive side of the scale, and then turned around and walked toward and into the negative side. In the following excerpt, Kevin revealed that he focused on and noticed displacement distance: "the starting point for us was -9, and he ended up at -41 meters, -9 meters and he ended up at -41 meters. So, we thought the displacement was 32 meters."¹

The second aspect of distance of DTQFs students sometimes focused on, in addition to the accumulating distance, was the *total distance*. For example, in the same DTQF animation referred to above, Rashana focused on the total distance as the animation went from -7 up to +44 and back down to -41: "136 [total distance] is the distance in the forest space . . . it's from -7, because that's when it started . . . all the way to 44. And then from 44 again, all the way to -41" (note that Kevin and Rashana noticed different starting distances). In sum, two aspects of distance, namely the displacement distance and the total distance, appeared to be competing with the accumulating distance for what students focused on.

Noticing that the accumulating distance changed direction. A teacher goal in these episodes for student noticing was for students to notice that the accumulating distance changed direction (i.e., that somewhere on a DTQF the accumulating distance grew and then shrunk or vice versa). Our analysis revealed that, although students' outcomes for noticing distances sometimes matched the teacher's goals, not all students appeared to notice that the accumulating distance changed direction and noticed another feature instead.

The aspect of the accumulating distance of DTQFs students sometimes noticed instead of noticing that the accumulating distance changed directions, was that the accumulating distance changed sign. To illustrate, consider the following exchange between the teacher and Yolanda, in which the teacher wanted students to notice the change in direction of the accumulating distance:

T: We're focusing specifically on how distance is changing. And someone from Group 3 is gonna come up and share with us their group was thinking about distance.

Y: We said the distance means amount of meters that Rover walked towards and moved away from the forest . . . So, like, you know the ruler thing? That would be like -5, then the distance from—is a negative, but then when it gets to like 1 or 2 which is positive.

T: One thing that I want to just connect with what you said. Did people notice that the distance quantity changed directions?

Y: I don't think so.

In this excerpt, a teacher goal was for students to notice the accumulating distance changed direction. However, Yolanda instead noticed that the accumulating distance changed sign. In this instance, the teacher's goal for student noticing and what they actually noticed did not align.

Noticing the speed that accompanies focusing on and noticing accumulating distances.

Another teacher goal for noticing was for students to notice aspects of distance and to ignore speed (i.e., to delay focusing on and noticing speed until future episodes). However, our analysis revealed that when students were supposed to focus on and notice accumulating distance, sometimes speed appeared to grab their attention. In other words, when they focused on and noticed distances, they often also reported noticing an aspect of speed. For example, as Yolanda explained, "we noticed that it ends at 40 meters, and it starts at 70 meters. The least amount of distance is -25 . . . it slows, or speeds up based on the negative and positive side, on the ruler." In this example, Yolanda linked what she noticed about the accumulating distances of the animation with how fast or slow the animation was going.

Focusing On and Noticing Time

The third theme that emerged with respect to the teacher's goals for focusing and noticing and students' outcomes for noticing and focusing was also quantity-specific and involved *time* (i.e., the independent variable of a DTQF). The theme about focusing and noticing time had two parts, (a) focusing on the accumulating time, and (b) noticing the speed that accompanies the accumulating time.

Focusing on accumulating time. A teacher goal in these episodes for focusing in the context of DTQFs was for students to focus on the *accumulating time*. Our analysis revealed that, although students' outcomes for focusing with respect to the accumulating time sometimes aligned with the teacher's goals, there was an additional aspect of the time of DTQFs not part of the teacher's goals on which students sometimes focused.

The aspect of the time of DTQFs on which students sometimes focused, in addition to focusing on accumulating time, was *total time*. For example, consider Damarcus's statement:

We noticed that time starts at a negative number . . . that it's from negative 4.5 to 8.7 . . . and we said that the total distance was 13.2, er what—not total distance, but total time is 13.2 seconds.

In this and other instances, whereas the teacher's goal for focusing was to focus on accumulating times, students sometimes also focused on total time. In other words, total time appeared to be competing with the accumulating time for what students focused on.

Noticing the speed that accompanies accumulating times. As stated above, a teacher goal for noticing was for students to ignore speed. However, our analysis revealed that sometimes speed appeared to grab students' attention, not only when students were to focus on and notice accumulating distance as described above, but also when they were to focus on and notice accumulating time.

In the following example, Natasha stated what her group noticed about accumulating times:

We noticed that the time continues as he turns around to go away from the forest . . . like when he turns around, he pauses, but the time still continues. He enters the forest, when it's zero . . . it seems like his speed is consistent, looking at it, we think he's like moving at a consistent pace, except when he turns around because he pauses. It starts at negative 4.5 seconds and ends at 8.7 seconds.

In sum, this excerpt further supports our claim that speed, more than time and distance, was an attention-grabbing quantity in a DTQF context.

Focusing On and Noticing Speed

The fourth theme that emerged with respect to the teacher's goals for focusing and noticing and students' outcomes for noticing and focusing was also quantity-specific and involved *speed* (i.e., the rate of change of a DTQF). The theme about focusing and noticing speed had two parts, (a) not ignoring speed, and (b) noticing speed as constant.

Not ignoring speed. As stated above, the teacher's goal for these episodes was for students to ignore speed and instead focus on distance and time. However, as explained earlier, speed was not always ignored. As further illustrated in Tables 1 and 2, in three of the four episodes students focused on speed and noticed features of speed. In E2B, the teacher intended for students to notice features of accumulating distances and times for two characters. While presenting what his group noticed, Bryan explained, "And then ah, for the Frog, it travelled about 28 meters in 1 second, umm and that was at its top speed." In E3A, the teacher intended for students to notice features of time. During the whole-group discussion, Demarcus shared, "...so, ah we said the time was the distance times speed." In E3B1, the teacher intended for students to notice features of distance. While explaining what her group noticed, Yazmin said, "...and then, it slows or speeds up based on the negative and positive side, on the ruler."

Noticing speed as constant. A feature of speed that students sometimes noticed was that the speed was constant (i.e., the rate of change was constant). Of course, for all quadratic functions, the rate of change is always changing (i.e., not constant). However, that is not what students always noticed; sometimes students noticed constant speed. For example, as stated above Natasha noticed that "he's like moving at a consistent pace." Similarly, Bob noticed the following about one of the DTQF animations:

And then ah, for the Frog, it traveled about 28 meters in 1 second and that was at its top speed. So that was about a pretty constant pace. So, we just used that throughout the rest of the time when ah for the Frog went.

This was not always true because sometimes students noticed that the speed was changing, such as Halima who said, "Another thing that I noticed with the Clown is, each time it moves, it travels way greater than it did last frame." Thus, not only did speed grab students' attention, but what students noticed about speed in terms of whether it was constant or changing was not consistent. Next, we explain the significance of these findings and implications for research and practice.

Discussion

We presented four themes emerging from our analysis that are specifically about students focusing on and noticing aspects of quadratic functions. Our results suggest that what students focus on and notice about DTQFs to put them in position to understand what makes DTQFs quadratic can be shaped by the amount of structure for focusing and noticing in the instructional

activities (e.g., clear goal, leveraging patterns). The results also suggest that the teacher goal that students should focus on and notice DTQFs' accumulating times, accumulating distances, or both, may also be accompanied by noticing other quantities (e.g., total time, displacement distance) or other features of quantities (e.g., constant speed). These results could be useful to practitioners who teach quadratic functions with a covariational approach and who use dynamic DTQFs and tabular representations.

In addition to these content-specific insights, we also have three hypotheses about how our themes may have applicability to focusing on and noticing of other mathematics topics. First, we hypothesize that in other content areas, some quantities attract student focusing and noticing more than others, just as speed did in our study. Second, we hypothesize that sometimes students may focus on the quantities their teachers want them to focus on, but not the variation of the quantities the teacher intended just as, for example, students in our study focused on total time instead of accumulating time. Third, we hypothesize that, in some cases, what students actually notice about aspects of mathematics content, may actually not be present just as, for example, students in our study noticed that the speed was constant when, in actuality, it always changing. Of course, these are exploratory hypotheses that will require testing in future studies. We hope this report will serve as a catalyst for new research on student focusing and noticing.

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