RESPONSIVE EVALUATION OF STUDENT WORK: A THEORETICAL PROCESS

Julien Corven
Illinois State University
jccorve@ilstu.edu

To base teaching on student thinking requires analyzing and interpreting students' thinking, key components of the construct of professional noticing (e.g., Jacobs et al., 2010). Although substantial research has been conducted using this construct, less attention has been paid to teachers' evaluations of student work based on these analyses and interpretations. In this theoretical report, I argue that evaluation of student thinking is a key prerequisite for effective instructional decision making for responsive teaching. I then present a synthesized framework of evaluation criteria and an elaborated process for effective and responsive evaluation of student written work using these criteria. Finally, I analyze excerpts from two interviews with elementary prospective teachers to demonstrate the utility of these products.

Keywords: Teacher Noticing, Preservice Teacher Education, Assessment

Effective mathematics teaching requires basing instruction on students' thinking (e.g., National Council of Teachers of Mathematics, 2014). To engage in such responsive teaching, teachers need the skills of eliciting (e.g., Shaughnessy & Boerst, 2018b), understanding (e.g., Association of Mathematics Teacher Educators, 2017), evaluating (e.g., Dyer & Sherin, 2016), and productively responding to (e.g., Jacobs et al., 2010) student thinking. Therefore, teacher preparation programs must provide multiple opportunities for prospective teachers (PTs) to learn how to enact these skills. This theoretical report contemplates PTs' ability to evaluate student mathematical thinking as reflected in written mathematical work, a construct which is underdeveloped in the literature. I argue that thoughtfully evaluating student work and thinking is an essential aspect of responsive teaching because it shapes how teachers can productively respond to students' mathematical understandings and ideas (Dyer & Sherin, 2016; Jacobs et al., 2010; van Es & Sherin, 2021). In this report, I address the following research questions: *What criteria for evaluation of student work and/or thinking have been documented in the literature?* and *Can these criteria be synthesized into a process that can be used to evaluate students' written mathematical work in a systematic way supportive of responsive teaching?*

Evaluating Students' Thinking in Responsive Teaching

One essential meaning of the verb 'evaluate' is "to determine the significance, worth, or condition of… usually by careful appraisal and study" (Merriam-Webster, n.d.). Thus, evaluation is necessarily judgmental, but it should be thoughtful (i.e., based on evidence and/or reasoning). I define "evaluation of student thinking/work" as the process teachers use to make reasoned judgments about students' mathematics using relevant criteria and against relevant standards to inform their next steps for teaching, including providing feedback and responding to students.

Robertson et al. (2015) defined responsive teaching as teaching that embodies three characteristics: (1) foregrounding the substance of students' ideas, (2) connecting students' ideas to key ideas within the discipline, and (3) taking up and pursuing students' ideas. Evaluation of student thinking plays a critical role in applying these tenets. For example, to connect a student's idea to a key idea within the discipline, a teacher must evaluate how closely the student's idea aligns with it. Ideally, this evaluation would help the teacher consider how to help move the student's thinking towards that key idea. Also, during typical classroom instruction, due to time constraints, teachers must choose which ideas will be pursued by the whole class (Stein et al., 2008; Stockero et al., 2017). To make such decisions, teachers must evaluate students' ideas in reference to their relevance to the learning goal (Stein et al., 2008).

Potential Criteria for Evaluating Students' Written Work

From a synthesis of the literature, I define six criteria that teachers could use to evaluate students' written mathematical work to support instruction aligned with the tenets of responsive teaching and a seventh criterion, personal preference, that could be detrimental to responsive teaching if used by a teacher to evaluate students' written work (Table 1).

Table 1: Definitions of Potential Criteria for Evaluating Students' Written Work

Criterion	Definition
Strategic competence	The extent to which a student's strategy was appropriate for solving the problem, including discussions of mathematical limitations of the strategy for solving this or similar problems.
Logical validity	The extent to which a student's reasoning forms a logically valid mathematical argument.
Conceptual understanding	The extent to which a student's work demonstrates understandings of mathematical concepts related to the topic of the problem.
Efficiency/ sophistication	The extent to which a student's strategy and/or work for the problem is efficient or mathematically sophisticated, including judgments related to the placement of the strategy within a developmental progression.
Clarity	The extent to which a student's thinking, reasoning, and work are communicated in ways others can easily understand.
Accuracy/procedural understanding	The extent to which a student obtains a correct answer, correctly completes computations, and/or knows/follows steps of a procedure.
Personal preference	The extent to which the evaluator prefers a student's strategy for solving the problem due to their own relationship with the strategy.

In the next sections, I will describe and synthesize literature that implicates the first six of the criteria in Table 1 and comment on how teachers' evaluations of student work or thinking on each criterion could contribute to responsive teaching.

Strategic Competence

Building from one of the strands for mathematical proficiency from the *Adding it up* report (National Research Council [NRC], 2001), Copur-Gencturk and Doleck (2021) defined three aspects of strategic competence. First, the student must devise a valid solution strategy for solving the problem. Second, the student must translate the problem and/or strategy into a useful mathematical representation. Finally, the student must execute their chosen solution strategy accurately. However, when evaluating student work for a non-computational problem, I consider accuracy to be a separate criterion for evaluation. An appropriate strategy that contains a minor calculation error in its application in a written solution could be seen by a teacher as strategically competent despite its inaccuracy.

Monitoring students as they work on a mathematical task is one of the five practices for orchestrating a productive mathematical discussion (e.g., Stein et al., 2008). Notably, the goal of monitoring is "to identify the mathematical learning potential of particular strategies or representations used by the students, thereby honing in on which student responses would be important to share with the class as a whole" (Stein et al., 2008, p. 326). Thus, as teachers circulate, they should be evaluating the student work they see for its strategic competence, which will then inform their decision for which strategies may be best to share with the class. Such decisions align with tenet (3) of responsive teaching: taking up and pursuing students' ideas.

Logical Validity

Although logical validity could also be viewed as an aspect of strategic competence, I consider strategic competence to be a criterion for evaluating the reasonableness of a student's strategy selection. Logical validity is more fine-grained, as it considers the validity of each step of reasoning (implicit arguments) in the student's work. As Özdemir and Pape (2012) noted, teachers can support students by asking them to explain their reasoning for selecting a strategy (strategic competence) and justifying why it works (logical validity).

As part of the process for selecting student solutions to present to the class (Stein et al., 2008), a teacher may decide to have a student share work that illustrates a common error in reasoning to help the class understand why it is not valid (Ayalon & Rubel, 2022). Doing so helps students connect their ideas to key disciplinary ideas, tenet (2), through contrast. However, Morris (2007) suggested that PTs may have difficulty applying this criterion, as they will often fill in logical gaps in students' arguments with their own reasoning.

Conceptual Understanding

Teachers often have a particular learning goal in mind when they plan a lesson and the tasks or activities contained within it. Hiebert et al. (2018) and Morris et al. (2009) described the process of decomposing the learning goal into component mathematical subconcepts. These subconcepts are specific conceptual understandings that students may have an opportunity to demonstrate as they work on a task. Evaluating student work to determine which conceptual understandings the student has demonstrated is therefore important for considering next steps.

Cohen and Benton (1988) recommended that teachers analyze students' thinking for evidence of specific conceptual understandings as they circulate while monitoring student work. As Stein et al. (2008) noted, one key goal of monitoring student responses is to ensure that approaches to the task aligned with the learning goal (i.e., that demonstrate specific conceptual understandings) will be available for class discussion. Diagnosing these specific conceptual understandings foregrounds the substance of students' ideas, tenet (1).

Efficiency/Sophistication

To orchestrate an effective mathematical discussion, teachers need to the sequence of solutions presented to the class. Stein et al. (2008) suggested one possible effective sequence is to present a strategy that is more accessible to students before presenting a more complex strategy. Thus, it would be important for teachers to evaluate students' solutions on an axis of efficiency, sophistication, or placement on a developmental progression. A study by Ayalon and Rubel (2022) confirmed that PTs often sequence strategies in increasing order of complexity, and PTs justified this decision by discussing students' ability to access more strategies. Additionally, several studies (e.g., Clements & Sarama, 2021; Moreno et al., 2021; Schack et al., 2013) have examined how instruction on learning trajectories succeeded in supporting PTs and teachers to learn to notice students' mathematical thinking, a prerequisite to tenets (1) and (2). Therefore, evaluation on this criterion may support responsive teaching by improving teachers' noticing.

Clarity

During a classroom discussion, students may make mathematical contributions, either verbally or through presenting their written work. Van Zoest et al. (2020) provided a framework that teachers can use to think about such contributions. According to the framework, a student's contribution can be non-mathematical, clarification-needed, inference-needed, or standalone (Van Zoest et al., 2020). A teacher's evaluation of a mathematical contribution on this framework could help them determine their next teaching move. For example, if the teacher evaluates the contribution as "inference-needed," they may ask a student to restate the contribution with additional context and precision before asking the class to engage with it. In this process, the student is given an opportunity to clarify their meaning if an incorrect inference was made. By making an evaluation of the clarity of a student's contribution and responding appropriately, the teacher can support the class in understanding the connection of the idea shared to key conceptual understandings or disciplinary concepts, tenet (2).

Accuracy/Procedural Understanding

For some procedural tasks, the learning goal concerns students' ability to demonstrate accurate computation and ability to follow the steps of a procedure. However, for other tasks, although teachers may wish to take note of computational or procedural errors, evaluations on this criterion may be of a lower priority in terms of responding productively to students' ideas. Nevertheless, as I explain below, it has been an evaluation criterion observed in prior research.

Prior Studies on Considering Students' Written Work

Several studies have found that PTs and teachers often attend to the accuracy of students' written work. For example, in a study by As'ari et al. (2019), most algebra teachers' ideal solutions for understanding student thinking employed accurate symbol manipulation after translating the given information into algebraic equations. As'ari et al. (2019) concluded the teachers in this study valued accuracy and procedural understanding highly. However, because the teachers also valued correctly mathematizing the given information into an algebraic representation, these teachers may also have valued solutions that demonstrated strategic competence (NRC, 2001). Additionally, PTs often assume a correct answer in student work indicates conceptual understanding (e.g., Bartell et al., 2013; Shaughnessy & Boerst, 2018a; Spitzer et al., 2011). Studies by Son (2013) and Lee (2021) concluded that, although PTs often successfully attended to errors in students' work, few PTs developed clear interpretations of the student's underlying mathematical difficulty. These studies strongly suggest that procedural understanding and accuracy are criteria that PTs and teachers use to evaluate students' work.

Prior qualitative research has provided some additional nuance to these findings. For example, Michael (2005) claimed that elementary PTs value solutions where all information and steps are shown, and their evaluations of solutions appeared to be positively influenced by features that made the work easy to follow. Corven (2021) also found that elementary PTs often preferred work that was clear to them, regardless of the solution's accuracy or validity.

Personal preferences may also affect PTs' evaluations of student work. For example, Van Dooren et al. (2002) observed that PTs generally prefer student solutions that match the one they would have used to solve a problem when asked to choose from among several correct examples of student work. A replication and expansion of this study by Michael (2005) noted a statistically significant, positive correlation between PTs' frequency of using a particular strategy to solve word problems and their evaluation scores for student work using a similar strategy.

Processes for Evaluating Student Work

Philipp's (2018) study on diagnostic competence provided insights into how mathematics teachers think about students' written solutions to mathematics problems. According to Philipp, after reconstructing a student's solution from their perspective, a teacher evaluates the student's work in terms of its strengths and deficits. However, Philipp's process is problematic for several reasons. First, although "deficits" may be an accurate characterization of how teachers in Philipp's study viewed aspects of students' work or thinking, a growing body of research (e.g., Louie et al., 2021) encourages the use of anti-deficit approaches to such analyses. Second, Philipp (2018) specifically defined deficits as "errors" (p. 121), but other aspects of the work could also raise concerns for a teacher (e.g., gaps in logical reasoning, unclear parts of the work, or not demonstrating an important subconcept of the learning goal). Finally, Philipp's process ceases after strengths are identified, but interactions with a student after teachers attend to and interpret their mathematics can help develop the student's thinking (e.g., van Es & Sherin, 2021), even when no errors or concerns are identified. To address these concerns, I created Figure 1, which is a refinement and elaboration of Philipp's (2018) idealized diagnostic process.

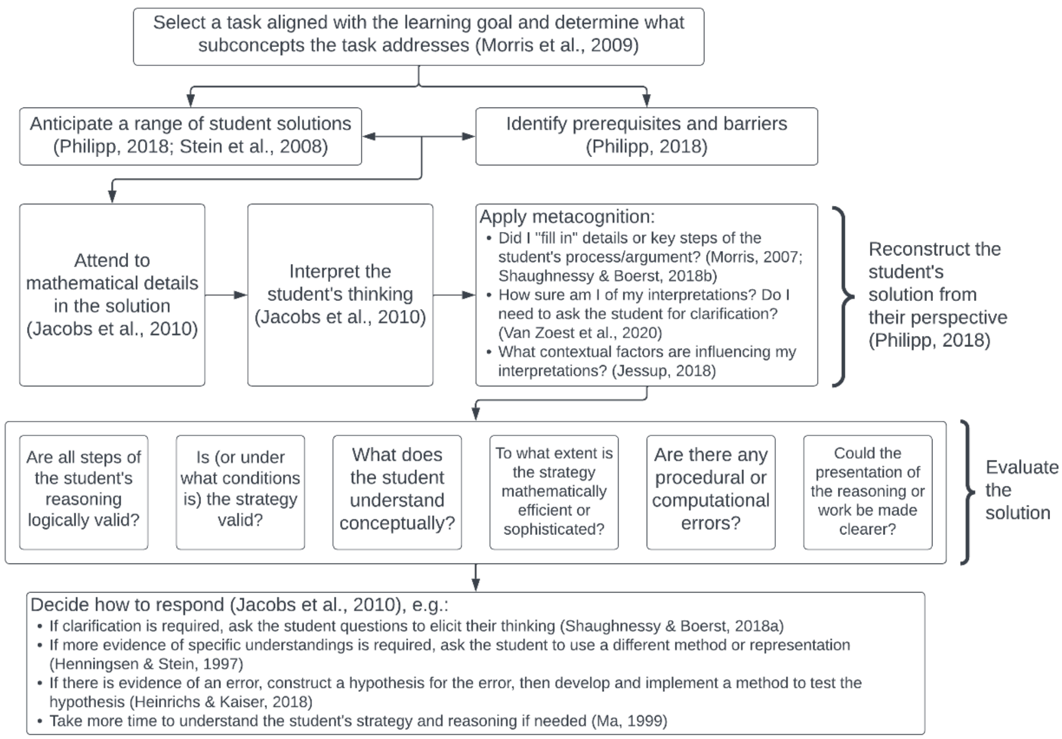

Figure 1: An Elaborated Process for Evaluating Students' Written Mathematical Work

The row labeled "Evaluate the solution" in Figure 1 poses questions aligned with the six productive criteria for evaluation in Table 1. However, some solutions may not contain sufficient evidence to answer all the questions in the "Evaluate the solution" row. For example, Spitzer et al. (2011) state that, for a procedural solution, an appropriate answer to the question "What does the student understand conceptually?" would be that there is not enough evidence to substantiate any such claim. The "Evaluate the solution" row in Figure 1 is also not meant to be followed strictly from left to right. Expert teachers will apply the evaluation criteria implicated by those questions flexibly, strategically, and purposefully depending on the details in a student's work.

Application of the Elaborated Process

I will now demonstrate an application of the framework in Figure 1. Below is an excerpt from an interview with Charlotte (pseudonym), an elementary PT who had just finished her first mathematics content course. Charlotte was asked to reconstruct the student's thinking shown by the solution in Figure 2, which was a response to the story problem, "A licorice rope was 36 inches long. Tammy cut the entire rope into pieces 4.5 inches long to share with her friends. How many pieces did she make?" (adapted from Moore et al., 2020).

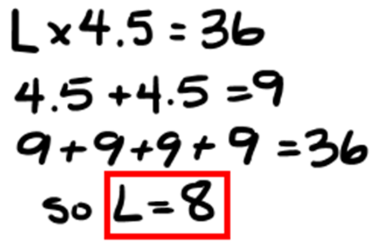

Figure 2: A Missing Factor Approach to Solving a Division Story Problem

Charlotte: So, I think they understand, just by starting out, I think they understand the correlation between division and multiplication. So, they switched the 36 divided by 4.5 to a multiplication problem. So, they understood that 36 inches [*trails off*]…. [T]hey don't know yet how many groups of 4.5, so they switched it to the problem L groups of 4.5 inches equals 36. So, to make it easier for them, they did 4.5 plus 4.5 to get a whole number, to get 9, and they understand that 9 goes into 36 four times. So, 9 plus 9 plus 9 plus 9 equals 36…. [B]y doing the 9 plus 9 plus 9 plus 9 is just saying like parentheses, 4.5 plus 4.5, close parentheses four times, so they can see that L equals 8….

Interviewer: So, they had the idea that each of these 9's represents two copies of 4.5?

Charlotte: Yes, and it looks like they could have made their work clearer by doing, like under that maybe, parentheses 4.5 plus 4.5, and do that across the line…. But it seems like they're able to understand it [so] that they don't need the work. If they were struggling and not getting the right answer, as a teacher, I might have been like, well, show your idea behind the 9. But, since they have the work above it, it seems like they understand what they're trying to say…. Maybe when they were first learning this, they did put that line under it. But maybe they're further along in the process where they understand now that, with the work above it, that 4.5 and 4.5 will give the 8 instead of 4….

Interviewer: In your opinion, how would you evaluate the reasoning shown in this solution?

Charlotte: I think that it's good that they understand the concept that division and multiplication relate to each other with switching the problem up like that. And then, with the 4.5 plus 4.5 equaling 9, it's good that they made it a whole number so they're not messing around with the decimals and everything. So, I think they showed the work they needed for this problem. I understood it, and it seems like they understood it as a student. So, overall, I think they understand the concept of division.

Charlotte's engagement with the student work was generally aligned with Figure 1. Charlotte interpreted the student's work as showing that "they understand the correlation between division and multiplication," and she cited mathematical details from the student's work ("L groups of 4.5 inches equals 36") to support her inference. Thus, Charlotte both attended to and interpreted

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1). University of Nevada, Reno.

mathematical details in the student's work (Jacobs et al., 2010). Further, she recognized where she had filled in a step in the student's reasoning ("they could have made their work clearer by doing, like under that maybe, parentheses 4.5 plus 4.5") that was implicit in the work (metacognition; Morris, 2007). Charlotte did not indicate a need to elicit the student's thinking for confirmation, suggesting she considered the contribution as standalone (Van Zoest et al., 2020). Charlotte also compared the work to (hypothesized) past performance (Jessup, 2018) when she said, "Maybe when they were first learning this, they did put that line under it. But maybe they're further along in the process [now]" (contextual factor). Thus, Charlotte was generally able to reconstruct the solution from the student's perspective to prepare for evaluation.

Charlotte considered and evaluated the logical validity of the student's reasoning, though she did not make an explicit strategic competence evaluation. Nevertheless, she did diagnose a specific conceptual understanding (the relationship between multiplication and division) demonstrated by the student's work. Given the relationship between strategic competence and mathematical knowledge for teaching described by Copur-Gencturk and Doleck (2021), this statement could be interpreted as an implicit evaluation of the strategy as valid. However, Charlotte did not conduct a full breakdown of the learning goal into subconcepts and overgeneralized the student's conceptual understanding. At the conclusion of the evaluation, Charlotte stated, "overall, I think they understand the concept of division," but she did not elaborate on what other specific understandings of division the work demonstrated. Charlotte did consider the efficiency of the strategy in the solution in terms of ease for the student by stating, "with the 4.5 plus 4.5 equaling 9, it's good that they made it a whole number so they're not messing around with the decimals." Additionally, Charlotte commented that the student's final answer was correct, indicating an evaluation of accuracy (no errors). Finally, Charlotte discussed ways in which the work could have been made clearer, but claims, "I think they showed the work they needed for this problem. I understood it, and it seems like they understood it as a student." Charlotte's response exemplifies a generally productive application of the process in Figure 1.

However, not all PTs I interviewed exhibited this level of alignment with the process. The transcript below shows Jane's (pseudonym) response to the solution in Figure 2.

> Jane: We started out with an equation of L times 4.5 equals 36, with 36 being how much of the rope we have, and the 4.5 being how much we want to cut it into, and L being our pieces. So, we know that 4.5 plus 4.5 would be 9, so we would add up 9 until we get relatively [close] or to the exact answer of 36, which they did. And we also know that 9 was the sum of 4.5 plus 4.5, so, because we had to use four 9's in order to get the 36, our answer would then be 8 pieces.
>
> Interviewer: So because we had four 9's and each of those 9's was two 4.5's, that's where they got the 8 from?
>
> Jane: Yes.
>
> Interviewer: So, how would you evaluate the reasoning shown in this solution?
>
> Jane: I think this is a really good method to use. It's quick and it seems pretty simple.
>
> Interviewer: Okay. Any other thoughts or comments about this solution?
>
> Jane: No.

Jane's reconstruction of the student's work had some similarities to Charlotte's in terms of attending to the mathematical details in the solution. However, unlike Charlotte, Jane did not apply metacognition to recognize that she filled in missing steps in the student's work and inferred that the student thought the same way. Also, although Jane described the meaning of each quantity in the original problem, she did not describe any conceptual understandings that the student demonstrated through their solution. Most of Jane's reconstruction is recitation of the steps the student did rather than interpreting the student's thinking, aligning with Level 1 in Fernández et al.'s (2013) framework: "interpretations of students' answers mainly rely on the description of the operations carried out and not on the meanings" (p. 453).

When I asked Jane to evaluate the student's work, she made evaluations of strategic competence ("I think this is a really good method to use") and efficiency/sophistication ("It's quick"). However, she did not provide any rationales for these evaluations, even after prompting. Specifically, she did not draw on any details from the work to justify her evaluations. In line with Bartell et al. (2013) and Spitzer et al. (2011), PTs may need more support to use the process in Figure 1 effectively and consistently (e.g., basing their evaluations on mathematical details).

Implications

Teacher Education

In the context of elementary teacher preparation, opportunities for PTs to evaluate written work (e.g., Fernández et al., 2013) and/or participate in simulated student interviews (e.g., Shaughnessy & Boerst, 2018a) can serve as initial exposures for PTs to consider how their future instruction can center student thinking. Such activities could be implemented in early content courses without a field experience component. The process in Figure 1 can also help MTEs understand how their PTs are engaging in productive (or unproductive) evaluations of student work and thinking. MTEs can then consider teaching PTs about evaluation criteria that are important but have not yet surfaced. For example, MTEs can purposefully select student work that is designed to elicit certain criteria for evaluation (e.g., a solution that does not clearly model the situation, yet reaches a correct numerical answer) as a basis for a classroom discussion.

The primary contribution of the evaluation process in Figure 1 is the elaborated synthesis of the construct of professional noticing of children's mathematical thinking (Jacobs et al., 2010) with Philipp's (2018) framework of teachers' diagnostic processes and the productive evaluation criteria from review of prior research in Table 1. Although professional noticing was one of the theoretical bases for Philipp's (2018) process, the elaborated framework clarifies the relationships between these ideas in a way that is useful to MTEs and teachers. MTEs can share this process with their PTs to guide them towards examining student work in ways that productively center student thinking. Additionally, in-service teachers can compare their own evaluation processes to the one in Figure 1 to help them develop new ways of thinking about student work. The questions listed in the "Evaluate the solution" row of Figure 1 could also serve as prompts during a professional development centered on examining student work.

Future Research

Figure 1 can also be used to characterize how PTs and in-service teachers evaluate students' mathematical work. Such investigations could lead to further refinements of this process. For example, tasks that ask students to execute a specified procedure may not provide any evidence of a student's conceptual understanding (Spitzer et al., 2011), so skipping some of the questions in the "Evaluate the solution" row would be appropriate. Additional arrows that explicate more specific paths through the process based on the nature of the underlying task could then be added to Figure 1. Creating and analyzing flowchart diagrams of PTs' reconstructions and evaluations could better emphasize the relationship between this process and evaluation criteria.

Evaluating students' work in ways that honor and center their thinking is complex. I hypothesize that opportunities to reconstruct and evaluate students' written work alongside instruction on processes for doing so could support PTs developing responsive teaching practices. However, more research is needed to ascertain how effective such instruction would be. Additionally, to tailor such instruction for PTs, it would be important to know the extent to which PTs' individual attributes (e.g., mathematical knowledge or beliefs) influence their selection or application of evaluative processes. Although these questions are the domain of future empirical studies, this work establishes a sound theoretical base for such research.

References

As'ari, A. R., Kurniati, D., & Subanji, S. (2019). Teachers expectation of students' thinking processes in written works: A survey of teachers' readiness in making thinking visible. *Journal on Mathematics Education*, *10*(3), 409–424. https://doi.org/10.22342/jme.10.3.7978.409-424

Association of Mathematics Teacher Educators. (2017). *Standards for preparing teachers of mathematics*. Available online at https://amte.net/standards.

Ayalon, M., & Rubel, L. H. (2022). Selecting and sequencing for a whole-class discussion: Teachers' considerations. *The Journal of Mathematical Behavior*, *66*, 100958. https://doi.org/10.1016/j.jmathb.2022.100958

Bartell, T. G., Webel, C., Bowen, B., & Dyson, N. (2013). Prospective teacher learning: Recognizing evidence of conceptual understanding. *Journal of Mathematics Teacher Education*, *16*(1), 57–79. https://doi.org/10.1007/s10857-012-9205-4

Clements, D. H., & Sarama, J. (2021). Sustainable, scalable professional development in early mathematics: Strategies, evaluation, and tools. In Y. Li, R. E. Howe, W. J. Lewis, & J. J. Madden (Eds.), *Developing mathematical proficiency for elementary instruction* (pp. 221–238). Springer. https://doi.org/10.1007/978-3-030-68956-8_10

Cohen, E. G., & Benton, J. (1988). Making groupwork work. *American Educator*, *12*(3), 10–17.

Copur-Gencturk, Y., & Doleck, T. (2021). Strategic competence for multistep fraction word problems: An overlooked aspect of mathematical knowledge for teaching. *Educational Studies in Mathematics*, *107*(1), 49–70. https://doi.org/10.1007/s10649-021-10028-1

Corven, J. (2021). Prospective elementary teachers' evaluations of student solutions to division story problems. In D. Olanoff, K. Johnson, & S. Spitzer (Eds.), *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1134–1135). Philadelphia, PA.

Dyer, E. B., & Sherin, M. G. (2016). Instructional reasoning about interpretations of student thinking that supports responsive teaching in secondary mathematics. *ZDM – Mathematics Education*, *42*(1), 69–82. https://doi.org/10.1007/s11858-015-0740-1

Fernández, C., Llinares, S., & Valls, J. (2013). Primary school teacher's noticing of students' mathematical thinking in problem solving. *The Mathematics Enthusiast*, *10*(1–2), 19. https://doi.org/10.54870/1551-3440.1274

Heinrichs, H., & Kaiser, G. (2018). Diagnostic competence for dealing with students' errors: Fostering diagnostic competence in error situations. In T. Leuders, K. Philipp, & J. Leuders (Eds.), *Diagnostic competence of mathematics teachers* (pp. 79–94). Springer. https://doi.org/10.1007/978-3-319-66327-2_4

Henningsen, M., & Stein, M. K. (1997). Mathematical tasks and student cognition: Classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education*, *28*(5), 524–549. https://doi.org/10.2307/749690

Hiebert, J., Morris, A. K., & Spitzer, S. M. (2018). Diagnosing learning goals: An often-overlooked teaching competency. In T. Leuders, K. Philipp, & J. Leuders (Eds.), *Diagnostic competence of mathematics teachers* (pp. 193–206). Springer. https://doi.org/10.1007/978-3-319-66327-2_10

Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education*, *41*(2), 169–202. https://doi.org/10.5951/jresematheduc.41.2.0169

Jessup, N. A. (2018). *Understanding teachers' noticing of children's mathematical thinking in written work from different sources* (Publication No. 10838739) [Doctoral dissertation, University of North Carolina at Greensboro]. ProQuest Dissertations & Theses Global.

Lee, M. Y. (2021). Using a technology tool to help pre-service teachers notice students' reasoning and errors on a mathematics problem. *ZDM – Mathematics Education*, *53*(1), 135–149. https://doi.org/10.1007/s11858-020-01189-z
Louie, N., Adiredja, A. P., & Jessup, N. (2021). Teacher noticing from a sociopolitical perspective: The FAIR framework for anti-deficit noticing. *ZDM – Mathematics Education*, *53*(1), 95–107. https://doi.org/10.1007/s11858-021-01229-2
Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Lawrence Erlbaum. https://doi.org/10.4324/9781410602589
Merriam-Webster. (n.d.). Evaluate. In *Merriam-Webster.com dictionary*. Retrieved January 17, 2023, from https://www.merriam-webster.com/dictionary/evaluate.
Michael, M. D. (2005). *Arithmetic and algebra word problems: Preservice teachers' content knowledge, attitudes, and appreciation of students' strategies* (Publication No. 3191240) [Doctoral dissertation, West Virginia University]. ProQuest Dissertations & Theses Global.
Moore, S. D., Morrow-Leong, K., & Gojak, L. M. (2020). *Mathematize it! Going beyond key words to make sense of word problems: Grades 3-5*. Corwin.
Moreno, M., Sánchez-Matamoros, G., Callejo, M. L., Pérez-Tyteca, P., & Llinares, S. (2021). How prospective kindergarten teachers develop their noticing skills: The instrumentation of a learning trajectory. *ZDM – Mathematics Education*, *53*(1), 57–72. https://doi.org/10.1007/s11858-021-01234-5
Morris, A. K. (2007). Factors affecting pre-service teachers' evaluations of the validity of students' mathematical arguments in classroom contexts. *Cognition and Instruction*, *25*(4), 479–522. https://doi.org/10.1080/07370000701632405
Morris, A. K., Hiebert, J., & Spitzer, S. M. (2009). Mathematical knowledge for teaching in planning and evaluating instruction: What can preservice teachers learn? *Journal for Research in Mathematics Education*, *40*(5), 491–529. https://doi.org/10.5951/jresematheduc.40.5.0491
National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all*. Author.
National Research Council [NRC]. (2001). *Adding it up: Helping children learn mathematics*. The National Academies Press. https://doi.org/10.17226/9822
Özdemir, I. E. F., & Pape, S. J. (2012). Supporting students' strategic competence: A case of a sixth-grade mathematics classroom. *Mathematics Education Research Journal*, *24*(2), 153–168. https://doi.org/10.1007/s13394-012-0033-8
Philipp, K. (2018). Diagnostic competences of mathematics teachers with a view to processes and knowledge resources. In T. Leuders, K. Philipp, & J. Leuders (Eds.), *Diagnostic competence of mathematics teachers* (pp. 109–127). Springer. https://doi.org/10.1007/978-3-319-66327-2_6
Robertson, A. D., Atkins, L. J., Levin, D. M., & Richards, J. (2015). What is responsive teaching? In A. D. Robertson, R. Scherr, & D. Hammer (Eds.), *Responsive teaching in science and mathematics* (pp. 1–35). Routledge. https://doi.org/10.4324/9781315689302
Schack, E. O., Fisher, M. H., Thomas, J. N., Eisenhardt, S., Tassell, J., & Yoder, M. (2013). Prospective elementary school teachers' professional noticing of children's early numeracy. *Journal of Mathematics Teacher Education*, *16*(5), 379–397. https://doi.org/10.1007/s10857-013-9240-9
Shaughnessy, M., & Boerst, T. (2018a). Designing simulations to learn about pre-service teachers' capabilities with eliciting and interpreting student thinking. In G. J. Stylianides & K. Hino (Eds.), *Research advances in the mathematical education of pre-service elementary teachers: An international perspective* (pp. 125–140). Springer. https://doi.org/10.1007/978-3-319-68342-3_9
Shaughnessy, M., & Boerst, T. A. (2018b). Uncovering the skills that preservice teachers bring to teacher education: The practice of eliciting a student's thinking. *Journal of Teacher Education*, *69*(1), 40–55. https://doi.org/10.1177/0022487117702574
Son, J.-W. (2013). How preservice teachers interpret and respond to student errors: Ratio and proportion in similar rectangles. *Educational Studies in Mathematics*, *84*(1), 49–70. https://doi.org/10.1007/s10649-013-9475-5
Spitzer, S. M., Phelps, C. M., Beyers, J. E. R., Johnson, D. Y., & Sieminski, E. M. (2011). Developing prospective elementary teachers' abilities to identify evidence of student mathematical achievement. *Journal of Mathematics Teacher Education*, *14*(1), 67–87. https://doi.org/10.1007/s10857-010-9141-0
Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning*, *10*(4), 313–340. https://doi.org/10.1080/10986060802229675

Stockero, S. L., Leatham, K. R., Van Zoest, L. R., & Peterson, B. E. (2017). Noticing distinctions among and within instances of student mathematical thinking. In E. O. Schack, M. H. Fisher, & J. A. Wilhelm (Eds.), *Teacher noticing: Bridging and broadening perspectives, contexts, and frameworks* (pp. 467–480). Springer. https://doi.org/10.1007/978-3-319-46753-5_27

Van Dooren, W., Verschaffel, L., & Onghena, P. (2002). The impact of preservice teachers' content knowledge on their evaluation of students' strategies for solving arithmetic and algebra word problems. *Journal for Research in Mathematics Education*, *33*(5), 319–351. https://doi.org/10.2307/4149957

van Es, E. A., & Sherin, M. G. (2021). Expanding on prior conceptualizations of teacher noticing. *ZDM – Mathematics Education*, *53*(1), 17–27. https://doi.org/10.1007/s11858-020-01211-4

Van Zoest, L. R., Stockero, S. L., Leatham, K. R., Peterson, B. E., & Ruk, J. M. (2020). Articulating the student mathematics in student contributions. In A. I. Sacristán, J. C. Cortés-Zavala, & P. M. Ruiz-Arias (Eds.), *Mathematics education across cultures: Proceedings of the 42nd meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 2105–2109). Mexico. https://doi.org/10.51272/pmena.42.2020-354