PROBLEM POSING IN INVESTIGATIONS: THE CASE OF BRANDI AND BEN

Andrew Kercher
Simon Fraser University
andrew_kercher@sfu.ca

Canan Güneş
Simon Fraser University
canan_gunes@sfu.ca

Rina Zazkis
Simon Fraser University
zazkis@sfu.ca

Extant research has demonstrated that problem-posing and problem-solving mutually affect one another. However, the exact nature and extent of this relationship requires a detailed elaboration. This is especially true when adidactical problem-posing arises within a problem-solving context. In this study, we analyze the scripting journey used by two students to record their investigation of sums of consecutive integers. We analyze the adidactical problem-posing found within the scripting journey using three facets of a problem posing framework: mathematical knowledge base, problem-posing heuristics, and individual considerations of aptness. Our analysis reveals how these aspects of problem-posing emerge within a mathematical investigation, how they are related to surrounding problem-solving, and what types of activity act as catalysts to promote further problem-posing activity.

Keywords: Problem Solving; Undergraduate Education; Preservice Teacher Education

Introduction

"How are problem-posing skills related to problem-solving skills?" (Cai et al., 2015, p. 14). According to Cai et al. (2015), this question is yet unanswered, but exploring it could lead to advances in the collective understanding of students' mathematical activity. We note that this is not a question of the existence of a relationship; at the time, extant research already supported the hypothesis that successful problem-posers were also successful problem-solvers, and vice versa (cf. Silver & Cai, 1996; Cai & Hwang, 2002; 2003). Instead, the question suggests that it is the fundamental character of this relationship that remains unclear—and it calls for mathematics education researchers to explore the nature and extent of the connection between problem-posing and problem-solving.

This call was not ignored. Liljedahl and Cai (2021) reported on advances in both problem-solving and problem-posing from the intervening years—and in particular, those studies that sought to understand better how the two fields intersect. For example, Elgrably & Leikin (2021) discovered that initial problem-solving efforts can spark later problem-posing creativity; conversely, Hartmann et al. (2021) noted that initial problem-posing can lead to unexpected student success in later problem-solving.

Although the two studies cited above combined problem-solving and problem-posing, in each case, all problem-posing activity was incited directly by the task itself. That is, participants were explicitly instructed to pose mathematical problems. However, Koichu (2020) exemplified that problem-posing can be instigated *adidactically*, "as an activity necessitated for the posers by the need to find or create problems that would serve another goal" (p. 3). In Koichu's work, this other goal was pedagogically oriented; problem-posing arose adidactically as participants prepared to teach a difficult topic. We wondered to what extent adidactical problem-posing might naturally arise when the ultimate goal is not to teach a particular mathematical idea, but instead, to investigate a particular mathematical phenomenon through problem-solving.

To this end, we developed an investigative task without an explicit problem-posing component. The data in this study is a self-reported dialogue inspired by participants' firsthand experiences as they use problem-solving to explore the task; we call this type of dialogue a scripting journey. We address the following research question: *How does adidactical problem-posing emerge when engaging with a problem-solving investigation?* A framework for the analysis of problem-posing, an overview of scripting journeys, and the task itself are provided in the following sections.

A Theoretical Framework for Problem-Posing Analysis

To address our research questions, we draw on a framework proposed by Kontorovich et al. (2012). The framework consists of attributes that attend to the cognitive, affective, and social dimensions of problem posing. In this paper, we analyze episodes of adidactical problem-posing activity using three of these attributes: mathematical knowledge base, problem-posing heuristics and strategies, and individual considerations of aptness.

A problem poser's *mathematical knowledge base* includes mathematical definitions, facts, procedures, prototypical problems, and competencies related to mathematical discourse and writing. *Problem-posing heuristics and strategies* refer to the systematic approaches that a problem poser adopts to analyze and transform a mathematical situation and, later, to pose problems. Drawing on previous research, Kontorovich et al. (2012) composed a provisional list of such strategies, three of which are of interest to this report:

1. *Numerical manipulation*: Posing a new problem by assigning different numerical values to given constraints.
2. *What-if-notting*: Posing a new problem by removing or changing either a given constraint or an underlying assumption about the mathematical setting.
3. *Generalization*: Posing a new problem "for which the given problem is a special case" (Kontorovich et al., 2012, p. 152).

Finally, throughout a problem-posing task, problem posers consider the suitability of posed problems for a particular audience. This audience might include themselves, a real or hypothetical evaluator, or an intended audience who will be tasked with solving the problem. *Individual considerations of aptness* are the problem posers' conceptualizations of the explicit and implicit criteria by which this audience will judge the posed problem (or by proxy, the problem-poser) and how necessary it may be to meet these criteria. For example, when considering whether a problem is appropriate for a later problem solver, the problem poser could try to anticipate whether the problem will be mathematically challenging, engaging, or capable of successfully teaching a desired concept to the solver.

The Task

Scripting Tasks and Scripting Journeys

A scripting task is an activity centered around the construction a mathematical dialogue, typically involving some combination of teacher- and student-characters. Sometimes, a scripting task provides a *prompt*—a few introductory lines of dialogue that introduce the topic of the script. When a prompt is included, it typically introduces a mathematical question of a student (e.g., Bergman et al., 2022; Kontorovich & Zazkis, 2016; Marmur & Zazkis, 2018), a misunderstanding (e.g., Zazkis et al., 2013), or a disagreement (e.g., Marmur et al., 2020; Zazkis & Zazkis, 2014) that the script should attend to and eventually resolve.

Recently, researchers have explored the application of scripting tasks that produce a particular type of dialogue referred to as a *scripting journey* (Kercher et al., in press). Unlike dialogues resulting from other types of scripting tasks, a scripting journey is not a continuation of a prompt; rather, the scriptwriters use their own mathematical activity as a model for constructing the scripting journey. Kercher et al. (in press) observed that student-written scripting journeys contain robust mathematical activity and are thus appropriate for analysis in mathematics education research. With this result in mind, we leverage scripting journeys to capture and analyze the emergent adidactical problem-posing activity within a problem-solving investigation.

The Consecutive Integers Task

The activity used to stimulate adidactical problem-posing was the *Consecutive Integers task* (CI task). Its introductory instructions are presented in Figure 1.

> The number 294 has the following interesting property:
>
> $$294 = 39 + 40 + 41 + 42 + 43 + 44 + 45$$
>
> That is, it can be written as the sum of 7 consecutive integers.
>
> **Your task is to investigate this property and its variations.** That is, your task is to investigate the sums of consecutive integers. The following questions can guide the beginning of your investigation. **You do not need to address all the suggested questions.** You can choose a few or you can proceed with your own problems/ questions.

Figure 1: The introduction of the Consecutive Integers task.

Participants were then provided with a selection of example questions to direct their explorations, which included: Consider any number of your choice. What values are possible for K? Can you find all of them? Can you generalize further? That is, given any natural number, can it be written as the sum of consecutive integers? If so, in what ways? If not, why?

Thus, the CI task can be thought of as a modular, semi-structured investigation in which participants were free to select from a number of smaller problem-solving tasks of varying degrees of open-endedness and mathematical sophistication. The task also invites participants to explore sums of consecutive integers independently of the suggested problems, and in doing so, guides participants towards posing and solving their own problems. Completion of the CI task required participants to record their mathematical activity as a scripting journey.

Participants, Data Collection, and Analysis

The data in this study is part of a larger study on the adidactical problem-posing of prospective elementary school teachers enrolled in a mathematics methods course. We focus on the work of Brandi and Ben (pseudonyms), who completed the CI task outside of usual classroom hours. Both scripting tasks and mathematical investigations were a typical component of the course, but the CI task was the first assignment in which the participants had been required to record their work as a scripting journey. The course did not include explicit problem-posing activities or instruction.

The research team first read and reread Brandi and Ben's scripting journey, in which they featured themselves as characters, to become familiar with their investigation. Then, the first author coded the script for instantiations of adidactical problem-posing. Considering that the participants were not asked to set aside a comprehensive list of what they considered to be the problems that they posed over the course of their investigation, two different methods were used to distinguish posed problems within the dialogue. First, we identified some explicit questions as posed problems—such as when Brandi asks, "But if 294 is X, what happens when X is unknown? For example, $7N + 21 = X$?" On the other hand, some posed problems were inferred from statements of intent or from a character's musings. These problems did not always appear in the form of a question, but the remark was coded as a posed problem if it was suggestive of particular constraints and goals that the speaker had in mind. For example: "I wonder if this can be written as a formula to work with other patterns."

The second author then independently coded the scripting journey for adidactical problem-posing, and these codes were compared with the first author's codes. Discrepancies were resolved by discussion until complete agreement was reached. Brandi and Ben's scripting journey was then subjected to analysis using the framework of Kontorovich et al. (2012). Throughout the application of the analysis framework, the entire authorial team met regularly to discuss their interpretations of the observed problem-posing activity.

Findings: Brandi and Ben

In this section, we first present a summary of Brandi and Ben's scripting journey in the form of a short vignette. Following the vignette, we analyze the recorded problem-posing behavior using the framework of Kontorovich et al. (2012).

Vignette

Immediately upon beginning their investigation, Brandi and Ben work together to establish a way of representing 294 as the sum of 7 consecutive integers algebraically. First, Brandi wonders if the property that 294 can be written as the sum of 7 consecutive integers "can be written as a formula to work with other patterns." In response, Ben proposes a formulation using the variable N:

> I think it should only go to $N + 6$ because N is the first integer and $N + 1$ is the second. Therefore, seven consecutive integers would end at $N + 6$. It should be: $N + (N + 1) + (N + 2) + (N + 3) + (N + 4) + (N + 5) + (N + 6)$.

Note that Ben uses the variable N to represent the smallest integer in the sequence even though the exact sequence of integers is given by the task (see Figure 1). This behavior, taken in context with Brandi's stated desire to work with "other patterns," suggests that the student-characters in this excerpt are working to solve an implicitly posed problem: namely, they are attempting to discover an algorithm that will allow them to locate a sequence of integers that add to a given sum without guessing and checking.

In support of this overarching problem, Brandi and Ben go on to pose a number of follow-up problems that they anticipate will help them better understand the functionality of their developing algorithm:

Table 1: Posed problems in response to the developing algorithm

	Speaker	Dialogue
[1]	Brandi	But if 294 is X, what happens when X is unknown? For example, $7N + 21 = X$.
[2]	Ben	[After noticing the sum is divisible by the number of divisors] Do you think that can work if the number of consecutive integers is an even number?
[3]	Ben	What happens when we solve for N and it's not a round whole number?

In the process of answering these problems, Ben eventually realizes procedure they have been using can be improved by simply dividing the desired sum by the number of consecutive integers. Then, "any quotient that is a whole number will have an odd number of consecutive integers, [...] and any quotient that is a half number will have an even number of consecutive integers." The student-characters decide to leverage technology and construct a spreadsheet that will handle these computations. Freed from the responsibility of manual arithmetic, Brandi and Ben pose a sequence of problems that attend to more abstract concerns. These include:

Table 2: Posed problems in response to the creation of the spreadsheet

	Speaker	Dialogue
[4]	Brandi	What are you going to put in the table?
[5]	Brandi	How will you know which numbers [of consecutive integers] to check?
[6]	Brandi	Can it go to an unlimited number of even or odd consecutive integers?
[7]	Brandi	What happens when we have a huge number of consecutive integers?
[8]	Brandi	Well, I guess we can go to negative numbers?

Brandi wonders first about the information that should be provided to the spreadsheet; that is, she considers which information in the task should be considered a constraint and which should be a goal. The student-characters then explore the boundaries of what reasonable values for K, the number of consecutive integers, might be. In particular, Brandi wonders about a hypothetical upper bound on K; in response, the student-characters consider the inclusion of negative numbers and zero as one avenue for generating a "huge number of consecutive integers."

After attending to these considerations and building the spreadsheet (Figure 2), Brandi suggests that they test it with a newly posed problem: to find a list of all possible values of K for a completely different number, 165. Using their spreadsheet, they divide 165 by consecutive integers starting at 2, consider the quotient, and, where possible, produce the sequence of consecutive integers that sum to 165 by using the quotient as the midpoint of the sequence.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	AA	AB	AC	AD	AE	AF	AG	AH
1	Dividend	Divisor/ Number of Consecutive Integers	Quotient	Should this work?	Sum																													
2						1		2		3		4		5		6		7		8		9		10		11		12		13		14		15
3	165	2	82.5	Y	165	82	+	83																										
4	165	3	55	Y	165	54	+	55	+	56																								
5	165	4	41.25	N	0																													
6	165	5	33	Y	165	31	+	32	+	33	+	34	+	35																				
7	165	6	27.5	Y	165	25	+	26	+	27	+	28	+	29	+	30																		
8	165	7	23.5714	N	0																													
9	165	8	20.625	N	0																													
10	165	9	18.3333	N	0																													
11	165	10	16.5	Y	165	12	+	13	+	14	+	15	+	16	+	17	+	18	+	19	+	20	+	21										
12	165	11	15	Y	165	10	+	11	+	12	+	13	+	14	+	15	+	16	+	17	+	18	+	19	+	20								
13	165	12	13.75	N	0																													
14	165	13	12.6923	N	0																													
15	165	14	11.7857	N	0																													
16	165	15	11	Y	165	4	+	5	+	6	+	7	+	8	+	9	+	10	+	11	+	12	+	13	+	14	+	15	+	16	+	17	+	18
17	165	16	10.3125	N	0																													
18	165	17	9.70588	N	0																													
19	165	18	9.16667	N	0																													
20	165	19	8.68421	N	0																													
21	165	20	8.25	N	0																													

Figure 2: The first 20 of the 450 rows in Brandi and Ben's spreadsheet.

Compiling this list of ways in which 165 is expressed as the sum of integers inspires Brandi to pose a new collection of problems:

Table 3: Posed problems in response to the application of the spreadsheet to an example

	Speaker	Dialogue
[9]	Brandi	Is there something to do with the factors of the number?
[10]	Brandi	Why won't that [choosing $K = 18$] work?

In what follows, the student-characters attempt to uncover an underlying mathematical justification for the acceptable values of K identified by the exhaustive search they undertook with the aid of their spreadsheet. For example, the student-characters readily accept that 18 consecutive integers will not sum to 165 after consulting their spreadsheet; to supplement this conclusion, they examine the prime factorization of 165 and attempt to rationalize why 18 did not work but, on the other hand, 15 did.

Analysis

Mathematical knowledge base. Brandi and Ben used mathematical knowledge when they addressed the validity of using negative numbers in their sequence of consecutive integers [Problem 8]. Furthermore, much of Brandi and Ben's work before the creation of their spreadsheet leveraged their knowledge of divisibility rules and properties. After the spreadsheet, they used their knowledge of divisibility and prime factorizations to attempt to find a relationship between valid choices of K for a number and its prime factors [Problems 9, 10].

In addition to their knowledge of mathematical content, Brandi and Ben also demonstrated an awareness of mathematical norms for justification, generality, and efficiency. For example, problems in the dialogue attended to multiple cases [Problems 2, 6], and Ben's initial suggestion to work with the variable N can be seen as a way of ensuring that these separate cases can be united under a sufficiently general notation. We also draw attention to the characters' attempts to streamline their spreadsheet application [Problems 4, 5, 7], which we take as evidence of their desire to avoid an impractical guess-and-check approach for locating possible values of K. Brandi and Ben seem to feel that such an approach would not sufficiently address the mathematical goal of the task.

Problem-posing heuristics and strategies. Brandi and Ben engaged with the *generalization* heuristic when they attempted to capture the mathematical situation using equations and variables; their attempts to unite multiple cases under a single algorithm are another form of generalization. Considering different cases typically required the *numerical manipulation* heuristic, as did testing conjectures on different examples [Problem 10]. The symbolic representation proposed by Ben also allowed the characters to easily manipulate not only the number of consecutive integers but also the value of the target sum [Problem 1]. Later, the spreadsheet would serve the same goal of allowing Brandi and Ben to apply the numerical manipulation heuristic as quickly as possible.

Brandi and Ben also employed the *what-if-not* strategy to challenge their implicit assumptions about the CI task. First, when they realized that they had taken for granted that K must be less than the desired sum and posed problems questioning whether that really was its upper bound [Problems 5, 7]; and second, when they wondered whether they should be allowed to use a sequence of consecutive numbers that included negative values [Problem 8]. Both of these applications of the what-if-not strategy were instigated by the use of the spreadsheet, which removed the computational barrier and enabled them to examine the reasonable bounds of K.

Individual considerations of aptness. The primary consideration of aptness made by Brandi and Ben concerned whether or not a problem would be immediately appropriate for furthering their understanding of the mathematical situation at hand. That is, a good problem in the context of the scripting journey was one that would yield progress in the ongoing investigation. In this sense, Ben's suggestion to consider even numbers [Problem 2] was apt in that it addressed a case that the group had not yet attended to; similarly, Brandi's problem attempting to integrate prime factorization [Problem 9] was considered apt because it allowed the characters to refine their procedure for locating appropriate values of K. This latter example highlights that Brandi and Ben valued efficiency as an appropriate criterion for their posed problems.

Brandi and Ben also considered a problem apt when it invoked a symbolic representation of the CI task, as demonstrated early in the vignette. Although neither character said outright why a symbolic representation was valuable, we might interpret this preference in multiple ways. First, a symbolic representation might be considered apt in that it is a more efficient way to test possible values of K; this interpretation is in line with Brandi and Ben's other evident priorities and explains why their symbolic system was abandoned once they had constructed the more efficient spreadsheet. Second, it could be that the characters expected that the required "general solution" to the CI task would only be sufficiently general if its steps could be demonstrated symbolically. In this sense, the characters would consider representing the task symbolically to be apt not only because it furthered their understanding but because it was an implicit requirement of a correct solution.

Discussion

The research question guiding our study was: *How does adidactical problem-posing emerge when engaging with a problem-solving investigation?* In addressing this research question, we focus on the process of problem-posing, the purpose of problem-posing, and the conditions under which problem-posing appears.

When we interpret this question in terms of the mechanical process of forming a novel mathematics problem, we see that there are many similarities between didactical and adidactical problem posing—in both cases, the problem poser relies on their mathematical knowledge base and a selection of heuristics to construct a problem that is relevant to the mathematical setting. This report illustrates the nuanced relationship between these three components of the analysis framework. For example, Brandi and Ben applied the what-if-not heuristic when they questioned their implicit assumption that the sequence of consecutive integers should consist of only positive numbers. But in order to apply this heuristic, they first had to consult their mathematical knowledge base and clarify the elements of the set of integers. Finally, they considered whether a problem that makes use of negative integers would even be appropriate for making progress in the investigation at hand.

Another aspect of addressing our research question lies in examining not only *the means by which* problems are posed but also *for what purpose* they are posed in an investigation. Certainly, one purpose of posed problems was to complete the CI task. We note that this goal necessarily reduced the amount of variation in problem type that Brandi and Ben felt compelled to explore. That is, participants' problem-posing activity was limited by the aptness of a problem for contributing to their ongoing investigation into sums of consecutive integers. We note that Problem 6 received very little follow-up in the scripting journey; this could be because Brandi and Ben recognized that sequences of odd or even numbers was beyond the scope of their investigation of sums of consecutive integers.

Although some of the constraints of the consecutive integer task were immutable, participants decided independently when their investigations had reached a natural conclusion. The point at which a group found their work on the CI task to be satisfactory illuminated what kinds of mathematical artifacts they perceived to be normatively valued within their mathematics course. Brandi and Ben were not satisfied with their work until they had answered some subset of the example problems provided by the task; however, they also posed and endorsed their own problems that dealt with formal mathematical representations, the efficiency or reliability of algorithms and formulae, or justifications that demonstrate generalizability. In this way, the CI task prompted problem-posing to emerge not only *in service of* solving a given problem but also *as a consequence* of solving that problem.

Finally, we address our research question by considering *when* in the course of problem-solving it is likely that students will engage in a didactical problem-posing. We observed that the creation of the groups' spreadsheet engendered a flurry of problem-posing unrestrained by computational limitations. These posed problems explored how the new spreadsheet might become even more efficient with a better understanding of the mathematical setting [Problems 4-8]. Because it triggered a burst of novel problem-posing activity, we identify the creation of the spreadsheet as an example of what we call a *problem-posing catalyst*. A problem-posing catalyst is a shift in perspective brought about by the removal of constraints or a mathematical realization. The problem-posing activity that follows a catalytic event is more concentrated because the catalyst creates a "fresh" problem space; problems occur more easily to the posers because they appear from previously unexplored directions. Consequently, the fact that post-catalyst problem-posing takes place within a newly conceptualized mathematical setting means more potential for further insights—that is, more catalysts. In this way the cycle of didactical problem-posing fuels itself.

Concluding Remarks

This report contributes to problem-posing literature by describing different ways in which a problem-poser's mathematical knowledge base, problem-posing heuristics and strategies, and individual considerations of aptness might play a role in the didactical problem-posing they exhibit during an investigation involving problem-solving activity.

Additionally, we introduce the construct of a problem-posing catalyst to provide a touchstone for future explorations of a didactical problem posing. This study illustrates problem-posing catalysts that arise through investigative problem-solving activity; however, we note that because catalysts are characterized by a shift in perspective, they might emerge in other settings. Future studies could describe catalysts in other types of tasks with potential fora didactical problem-posing, such as when students must generate a variety of examples or search for visual patterns.

Because scripting journeys are participants' self-reported retellings of problem-solving activity, conclusions that we are able to draw about their problem-posing come with caveats. By attempting to capture their engagement with the CI task as a narrative dialogue, the participants could be expected to selectively include only those problems that they perceived as contributing to their mathematical progress. This can be seen as a limitation. That is, the scripting journeys may have included only those problems, both solved and unsolved, which already met the scriptwriters' individual considerations of aptness. Future research might use other methods of monitoring problem-posing activity, such as video recordings of group work, to capture problems that were posed by the group but not included in their scripting journey.

References

Bergman, A. M., Gallagher, K., & Zazkis, R. (2022). Prospective teachers' responses to students' dialogue on fractions: attribute substitution and heuristic approaches. *Research in Mathematics Education, 25*(1), 85-104. https://doi.org/10.1080/14794802.2021.2020155

Cai, J., & Hwang, S. (2002). Generalized and generative thinking in US and Chinese students' mathematical problem solving and problem posing. *The Journal of Mathematical Behavior*, *21*(4), 401-421. https://doi.org/10.1016/S0732-3123(02)00142-6

Cai, J., & Hwang, S. (2003). A perspective for examining the link between problem posing and problem solving. *International Group for the Psychology of Mathematics Education*, *3*, 103-110.

Cai, J., Hwang, S., Jiang, C., & Silber, S. (2015). Problem-posing research in mathematics education: Some answered and unanswered questions. In *Mathematical Problem Posing* (pp. 3-34). Springer. https://doi.org/10.1007/978-1-4614-6258-3_1

Elgrably, H., & Leikin, R. (2021). Creativity as a function of problem-solving expertise: Posing new problems through investigations. *ZDM - Mathematics Education*, *53*(4), 891-904. https://doi.org/10.1007/s11858-021-01228-3

Hartmann, L. M., Krawitz, J., & Schukajlow, S. (2021). Create your own problem! When given descriptions of real-world situations, do students pose and solve modelling problems?. *ZDM - Mathematics Education*, *53*(4), 919-935. https://doi.org/10.1007/s11858-021-01224-7

Kercher, A., Bergman, A. M., Zazkis, R. (in press) Counting the stars: Advancing mathematical activity in a scripting journey. *Mathematical Thinking and Learning*.

Koichu, B. (2020). Problem posing in the context of teaching for advanced problem solving. *International Journal of Educational Research*, *102*. https://doi.org/10.1016/j.ijer.2019.05.001

Kontorovich, I., Koichu, B., Leikin, R., & Berman, A. (2012). An exploratory framework for handling the complexity of mathematical problem posing in small groups. *The Journal of Mathematical Behavior*, *31*(1), 149-161. https://doi.org/10.1016/j.jmathb.2011.11.002

Kontorovich, I., & Zazkis, R. (2016). Turn vs. shape: Teachers cope with incompatible perspectives on angle. *Educational Studies in Mathematics*, *93*(2), 223-243. https://doi.org/10.1007/s10649-016-9699-2

Liljedahl, P., & Cai, J. (2021). Empirical research on problem solving and problem posing: A look at the state of the art. *ZDM - Mathematics Education*, *53*(4), 723-735. https://doi.org/10.1007/s11858-021-01291-w
Marmur, O., & Zazkis, R. (2018). Space of fuzziness: Avoidance of deterministic decisions in the case of the inverse function. *Educational Studies in Mathematics*, *99*(3), 261-275. https://doi.org/10.1007/s10649-018-9843-2
Marmur, O., Moutinho, I., & Zazkis, R. (2020). On the density of Q in R: Imaginary dialogues scripted by undergraduate students. *International Journal of Mathematical Education in Science and Technology*, 1-29. https://doi.org/10.1080/0020739X.2020.1815880
Silver, E. A., & Cai, J. (1996). An analysis of arithmetic problem posing by middle school students. *Journal for Research in Mathematics Education*, *27*(5), 521-539. https://doi.org/10.5951/jresematheduc.27.5.0521
Zazkis, R., Sinclair, N., & Liljedahl, P. (2013). *Lesson play in mathematics education: A tool for research and professional development*. Springer. https://doi.org/10.1007/978-1-4614-3549-5
Zazkis, R., & Zazkis, D. (2014). Script writing in the mathematics classroom: Imaginary conversations on the structure of numbers. *Research in Mathematics Education*, *16*(1), 54-70. https://doi.org/10.1080/14794802.2013.876157