OPPORTUNITIES FOR LEARNING: THIN SLICING CONTENT WITH VARIATION

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Building Thinking Classrooms (Liljedahl, 2021) provides teachers with a new method of designing and sequencing tasks called "thin slicing," which emerged from variation theory. The results of the present study indicate that an analysis of the dimensions and ranges of variation within such a task offers insights into learning opportunities available. Specifically, identifying instances where variation has not been adequately positioned against a background of sameness can highlight potentially limited opportunities for students to notice the intended mathematics. The results of this analysis can inform design decisions and modifications to the task before implementation increasing the potential of the task to support student learning.

Keywords: Instructional Activities and Practices, Curriculum, and Professional Development.

Building Thinking Classrooms (Liljedahl, 2021) is rapidly growing in relevance and impact in mathematics education across North America. Experienced teachers and preservice teachers are studying the book, sharing implementation experiences at conferences, and participating in professional development to learn new strategies for increasing student thinking in the classroom. In the summer of 2022, we selected the text as a tool for professional development for a cohort of experienced teachers. In this paper, we focus on one pedagogical strategy presented in the book called *thin slicing*. Designing thin sliced tasks involves writing a carefully sequenced series of problems utilizing small, incremental changes to support students' development of new mathematical knowledge building from their current ways of understanding. Liljedahl (2021) provides guidance for designing these task sequences grounded in variation theory (Marton et al., 2004), but much is left for teachers to work out in their classrooms. We, as mathematics teacher educators, are interested in how teachers are taking up this practice. Specifically, we are interested in the variations teachers use when designing thin sliced tasks and the possible learning opportunities afforded by this task design.

Literature Review and Framework

The National Council of Teachers of Mathematics (NCTM) recommends the use of intentionally sequenced tasks to build procedural fluency from conceptual understanding by informally drawing on students' prior knowledge, assessing students' preconceived ideas that may serve as intellectual motivation for the concepts being learned, or transitioning students from simple, concrete representations to more complex and abstract representations (Boston et al., 2017). A practice that aligns with NCTM's recommendation is the use of problem or number strings. Problem strings are carefully sequenced tasks for the purposes of facilitating students' understanding of mathematical relationships to develop certain numeracy strategies (Carpenter et al., 2003; DiBrienza & Shevell, 1998; Fosnot & Dolk, 2002; Harris, 2011) and providing rich discussion opportunities in classrooms (Bofferding & Kemmerle, 2015).

While problem strings originated at the elementary level, the practice is extending to secondary classrooms (Harris, 2011; Liljedahl, 2021; Wieman et al., 2021). Liljedahl (2021) introduces thin slicing as a method of task design similar to problem strings that uses principles of variation theory (Marton et al., 2004) to foster student thinking around curricular content. Liljedahl distinguishes the sequencing of tasks as *thin slicing*, in which there are incrementally small increases in challenge from one task to the next as students' abilities increase, from *thick slicing*, where the increase in challenge between tasks is much greater (see Liljedahl, 2021, p.151). Liljedahl suggests that teachers can design thin sliced tasks using their current curricular resources to move students from their current mathematical conceptions to deeper understandings while keeping students in the *flow* of learning, avoiding the disruptions of frustration and boredom.

In mathematics education research, task design and sequencing are often informed by learning trajectories and learning progressions which describe ways students may develop new mathematical conceptions (Battista, 2011; Confrey, 2012). A *hypothetical* learning trajectory (as opposed to an *actual* learning trajectory) is a description of the learning goal, the learning activities and a prediction of how students' thinking will develop by engaging the activities (Simon, 1995). For some topics, there are collections of tasks for which the design and sequence have been shown to effectively support students as they develop more sophisticated mathematical reasoning and understanding in relation to a specific learning goal (e.g., Battista, 2012). However, not all mathematics topics in the K-12 curriculum have been studied from the perspective of learning trajectories, and existing learning trajectories continue to be refined (Confrey, 2012). Furthermore, each learning trajectory must assume a starting point and offers one possibility among many (Rich et al., 2017).

Teachers are in the classroom each day implementing tasks and determining what works for their students. The design principles of thin sliced tasks and problem strings offer teachers a way to create tasks based on a sequence they think will support students to advance their understanding of a particular mathematical idea, defining a hypothetical learning trajectory that is then tested in the classroom. Because thin slicing is a new method of task design for many teachers, additional resources to support the design, analysis, and implementation of such tasks are not readily available.

Variation Theory

In *Building Thinking Classrooms* (BTC), readers are introduced to two principles of variation theory to guide the development of thinly sliced curricular tasks: learners see variation against the backdrop of sameness, and a required condition for learning is that only one thing is varied at a time (Liljedahl, 2021). Variation theory describes the necessary conditions for learning, with the aim of enabling the learner to engage in novel situations in powerful ways (Marton et al., 2004). Learning "implies seeing or experiencing critical aspects of an object of learning," where the *object of learning* defines the content and the objective to be learned (Kullberg et al., 2017, p. 560). The teacher's goal is to provide learning opportunities that offer students different ways to see or experience the critical features of the object. What the learner sees or notices during the learning process is impacted by past experiences and social, cultural, environmental, and mathematical dispositions and practices (Watson & Mason, 2006). If a learner does not learn the intended objective, it is because they were not able to discern the critical features of the object of learning which can only occur when learners have experienced variation against a background of sameness.

In mathematics education, Gu et al. (2004) propose two forms of variation, *conceptual variation* and *procedural variation*, to provide a deliberate and reasoned way to utilize variance and invariance in mathematics teaching. Conceptual variation is the use of examples of a concept to discern its features and non-examples of the concept to distinguish it from others. Procedural variation is dynamic and engages students in a sequence to see connections among concepts and processes and to facilitate problem solving strategies and solutions. Procedural variation includes varying a problem to provide scaffolding for higher level concepts or extensions resulting in generalizations; developing multiple methods for solving a problem; and applying the same solution method to a group of similar problems.

Variation Theory and Task Analysis. Variation theory can be used to analyze mathematical tasks or chosen examples to determine opportunities for learning (e.g., Kullberg et al., 2017; Watson & Mason, 2006). Aligning with procedural variation (Gu et al., 2004), Watson and Mason (2006) advocate for the use of variation to design exercises that foreground mathematical structure leading to generalizations. Students enact a procedure on intentionally varied problems as part of a mathematical exercise and then reflect on the results to generalize a new mathematical relationship. Before implementation, the exercises can be analyzed for the potential dimensions of variation, which are aspects of the task that may be varied. Identifying what is available for the learner to notice through variation exposes the underlying mathematical structure and learning opportunities present in a task. For a given task or exercise, the analysis of what is varied (*dimensions of variation*) and how it is varied (*range of permissible variation*) helps the teacher predict opportunities for student learning, a helpful tool for lesson planning. The predicted learning outcomes from task analysis sometimes align with what is learned by the student, but it is possible that the task designer's choices of variation do not always produce the desired student learning (Watson & Mason, 2006). While discrepancies between the intended object of learning and what is actually learned can be caused by the learning environment or the learner's prior knowledge and experience, the task itself can also impact the learner's ability to discern the critical aspects of the object of learning. If there is no variation with respect to a particular object of learning, then it is guaranteed that students will not have an opportunity to learn it (Gu et al., 2004; Kullberg et al., 2017). As teachers experiment with the principles of variation theory to design thin sliced tasks, the potential discrepancies between the intended learning objective and actual learning may result in frustration or discouragement for both teachers and students, possibly leading to time-consuming cycles of task implementation and refinement or abandonment of the task design method completely.

To support teachers in this type of design work, we examined a set of initial thin sliced designs from a group of experienced secondary math teachers. Following Watson and Mason's (2006) example of analyzing tasks before implementation, we conducted a similar analysis to better understand the features of the initial task designs by the group of teachers. The results of our analysis support the design of thin sliced tasks. The purpose of this paper is to illustrate how the tools of variation theory can be used to analyze thin sliced tasks to answer the following questions: what variations are present and what opportunities for learning are available to students in thin sliced tasks designed by teachers using the BTC framework?

Methodology

Participants & Data Collection

The teachers in the present study are part of a larger, five-year teacher development program that advances mathematics teacher leadership. All teachers have at least five years of experience teaching secondary mathematics and were nominated by their district for the program. For this report, we draw on 19 teachers' submissions from the first summer of professional development coursework. The course assignment we analyzed involved teachers working in groups of two or three to design thin sliced mathematics tasks for topics of their choice using variation theory and the BTC framework as a guide. Teachers were directed to select topics in their current curriculum for which they wished to consider alternative teaching methods. While the tasks were designed in groups, each teacher wrote their own analysis of one of the group's tasks. Both the tasks and teacher reflections were analyzed. Consistent with Liljedahl's (2021) description of how teachers can design thin sliced task sequences, the assignment asked teachers to reflect on how to ensure the goal of the task sequence was clear, how they could "slice" the mathematical content to incrementally increase the challenge for students as their abilities increase, and how to extend student learning by varying only one thing at a time.

Data Analysis

The research team analyzed a total of eight tasks and the corresponding teacher reflections. Analyses involved iterative stages of qualitative coding by the four authors of this paper (Saldaña, 2015). In the first stages of coding, the team utilized open coding with teachers as the unit of analysis to understand teachers' design decisions. The team first examined the mathematics in each task, elements of variation, and teachers' learning goals. To support reliability in coding, all four team members coded one teacher's submission together as a group and then pairs of team members double coded the remaining 18 teachers' submissions. Building on these early observations, the team shifted the unit of analysis to be individual mathematics tasks, rather than teachers. Subsequently, the team engaged in refined coding of the eight tasks looking for themes related to the objective or object of learning for each task; the type of variation; and whether the object of learning could be reached based on the variation observed. Emergent themes, such as variation for the purpose of practice or for illuminating patterns, were recorded in matrices (Saldaña, 2015). Throughout this process, the team specifically looked for disconfirming evidence (Creswell & Miller, 2000).

In the final stage of coding, the team refined codes based on the work of Watson and Mason (2006). Tasks were coded for dimensions of the task that could be varied; of those dimensions, which ones were fixed; of those dimensions, which ones were open for variation; what was the possible range of variation; and what range of variation was chosen by the task designer. The team found that by distinguishing the possible range of variation from the chosen range helped to characterize the learning opportunities available to students. Again, all tasks were double coded by pairs of researchers with the lead author coding all eight tasks. The research team met together to discuss and came to consensus on codes related to dimensions and ranges of variation.

We acknowledge that the analysis of the tasks depends on our own mathematical perspectives and teaching experience. All researchers identify as university-based mathematics teacher educators, and most have taught in K-12 schools. Two members of the research team are faculty in mathematics departments and two are members of education departments.

Findings

We observed a wide range of objectives and task structures as the teachers applied BTC recommendations to create their first thin sliced tasks. For this paper, we focus on the analysis of two tasks because they are instructive in revealing the relationship between the objective and the chosen dimensions of variation. One task was designed to help students develop new mathematical knowledge (Task 1: Exploring Negative Exponents), and one task was designed for skill practice (Task 2: The Quadratic Formula). We analyzed the open dimensions of variation and the chosen range of variation leading to the opportunities for learning in each task.

Task 1: Exploring Negative Exponents

The teachers' stated goal for the Exploring Negative Exponents task is to use patterns to help students draw "their own conclusions about the reciprocal nature of negative exponents." The teachers thinly sliced the concept of negative exponents into five "task cards" intended to be given to students in sequence. Card 1 (Figure 1) is a review of prior knowledge to establish the meaning of a base and exponent. The equations are exponential, with the open dimensions of variation being the base, exponent, and its equivalent value. The chosen range of variation for each of these dimensions is positive integers. The opportunity for learning is to recall the relationship between base, exponent, and the equivalent value they define.

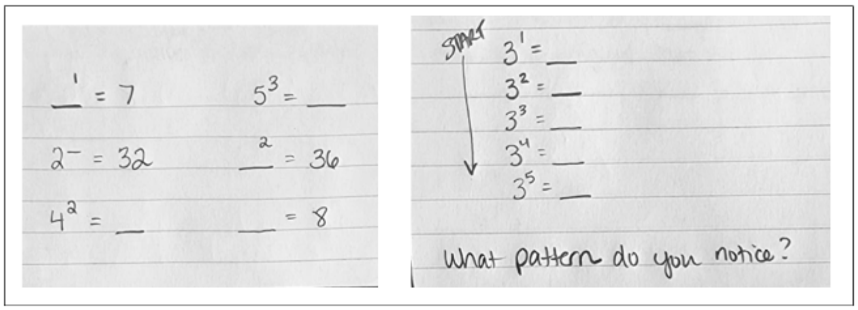

Figure 1. Exploring Negative Exponents Task Cards 1 and 2

In Card 2 (Figure 1), two dimensions of variation are closed: the value of the base and the unknown value. The base of the exponential expression is always 3, and the unknown is always the equivalent value of the exponential expression. The exponent is an open dimension of variation for which the range is constrained to positive integers from 1 to 5. It is intended that students complete the task sequentially beginning with 3^1, and students are encouraged to look for a pattern. The opportunity to learn is that the values of the equivalent expressions for each exponential increase by a factor of 3 as the exponent value increases by 1.

In Card 3 (Figure 2), three aspects are varied relative to the previous card: the base, the order of the exponents, and the inclusion of 0 as an exponent. Students are encouraged to complete the pattern sequentially beginning with 2^5. With these changes, the opportunity to learn now includes a recognition that the pattern in the equivalent values for each expression is multiplied by a factor of ½ as the exponent decreases by a value of 1. Students also have the opportunity to conjecture that 2^0 follows the same pattern, if this was not already known.

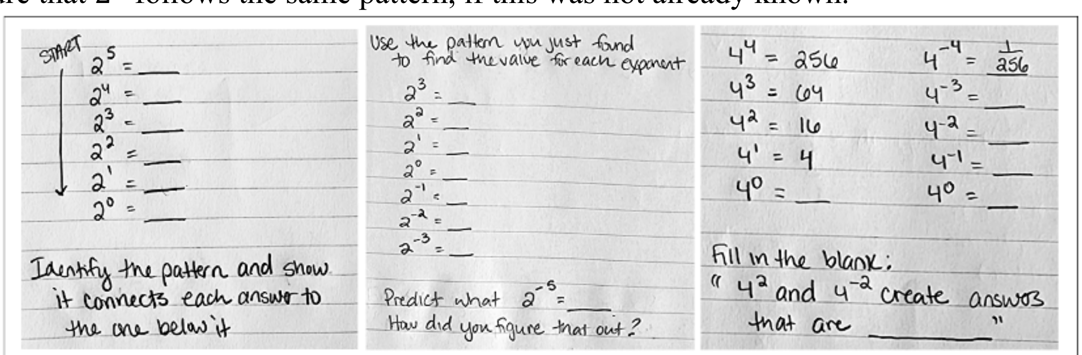

Figure 2. Exploring Negative Exponents Task Cards 3, 4, and 5

In Card 4 (Figure 2), only one aspect is varied relative to the previous card: the value of the exponents. By extending the range of variation to include negative exponents, students have the opportunity to conjecture the equivalent values for 2^{-1}, 2^{-2}, and 2^{-3}. Students are then asked to predict 2^{-5}. Since there are no explicit instructions for how to do this, some students may continue to extend the pattern, concluding that 2^{-5} will be $1/32$.

In Card 5 (Figure 2), there are four aspects that are varied from the previous: the base, the value of the exponents, some equivalent expressions are provided, and the position of the problems into two columns rather than one. The juxtaposition of the two expressions with a base of 4 with exponents of *n* and $-n$ opens the opportunity to notice the reciprocal relationship between 4^n and 4^{-n}. Because the students have been prompted to discover patterns, they have a way to determine the missing information if they do not yet see the reciprocal relationships.

If students use a calculator for Cards 4 and 5, the reciprocal relationship may not be available for them to notice. While this means that students could complete Card 4 without observing the intended relationship, the implementation of the task in the classroom could provide a rich discussion of the equivalence of fraction and decimal representations and the recognition that one representation is more advantageous in this instance for pattern recognition.

Task 2: The Quadratic Formula

For the Quadratic Formula task, the teachers' stated goal is to help students "feel confident substituting and then following the order of operations" to find the solution(s) to the quadratic equation. Each equation in Figure 3 is one "slice" presented to students sequentially. The open dimensions of variation for the task include the form of the equation (set equal to 0 or not); the values for coefficients *a*, *b*, and *c*; the number of unique solutions; and whether the solutions are rational or irrational. For each slice, the *a*, *b*, and *c* values are varied, and the range of variation only includes integers.

Solve for x using the quadratic formula:

1. $2x^2 + 7x + 3 = 0$
2. $x^2 + 7x + 10 = 0$
3. $x^2 - 4x + 3 = 0$
4. $-2x^2 - 4x + 3 = 0$
5. $9x^2 - 6x + 1 = 0$
6. $x^2 + 2x - 1 = 2$
7. $5x^2 - 80 = 0$

Figure 3. The Quadratic Formula Task

For Slices 1-5, the form of the equation remains consistent with 0 on one side of the equal sign. In Slice 1, the choices for *a*, *b*, and *c* result in solutions that are rational (in this case one integer and one non-integer). There is an opportunity for students to identify where *a*, *b*, and *c* are located in a quadratic equation set equal to zero with the quadratic, linear, and constant terms in descending order. There is also an opportunity to notice that values of *a*, *b*, and *c* can be positive integers, and solutions can be integer or non-integer rational numbers. In Slice 2, the only aspects varied relative to the previous slice are the values for *a* and *c*. There is an opportunity for students to recognize, conjecture, or ask the meaning of an implicit coefficient of 1 and to recognize the effect of $a = 1$ in the quadratic formula. In Slice 3, the value of *b* is a negative integer, while *a* remains 1, and *c* is a positive integer. There is an opportunity for students to see that the *b*-value can be a negative integer. There is also the opportunity for students to notice the effect of a negative *b*-value in the quadratic formula. In Slice 4, the choices of a, *b*, and *c* yield irrational solutions, giving students an opportunity to notice that irrational solutions are possible for quadratic equations. In Slice 5, the choices of a, *b*, and *c* yield one unique solution giving students an opportunity to notice that there may only be one unique solution for a quadratic equation, and the one solution can be rational (non-integer).

In Slice 6, the form of the equation is varied because it does not have 0 on one side of the equal sign. There is an opportunity for students to notice that quadratic equations are not always given in the same form. Students may have an opportunity to notice that determining the *c* value will require computation, but this observation is not certain. The introduction to the lesson (not shown in Figure 3) includes a discussion of how to identify a, *b*, and *c* when the equation is set equal to 0, but there is no indication that the form of the equation will be varied during the discussion. While students may notice that the equation is not equal to 0, there is not an explicit opportunity to notice how this impacts the value of *c*. Students are encouraged to check their solutions by substitution, which could provide the opportunity to recognize that something is incorrect if they choose $c = -1$ (rather than $c = -3$). It is also possible that students recognize that $c = -1$ is incorrect by graphing the left and right sides of the equation as two functions and examining the intersection points. However, if students graph only the left side and examine the *x*-intercepts, they will not have an opportunity to recognize that the solution is incorrect.

In Slice 7, there are two varied aspects from the previous slice: *b* is 0 (as opposed to a positive or negative integer) and the equation is set equal to 0 again. There is an opportunity for students to notice *b* can be 0. There is also the opportunity for students to notice the effect of $b = 0$ in the quadratic formula.

Discussion

When designing a task with variation theory, the open dimensions of variation and the constraints on the range of variation impact what students have the opportunity to notice. In the present study, the teachers chose constraints on the range of variation for different reasons, such as illuminating mathematical patterns or anticipating student difficulties. For the Exploration of Negative Exponents task, the teachers' goal was for students to recognize the reciprocal relationship between positive and negative exponents of 2 and generalize this relationship to a base of 4, concluding that 4^n and 4^{-n} are reciprocals. The choice of constraints on the dimensions of variation and ranges of variation was guided by the teachers' desire for the students to recognize a pattern resulting in a generalization. For instance, the teachers explained that Cards 2 and 3 were intended to help students "recognize that increasing the exponent by one creates a subsequent answer that's a scalar multiple of the base." This targeted goal of pattern recognition and generalization resulted in very few open dimensions of variation and a small range for variation of open dimensions. Consequently, the task adequately provided variation against a background of sameness. This task provides an example of how students can enact a procedure on carefully varied problems illustrating a pattern that ultimately leads to a mathematical generalization (Gu et al., 2007; Watson & Mason, 2007).

In contrast, the Solving Quadratic Equations task was designed to increase students' "confidence substituting in numbers and then following the order of operations," resulting in different reasons for the teachers' selection of constraints of variation than the exponent task. The teachers' reported design decisions indicated that they wanted students to attend to (1) coefficients of the quadratic: positive, negative, and zero values for *b*; positive and negative values for a, including a = 1; (2) solutions to a quadratic equation can be rational and irrational; (3) there can be one or two unique solutions; and (4) quadratic equations are not always set equal to 0. These ranges of variation do not account for all variation in quadratic equations. For instance, it is possible that $c = 0$, but $a \neq 0$, and a quadratic equation can have zero real solutions. We point out these additional variations not to argue that this task should include all of them, but rather to highlight that the teachers' decisions about what to constrain were not focused on illustrating the full range of variation related to quadratic equations. We conjecture that the teachers intentionally chose constraints to address students' challenges with numerical computation. For instance, the teachers included an implied $a = 1$, likely because this is something they want to reinforce with students, but they did not intentionally choose $b = 1$ or $c = 1$. Additionally, b is varied in Slices 2 and 3 from a positive integer to a negative integer, and in Slice 7, $b = 0$. We conjecture that the teachers wanted to highlight various b-values to address common errors when evaluating the expressions $-b$ and b^2 in the quadratic formula. Variation theory suggests that varying too many aspects of a problem unrelated to the targeted concept might not provide the background of sameness needed to highlight the intended concept. For example, if students have difficulty understanding the meaning of $-b$ when b is negative, they will likely get an incorrect answer in the quadratic formula but not have an opportunity to focus on the concept that led to the incorrect answer because there are many places where the computation error could have occurred. For this task, the goal of learning how to use the quadratic formula had a subgoal of accurately evaluating expressions for positive and negative integers. Since the variation of b values occurred at the same time other dimensions were varied (e.g., form of equation, values of a and c), students may not have had the opportunity to observe patterns in evaluating $-b$ and b^2 for positive and negative b-values. Another draft of this thin-sliced task might highlight the nuances of working with $-b$ and b^2 by offering a more static background by closing some dimensions of variation. This task provides an opportunity for some students to develop efficiency and accuracy with the quadratic formula, which are components of procedural fluency (NRC, 2001), but opportunities for developing fluency may be missing for students who have difficulty with the necessary numerical computations.

Conclusion

Well-designed thin sliced tasks can provide opportunities for students to learn curricular content while developing their mathematical thinking and reasoning (Liljedahl, 2021). With any task design method, there is much to learn about the nuances of design and possible learning opportunities afforded in the task as well as limitations in the design. This report illustrates how the elements of variation theory (Watson & Mason, 2006), specifically the juxtaposition of sameness and variation within and across dimensions, can be used to highlight the learning opportunities available in thin sliced tasks. The analytic approach used here to identify the dimensions of variation along with the ranges of variation seems a promising tool for teachers and mathematics teacher educators to reflect on their thin sliced task designs. Such reflection before implementation may inform task revisions resulting in increased opportunities for students to learn the intended objectives. Additionally, by examining why particular dimensions of variation were constrained and by considering the resulting opportunities to learn, teachers may identify important mathematics to discuss or call attention to when consolidating ideas (Liljedahl, 2021) at the conclusion of the learning activity.

Acknowledgments

The research described in this paper was supported by the National Science Foundation Robert Noyce Teacher Scholarship Program under Grant Number 2050659. The opinions, findings, conclusions, and recommendations, however, are those of the authors and do not necessarily reflect the views of the National Science Foundation.

References

Battista, M. T. (2011). Conceptualizations and issues related to learning progressions, learning trajectories, and levels of sophistication. *The Mathematics Enthusiast, 8*(3), 507-569.

Battista, M. T. (2012). *Cognition-based assessment and teaching*. Heinemann.

Bofferding, L. & Kemmerle, M. (2015). Elementary teacher candidates' use of number strings: creating a math-talk learning community. *Mathematics Teacher Educator, 3*(2), 99-115.

Boston, M., Dillon, F., Smith, M. S., & Miller, S. (2017). *Taking action: Implementing effective mathematics teaching practices in grades 9-12.* National Council of Teachers of Mathematics.

Carpenter, T. P., Franke, M. L, & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic & algebra in elementary school*. Heinemann.

Confrey, J. (2012). Articulating a learning sciences foundation for learning trajectories in the CCSS-M. In L. R. Van Zoest, J.-J. Lo, & J. L. Kratky (Eds.), *Proceedings of the 34th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 2-22). Kalamazoo, MI: Western Michigan University.

Creswell, J. W. & Miller, D. L. (2000). Determining validity in qualitative inquiry. *Theory Into Practice, 38*(3), 124-130.

DiBrienza, J. & Shevell, G. (1998). Number strings: Developing computational efficiency in a constructivist classroom. *The Constructivist, 13*(2), 21-25.

Fosnot, C. T., & Dolk, M. (2002). *Young mathematicians at work: Constructing fractions, decimals, and percents*. Heinemann.

Gu, L., Huang, R., & Marton, F. (2004). Teaching with variation: A Chinese way of promoting effective mathematics learning. In L. Fan, N.-Y. Wong, J. Cai, & S. Li (Eds.), *How Chinese learn mathematics. Perspectives from insiders.* World Scientific.

Harris, P. W. (2011). *Building powerful numeracy for middle and high school students*. Heinemann.

Kullberg, A., Kempe, U. R., & Marton, F. (2017). What is made possible to learn when using the variation theory of learning in teaching mathematics? *ZDM Mathematics Education, 49*, 559-569. DOI 10.1007/s11858-017-0858-4

Liljedahl, P. (2021). *Building thinking classrooms in mathematics grades K-12: 14 teaching practices for enhancing learning.* Corwin Press.

Marton, F., Tsui, A. B., Chik, P. P., Ko, P. Y., & Lo, M. L. (2004). *Classroom discourse and the space of learning*. Routledge.

National Research Council. (2001). *Adding it up: Helping children learn mathematics*. J. Kilpatrick, J. Swafford, & B. Findell (Eds.) Mathematics Learning Study Committee, Center for Education, Division of Behavioral and Social Sciences and Education. National Academy Press.

Rich, K. M., Strickland, C., Binkowski, T. A., Moran, C., & Franklin, D. (2017). K-8 learning trajectories derived from research literature: Sequence, repetition, conditionals. In *Proceedings of the 2017 ACM conference on international computing education research* (pp. 182-190).

Saldaña, J. (2015). *The coding manual for qualitative researchers* (3rd ed.). Thousand Oaks.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*(2), 114-145.

Watson, A. & Mason, J. (2006). Seeing an exercise as a single mathematical object: Using variation to structure sense-making. *Mathematical Thinking and Learning, 8*(2), 91-111.

Wieman, R., Freedman, L., Albright, P., Nolen, D., & Onda, J. (2021). From notes to strings. *Mathematics Teacher: Learning & Teaching PK-12, 114*(12), 956-963.