CHANGING TEACHERS' CONCEPTION OF PROOF

Óscar Chávez
Illinois State University
ochave2@ilstu.edu

David Barker
Illinois State University
dbarker@ilstu.edu

Alicia Erwin
Illinois State University
amerwi2@ilstu.edu

Seyedehkhadijeh Azimi
Virginia State University
sazimi@vsu.edu

Proof is a fundamental aspect of mathematics. However, in the high school curriculum, it often receives uneven attention that is focused on form rather than understanding. One avenue for addressing this issue is to change and strengthen teachers' conceptions of proof. To explore this idea, we followed a group of teachers as they participated in a summer mathematics research experience. During this experience, proof was not an isolated exercise but part of the mathematical process of discovery. In this study, we analyzed pre- and post-survey data and participants' critique of proofs to uncover the influence of the mathematics research experience on their concept of proof. We present data on the criteria participants used to evaluate proofs, their conception of proof, and how the mathematics research experience changed their conception of proof.

Keywords: Reasoning and Proof, Preservice Teacher Education, Teacher Knowledge, High School Education

Introduction

Since proof is critical to mathematical activity, both educators and mathematicians agree that it should be part of the mathematics curriculum for all students. Stylianides et al. (2017) pointed out that even though proof is central to the learning of mathematics, it "has a marginal place in ordinary mathematics classrooms" (p. 237). The National Council of Teachers of Mathematics (NCTM,1989, 2000) has long recommended that students at all grade levels recognize reasoning and proof as a fundamental aspect of mathematics. Unfortunately, this has been unevenly implemented in topics outside of Geometry (Thompson et al., 2012).

Historically, proof has been part of the high school geometry curriculum in the United States. According to Harel and Sowder (1998) it was only with the *New Math* that proofs were included in secondary algebra courses. But then, "the death of the 'New Math' almost put an end to algebra proofs in school mathematics" (p. 234). Historical precedent, textbook design, and a narrow interpretation of the standards continue to influence how proof is taught and learned in schools, particularly in secondary schools. Thus, most students in American secondary classrooms experience proof first, and sometimes only, in the context of a Geometry course (Zaslavsky et al., 2012). Even then, proof tends to be taught as a separate topic, not as an integral component of how we do mathematics (Knuth, 2002b).

CadwalladerOlsker (2011) argued that the way in which mathematical proof is introduced in high school is implicitly focused on the formal act of proving at the expense of the construction of knowledge. Furthermore, teachers' experiences in college might reinforce this formalistic view of proof (Selden, 2012). Reid and Knipping (2010) summarized studies that examined teachers' understanding of proof. They noted,

> […] there seems to be not much difference when compared to students' understanding of proof. If this is the case then the level of students' understanding might be best improved by addressing teachers' understanding of proof itself, rather than exposing them to new methods of teaching about proofs and proving. (p. 71)

In this study, we examined the views of proof held by teachers who participated in a summer mathematics Research Experience for Undergraduates (REU). Although the purpose of the REU program was not to modify participants' views of proof, we conjectured that the experience of doing mathematics research would have some effect in this direction. Teacher's beliefs about proof are relevant to their practice. As Stylianides and Stylianides (2022) stated, "teachers' knowledge and beliefs about proof shape their readiness, willingness, and capacity to support students' engagement with proof" (p. 1). They further argued that our field lacks productive ways to introduce students and prospective teachers to a notion of proof that would help them see proof as relevant and important. Our research provides a potential example of engaging teachers with proof. With this in mind, we addressed the following research questions: 1) What criteria did REU participants use to evaluate proof? 2) What are the REU Participants' conceptions about the role of proof? 3) How did the REU program influence their conception of proof?

Literature

Researchers have found it useful to make distinctions between the different roles of proof in the classroom. For instance, Hanna (1990) distinguished between proofs that *prove* and proofs that *explain.* She stated that, "in the classroom the key role of proof is the promotion of mathematical understanding" (p. 5). Similarly, Schoenfeld (2009) argued that "Proofs are hardly 'mere' confirmation, verifying one's intuitions. For the mathematician, proof is a way to figure out how things work" (p. xiv). Looking at the function of proof in the work of mathematicians, de Villiers (1999) enumerated various roles of proof in mathematics: to verify that a statement is true, to explain why a statement is true, to communicate mathematical knowledge, to discover or create new mathematics, and to systematize statements into an axiomatic system.

Nevertheless, many teachers hold a narrow view of proof. Ko (2010), summarizing studies on teachers' views of proof, found that teachers,

> […] do understand that proof serves as a means of verifying the truth of a statement. Only a few teachers stated that proof serves the functions of communicating mathematics, helping students make discoveries, and systemizing results, and none of the teachers mentioned proof as a means of providing intellectual challenge. (p. 1115)

Moreover, when evaluating proofs, teachers and students seem swayed by the appearance of a proof rather than the logic of it. When teachers were asked to judge student proofs, about half the teachers rejected correct proofs written as verbal justifications (Tabach et al., 2010). In addition, understanding is not always seen as a goal of a proof. Stylianou et al. (2015) found that college students selected proofs written as deductive arguments as best, even though they believed they were not helpful in providing understanding.

In a study aimed at examining secondary mathematics teachers' conceptions of proof, Knuth (2002a, 2002b) interviewed 16 in-service mathematics teachers. Knuth investigated (a) teachers' conceptions about the role of proof, (b) what constitutes proof for teachers, and (c) what teachers found convincing. Following de Villiers (1999), Knuth classified these teachers' views of proof as a means of verifying, explaining, communicating, creating, and systematizing knowledge. In addition, Knuth found two additional roles of proof in the secondary classrooms: developing thinking and displaying thinking. Most teachers identified developing thinking skills as a primary role of proof, but none identified promoting understanding as a role. Knuth also asked teachers about the centrality that proof should have in the secondary mathematics curriculum. Although their views were diverse, most teachers thought proof was more appropriate for advanced mathematics courses. Knuth reported that teachers with a more formal interpretation of proof were more likely to limit student exposure to proof. Most teachers stated that proof should be addressed in geometry courses (Knuth, 2002b).

The teachers in Knuth's study primarily used four criteria to determine whether an argument constituted proof: valid methods, mathematically sound, sufficient detail, and knowledge dependence. In addition, features of the argument, familiarity with the argument, or the method of proof used, rather than the mathematical substance of the argument, often determined whether an argument was convincing for teachers. This is consistent with other studies. Harel and Sowder (1998) found that college students focus their attention on the format of proof, not on the content. Stylianou et al. (2015) found that undergraduate students' proof choices were strongly influenced by surface characteristics.

It is also important to point out that what teachers identify as proof and what they find convincing is not always the same. In Knuth's (2002a) study, "teachers seemed to reach a stronger level of conviction regarding the truth of a proof's conclusion by testing it with empirical evidence" (p. 410). Lesseig et al. (2019) found that preservice teachers used different criteria to evaluate arguments than when they decided what proofs they would present to students. Ko (2010) argued that teachers should be provided more opportunities to engage in proof. Moreover,

> Since mathematics teachers' conceptions of mathematical proof influence not only the experiences they provide for their students but also the expectations they hold for their students in learning proof (Knuth & Elliott, 1997), having a robust understanding of proof is important for teachers. (Ko, 2010, p. 1124).

Methodology

The work of Knuth (2002a, 2002b) served as a foundation for our study. The primary distinction was that the four secondary mathematics teachers and 11 pre-service teachers in our study were participants in a mathematics Research Experience for Undergraduates (REU). The eight-week REU program explored research topics in graph theory that allowed participants to explore research questions, generate examples, form generalizations and conjectures, and to provide careful justification. The different conceptions of proof were not directly addressed during the program, but rather, proof was experienced as an integral part of the mathematics research process. An education component was implemented each week to help participants translate their mathematics research experience to their classrooms. According to the Conference Board of the Mathematical Sciences (2012), "Teachers who have engaged in a research-like experience for a sustained period of time frequently report that it greatly affects what they teach, how they teach, what they deem important, and even the ability to make sense of standard mathematics courses" (p. 65).

The first data source used in this study was a survey instrument given at the beginning and end of the REU that included the following prompts: "What is proof? What is the role of proof in Mathematics? and What is the role of proof in the classroom?" The post-survey included the additional prompt, “Describe any changes since the beginning of the program and why your thinking has changed.” The second data source was a proof instrument given during the fourth week of the REU that asked participants to provide the strengths and weaknesses of a series of proofs. It also included the question, “What other roles does proof play in the high school classroom?” Four proofs from this instrument (paragraph, two-column, algebraic, and visual) were analyzed. The paragraph and two-column proof were correct geometric proofs of the proposition that complements of congruent angles are congruent. Both were based on the same argument but presented differently. The other two were informal arguments (one algebraic and one visual) showing that the sum of the first n consecutive positive integers equals $n(n-1)/2$.

To answer the first research question (What criteria did REU participants use to evaluate proof?) each critique was coded using the criteria outlined by Knuth (2002a) —valid methods, mathematically sound, detail, and knowledge dependent. We used the constant comparative method (Glaser & Strauss, 1967) to refine and adjust our definitions to account for statements that could not be coded using Knuth’s original criteria. This process continued until all disagreements were resolved. Once each critique was coded in terms of the criteria used, we determined whether the criterion was used to identify a strength or weakness and then further analyzed the data for patterns. A similar approach was used to answer the second research question (What are the REU Participants’ conceptions about the role of proof?). For this question, the survey prompts were initially coded using the seven roles identified by Knuth (2002a): *to verify that a statement is true, to explain why a statement is true, to communicate mathematical knowledge, to discover or create new mathematics, to systematize statements into an axiomatic system, to develop logical thinking skills, and displaying thinking.*

To answer the final research question (How did the REU program influence their conception of proof?), we identified statements in their initial and final responses where they discussed the nature of proof and compared those responses. In addition, we used open coding to generate codes for their responses to the prompt, “Describe any changes since the beginning of the program and why your thinking has changed.” We then used those codes to look for themes across the participants.

Findings

The findings of this study will be presented according to the three research questions addressed. Each section provides a perspective on the conception of proof that teachers with mathematics research experience hold and whether these experiences have the potential to change teachers’ conceptions of proof.

Criteria Used to Evaluate Proof

The teachers in Knuth’s study primarily used four criteria to determine whether an argument constituted proof: valid methods, mathematically sound, sufficient detail, and knowledge dependent. We found all these criteria in our analysis, but to different degrees. The knowledge-dependent criterion was primarily used to evaluate a proof by induction that was not included in the data for this paper, so it was removed. In addition, detail was a common theme in our participants’ critiques, but often in connection with other categories such as mathematical soundness or communication. Hence, detail was used as a subcode. For instance, if a teacher stated that a proof needed more detail in the mathematical argument, it would have been coded mathematically sound with the detail subcode. It would have also been identified as a weakness. The other criterion that we found (communication, understanding why, concreteness, and generality) were also found in the work of Knuth, but are included here because they were prominent in the critiques provided by our teachers. Table 1 provides the main criteria used by our teachers to evaluate proof and the associated definitions.

Table 1: Criteria Used to Evaluate Proof

Criteria	**Definition**
Methods	"The focus of the teachers who applied this criterion was primarily on the method (or perhaps the form) used in producing an argument rather than on the reasoning behind it" (Knuth, 2002a, p. 395).
Mathematically Sound	"These teachers focused explicitly on the validity of the reasoning presented in an argument" (Knuth, 2002a, p. 396)
Communication	The teacher focused on communicating mathematical ideas to the audience.
Understanding Why	The teacher's focus went beyond just proving a statement is true to focusing on why it is true or to student conceptual understanding
Concreteness	The teacher commented on the inclusion of a concrete feature, visual reference, or specific example that helped the reader.
Generality	The teacher commented on "arguments that established the truth of a statement for all relevant cases" (Knuth, 2002a, p. 399).

Table 2 provides the criteria used by the participants to evaluate the four proofs (visual, algebraic, paragraph, two-column) analyzed for this paper. The 2 in the upper left-hand corner of the table represents two participants who made a positive statement about methods when critiquing the visual proof.

Table 2: Distribution of Criteria Used

	Visual		Algebraic		Paragraph		Two-Column		Total
	Pos	Neg	Pos	Neg	Pos	Neg	Pos	Neg	
Methods	2	4	8	1	2	6	6	3	32
Mathematically Sound	1	2	2	4	1	6	4	3	23
Communication	1	1	1	7	4	5	7	2	28
Understanding Why	1		1	2	1	2	1	1	9
Concreteness	10								10
Generality		4	1		1				6

Note, the values represent the number of teachers who used each criterion for a particular proof. There were teachers who used a criterion to make both a positive and negative statement about a proof, the teacher in this case would have been included under both the positive and negative. In addition, data for the pre-service and in-service teachers were merged due to the small number of in-service teachers and our focus on other patterns within the data.

In terms of the criterion used, we found evidence that participants were often focused on the formal aspects of proving, which is consistent with the findings of CadwalladerOlsker (2011).

There were 32 instances where participants used the method of proof as a criterion, 16 of these specifically referenced format. In contrast, there were nine instances where understanding why was used as a criterion. Communication (28) and mathematical soundness (23) were also used often. Although there was a strong emphasis on format, the teachers in this study attended to students' abilities to communicate and understand through proof. This contrasts the finding of Ko (2010) that "few teachers stated that proof serves the function of communicating mathematics" (p. 1115) and Knuth (2002b) who found that none of the teachers in his study identified understanding as a role. We do not know whether this difference is a result of their experience with mathematics research, but this possibility deserves the attention of further research.

Knuth (2002b) found that features such as familiarity and method, rather than the substance of the argument, often determined whether an argument was convincing. This finding was also observed in our data. Consider the data from the two-column and paragraph proof, which were based on the same argument, but presented differently. Participants identified the paragraph proof as having the most weaknesses. Six participants made negative comments about the methods of the proof with one student stating that the proof "lacked structure and a more formal foundation." One participant stated that it left "some logical steps to the reader." The participants were divided on whether communication was a strength or weakness for the paragraph proof. Four felt that it was a strength given that the "vocabulary was very simple" and that it would not confuse a reader who was not strong in mathematics. On the other hand, five felt that it was a weakness because it did not "use mathematical language."

Contrast this with the critique of the two-column proof, which had more positive comments. Seven participants mentioned communication as a strength, with one stating that it made "every step clear to the reader." Six participants mentioned methods as a strength of the proof, with five of these referencing the formalism of the proof. "Utilizing the given information to develop a clear outline" and "Laying out each statement clearly with reasoning on the side" were among the features of the proof that were considered positive.

Method, communication, and mathematical soundness were more often considered a strength of the two-column proof and a weakness of the paragraph proof, even though both proofs contained similar information. This supports the claim that features such as familiarity and method, rather than substance, determine whether an argument is convincing. These data suggest an interplay between participants' conceptions of proof, their evaluations of proof, and their prior experiences with proof. It may not be enough to simply change teachers' conceptions of proof or the criteria they use to evaluate proof, without also addressing deeply engrained beliefs and traditions regarding proof.

The criteria used to evaluate a proof differs depending upon the type of proof. For instance, observe the data for the visual proof. Ten of the 12 participants who critiqued the visual proof commented that the concreteness was a strength, with one commenting that using specific examples "made the logic clearer." Concreteness was not used as a criterion for the other proofs. When discussing weaknesses of the proof, participants were concerned about the generality of the proof (four participants). The participants still commented on the communication, methods, and soundness of the visual proof, but their comments also included the additional criteria of concreteness and generality, which is quite different from the distribution for the other proofs. The variety observed in our data not only illustrates that teachers evaluate proofs using different criteria that may reflect different conceptions of proof, but that proofs differ in terms of the characteristics and potential roles they project.

Role of Proof

At the beginning of the research experience, 14 participants gave initial responses to the prompt, “What is the role of proof in Mathematics?” We found evidence of all five of Knuth’s (2002b) original roles of proof with all 14 participants suggesting that one of the roles was to verify that a statement is true. Five participants cited communication as a role of proof, and only two participants saw the role of proof as explaining why a statement is true.

Interestingly, in their initial responses to “What is the role of proof in the classroom?”, more participants (five) said that explaining why a statement was true was important. Knuth (2002b) noted that the teachers’ comments in his study suggested that they were talking about understanding the steps of the proof, rather than the underlying concepts. In our data, two of the five, both pre-service teachers, suggested that proofs promote understanding. The other three emphasized that explanation was helpful to “show students” why things work and are true.

All fifteen participants responded to the prompt about the role of proof in the classroom at the conclusion of the REU. We again found evidence of all five of Knuth’s (2002b) original codes and one of his additional codes related to teaching (developing logical thinking skills, Knuth, 2002a). Similar to the initial responses, the two roles of proof cited most often were to verify a statement was true (10 participants) and to explain why a statement is true (seven participants).

There were two key differences in the statements coded as proof as explanation in the pre- and post-survey data. First, more participants emphasized understanding of mathematics in the post-survey. Second, while some participants initially suggested the importance of “showing students” why, their focus on the post-survey shifted away from a teacher-focused model to discussing how proofs can help students deepen their own understanding of mathematics.

The finding of teachers citing proof as verification was consistent with Knuth’s (2002b) findings. However, the role of proof as explanation was mostly absent in responses from teachers in Knuth’s (2002b) study, but emphasized by many participants in our study, especially after the completion of the REU. In addition, we found two roles of proof cited by our participants that were not present in Knuth’s (2002a, 2002b) studies: proof to foster abstract thought and proof as a means of giving students mathematical authority. Two participants discussed the role of giving students mathematical authority, with one participant stating, “it (proof) allows more mathematical independence because you are able to determine for yourself something is true.”

Impact of the Summer Mathematics Research Program

At the beginning and end of the REU, participants were asked to respond to the prompt, “What is proof?” In their initial responses, the participants viewed proof to be a static end-product. They made statements like, “proof is a way of definitely stating some idea to be true,” “proof is the formalization of a process,” and “proof is a demonstration of a mathematical theory.” In their final responses, the participants seemed to view proof as more of a process rather than an end-product. They emphasized that proof was a way of reasoning and communicating. One participant said, “rather than memorizing properties or theorems, proofs should be a natural way to describe a process.” Another participant stated, “a proof is the reasoning we use for mathematics.”

On the final prompt, participants were asked how their thinking had changed since the beginning of the REU. Two themes arose that gave insight into these shifts in views of proof. First, several participants referred to their work on generalization during the REU; generalization was a critical aspect of the process of doing mathematics research. One participant stated that their research helped them realize that “not all proofs are about generalization, but all generalizations necessitate proof to explain why they are correct." This quote suggested a shift to viewing proof as part of the larger process of doing mathematics. Schoenfeld (2009) stated that, "for the mathematician, proof is a way to figure out how things work" (p. xiv). The participants started to view proof as a useful tool in their work of developing generalizations.

Second, several participants commented at the beginning of the REU that they preferred formal proofs due to their prior mathematical experiences. At the conclusion of the REU they saw the value of informal proofs as they can often "best represent student thinking and can be less confusing." Ko (2010) argued that instruction in proof is often influenced by the teacher's conception of proof. In this case, one might conjecture that as our participants' views of proof started to shift during the REU, their view of proof instruction also changed.

Discussion

Our findings are consistent with previous studies on teachers' and undergraduate students' conceptions of proof. Specifically, we found that many participants emphasized the form or appearance of a proof over its substance and prioritized verification among the roles of proof. Knuth (2002b) suggested that one way to help give teachers a robust understanding of proof was for them, "*as students*, to experience proof as a meaningful tool for studying and learning mathematics" (p. 403). The evidence reported in this paper suggests that the REU may have contributed to qualitative changes in teachers' perception of proof and the role of proof in the mathematics classroom. The REU seems to have helped participants understand that students can use proof for themselves rather than simply exposing students to proof through direct instruction. Indeed, participants reported seeing proof as a process used to explain and promote understanding. This new perspective on the role of proof may explain how participants viewed the role of informal proof and the value of using them with high school students. Nevertheless, participants still favored proofs that were more formal when evaluating them. This conflict is not surprising given the observations of other researchers (e.g., Tabach et al., 2010).

Our study was a small, self-selected sample of teachers that cannot be easily generalized. Nevertheless, we believe we have documented that authentic mathematical experiences for teachers can shift their understanding and conceptions of proof. Research evidence (Stylianides & Ball, 2008; Ko, 2010) suggests that these shifts in conceptions will improve the instruction that their future students' experience with proofs and proving.

References

CadwalladerOlsker, T. (2011). What do we mean by mathematical proof? Journal of Humanistic Mathematics, 1(1), 33–60. https://doi.org/10.5642/jhummath.201101.04

Conference Board of Mathematical Sciences [CBMS]. (2012). The Mathematical Education of Teachers II. Providence, RI: American Mathematical Society.

De Villiers, M. (1999). The role and function of proof with Sketchpad. Rethinking Proof with Sketchpad, 3–10.

Glaser, B., & Strauss, A. (1967). The discovery of grounded theory: Strategies for qualitative research. Sociology Press. https://doi.org/10.1097/00006199-196807000-00014

Hanna, G. (1990). Some pedagogical aspects of proof. Interchange, 21(1), 6–13. https://doi.org/10.1007/BF01809605

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. American Mathematical Society, 7, 234–283. https://doi.org/10.1090/cbmath/007/07

Knuth, E. J. (2002a, March). Teachers' conceptions of proof in the context of secondary school mathematics. Journal of Mathematics Teacher Education, 5, 61–88.

Knuth, E. J. (2002b, November 1). Secondary school mathematics teachers' conceptions of proof. Journal for Research in Mathematics Education, 33(5), 379–405. https://doi.org/10.2307/4149959

Ko, Y. Y. (2010). Mathematics teachers' conceptions of proof: Implications for educational research. International Journal of Science and Mathematics Education, 8, 1109–1129. https://doi.org/10.1007/s10763-010-9235-2
Lesseig, K., Hine, G., Na, G. S., & Boardman, K. (2019). Perceptions on proof and the teaching of proof: a comparison across preservice secondary teachers in Australia, USA and Korea. Mathematics Education Research Journal, 31, 393–418. https://doi.org/10.1007/s13394-019-00260-7
National Council of Teachers of Mathematics (1989). Curriculum and Evaluation Standards for School Mathematics. Reston, VA. NCTM
National Council of Teachers of Mathematics. (2000). Principles and standards for school mathematics. Reston, VA. NCTM
Reid, D. A., & Knipping, C. (2010). Proof in mathematics education: Research, learning and teaching. Brill. https://doi.org/10.1163/9789460912467
Schoenfeld, A. H. (2009). Series editor's foreword: The soul of mathematics. In D. A. Stylianou, M. L. Blanton, & E. J. Knuth (Eds.), Teaching and Learning Proof Across the Grades: A K-16 Perspective (xii-xvi). Routledge. https://doi.org/10.4324/9780203882009
Selden, A. (2012). Transitions and proof and proving at tertiary level. In G. Hanna & M. de Villiers (Eds.), Proof and proving in mathematics education (pp. 391–420). The Netherlands: Springer. https://doi.org/10.1007/978-94-007-2129-6
Stylianides, A. J., & Ball, D. L. (2008). Understanding and describing mathematical knowledge for teaching: knowledge about proof for engaging students in the activity of proving. Journal of Mathematics Teacher Education, 11(4), 307–332. https://doi.org/10.1007/s10857-008-9077-9
Stylianides, A. J., & Stylianides, G. J. (2022). Introducing students and prospective teachers to the notion of proof in mathematics. The Journal of Mathematical Behavior, 66, 100957. https://doi.org/10.1016/j.jmathb.2022.100957
Stylianides, G., Stylianides, A., & Weber, K. (2017). Research on the teaching and learning of proof: Taking stock and moving forward. In J. Cai (Ed.), Compendium for research in mathematics education (pp. 237–266). Reston, VA: National Council of Teachers of Mathematics.
Stylianou, D. A., Blanton, M. L., & Rotou, O. (2015). Undergraduate students' understanding of proof: Relationships between proof conceptions, beliefs, and classroom experiences with learning proof. International Journal of Research in Undergraduate Mathematics Education, 1(1), 91-134. https://doi.org/10.1007/s40753-015-0003-0
Tabach, M., Barkai, R., Tsamir, P., Tirosh, D., Dreyfus, T., & Levenson, E. (2010). Verbal justification—is it a proof? Secondary school teachers' perceptions. International Journal of Science and Mathematics Education, 8(6), 1071–1090. https://doi.org/10.1007/s10763-010-9230-7
Thompson, D. R., Senk, S. L., & Johnson, G. J. (2012). Opportunities to learn reasoning and proof in high school mathematics textbooks. Journal for Research in Mathematics Education, 43(3), 253–295. https://doi.org/10.5951/jresematheduc.43.3.0253
Zaslavsky, O., Nickerson, S. D., Stylianides, A. J., Kidron, I., & Winicki-Landman, G. (2012). The need for proof and proving: Mathematical and pedagogical perspectives. In G. Hanna & M. de Villiers (Eds.), Proof and proving in mathematics education (pp. 215–229). The Netherlands: Springer. https://doi.org/10.1007/978-94-007-2129-6