THE ROLE OF BELIEFS, VISUALIZATION AND TECHNOLOGY IN TEACHING AND LEARNING PROOF: THE CASE OF SKYLAR

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Bramlett and Drake (2013) suggest that the ability of teachers to teach proof is crucial for students to learn and develop formal and informal proofs. Teachers need to be involved in the process of proving and have a firm understanding of the critical role of proofs in order to effectively engage their students in proving activities. It is unrealistic to expect K-12 teachers to educate students on proof if they themselves are not given opportunities to engage deeply in the process of proving and understand its significance (Bramlett & Drake, 2013). According to McClain (2010), it is important for teachers to help students understand proof and engage in proving tasks. However, there has been little research on how to teach proof in secondary school mathematics. Steele (2012) suggests that teachers should have the knowledge and skills to identify whether mathematical arguments constitute proofs or not and determine what counts as proof across different representations. Pedagogical beliefs toward proof may be influenced by teachers' beliefs about the nature and role of proof in mathematics, its role in school mathematics, and their beliefs about themselves as mathematical thinkers. In this case study, the researchers leveraged a qualitative method – case study – to study a pre-service teachers' belief about the process of proof and the role of visualization and technology in teaching proof.

Keywords: Reasoning and Proof, Calculus, Preservice Teacher Education, Technology

Introduction

This study aims to investigate the ways in which the integration of dynamic geometry software (DGS) and visual representations can enhance the understanding of proof in calculus concepts among pre-service secondary mathematics teachers. The study focuses on the teaching and learning of proof by using technology and aims to contribute to the field of mathematics education. The research question guiding the study focuses on the ways that dynamic technology integration and visual representations can support teachers' experiences and beliefs regarding the process of proving calculus concepts. The study is inspired by the works of Raman (2003), Stylianides (2009), Abbaspour (2022), and the National Council of Teachers of Mathematics (NCTM) *Catalyzing Change in High School Mathematics* (2018) and aims to investigate the effect of visual representations connected to key ideas through technology on teachers' understanding of the proof process. (Raman, 2003; Stylianides, 2009; Abbaspour & Safi, 2022, NCTM, 2018).

Rationale of the Study

The National Council of Teachers of Mathematics (NCTM) has consistently emphasized the importance of proof in mathematics education in its *Principles and Standards for School Mathematics* (2000) and *Catalyzing Change in Middle School Mathematics* (2020). However, the Association of Mathematics Teacher Educators (AMTE) Standards for Preparing Teachers of

Mathematics (SPTM) does not provide clear guidance on how to teach proof in school mathematics, and in fact only mentions proof six times. Although the document highlights the importance of argumentation, justification, and teachers' roles in helping students understand the limitations of notions, it does not provide specific standards related to proof. Generally, additional research is needed in ways to engage current and future teachers more intentionally (Safi, 2020) more effectively. There is a need for more research on this topic as the current standards, suggestions, and current practices may not be sufficient for preparing teachers to learn and teach proof (AMTE, 2017; NCTM, 2000, 2020). After considering the guidelines and practices related to teaching proof in mathematics, it becomes apparent that there is a need to improve the content of teacher preparation programs in this area. Since there is a lack of specific recommendations for preparing teachers to teach and learn proof, it is reasonable to conclude that further research is required to identify how to address this issue.

Methodology

This research study aims to investigate the impact of using DGS as an intervention to enhance the understanding and construction of proofs related to calculus concepts and theorems at the secondary school level (Abbaspour Tazehkand, 2022). Since this topic has not been explored before, the study employs a qualitative research approach (Creswell, 2014) using a multiple case study design (Merriam, 1998; Creswell, 2013). This approach allows the researcher to achieve an in-depth understanding of the subject matter by examining a variety of cases. To ensure that the study captures a range of perspectives, the maximum variation method is used to select participants (Creswell, 2013;Abassian, 2018). Multiple sources of information are collected from each participant to obtain a comprehensive understanding of each case.

The study planned to use various methods to collect data, including questionnaires, audio and video recording, artifact collection, and interviews. Participants provided consent and were asked to participate. Data collection consisted of one *pre*-interview and three *post*-interviews, as well as recorded videos of students working on assignments, student artifacts submitted online, and audio and video recordings of whole-class and group discussions. Three assignments were required for the study, and *post*-interviews were conducted after each assignment. The interviews were semi-structured, allowing for flexibility in question selection and modification based on student responses to class activities and assignments.

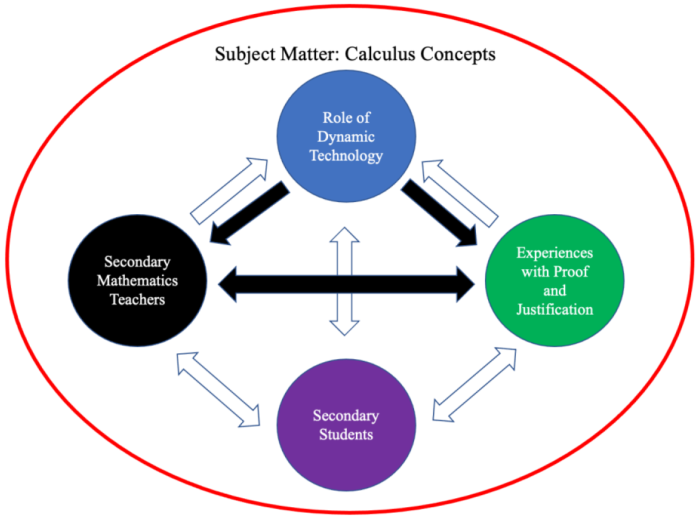

Figure 1: The Conceptual Framework

This study focuses on the interplay between preservice and in-service secondary mathematics teachers, dynamic interactive technology such as GeoGebra, and exploring calculus concepts and theorems. Teachers are the main focus of the study because achieving the goal of helping students make sense of calculus concepts requires teacher preparation programs and pedagogical support. The study fuses different aspects of mathematical teaching and learning to investigate secondary mathematics teachers' understanding of proof and reasoning, including justification and proof, teacher content knowledge, and technology and dynamic visual representations. Technology-enabled peer feedback activities, coupled with socialization opportunities, promote enhanced interaction and feedback generation (Cuocci et al., 2023). The study will analyze the interplay between technology and participants' experiences with proof and justification, as shown in Figure 1, to dig deeper into the dynamic setting of teaching proof. The study's definition of mathematical reasoning and proof is directly linked to mathematical activities, including identifying patterns, making conjectures, checking the validity of the conjectures, and proving arguments. The framework developed by Stylianides (2008) is suitable for the study's theoretical framework, which guides the selection and design of tasks and interventions. The study's focus on pre-service and in-service secondary mathematics teachers benefits from the pedagogical component of the framework. Additionally, the study examines the distinction between inductive and deductive reasoning and the relationship between the two. The framework provides a way to connect different modes of reasoning to the process of proving. Stylianides (2005) has analyzed the curriculum to identify opportunities for reasoning and proving tasks, categorizing them into two groups: those that begin with widely accepted truths and proven arguments using inductive reasoning, and those that begin by observing limited cases, identifying patterns, making conjectures, and ultimately generalizing mathematically.

The case of Skylar

Skylar is a recent graduate student who is new to teaching mathematics. Prior to this study, she had only supervised AP classes and had no formal teaching experience. Unlike some of her peers in the Mathematics Education master's program, Skylar did not have any familiarity with NCTM's Principles and Standards for School Mathematics. However, Skylar's mother is a mathematics teacher who specializes in Calculus and Algebra for college-bound seniors. Skylar

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1). University of Nevada, Reno.

shared that she has always enjoyed math and consistently earned high marks in advanced courses. When asked about her experience with algebra, Skylar expressed a fondness for the subject's logical structure and the satisfaction of finding solutions to algebraic problems.

During the interview, Skylar was asked about her understanding and beliefs regarding mathematical proofs. Initially, she defined proof as "showing how you get the answer." However, when asked about the difference between justification and proof, Skylar seemed uncertain and used the terms interchangeably, with no clear distinction between them in her explanation. Despite providing definitions, her descriptions of the two terms were very similar:

> *I think proof is more numbers, more number based and then the justification is okay well how did you use those numbers and why did you use those numbers, why did you use those steps to get the answer… and you're proving why you got those numbers.*

Skylar's view on mathematical proof was discussed in the second half of the interview, where she defined it as explaining the reasons for arriving at a solution to a numerical problem, but struggled to distinguish between proof and justification:

Researcher:	*You use the word "because".*
Skylar:	*Uhum.*
Researcher:	*So, we are talking about the whys?*
Skylar:	*Right.*
Researcher:	*But in the beginning you had a different definition. You told me that proof means that we need to know the hows… Has something changed in the past 20-25 minutes?*
Skylar:	*I think it is the same thing. The hows and the whys are closely related… This is how I did it, and this is why I did it. If you answer both questions, you are proving yourself.*

To understand Skylar's perspective on mathematical proofs, she was asked about the definition of proof and when it can be considered complete. Skylar demonstrated a strong understanding of proof construction and validity, and recognized the social element of evaluating proofs. She believed that all students can learn mathematical proofs and suggested that middle school is the ideal time to start learning. Skylar had encountered more proofs in Calculus, but as a K-12 student, she worked with proofs more in Geometry and had to memorize several of them. Although proving required a lot of memorization, she emphasized the importance of understanding the reasoning behind each proof. Skylar preferred symbolic representations over visual ones, but acknowledged the usefulness of visual representations in verifying statements or reasoning. She believed that students are often taught to rely more on numbers and variables than diagrams, and admitted that she was not well-prepared to help students work with visual representations:

> *I think it definitely needs to be practiced, but I don't know. I think it's hard, because I know for me like if I was told I have to teach visually I wouldn't necessarily even know how… I would definitely need help.*

Skylar was asked about generalization and its relationship to mathematical proofs. She displayed a solid grasp of what generalization means, describing it as the process of making a statement more widespread and applicable to a broader range of categories. However, Skylar's response to the question surprised the researcher as it contradicted the study's framework. Skylar believed that generalization is the opposite of proof, as it involves creating an overarching umbrella statement, while proofs require finding specific details. While Skylar did not recall engaging in any activities related to constructing mathematical conjectures, they stressed the importance of understanding proofs as a matter of justifying why one is undertaking a particular approach, with a proof being considered complete if all elements of a statement can be justified.

Skylar's work on the geometric series assignment showed a reliance on external resources such as course materials and the internet. She did not provide any visual representations and substituted numbers into formulas without explaining why they worked. However, she demonstrated an ability to construct conjectures by connecting the formula for the sum of the geometric series with the series given in the assignment. Skylar was less confident in her ability to construct a conjecture for the general case, but she identified the denominator of the first fraction in the assignment and wrote that the sum could be calculated by plugging in the denominator as x. Her reliance on memorization and external resources demonstrated her perception of how mathematics was taught in colleges and universities.

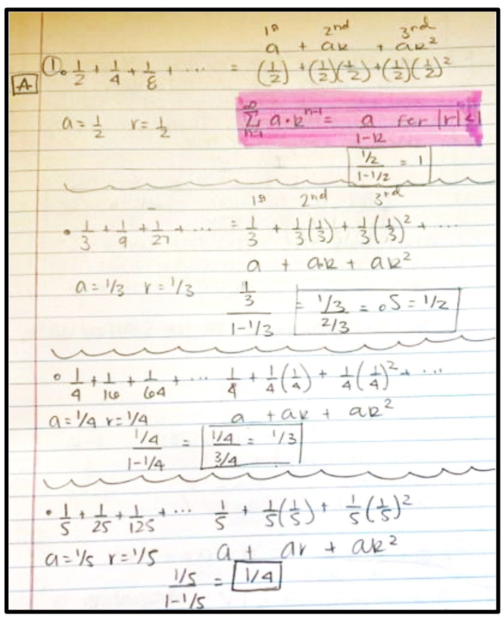

Figure 2: Skylar's response to the conjecture task

The students were given a DGS to interact with, connecting it to the general form of the geometric series to calculate the series sum. Skylar understood the DGS and interpreted the sliders, connecting it to the general form and deepening their understanding. During an interview, Skylar explained that they relied on memory for the conjecture problem and found the formula by going back to notes and looking online. Skylar initially struggled with visualizing the series but eventually understood after group discussions. They expressed a desire to teach geometric series visually to their future students.

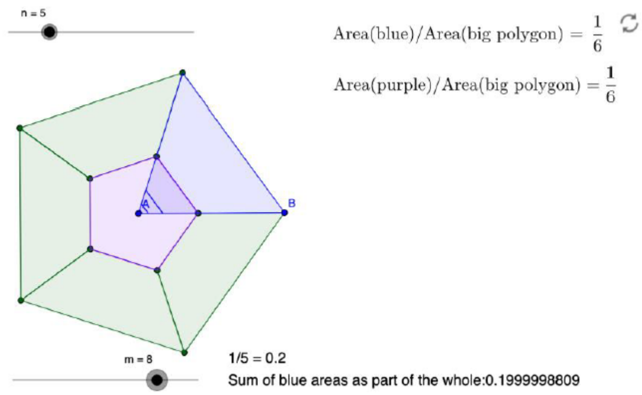

Figure 3: The conjecture GeoGebra task

Skylar found it difficult to visualize the geometric series and was unsure how visual representations were created. After discussing with her class group, she began to understand the visuals for other examples. During the interview, she expressed that she would use visual representations to teach the geometric series to her future students.

> *I think I would start with the visual and be like okay so what do we think… how do we solve that… You can either give them the equation at that point and be like Okay, this is the equation that most people use let's see how they compare, or you can try and work through it, I would need more practice with how to work through… I think it's important as a teacher to give both (numerical and visual). I know that's definitely something where I always am like let's just get to the math, but I think as a teacher, you always have to, try to do both mathematics, or the numerical values and then also a visual representation… I think, also with series and how numbers are formed, I think giving both options give every different type of learner a different way of finding that example or finding out how. It would be so I think that I would start with the visual. And then maybe teach the numerical side of it and then go back to the visual.*

During the investigation, the researcher asked Skylar why she didn't describe visual representations as mathematical. Skylar explained that she always thought of numbers and equations built by variables when working on a problem. She emphasized that this was because she was taught math that stressed the use of representations other than visuals. She noted that in school, students were required to prove everything symbolically and then might be given a visual representation to certify their work. While Skylar recognized the benefits of working with visual representations, her experiences as a student and the emphasis on symbols were hindering her ability to visualize mathematical ideas:

> *I think mathematics, in my mind is numerical. There's no other way… I was taught and there is no other, like I was taught numerically… when I think mathematics, I think numbers, paper and pencil.*

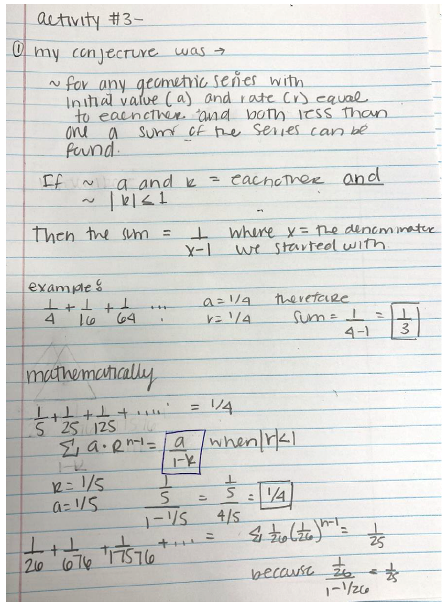

Figure 4: Skylar's response to the proof activity

Skylar's submitted work on proof did not focus on proving her conjecture, instead, she simply repeated what she had previously written in response to task 1 of assignment II. The work only demonstrated that the formula she presented worked for a specific numerical case. Skylar used two formulas, one of which she discovered by referring to her notes from Calculus II, and the other she developed by working with DGS and connecting it to the denominator of the first term of the geometric series. In attempting to prove the validity of the formula she remembered; Skylar provided an example that worked. The researcher noted that Skylar needed to be reminded about the connection between generalizations and proofs and connected her work to her original perception that proof was the opposite of generalization. If Skylar had accepted generalization as part of the process of proving a statement, she may have realized that demonstrating one case did not establish its validity in general form, and that it was impossible to numerically check if her conjecture and remembered formula worked for all natural numbers since there are infinitely many cases.Skylar was tasked with using a DGS to understand a dynamic visual representation of the geometric series and connect it to the proof. While Skylar initially found the DGS confusing, she eventually realized the role of each slider and connected it to the series. However, she tended to focus more on the symbolic representation at the bottom of the DGS than the visual representation, and her understanding of the DGS was only superficial. Despite this, she successfully answered the first two prompt questions for the task.

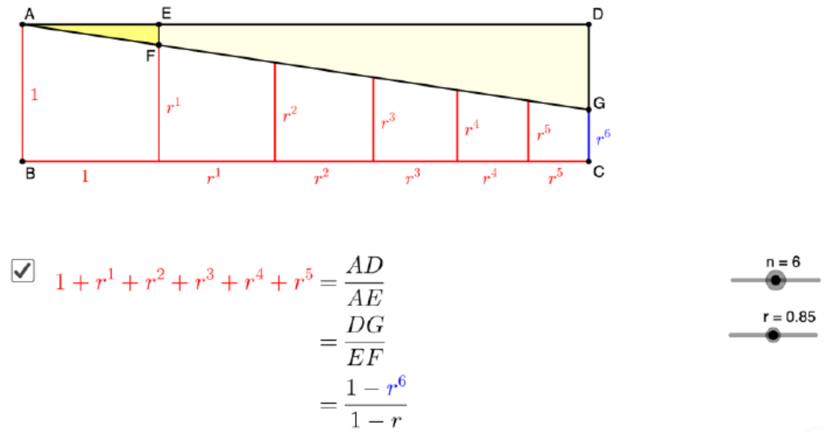

Figure 5: The proving GeoGebra task

She found it challenging to make sense of the DGS this time as it did not use area of shapes, which was different from previous DGS. Working with the DGS on her own did not help her find the similar triangles. However, after breaking down each part and studying them one by one in class discussions, she understood the proof and the similar triangles. Skylar mentioned that she was not a visual learner and did not trust visual proofs, but acknowledged the importance of visual representations and classroom discussions for learning:

> *They (teachers) are good at explaining how this applet works for what we're learning, and I think that's really cool, and I think it like, if you look at the bigger picture it's okay, these teachers are helping other teachers. A lot of your information doesn't come by working by yourself, it actually comes from learning and talking to other teachers as well. There are ways of proving we're talking about in an Algebra class, and you think algebra, you think numerical values… but you can take it more to a geometry standpoint. More of shapes and stuff like that, and I think that's the hard part, especially like I know in my curriculum it goes Algebra and then Geometry like Algebra one, Geometry, Algebra two... I know my… my Algebra one kids would never understand that.*

Skylar reported that the three assignments did not significantly alter her understanding of geometric series and proofs. She expressed frustration at what she perceived as repetitive problem-solving. However, she found exploring different visuals and connecting them to other representations a useful challenge. Skylar believed that using multiple representations helps in making sense of proofs and answering "why" questions. She thought technology could facilitate group discussions related to proving activities. Skylar's belief in using visual representations developed to the point where she expected her future students to make sense of DGS and visual representations as well as numerical and symbolic representations.

Conclusion

Skylar demonstrated a good understanding of mathematical proof as a means to support a claim through reasoning. Initially, Skylar equated proof with explanation and justification, but later she connected it to providing reasons when faced with "why" questions. However, she acknowledged the need to improve her understanding of different types of reasoning and their connection to mathematical proofs, such as inductive and deductive reasoning. Skylar's case illustrates how a student's experiences can shape their beliefs and perspectives towards mathematical concepts. Initially, Skylar valued numerical and symbolic representations more than visual representations, but through interventions, she came to recognize the importance of visual representations in understanding mathematical concepts. She even expressed interest in using visual representations in her future teaching but needed help in implementing them.Skylar struggled with recalling instances where she was required to construct conjectures as a K-12 student, which affected her confidence in coming up with conjectures. Nonetheless, she showed her ability to connect key ideas to visual representations, even though she had difficulty constructing the visuals herself. Skylar found dynamic geometry software (DGS) helpful in visualizing geometric series concepts, particularly the first two DGS. Although she found the third DGS less helpful, she still made sense of the proof of the sum of the geometric series by working with the DGS connected to the proof. Overall, Skylar believed that DGS played a positive role in visualizing dynamic concepts and enhancing understanding.

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