SOCIOMATHEMATICAL SCAFFOLDING AS STUDENTS ENGAGE IN DISCIPLINARY PRACTICES

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Instructors manage several tensions as they support students to engage in mathematical disciplinary practices such as defining, conjecturing, and proving. These tensions include honoring students' contributions while simultaneously apprenticing students to following mathematical norms. I present a case study of a teacher-researcher in a laboratory setting who was particularly skilled at this endeavor. I found that the teacher-researcher engaged in a pattern in which the teacher-researcher cycled between inquiring into the students' thinking about their draft of a definition, conjecture, or proof and then engaged in scaffolding, including scaffolding of mathematical norms. I exemplify this pattern with an episode of students writing a conjecture equivalent to the Archimedean Property that served as a warrant for one of their proofs. I close the paper discussing complexities of apprenticeship into the norms of the discipline.

Keywords: Instructional Activities and Practices, Reasoning and Proof, Undergraduate Education

One common goal of many undergraduate mathematics classes is to engage students in disciplinary practices that support them in creating definitions, conjectures, and proofs. This allows students to experience mathematics as a creative activity, rather than a static product handed down to them by their instructor. For example, a student from a class that used curriculum materials that were specifically designed for this goal commented on their experience:

> "You're not being handed a finished package and being told to just take it as is. You're building it. It's the difference between being given an assembled Lego set and building it yourself…"

Many scholars have discussed the tensions that instructors manage as they support students to 'build it themselves' (e.g., Ball, 1993; Melhuish et al., 2022; Rasmussen & Marrongelle, 2006; Speer & Wagner, 2009). At the heart of this complex work is honoring the students' contributions, or using the Lego analogy, valuing what the students do with the Legos. At the same time, instructors aim to support students to make progress on the mathematical goals that are set by the course. Using the Lego analogy, this means the "it" in "build it themselves" is a specific structure that the instructor has in mind such a Lego-car. Further, instructors are often obligated to support students to adopt norms from the mathematics discipline. Put in another way, instructors are compelled to support students to build their Lego-car with four wheels touching the ground, for instance, rather than fulfilling the wheels' purpose in some other inventive way.

There have been notable discussions around the teaching and learning of mathematics discipline's norms (Dawkins & Weber, 2017; Weber & Melhuish, 2022). Mathematical norms have been discussed as a matter of access that can operate to equip or exclude students in mathematics and mathematics classrooms (Dawkins & Vroom, in press). And thus, students who desire to interact with the mathematics discipline should be given meaningful experiences with them, such as those that support students to see norms as reasoned choices (Larsen et al., 2022; Vroom, 2022). Mathematical norms have also been discussed in relation to identity since the adoption of disciplinary norms may overshadow students' personal, cultural, or linguistic capacities (Goffney et al., 2018). And so, an instructor must "strike a balance between opportunities to reflect on oneself and others as part of the mathematics learning experience" (Gutiérrez, 2009, p. 5). The complexities of apprenticeship into the norms of the discipline warrant more research on the different ways this might be enacted. This research could better inform when this apprenticeship is (and is not) appropriate, how instructors can craft meaningful experiences with disciplinary norms, and how instructors can strike a balance between providing opportunities to reflect on oneself and others.

This paper makes some progress on understanding apprenticeship into mathematical norms. I present a case study of a teacher-researcher in a laboratory setting who was particularly skilled at building on students' initial drafts of definitions, conjectures, and proofs. He supported the students to refine the drafts so that they both captured the students' ideas and followed mathematical norms. I investigated what the teacher-researcher was doing to support the students in this way. In what follows, I discuss the constructs that I used to describe the teacher-researcher's moves, including sociomathematical scaffolding. Then, I present a cyclic pattern that I noticed in the teacher-researcher's moves and exemplify it with an episode from the laboratory experiment.

Theoretical Perspective

I take scaffolding to mean the process that enables students to carry out a task that they may have not otherwise been able to accomplish without assistance (Wood et al., 1976). In this report, I investigate the teacher-researcher's scaffolding of students' creation of definitions, conjectures, and proofs that are consistent with mathematical norms. Mathematical norms are expectations on practice accepted by the mathematics community to uphold mathematics community's shared orientations and goals that underlie shared activity (Dawkins & Weber, 2017). For instance, conventions in formal mathematical language are examples of norms that uphold the goal of effectively communicating and using definitions, theorems, and proofs. Norms can also include expectations such as excluding irrelevant statements from a proof in order to uphold that proofs should increase understanding of the mathematical theory involved (Dawkins & Weber, 2017).

Williams and Baxter (1996) distinguished between analytic and social scaffolding. *Analytic scaffolding* is "the scaffolding of mathematical ideas for students" (p. 24) and is a tool to support students to create mathematical knowledge. Whereas *social scaffolding* is "the scaffolding of norms for social behavior and expectations regarding discourse" (p. 24) and influences the "ritual of classroom life" (p. 36). Nathan and Knuth (2003) note that social scaffolding is not particular to mathematics and can apply to any instruction regardless of the subject. An example of social scaffolding is requesting students to take turns sharing their rough draft ideas with their groupmates.

Inspired by Cobb and Yackel's (1996) emergent perspective, I distinguish a third category, *sociomathematical scaffolding*, as the scaffolding of mathematical norms. For instance, an instructor might let students know that it is normative in mathematics for "there exists" and "there exists at least one" to be interchangeable as students interpret a mathematical statement with the phrase (Vroom, 2022). Such scaffolding was included in Nathan and Knuth's (2003) analytic scaffolding; however, this differentiation is crucial as students engage in creating definitions, conjectures, and proofs as such products are heavily influenced by mathematical norms. It is a widely shared goal that students should be supported to create products that are conceptually reasonable (potentially supported by analytic scaffolding). But there is critical debate about the extent to which and how students should be supported to adopt the norms of the mathematics discipline (potentially supported by sociomathematical scaffolding).

There are a variety of ways in which instructors can scaffold. I highlight two categories of ways because of their relevance to the data that I share in the Results section: (a) creating problematic situations and (b) pedagogical content tools. *Creating problematic situations* is connected to Harel's (2008, 2013) theory that students should have an intellectual need to learn what we intend to teach them, where an intellectual need refers to a problematic situation that motivates the construction of the piece of knowledge. An instructor who aims to create intellectual needs to refine students' definitions, conjectures, and proofs would focus efforts on shining light on an issue with the draft (i.e., creating a problematic situation).

Pedagogical content tools (PCTs) can be powerful tools in instructor scaffolding. A PCT is "a device, such as a graph, diagram, equation, or verbal statement, that a teacher intentionally uses to connect to student thinking while moving the mathematical agenda forward" (Rasmussen & Marrongelle, 2006, p. 389). Rasmussen and Marrongelle discuss two types of PCT's: transformational records and generative alternatives. *Transformational records* are "notations, diagrams, or other graphical representation that are initially used to record student thinking and that are later used by students to solve new problems" (p. 389). *Generative alternatives* are "alternate symbolic expressions or graphical representations that a teacher uses to foster particular social norms for explanation and that generate student justifications for the validity of these alternatives" (p. 389). For instance, the teacher might introduce a teacher-initiated alternative to the class with the intention of soliciting justifications, or a teacher might highlight a student-initiated alternative and then engage students in a discussion about merits of the alternative.

Methods

The data that I present in this report comes from a laboratory design experiment (Cobb & Gravemeijer, 2008). The primary design goal was to refine tasks that supported students in reinventing foundational concepts in real analysis using the instructional design theory of Realistic Mathematics Education (Gravemeijer, 1999). A secondary goal was to test the conjecture that the task sequence could be leveraged to support students in learning about proof-related activity such as defining, conjecturing, and proving (see Larsen et al., 2022 and Vroom, 2020 for details about the task design). The experiment was ten 1.5-hour long teaching sessions.

The design experiment involved two students who I refer to as Chloe and Gabe, a teacher-researcher who I refer to as Bryan, an observer (myself), and a camera operator. Chloe and Gabe typically earned high marks in their mathematics coursework and were recruited from the last two courses of the calculus sequence at their university. Bryan was particularly experienced at teaching in such a way that built on students' thinking. He had previously conducted many teaching experiments in this role (in both laboratory and classroom settings) to develop inquiry-oriented curricula and instructor support materials. The reason I selected the design experiment for further investigation was because Bryan seemed to play a key role in supporting students to refine their contributions (draft of definitions, conjectures, proofs) in such a way that both captured the students' ideas and followed disciplinary norms.

The data analysis for this study began with creating content logs of each teaching session. The logs were a detailed chronological account of the students' activity and discourse. The content logs were supplemented with pictures of the students' work and selected transcribed excerpts. After the end of the experiment, I reread the logs and identified episodes of the students engaging in writing definitions, conjectures, and proofs. I considered an episode to be the activity that occurred between the students' first and last draft of a particular definition, conjecture, or proof. I then selected three focal episodes that varied in activity type (defining, conjecturing, proving) to further analyze based on Bryan's seemingly crucial role. To do so, I identified instances in which Bryan inquired into the students' thinking, engaged in scaffolding, created problematic situations, or used a pedagogical content tool. I used these codes because they were either instructional moves that Bryan and I specifically talked about while collecting data (inquiring into the students' thinking, PCTs) or I perceived there to be relevant afterwards (scaffolding, creating problematic situations). As I analyzed the data, I noticed an emerging pattern in Bryan's moves. After I analyzed the focal episodes, I then returned to the remaining episodes to both observe whether the pattern appeared in the other episodes and add more nuanced understanding of this pattern. In what follows, I first describe this pattern and then I illustrate it with one of the focal episodes.

Results

Throughout the teaching sessions, Bryan's moves followed a cyclic pattern in which he repeatedly inquired about the students' intended meaning of their draft (of a definition, conjecture, or proof) and then followed with scaffolding to support the students in refining their draft. I illustrate the cyclic patten in the episode that follows in which Bryan supported the students to write and refine a conjecture equivalent to the Archimedean Property by repeatedly engaging in sociomathematical scaffolding.

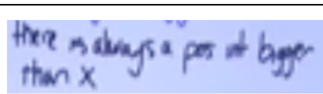
First Cycle

Prior to this episode, Chloe and Gabe engaged in writing a proof that the sequence $\{2^n\}$ tended to infinity. Their emerging proof let k be a real number, let $x = \log_2 k$, and then "let $m > x$ where m is a positive integer". The episode began with Bryan following up with the line "let $m > x$ where m is a positive integer" by asking why they could "pick a positive integer that is bigger than x". Chloe responded with the first draft of their conjecture (see Figure 1.)

"there is always a bigger number"

Figure 1. Draft 1.

Bryan began sociomathematical scaffolding, saying "Add a little more precision to this, what kind of number is a bigger number and what is it bigger than?" This question implicitly shared the mathematical norm that statements should be unambiguous. The question exposed a problematic situation in that Chloe's intended meaning was unclear because it did not state what type of number she was referring to, nor did it state what sort of number she was comparing it to. This led Chloe to refine the statement to Draft 2 (Figure 2).



"there is always a pos int bigger than x"

Figure 2. Draft 2.

Second Cycle

Next, Bryan inquired into Chloe's intended meaning of Draft 2 by asking clarifying questions such as: "And what is x?", "And what do we know about x?", "So, are you saying that there is always a positive integer that is bigger than that particular x, the one that is $\log_2 k$? Or is some more general principle of numbers that you're trying to say? So, is this x specific to our problem?". With this inquiry, Chloe revealed that she intended for the statement to be true for the particular x-value that they introduced in their proof ($x = \log_2 k$). She also explained that the statement was also true for an arbitrary x-value, explaining "but like it is applicable for any x".

The conversation continued with Bryan sociomathematical scaffolding by exposing a mathematical norm that the statement should justify the line in the proof:

Teacher-researcher: So, what kind of number should we say that it is in our statement of this property? If we said that 'there is always a positive integer that is bigger than x where x is an integer' would that be saying the same thing as saying 'where x is a real number'?

Chloe: No, but they are both true.

Teacher-researcher: And which are you using here?

Chloe: It doesn't matter.

Teacher-researcher: Well, let's test you on that. Write 'where x is an integer'.

(Chloe altered the statement as requested: "there is always a pos int bigger than x where x is an integer")

Teacher-researcher: Now if we were somehow able to prove that true, what you just wrote, would you be able to use that to explain why you can choose an m over here? (referring to "let $m > x$ where m is a positive integer" in the students' emerging proof)

Gabe: No.

Observer: Why not?

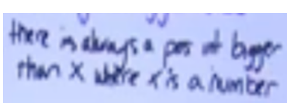
Gabe: Because this isn't necessarily going to be an integer (points to $x = log_2 k$).

Chloe: Yeah, true.

Gabe: And so it won't cover all the potential values.

Teacher-researcher: So, if you say that 'there is always a positive integer bigger than x where x is an integer' then that's not strong enough to use in this case.

(Chloe refined the statement to Draft 3, see Figure 3.)



"there is always a pos int bigger than x where x is a number"

Figure 3. Daft 3.

During the above exchange, Bryan leveraged a generative alternative by focusing the students' attention on two specification options for x: a real number or an integer. Chloe explained that both options were acceptable since both versions of the statement were true. Bryan then refocused the students to not only attend to the truth of the statement but also the need to explain the step in their proof. Using the alternative in relation to the line of the proof created a problematic situation: the version of the statement that specified x to be an integer was true but did not function as a warrant because “it won't cover all the potential values” of $x = \log_2 k$. This motivated Chloe to refine the statement by adding “where x is a number” to the drafted conjecture (Figure 3).

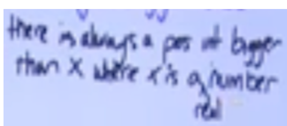
Third Cycle

Bryan continued the conversation by inquiring into the students’ thinking about Draft 3, asking “Do you mean real number? Or a…” Chole responded, “I don’t see why it matters what number it is because there is going to be a positive integer that is larger than it.” I interpret Chloe’s comment to mean that she didn’t see why there was a need to specify x to be real number, and that she thought her statement (that said ‘where x is a number’) captured her idea.

Bryan continued the discussion by sociomathematical scaffolding by sharing a mathematical norm. He explained:

> Well that’s why we say real numbers when we mean that. When we mean that it doesn’t matter what kind of number it is, we say real number. To do two things. To one to say that we mean it more than just naturals, or integers, or rationales, or irrationals, we mean it for all of them. That’s the set that we say when it doesn’t matter, we say it’s true for all real numbers. It also lets them know that we are not thinking of imaginary numbers.

With this comment, teacher-researcher shared that the mathematics community would use ‘real numbers’ instead of ‘numbers’ to communicate the students’ meaning. Chole responded with “ok” and then altered the draft by adding the word “real” before number. (See Figure 4.)



“there is always a pos int bigger than x where x is a real number”

Figure 4. Draft 4.

Fourth Cycle

Bryan then considered Draft 4 before he continued the conversation by sociomathematical scaffolding:

Teacher-researcher: Alright, so… [reads statement under breath]. Alright, so one thing that I want to do for practice if you go back to your definition we wrote that one (pointing to a definition that they previously wrote for a sequence converging to infinity) in terms of ‘for any real number there exists’, so it’s sort of like ‘for any… there exists’. But this one over here that Chloe just wrote doesn't quite... it seems like it is trying to say the same sort of thing but not using the same kind of language. So, can we write this with saying like ‘there exists’ and ‘for any’ or ‘for all’? I think that ‘for all’ is more common but ‘for any’ is also kosher.

(Chloe starts writing.)

Teacher-researcher: It's not that what you are saying is wrong, but if we kind of write them all in the same way then it is easier to see how we map them in proofs and stuff. And maybe if you use variables that would be cool too.

(Chloe and Gabe give teacher-researcher questioning look.)

Teacher-researcher: Like letters like you did before you had the $a_x's$ and the $k's$ (pointing to their previously constructed definition of sequence converging to infinity), things like that. So, you have an x for one of them but you didn't use a variable for your integer.

Chloe: Yeah.

Teacher-researcher: So maybe use a letter for both of them.

Chloe: Yeah.

(Crosstalk while Chloe writes)

(Chloe writes: for any real number x there exists a positive integer m that is larger than x)

Chloe: Happy now?

Teacher-researcher: Well I am always happy, I am also always happy to ask for more.

(Bryan and students laugh)

Teacher-researcher: So, is there a mathy way to say ‘is larger than’?

Gabe: Yeah. (Gabe finished refining the statement to Draft 5, see Figure 5.)

for any real numberx there there exists a pos int m Such that m>x	“for any real number x there there exists a pos int m such that $m > x$”

Figure 5. Draft 5.

In the previous segment, Bryan leveraged transformational records. First, Bryan continued to connect the students’ statements with a conventional version of the statement by suggesting that they use the phrases ‘there exists’ and ‘for any’ or ‘for all’. Then, Bryan suggested that they use symbols to encode ‘is larger than’. With these requests, Bryan suggested that such phrases are normative in mathematics discourse. These requests motivated the students to refine their draft to what’s in Figure 5. Bryan appeared to be satisfied with Draft 5 ending the refining activity.

Conclusion

In this report, I presented the cyclic pattern in which Bryan supported the students in refining their definitions, conjectures, and proofs. In the episode that I presented, Bryan cycled between inquiring into the students’ thinking about a draft of a conjecture and engaged in sociomathematical scaffolding. The episode that I offered exemplified that this is complex work. He drew on a variety of tools to support his scaffolding including creating problematic situations and using PCTs. I do not present this pattern as necessarily an idealistic way to support students to write and edit definitions, conjectures, and proofs. The appropriateness of this pattern is highly contextual and minimally depends on the students’ learning goals. Rather, I present the pattern to further contextualize the notion of sociomathematical scaffolding to promote critical discussions on the teaching of mathematical norms. I highlight serval points about Bryan’s sociomathematical scaffolding in what follows.

There were instances in which Bryan was able to create problematic situations to support his sociomathematical scaffolding (during the first and second cycle). Sociomathematical scaffolding seemed particularly productive when paired with creating problematic situations in the sense that such moves emphasize reasoning for such refinements. During the first cycle, this pairing revealed that Draft 1 was unclear what number was being reference and compared. During the second cycle, this pairing exposed that the version of the statement that specified x to be an integer was true but did not function as a warrant in the students' proof. Both of these instances focused on the need for a refinement that was connected to a mathematical norm. I suspect that sociomathematical scaffolding paired with problematic situations would promote student access in mathematics classrooms (the extent to which this instruction allows students to "play the game", Gutiérrez, 2009, p. 6). Future work could further explore this conjecture.

I also predict that the way in which sociomathematical scaffolding is enacted can promote or constrain the re-making of mathematical norms through negotiation with students and their contexts (the extent to which this instruction allows students to "change the game", Gutiérrez, 2009, p. 6). I conjecture that sociomathematical scaffolding paired with creating problematic situations would affirm students' identity in the mathematics classroom because creating problematic situations focuses on the need for a refinement, not the actual refinement itself. Thus, students' have the ability to negotiate how they want to (re-)make mathematical norms for their classroom. For instance, the second cycle highlighted the need for the statement to warrant a line in their proof. The students refined the draft so that it functioned as a warrant by stating that x was a number, which later it was clarified that the students intended "number" to mean how others might mean "real number". During the third cycle, Bryan choose to share the norm that mathematicians use "real number" when they reference the sort of numbers that the students were referring to. I argue that the appropriateness of this move depends on the students. If the students desired to abide by mathematical norms then this scaffolding seems appropriate. If the students did not have this desire, then it may be more advantageous to allow students to use "number" instead of "real number" to affirm students' own linguistic capacities in the mathematical setting.

I did not see evidence of the instructor engaging in social scaffolding as he supported students to refine their definitions, conjectures, proofs. This is likely because of the laboratory setting and the episodes that I analyzed were later in the sessions likely after social norms had been established. I did, however, find evidence of the teacher-research engaging in analytic scaffolding in other episodes that I analyzed. I suspect that the interplay between social, sociomathematical, and analytic scaffolding would be important in the whole class setting and future research could investigate the ways in which these interact.

This paper takes some first steps to document how the apprenticeship into mathematical norms can be enacted as students engage in defining, conjecturing, and proving. I hope that future work can build on it to further explore contexts for which this apprenticeship is (and is not) advantageous, the ways to create meaningful experiences with disciplinary norms, and how instructors can balance supporting students to play and change the game of mathematics through classroom mathematical norms.

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