FIRST YEAR UNIVERSITY STUDENTS WAYS OF REASONING AND ARGUMENTATION IN PROBLEM-SOLVING ACTIVITIES

FORMAS DE RAZONAR Y ARGUMENTAR EN ACTIVIDADES DE RESOLUCIÓN DE PROBLEMAS DE ESTUDIANTES DE PRIMER AÑO DE UNIVERSIDAD

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The aim of this study is to characterize ways of reasoning and arguing that first year university mathematics students exhibit in problem-solving activities from a course that emphasizes the importance of formulating conjectures and the search for different ways to support or validate them. The use of a Dynamic Geometry System in the representation of problems and in the formulation of conjectures or relationships that are important in the solution processes is highlighted. In this context, students have the opportunity to look for different ways to argue and support the relevance and validity of the conjectures. Results indicate that students extend their ways of reasoning so that allows them to move from empirical to formal arguments within problem solutions

Keywords: Undergraduate Education; Problem Solving; Reasoning and Proof; Technology

According to admission profiles for higher education, in Physical-Mathematical Sciences and Engineering, students are expected to have a solid knowledge of basic concepts from High School Calculus, Analytical Geometry and Algebra, as well as to be interested in problem solving. However, data from diagnostic exams for admission to the bachelor's degree in Mathematics of various institutions show that most students enter this area of study without an adequate understanding of basic mathematical concepts and with skills and strategies for problem solving focused on a basic level of reproduction. This shows that students will have significant difficulties during their integration into higher education.

This highlights the need to provide a mathematical education that encourages students to focus their interest and attention on the development of skills and strategies for the management of concepts, the resolution of mathematical problems and the elaboration of arguments that allow them to go beyond the reproduction of knowledge to build a robust and abstract mathematical thinking that enables them to formulate and solve different types of problems in any area of their academic, social and labor training.

With this in mind, it becomes important to continue researching on aspects and factors that have an influence on a successful or deficient integration at university level, mainly in the Mathematics area. Thus, in this research we sought to characterize, by means of a task focused on Problem Solving (PS) (Polya, 1965; Schoenfeld, 1985), which are the tools, skills and difficulties within the process of argumentation and mathematical reasoning that first-year undergraduate students in Mathematics show, therefore the following research question was posed How does the use of Dynamic Geometry System (DGS) within the PS promote the development of mathematical processes such as obtaining conjectures, arguing and validation, so this allows students to get in higher level mathematical activities?

Contextual framework

Research on the secondary-tertiary transition in Mathematics has considered different perspectives in order to explain and address the problems and stages that students go through during this transition period. In this regard, Leviatan (2008) states that students' difficulties are due to the fact that high school mathematics tends to focus on developing algorithmic skills for resolution of concrete and routine exercises, while at university, skills for abstraction and aspects of inquisitive questioning are required, while non-routine problem solving, and mathematical rigor are emphasized. On the other hand, Clark and Lovric (2010) define that transition from one level to another involves the process of a rite of passage divided into three phases: 1) Separation from the level, from the previous ways and routines of learning; 2) Liminality, a phase in which routines, beliefs and habits from high-school level still form part of students' attitudes within the new educational system not yet assimilated; and 3) Incorporation into the new environment. This transition implies a crisis that leads the interruption, modification, and distortion of previous routines, so this crisis is inevitable but necessary for students to develop advanced mathematical thinking and autonomy before their training. Considering this, it becomes feasible to investigate aspects that could smooth this process. Thus, Rach and Heinze (2016) identified 5 variables involved in academic success or failure during second-tertiary transition: 1) interest in mathematics, 2) self-concept as a mathematics learner, 3) previous achievements as a mathematics learner, 4) previous knowledge of mathematics, and 5) quality of learning strategies. For their part, Di Martino and Gregorio (2019) established five categories of causal attributes that lead to the difficulties presented during integration to university education: 1) Context factors; 2) Transition aspects; 3) Inadequate knowledge; 4) Inadequate way of thinking in/for mathematics; and 5) Comparison with peers. Considering these aspects allows us to generate alternatives to support the student in facing these problems.

As can be seen, problems that students will face during this transition have a multicausal nature. In this regard, Adelman (2006) suggests that students previously need examples of the activities performed during the first year of university education and the kind of future exams in order to have a better idea of what students are expected to do. Thus, it is important for students to have approaches to processes linked to argumentation and mathematical reasoning. For his part, Schoenfeld (2022) mentions that the educational challenge lies in creating robust learning environments that support students in developing not only the authentic knowledge and processes that underlie mathematics, but that promote the development of a sense of agency and authority to make sense of mathematical objects and practices within robust mathematical thinking.

Given the above, it becomes necessary to contribute to research on aspects, factors and practices that contribute to a more accessible and with greater opportunities to succeed during Mathematics education. The analysis presented in this paper is on the basis of, as mentioned before, the results of a task based on PR within a Geometry course whose methodology includes aspects related to the five dimensions to create powerful mathematics classrooms (Schoenfeld, 2014): 1) Mathematical content; 2) Cognitive demand; 3) Access to mathematical content; 4) Agency, authority and identity and 5) Use of assessments. Thus, first is given a general description of the teaching practices carried out in the course, followed by a descriptive analysis of the processes set in motion by the group of participating students. For the analysis of data obtained, we identified resources and heuristics (Schoenfeld, 1985), as well as conceptual and procedural tools (Melhuish, Vroom, et al., 2022) that students use and that bring them closer to the realization of authentic mathematical activities. For this purpose, we consider Authentic

Mathematical Proof Activity (AMPA) theoretical framework proposed by Melhuish, Vroom, et al. (2022), from the ten procedural tools expressed by the authors, we sought to identify: 1) refinement or analysis of a proof, a statement, or definition by focusing on the attainment of assumptions; 2) elaboration of formalizations, i.e., the process of translating informal ideas into formal or symbolic rhetorical forms; 3) elaboration of analogies, i.e., the process of importing proofs, statements or concepts across different domains adapting them to new schemas; 4) use of examples, a specific and concrete representation of a statement, concept or proof that represents a class of objects; and 5) elaboration of diagrams and visual representations of mathematical objects (statements, concepts or proofs) that capture structural properties.

Description of didactic methodology

Group G1 consisted of 8 undergraduate mathematics students who participated on a voluntary basis and who were taking a Modern Geometry I course from a public university.

Contents of the course were developed on the basis of problems or initial questions posed by the teacher, which students had to explore prior to the class. During the class, the teacher used the Dynamic Geometry System Geogebra (DGS) to explorate the proposed problems, as well as to simulate geometric straightedge-and-compass constructions and to verify initial conjectures, so that students were familiar with this technology and could use it to explore problems presented on their own.

The dynamics of the course sought to provide opportunities for students to have equal access to the contents developed, by means of course notes, suggested bibliography, interactive applets in Geogebra and the possibility of research via the internet. Students were encouraged to develop the ability to argue and not only the use of established formulas or algorithms; on the contrary, through the problems and questions posed, students were encouraged to generate conjectures and different ways to corroborate the validity of them, as well as the exchange of ideas in groups to meet the dimension of cognitive demand (Schoenfeld, 2014) and to generate both individual and collaborative commitment in the performance of the activities by students.

Thus, by the time the task was assigned, students had received training aimed at obtaining conjectures, arguing and exploration using Geogebra. In addition, by this time they had reviewed content related to triangles properties, triangles congruence and similarity criteria, inscribed angles and cyclic quadrilaterals properties.

Context of the Problem Solving Task Assignment

The group of students was given the following task: There is a square ABCD. If on the DA side you construct the midpoint E, then draw the segment BE and construct the perpendicular segment CF with F the perpendicular foot on BE. What kind of triangle do the points C, D and F form? Prove your conjecture in two different ways.

Students had the option of tackling the task individually or in pairs, as the latter modality of work prevailed in the dynamics of the course. Students took the problem home and had about three days for a first approach to it. Then, in a classroom class, space was provided for the group to present the conjectures obtained, the initial ideas for the demonstration of the conjecture and possible doubts or concerns. This class was part of the control elements of the RP (Schoenfeld, 1985), so the necessary feedback was given to the students.

To collect data, a logbook was requested to record the resolution processes, as well as the questions, ideas or actions that arose during the resolution of the task. For this purpose, the following elements were requested:

1. Description of the exploration, understanding of the problem and making a conjecture. For the analysis of data in this section, we sought to determine what means, instruments and processes students used or followed to explore and understand the problem, as well as to make a conjecture.
2. Description of the process of developing a plan or strategy for solving the problem. In this section we sought to identify whether students recognized the resources (concepts, mathematical content, evidence or previous results) they had or did not have to tackle the problem, so that this would lead them to determine a possible path to follow for the solution or to consider various sources of research.
3. Process of solving the problem. This section analyses arguments and processes that led to proving the conjecture obtained.
4. Problem extensions. This section analyses whether students pose new questions or problems to be solved based on what has already been solved.

From the analysis of these elements, the aim is to describe the type of reasoning and ways of acting that students put into practice in order to solve problems, as well as the difficulties they faced and the ways in which they overcame them.

Analysis of results

For the analysis of the results, it was considered the evaluation of three logbooks developed in pairs (E1, E2, E3) and one individually developed logbook (E4) of group G1. In this report, the tools and processes within the logbook developed by students are exemplified with short episodes. It should be noted that in the logs it is observed that the predominant way of working to understand and explore the problem was individual, as well as for the general writing of the log, while students worked collaboratively mainly to exchange and verify ideas during the planning of a strategy, in the process of solving it, and to obtain feedback from their peers. The results obtained in the team logbook sections are described below.

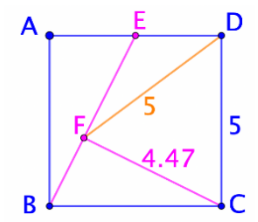
Analysis: Exploring and Understanding the Problem

Students in each team were very descriptive in terms of the acts they performed to understand the problem, and they were also open in expressing their way of acting and thinking about the processes during the RP, which allows us to identify, at least in a global way, their belief system in relation to this activity (Schoenfeld, 1985).

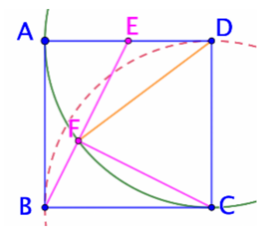
As can be seen in Table 1, all students used Geogebra to carry out the construction, and also used the software tools either to measure distances between points or the construction of circles to compare radii and thus compare lengths. The use of Geogebra helped students to represent and understand the problem, and thus to make a conjecture about the type of triangle generated in the construction.

Table 1. Responses from exploration and understanding phase

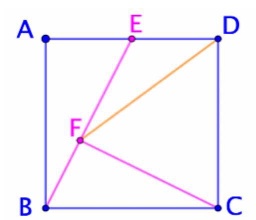
Problem representation	Processes followed by each team



E1. We first developed the construction that posed the problem with the help of the Geogebra plotter. When we made the model, we realised that the triangle CDF was isosceles, since its bisector, height and mediatrix of D coincided, which is characteristic of a triangle of this type, and we subsequently checked this by measuring the distances DF and DC with the help of the program.



E2. The first thing I did was to construct the figure in Geogebra to get a clearer idea of what was being drawn. Then when I looked at the triangle, at first glance it looked like an equilateral triangle, but after comparing the lengths of the sides, using circles, I realised that it was actually an isosceles triangle.



E3. First I traced [the figure] freehand, as I didn't get very far, I plotted the hypotheses in Geogebra and there I discovered that the triangle was isosceles.

E4. I decided to open Geogebra to build the construction step by step (I suspected it would be complicated). After building it I realized that the triangle FCD is isosceles. Now I wonder how to prove it, because I can't think how I know it is isosceles.

All the students conjectured that the triangle in question is isosceles. Thus we have that students implement the strategy of elaborating diagrams accompanied by the use of technology, and it is also observed that they have the necessary resources to elaborate the construction based on the structural characteristics of the mathematical objects. On the other hand, it is observed that students have acquired a certain degree of confidence both in making decisions regarding their actions as problem solvers and in the use of auxiliary tools or devices.

Analysis: Drawing up a Plan

In general, it can be observed that for the first demonstration of the conjecture, the students considered two possible ways, the first related to demonstrating that two sides of the triangle have the same length, the second, demonstrating that in the triangle there are two internal angles that measure the same, for which they mentioned that they could use congruence or similarity of triangles. One team highlighted as an important aspect the fact that both the angles formed by the perpendicular and those of the square are right angles, which allowed them to consider the CDEF quadrilateral and hence to consider the use of results relating to cyclic quadrilaterals. Below are fragments of what the students expressed in search of a first demonstration.

E1. At the beginning we came to the conclusion that we could find equal angles from the figures formed by the construction [...] and consider that there were congruent triangles.
E2. I felt that the best way to test this was to use triangle congruence.
E3. The fact that the sides [of the quadrilateral] measure the same and the angles are right angles is relevant, because in this way we see that it is a cyclic quadrilateral and so we

can relate this to other results such as congruence, similarity of triangles and angles inscribed in a circle.

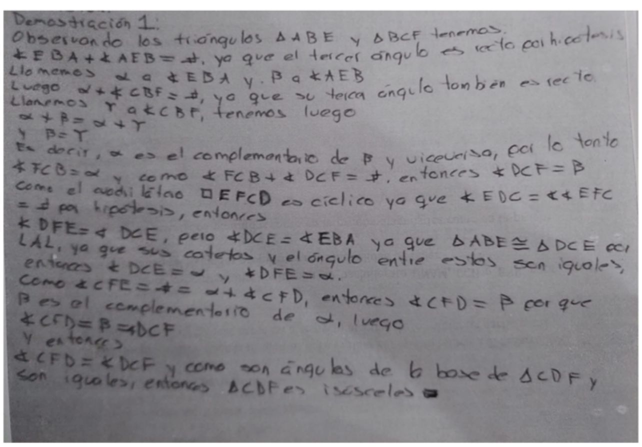
E4. First I will try to arrive at the equality of two angles of the triangle FCD.

It is worth mentioning that both in this stage and in the comprehension stage, the identification of the resources available to the students becomes evident, from which it is possible to draw up a suitable diagram and define a first approach or resolution plan.

Analysis: Problem-Solving Process

The students mentioned that after having obtained the conjecture, they faced several moments of frustration and despair, as they did not quickly find a way forward to prove their assertion. Some of them expressed "letting the ideas and frustrations rest" for a considerable time and then resuming with a calmer attitude, during this time a face-to-face class was held in which the initial ideas were expressed as a group lesson, which allowed the students to reconsider the resources they had and other possible ways of approaching the conjecture. Thus, for the demonstrations, teams E1, E2 and E4 also considered results relating to cyclic quadrilaterals and angles inscribed in a circle.

The resolution processes followed by two teams are described and analyzed below.



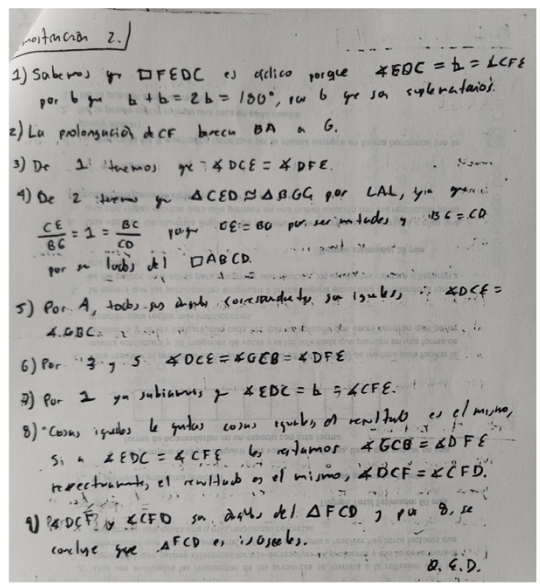


Figure 1. Answers corresponding to the resolution of the problem (on the left side demonstration 1 of E1, on the right side demonstration 2 of E4).

In general, it is observed that students used results related to characteristics of cyclic quadrilaterals, criteria of congruence and similarity of triangles, and inscribed angles in a circle. In addition, they expressed using Geogebra to verify the ideas that emerged in this process.

In the argumentation generated by E1, it can be observed that they use (although it is not explicitly mentioned) the property that the internal angles of a triangle add up to 180° together with the fact that the angles of the initial square are right angles in order to obtain the value of other angles. In addition, it is observed that one of the (almost immediate) ways of acting of the students is to obtain results by means of mathematical calculations even when the relevance of performing certain calculations has not been established, i.e. more calculations are performed than necessary (not for that reason incorrect), so that the process of monitoring and refining the elaborated demonstration could be improved. Subsequently, they use the fact that if in a convex quadrilateral its opposite angles are supplementary (add up to 180°) then the quadrilateral is cyclic, this result is also not explicitly expressed. Finally, they also implicitly use the property that in a cyclic convex quadrilateral the measure of an angle formed by one side and a diagonal is equal to that of the angle formed by the side opposite to the first and the other diagonal.

In E4's answer we can see that their argumentation takes as a starting point that the quadrilateral FEDC is cyclic, this is argued because the [opposite] angles of the quadrilateral are supplementary. On the other hand, in point 2, it is not argued why point G is the midpoint of the segment AB. Then, like E1, in (3) they implicitly use the fact that in a convex cyclic quadrilateral the measure of an angle formed by one side and one diagonal is equal to that of the angle formed by the side opposite to the first and the other diagonal. In (4) it is observed that although students handle the concept of congruence and the criteria that allow them to establish the congruence of triangles, they do not correctly use the notation to express this fact, which subsequently leads them to compare sides and corresponding angles correctly or incorrectly (point 5). In (8) it is established that by considering equal angles and subtracting other angles whose measure is equal, equal angles will be obtained, which will make it possible to establish the equality of two internal angles of the triangle in question in order to demonstrate that it is an isosceles triangle. Although the students' reasoning is correct and allows them to reach the desired conclusion, it is observed that the equality considered in (7), the expressions corresponding to the angles and the corresponding subtractions are incorrect. This shows that there is still a need to work on the process of transferring the ideas [oral or thought] to a written and formal argumentation.

Analysis: Problem extensions

In group G1 it was observed that the extension phase was not developed by most of the students as they omitted this section, those who elaborated an answer (E3) considered asking about some properties that are generated by other objects in the construction or if the circumcircle of the triangle DEB is considered, how many other circumcircles present in the construction are going to be cut by the first one? None of these questions are answered. On the other hand, one student in particular, expressed that by means of Geogebra she built (and replicated) the initial construction, with which she observed that "a repetitive figure" was formed, with which she asks herself if this construction forms a fractal, "how does it look like to repeat this process", "how is it demonstrated that point D is the midpoint of LK and that D is the vertex of ED? These questions are left open for further exploration.

Conclusions

The dynamics implemented in group G1 course encouraged students to use Geogebra as a means for exploration, understanding, obtaining conjectures, and verifying some initial ideas. This leads us to conclude that it is important to provide physical and temporal spaces in high-school and university mathematics courses for students to use GDS not only as a means of representation but also to manipulate the elements that make up the construction to obtain conjectures and verify them, and even more so, to obtain their own or additional results of the problem from this manipulation.

With regard to argumentation, on the one hand, it is important for students to first recognize the mathematical resources they have and then to be able to apply them in problem solving; on the other hand, it is also important that during mathematics education, spaces are created to develop oral and written arguments that allow them to develop an informal mathematical thinking, but with a logical sequence that brings them closer to the process of abstraction and demonstration. Thus, it was observed that the students had the opportunity to put into action various mathematical skills and seek different ways of arguing and supporting the relevance and validity of the conjectures. The results indicate that the students extended their ways of reasoning, allowing them to move from empirical arguments to formal arguments in the presentation of solutions to the problems.

On the other hand, it was observed that within the group dynamics there is a lack of space for students to pose their own problems, either based on those proposed by the teacher or not. Thus, the role of the teacher in guiding and providing feedback during the monitoring and control processes is considered to be of utmost importance.

Finally, it is crucial that dynamics in mathematics courses encourage students to make decisions regarding the way in which the mathematical problems presented are approached, the manipulation of auxiliary tools such as technologies for the exploration and construction of mathematical objects in order to obtain conjectures and create spaces for the generation of formal arguments, the use of mathematical notation and symbology and the refinement of proofs to strengthen mathematical reasoning during the transition to university.

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