BRIDGING SITUATIONAL AND GRAPHICAL REASONING TO SUPPORT EMERGENT GRAPHICAL SHAPE THINKING

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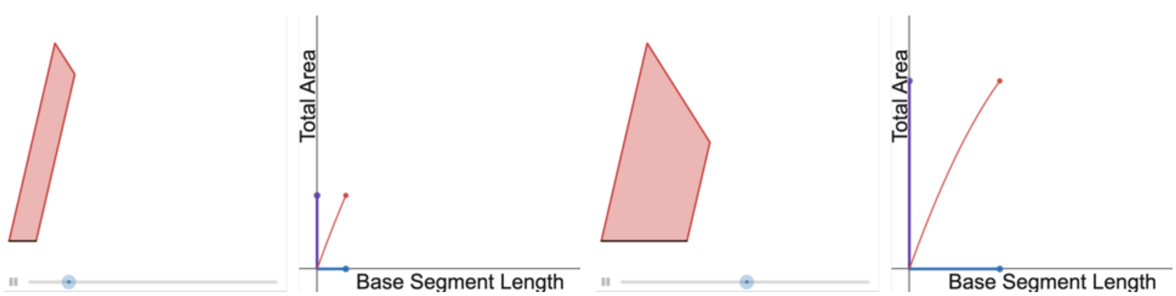
Emergent graphical shape thinking (EGST) entails conceiving a graph as being dynamically generated via the trace of a moving point constrained by two changing quantities. As such, Paoletti et al. (2023) argue that meanings for quantities within a situation and meanings for graphical representations must be connected, or bridged, to engage in EGST. In this report, we explore this bridging process through a case study investigating how two students made connections that bridge their situational and graphical meanings during their work on a mathematical task. We found that the pair's connections between situational and graphical meanings emerged most prominently only after recursive engagement with reasoning in both contexts. We discuss the implications of these findings for researchers and practitioners seeking to support students as they develop EGST.

Keywords: Algebra and Algebraic Thinking, Middle School Education, Learning Trajectories and Progressions; Reasoning

Students' graphical reasoning plays an important role in their learning across STEM fields and their participation as critical citizens (e.g., Glazer, 2011; Potgieter et al., 2008). In particular, emergent graphical shape thinking (EGST) can be useful, as it entails conceiving a graph as being dynamically generated via the trace of a moving point constrained by two changing quantities (Moore, 2021; Moore & Thompson, 2015). For instance, Figure 1 shows how a student can represent a conceived relationship between the dynamic quantities of the base segment length and area of a growing shape via a graph that is produced from the movement of a dynamic point. Paoletti et al. (2020) noted that EGST is important to interpret many graphs across STEM textbooks and practitioner journals in ways consistent with the authors' intentions. However, such reasoning is non-trivial. For example, less than 30% of the 121 U.S. teachers in Thompson et al.'s (2017) study provided evidence suggestive of EGST on a task that could potentially elicit such reasoning. Hence, there is a continued need to explore ways to support students' in developing EGST.



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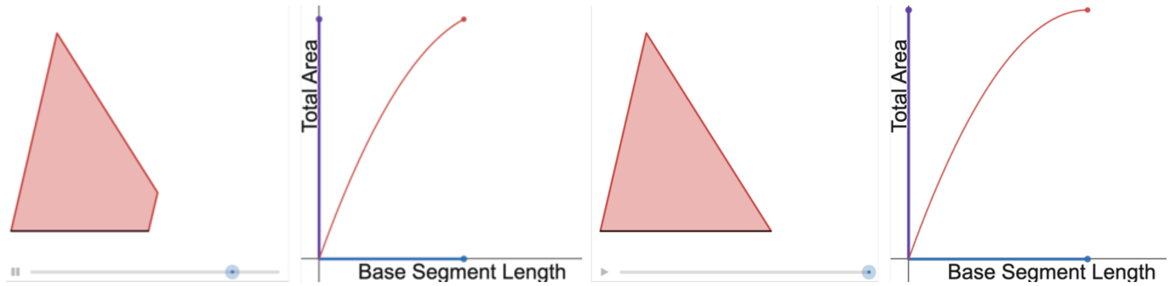


Figure 1: Four screenshots showing a point as a dynamic trace of the relationship between two situational quantities.

In order to engage meaningfully in EGST, Paoletti et al. (2023) argued that learners need to engage both in reasoning specific to the situation (real-world or abstract) and in reasoning about objects in graphical representations. However, the process by which students bridge their thinking in situational and graphical contexts to support EGST is still relatively unexplored.

In this paper, we present a case study (Yin, 2018) to explore learners' situational and graphical bridging process. We feature the activity of a pair of sixth-grade students as they engaged in a teaching experiment (Steffe & Thompson, 2000) to investigate how we might support students to engage in such bridging. To provide context for our case study, we first elaborate on the literature showing the relevance of situational and graphical representations to the development of EGST. Then, we present our methods and results, highlighting how students' iterative engagement with situational and graphical connections corresponded to greater specificity in both domains. Finally, we share implications for researchers and educators who wish to support their students in EGST.

Situations and Graphs in Emergent Graphical Shape Thinking

This case study builds upon the local instruction theory presented in Paoletti et al. (2023). The local instruction theory posited that students must engage in quantitative and covariational reasoning both with respect to situational quantities and with objects represented graphically prior to engaging in EGST. Reasoning quantitatively, both situationally and graphically, entails an individual constructing quantities to interpret their experiential worlds (Smith & Thompson, 2008; Steffe, 1991; von Glasersfeld, 1995). Covariational reasoning entails a learner mentally coordinating two varying quantities (Thompson & Carlson, 2017). Covariational reasoning often follows a developmental progression (Paoletti et al., 2023; Carlson et al., 2002) in which a student coordinates two quantities by thinking "of one, then the other, then the first, then the second, and so on" (Saldanha & Thompson, 1998, p. 299) until they have constructed a relationship that entails both quantities simultaneously being tracked for some duration. Researchers (Saldanha & Thompson, 1998; Thompson et al., 2017) refer to such a conception as a multiplicative object (i.e., a Cartesian product).

Although we often use mathematical representations (e.g., graphs, equations) to represent relationships between covarying quantities, covariational reasoning does not require such representations (e.g., Paoletti & Moore, 2018; Castillo-Garsow et al., 2013; Johnson, 2015). Learners often construct and coordinate covarying quantities to develop meanings for situational quantities and relationships between quantities. These *meanings for a situation* (M.S) can serve as the foundation for their mathematical activity (e.g., constructing a graph representing a relationship) or can result from their interpreting mathematical representations (e.g., developing a novel meaning for a situational relationship as they interpret a graph).

As a pre-requisite to engaging in EGST, learners must also engage in quantitative and covariational reasoning with respect to objects in a graphical representation (Moore, 2021; Paoletti et al., 2023). Paoletti et al. (2023) described a particular sequence of three *meanings in graphical representations* (M.R) that can support students' EGST. They described a learner must first consider how a segment length can represent a quantity's magnitude (M.R.1). Next, in a Cartesian coordinate system, a learner can consider changes in two orthogonal segment lengths in relation to two covarying quantities (M.R.2). Then, a learner can conceive of or anticipate a point in the coordinate system as a multiplicative object simultaneously representing the two segments' magnitudes (M.R.3).

Reflected in the above descriptions, to engage in EGST, learners must ultimately coordinate their situational (M.S) and graphical meanings (M.R) to conceive of a graph as being generated by the dynamic trace of a point. Paoletti et al. (2023) contended that there is likely a dialectal relationship between students' situational and graphing meanings, stating, "our LIT does not outline a single, linear, or developmental progression; rather, we call for repeated and connected occasions for students to engage in M.S, M.R, and [emergent reasoning] as they develop stable graphing meanings that entail EGST" (p. 205). However, the way in which students construct these connections has not been explored in detail.

Methods

We applied case study methodology (Yin, 2018) to respond to the following research question: *How might students make connections between situational quantitative and covariational reasoning (M.S) and graphical representations of covarying quantities (M.R) as they build toward EGST?* We drew our case from data collected during a teaching experiment (Steffe & Thompson, 2000) with one pair of students. We engaged the pair in tasks we intended to support their EGST. In this report, we present results from the students' activity during one session where we identified that students were engaged in reasoning to relate situational (M.S) and graphical (M.R) meanings in explicit ways.

Participants, Context, and Task

We conducted our teaching experiment with sixth grade students in a public charter school in the Northeastern United States. The first author facilitated the teaching experiment as the teacher-researcher (TR). In our selected case, our participants were Sebastian (age 11; self-identified as male and Black, Puerto Rican, and Latino) and Tom (age 11; self-identified as male and White). Both participant names are pseudonyms selected with student input.

In our case for this report, the pair collaborated on a task we designed in Desmos called *The Big Event*. We presented students with a dynamic scenario in which two teachers (Mr. K and Mrs. B) are walking away from a podium and post respectively to create a growing area where hypothetical students could stand for a presentation (the shape grew as shown in Figure 1). Throughout the task, we asked Sebastian and Tom to identify and coordinate Mr. K's distance from the podium (i.e., the horizontal base segment length) and the area of the shape formed (M.S). In this scenario, as Mr. K's distance increases, the total area of the shape increases and the amounts of change of area (hereafter AoC, see Carlson et al., 2002) decrease. Students had the option to toggle between views of the scenario changing smoothly (e.g., Figure 1) and changing in chunks (as in the triangular image in Figure 2).

Our report focuses on Sebastian and Tom's activity with respect to two prompts in the *Big Event Task* that we designed to transition students from reasoning situationally (M.S) to reasoning graphically (M.R). Prompt 1 presented students with three dynamic line segments (blue, orange, and purple) alongside the growing shape (Figure 2). Whereas the blue line segment grew by equal amounts for each jump of change in the shape, the orange segment grew by increasing amounts and the purple segment by decreasing amounts. Similar to the *Which One Task* (Liang & Moore, 2021), the text asked them to select a segment that could represent Mr. K's distance from the podium and the shape's area respectively (M.S & M.R.1). After selecting segments, Prompt 2 presented students with a blank coordinate plane and axes labeled with each focal quantity. The text requested that students consider plotting points along each axis (M.R.2) and then asked students to plot points in the plane (M.R.3).

Prior to engaging with this task, Sebastian and Tom had constructed and interpreted graphs during the teaching experiment. Specifically, they had completed the *Faucet Task*, wherein they coordinated the directional changes between the amount of water and temperature of the water depicted in a dynamic faucet applet (see also Paoletti, 2019; Paoletti & Vishnubhotla, 2022). They also graphed another scenario in the *Big Event Task* that showed a triangle growing by more each time.

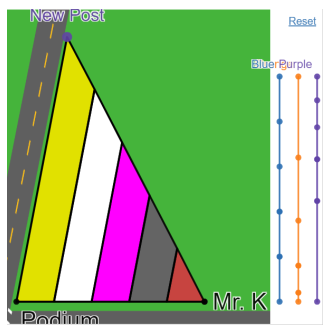


Figure 2: Screenshot from the *Big Event Task* showing the chunkily growing scenario alongside the segments (blue, orange, and purple) in Prompt 1.

Data Collection and Analysis

We audio- and video-recorded Tom and Sebastian's activity with a camera and screen capture tool. We conducted a conceptual analysis (Thompson, 2008) as we analyzed the data to build models of the students' meanings that viably explained their words and actions. With this goal in mind, after each session we constructed an event map (Green & Bridges, 2018) to organize the key events of each session. This process supported us to identify and bound our case. We transcribed the activity (i.e., dialogue, written activity, and gestures) from the segment we identified as informing our case analysis. Next, we analyzed the transcript at the utterance level, highlighting moments in which 1) the TR or text of the task prompted or 2) the pair engaged in building connections between M.S and M.R. We leveraged the patterns in our analysis to organize an account of activity that could address our research question.

Results

To explore how Tom and Sebastian might bridge their situational quantitative and covariational reasoning (M.S) with their developing graphical representations of covarying quantities (M.R), we presented prompts that we intended to provide consistent and progressively specific connections between these meanings. In this case, the students first proceeded through the task as designed (which we intended to support a shift from M.S. to M.R), which only supported limited connections between the situation and graph (Pass 1). The TR subsequently prompted Sebastian and Tom to consider more explicit connections between the situational quantities and graphical representations at each stage of their activity. Such prompts supported the students to construct a graph with increasing precision and justification (Pass 2). We illustrate how students' recursive and specific connections between situations (M.S) and graphs (M.R) across Pass 1 and Pass 2 prepared them for the transition to EGST.

Pass 1: Addressing the Task with a Linear Progression from M.S to M.R

As Tom and Sebastian engaged in their first pass through the task, they read Prompt 1 to select a segment corresponding to the situational quantities of Mr. K's total distance from the podium and the total area of the shape (i.e., to bridge M.S. with M.R.1). Sebastian identified the blue segment as representing Mr. K's total distance, gesturing between the "same jumps" he observed in the situational context (M.S) and in the blue segment. Subsequently, Tom added that the area of the triangle would match the purple segment "because the biggest jump [*gestures to Sit-A in Figure 3a*] is the first one [*gestures to Seg-A in Figure 3a*] and then it gets smaller every time." We note how Tom engaged in gestures indicative of connection between the changes in the quantity of area (M.S) and the changes in the partitions of his segment choice (M.R.1).

Next, the students advanced to Prompt 2, requesting a graph. The TR added an explicit request for students to consider how the segments on the previous slide (M.R.1) could support the construction of the graph itself (M.R.2 & M.R.3) and re-read the labels of the quantities on the axes (M.S). However, reflecting the non-trivial nature of bridging meanings across situations and graphs necessary to engage in EGST, neither Tom nor Sebastian explicitly referenced both situational quantities as they presented their strategies to graph points. For example, Sebastian described that points on the graph would begin in the top right area of the plane and proceed toward the origin to "go down smaller and smaller." Sebastian explained that this was because he expected the graph to move in the opposite direction as the increasing and concave-up graph they had constructed from the growing triangle portion of the *Big Event Task*. We note that Sebastian did not reference either of the situational quantities (Mr. K's total distance or the area) to support his conjectures.

Tom disagreed with Sebastian's strategy. Although Tom verbally described the situational quantities of Mr. K's total distance and area of the triangle, his gestures did not correspond to the quantity of area. His graphing activity focused on marking points in the situation in his constructed graph as opposed to representing the intended quantities (M.R.2 & M.R.3). First, engaging with the animation of the triangle, Tom explained:

> I would put…[Mr. K's] dot right there [*points as in (1) in Figure 3b*] and then for the area roped off, the dot right there [*points as in (2) in Figure 3b*] so I would, like, combine those [*gestures as in (3), reaching location (4) in Figure 3b*] and put it there.

To support his description, Tom plotted a first point on the graph, followed by a second point at the TR's request (Figure 3c). We note that the location of these points strongly mirrors the locations Tom identified in the situation (Figure 3b). As such, we conjecture Tom was reproducing an image (i.e., iconic graphical reasoning; see also Clement, 1989; Johnson et al., 2020) rather than engaging with explicit quantities (M.S). When the TR prompted more specifically about the match between the situational quantities of both distance and total area and the graph, Tom further acknowledged a disconnect: “Yeah, it’s not the same because it [*gestures to triangle from left to right*] increases over time.” Although we could interpret “it” in this statement to area, such an attribution is still unclear. However, the students explicitly acknowledging a disconnect provided an opportunity for them to further develop explicit connections between the situational quantities (M.S) and the graphical representation (M.R).

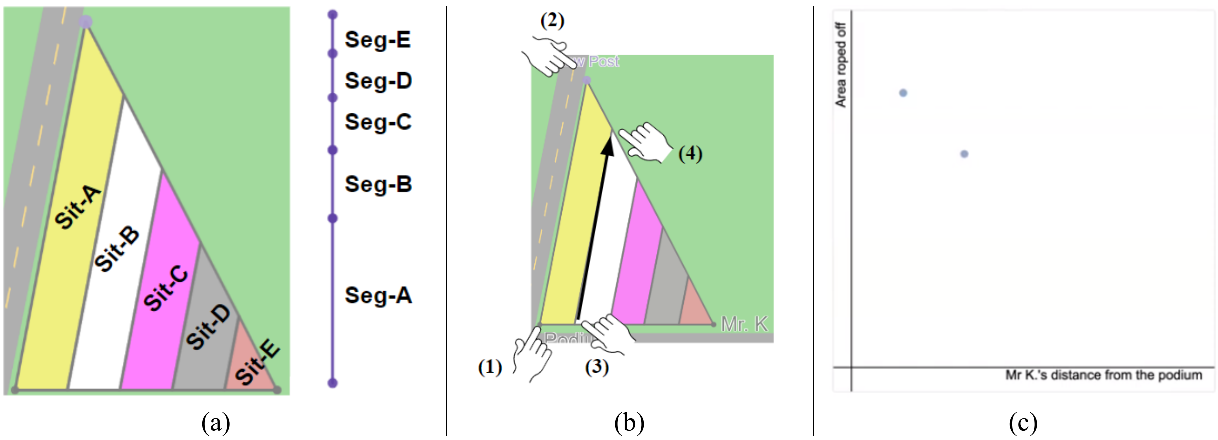


Figure 3: (a) A reference image for students’ gestures to the situation and purple segment and (b) a recreation of Tom’s gestures prior to (c) plotting initial points in the graph.

Pass 2: Addressing the TR’s Iterative Prompting to Engage M.R and M.S

The TR conjectured the students were not explicitly attending to the situational quantities (M.S). This led the TR to provide additional prompts to support increased specificity with respect to situational quantities before returning to the graph to support Sebastian and Tom developing meanings for EGST. Hence, the TR returned to Prompt 1 with the segments. On this pass, the TR’s prompts extended beyond asking students to identify a segment to match the total area of the triangle to a request to analyze how and why the segment matched in specific ways.

Viewing the screen in Figure 2, the TR focused first on a scenario in which Mr. K had only made three equal-sized jumps in his distance from the podium. The TR asked, “What would that purple segment look like?” Sebastian first described that “on the purple segment, it’s like, big [*puts hand in a “C”*] to little [*moves “C” upward while framing a smaller space*], to little [*repeats*].” The TR subsequently prompted for Sebastian to explain the parts of both the segment and the diagram he was “paying attention to.” Sebastian explained:

> Like the yellow right here [*gestures Sit-A in Figure 3a*] which is right here [*gestures from bottom to top to bottom of Seg-A in Figure 3a*] and the white [*gestures to Sit-B in Figure 3a*], right here [*points to Seg-B in Figure 3a*] and then the pink [*gestures to Sit-C in Figure 3a*], would be right here [*points to Seg-C in Figure 3a*].

We note the increasing specificity of Sebastian’s gestures between corresponding quantities in the situation (M.S) and portions of the segment (M.R.1), particularly related to distinguishing AoC of area. The TR followed Sebastian’s explanation by asking, “[If] I want to look at the total area…roped off at that point, what would we be looking at?” Tom repeated Sebastian’s connections between AoC of area and the segment components, and then described, “We’re looking at a line where it stopped…so it’d stop like [*points to the highest point of Seg-C in Figure 3a*] right there.” We highlight the explicitness of Tom’s connections between the area and the segment representation (M.S and M.R.1) in terms of both AoC and total amounts.

The TR then asked the pair to consider the AoC and total area in the situation and segment as Mr. K transitioned to his fourth jump in the diagram: "Is my [purple] line segment going to be bigger or smaller?" Tom began by gesturing across the triangle diagram from left to right: "So the triangle's going to get less but the total area is not… they are all combining." Sebastian further described that the activity felt like "reverse psychology," explaining:

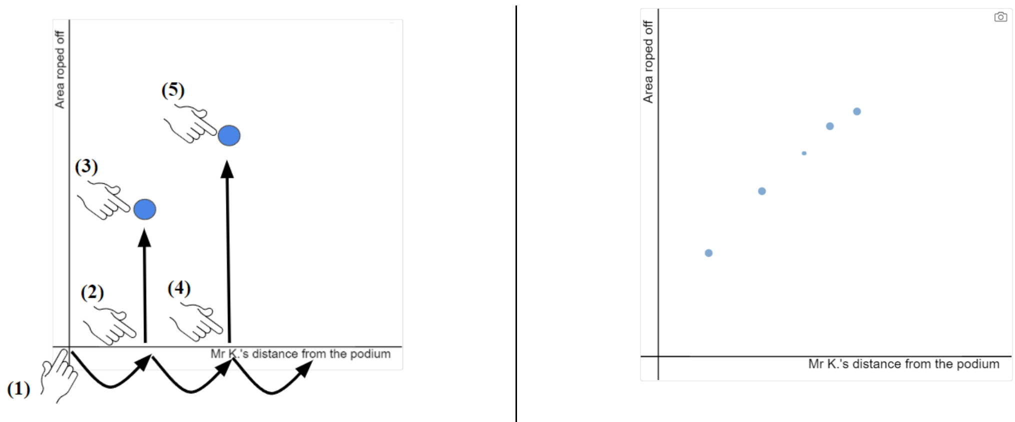
> We start off with big jumps [*gestures over Sit-A in Figure 3a*] and started to decrease with smaller jumps [*moves across triangle to Sit-E in Figure 3a*] but the area starts to get bigger as those jumps [*waves three times from left to right with hand*] go on.

We take Sebastian's detailed explanation as evidence of the productive shifts in the relationship he conceived between situational quantities (M.S) and corresponding segments (M.R.1). Given this specific evidence, the TR advanced again to the graphing slide. To set up the task, the TR reminded the students of prior activity graphing with dynamic, orthogonal segments in the *Faucet Task* (M.R.2) and repeated their conclusions from the activity on the previous slide (M.R.1). Tom constructed the initial graph for the pair:

> So, I think, each jump from Mr. K [*moves over 3 times as in (1) in Figure 4a*] is going to be the exact same, but first I am gonna start out with [*moves mouse to (2) in Figure 4a*] a big jump [*moves mouse up as in (3), plotting point*] like that, and then we're going to go here [*moves mouse to (4) in Figure 4a*] and then it's a smaller jump [*moves mouse up as in (5), plotting point*].

Tom continued to plot an additional three points in this way (see Figure 4b), explaining the jumps in area were "getting smaller every time we do it." Sebastian further interpreted Tom's graph: "As Mr. K's distance is increasing and the jumps are getting smaller, [the total area]'s still getting bigger [*places hands orthogonally*], no matter how far Mr. K walks [*motions horizontal hand from left to right, repeats twice more*]."

We consider Tom and Sebastian's activity collectively to emphasize how their iterative engagement with situational quantities (M.S) and segments (M.R.1) supported shifts in their graphing activity. First, the students displayed explicit indications of conceiving orthogonal segments along each axis (e.g., Sebastian's orthogonal hand gestures, M.R.2). Furthermore, the students also indicated the simultaneity of those segments in the points they constructed (e.g., Tom's over and up gesturing "getting smaller every time we do it," M.R.3). Importantly, these graphical objects were not described as contextless; rather, Sebastian, in particular, referenced situational quantities explicitly in his interpretation of the points (M.S). In response to the TR's prompts, the pair specifically engaged with AoC and total area as they drew connections directly to segments. This activity supported their graphing with greater justification and support.



(a) (b)

Figure 4: (a) A recreation of a subset of Tom's gestures in constructing his second graph and (b) Tom's final graph.

Discussion and Conclusion

In this paper, we presented a case study to address our RQ and describe how two students built connections between situational quantitative and covariational reasoning (M.S) and graphical representations of covarying quantities (M.R) as they progressed toward EGST. We note that the linear progression we initially designed from a situation to a graphical representation in Pass 1 did not provide sufficient opportunities for our students to reason in a coordinated way about how to represent the same two quantities across both contexts. In Pass 2, the students iteratively engaged with situations and graphs (that is, returning to M.S and repeatedly making connections between each component of M.R with M.S) in a way that resulted in more specific thinking. Thus, we conjecture that by engaging in both Pass 1 and Pass 2, the students developed robust connections between situational quantities (M.S) and graphs (M.R); we present Figure 5 to synthesize these results. Although we do not detail Tom and Sebastian's full construction of a smooth graph and EGST later in their activity here, we show evidence of robust connections between quantities in situations and graphs that prepared them for this next step in their reasoning.

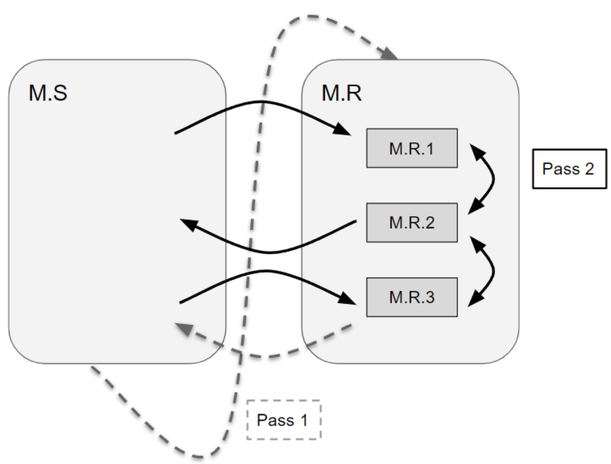


Figure 5: Visual presentation of the relationship between Pass 1 and Pass 2 in supporting students' connections between meanings in situations (M.S) and graphs (M.R).

In this report, we describe the process by which two students bridged their thinking in situational and graphical contexts. Reflecting the non-trivial nature of EGST (e.g., Thompson et al., 2017), our case provides evidence of the importance for students to have "repeated and connected occasions for students to engage in M.S [and] M.R," (Paoletti et al., 2023, p. 205) as they build towards EGST. Such findings have implications for practitioners and researchers intending to support learners' graphing meanings. Importantly, providing students recursive opportunities to make connections between meanings for situations (M.S) and graphical representations (M.R) is critical in the design of tasks and instruction. Future researchers may be interested in exploring other ways to students may connect situational quantities and graphical representations to support EGST. Such investigations could further efforts to improve the teaching and learning of graphing in ways that are attentive to the needs of STEM fields.

Acknowledgments

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