"I UNDERSTAND IT EVEN MORE!" PROMOTING PRESERVICE TEACHERS' RELATIONAL UNDERSTANDING OF FRACTIONS

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Preservice teachers (PSTs) are expected to possess a relational understanding (i.e., knowing how to do and why) of mathematics for ambitious instruction. This study aimed to shed some light on the possibilities of supporting PSTs' development of relational understanding of fractions through engaging them in writing collective argumentation. Drawing data from a larger project; we explored the development of a PST's understanding of fractions through the engagement of collective argumentation. The results indicated that the PST's relational understanding of fractions developed from both structural and content perspectives. Some educational implications for teacher education are discussed.

Keywords: Mathematical Knowledge for Teaching, Preservice Teacher Education, Number Concepts and Operations, Elementary School Education

Developing mathematical understanding is the core of mathematics education. Skemp (1976) conceptualized that one's mathematical understanding can be considered as a continuum moving from *instrumental* (processing procedures or mathematical facts without understanding how they are connected) to *relationa*l (knowing how and why mathematical facts and procedures are related). In the field of mathematics education, there is agreement that we should engage All students in meaningful mathematics by supporting students in developing a relational understanding of mathematics (Hiebert, 1997; Ma, 2020; NCTM, 2014; Van de Walle et al., 2020). Given teachers' mathematical knowledge for teaching (Ball et al., 2008) plays an influential role in affecting students' opportunities of engaging in developing a relational understanding of mathematics (Hill et al., 2005), it is essential to prepare preservice teachers (PSTs) with a relational understanding of mathematics (AMTE, 2017). This study serves this broad goal: promoting PSTs' relational understanding of mathematics, with a focus on fractions.

The data of this study draw from a larger project which was designed to promote PSTs' fraction proficiency via engaging them in *writing collective argumentations online* (WCAO), an approach asking PSTs to write their justification for how and why a certain mathematical strategy (e.g., the common denominator strategy for ordering fractions) works (Liu, 2021). The larger project focused on the topic of fractions and the population of PSTs because of the following three major reasons. First, fractions play a fundamental role in mathematical learning, which can predict students' overall mathematics achievement in upper elementary, middle school, and high schools (Bailey et al., 2012; Brown & Quinn, 2007; Hansen et al., 2017; Siegler et al., 2012; Torbeyns et al., 2015). Second, students are often struggled with fractions and lack an understanding of fractional procedures (Booth & Newton, 2012; Carpenter, 1981; Cramer et al., 2002; Gómez & Dartnell, 2019; Liu & Jacobson, 2022; NMAP, 2008; Siegler & Pyke, 2013; Stafylidou & Vosniadou, 2004). In other words, although fractions are especially important, students' understanding of fractions is limited, which makes improving students' fraction learning opportunities urgent. Scholars argued teachers' fraction knowledge might play a role in this situation, which is related to the third reason that teachers, including pre-and in-service teachers, possess inadequate fraction knowledge for teaching fractions for understanding (Olanoff et al., 2014).

Situating in the broad context of the large project, the current study aims to shed light on understanding the possibility of improving PSTs' relational understanding of fractions and exploring an effective analytical approach to capture the progress toward a more relational understanding. Specifically, we zoomed in on one case, a PST named Hope, in which she used the common denominator strategy to solve a fraction ordering task to answer the following research question: *What relational understanding has the PST developed after engaging in WCAO?*

Theoretical Background

A Continuum of Mathematical Understanding: From Instrumental to Relational

This study employed a constructivist perspective that individuals construct mathematical understanding and meanings by connecting what they know and the new information they encounter. Skemp (1976) proposed that one's mathematical understanding can be thought of as a continuum that progresses from instrumental to relational (see Figure 1). Instrumental understanding refers to a situation in which an individual's knowledge is primarily made up of isolated facts, meaningless procedures, and rules learned by rote (see the left circle in Figure 1). On the other hand, a relational understanding enables an individual to comprehend how and why mathematical concepts and procedures are logically related (see the right circle in Figure 1). Van de Walle et al. (2020) argued that "understanding is a measure of the quality and quantity of connections a new idea has with existing ideas. The greater the number of connections to a network of ideas, the better the understanding" (p. 20). Thus, a more relational understanding can be understood as a process of making more qualified connections between mathematical ideas to one's network.

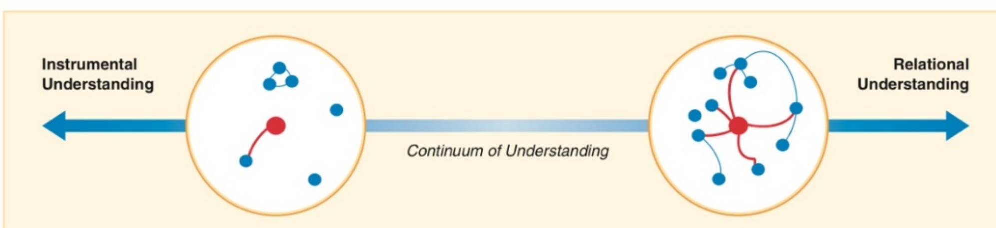


Figure 1: An Illustrative Model of Understanding Development (Van de Walle, 2020, p.20)

Toulmin's (1958/2003) Model for Collective Argumentation

As shown in Figure 2, according to Toulmin (1958/2003), an argument involves some combination of claims (statements whose validity is being established), data (support provided for the claims), warrants (statements that connect data with claims), rebuttals (statements describing circumstances under which the warrants would not be valid), qualifiers (statements describing the certainty with which a claim is made), and backings (usually unstated, dealing with the field in which the argument occurs).

In order to explore how a person's relational understanding of a certain mathematical object shifts, it is crucial to analyze both the quality and/or quantity of new connections an individual made between mathematical ideas. In this study, we adapted Toulmin's (1958/2003) model as an analytical framework because this model has been widely used in mathematics education literature to analyze how collective argumentation developed in a social setting concerning the content and structure of arguments (e.g., Krummheuer, 1995; Zhuang & Conner, 2022). In addition, Toulmin's (1958/2003) model provides us with a tool to identify how mathematical ideas are connected by examining how argument components (e.g., claims, warrants) are constructed.

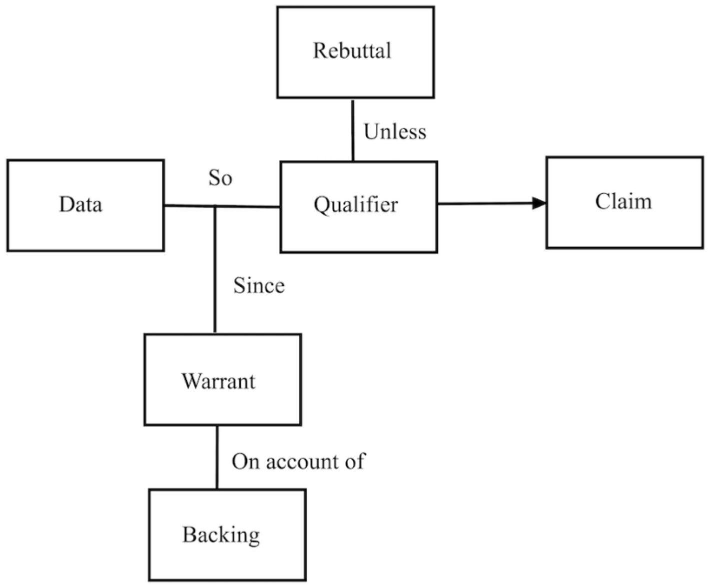


Figure 2: Structure of a Generic Argument (adapted from Toulmin, 1958/2003)

To frame one's understanding, this study followed the idea from Conner (2008) to use colors to denote different types of argument components: given data (green square), claim or data claim (blue square), explicit warrant (yellow circle), implicit (brown circle). This allows us to explore the structure of an argument and analyze the content of a specific argument component. In this way, from the lens of the structure of an argument, we assess the *quantity* of the connections according to the structure of an argument; and from the lens of the content, we assess the *quality* of the connections. These two lenses together help us to determine to what extent an individual develops a more relational understanding. For the purpose of this specific study, we focused on three main core components of argument (i.e., claim, data, warrant). If a component functions as both data in one argument and as a claim in a sub-argument, we label it as a data/claim. Sometimes, parts of an argument may not be explicitly stated but can be inferred were labeled as implicit.

Methodology

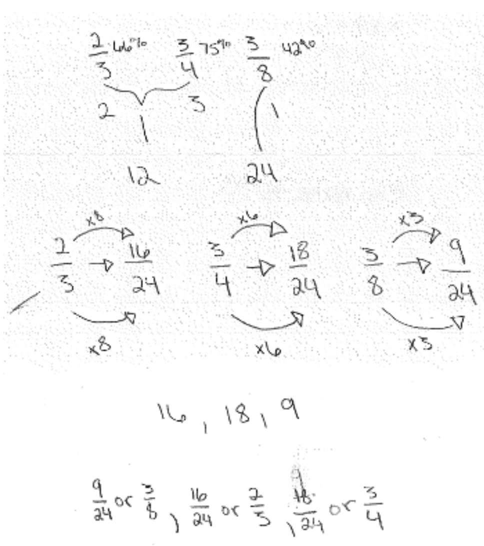
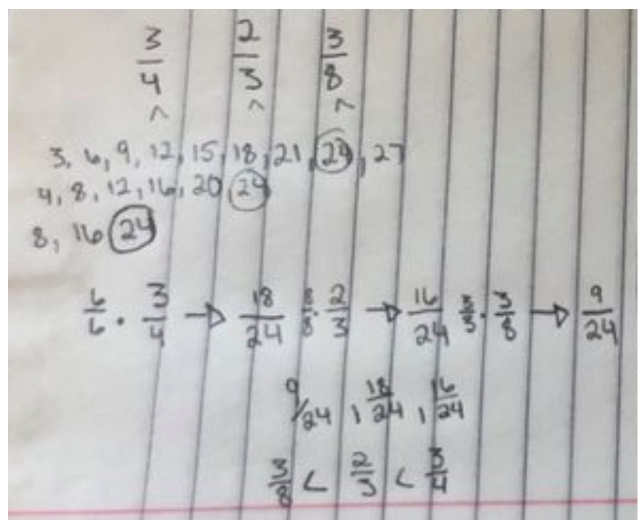
Setting and Participant

This study is part of a larger research project which aims to promote PSTs' fraction proficiency by engaging them in writing collective argumentations for each of the eight commonly used fraction comparison strategies (e.g., the common denominator strategy, the number line strategy, the fraction bar strategy) in an online setting (Liu, 2021). To assess the impact of the intervention on PSTs' fractional understanding, the larger project employed a pretest-post-test design. Each participant was asked to order 2/3, 3/4, and 3/8 from the smallest to the greatest using as many strategies as they could on a paper sheet and orally explain their solutions at the beginning and the end of the project (We refer to them as pre-and post-test). A reflective interview was also conducted at the end of the larger project.

We adopted Yin's (2014) selection procedure of typical cases (details see p. 46) and identified Hope as a typical case to focus on. Hope was a white female who engaged in a WCAO session using the common denominator strategy to order 2/3, 3/4, and 3/8. Hope was purposefully selected for this study because she expressed an interest in engaging in WCAO due to her fluency with fraction-related procedures but not confident with her conceptual understanding of fractions. Hope was in the second year of her undergraduate teacher education program and had completed two mathematics content courses (rational numbers and geometry) so far. We focused on examining Hope's understanding of the common denominator strategy because this strategy is the most wildly taught and used procedure-oriented strategy and was the strategy that Hope felt "the most comfortable with when we started this project" and "would just kind of default to it" before joining the project.

Data Source and Analysis

The data source used in this study includes the pre-and post-tests, where Hope used the common denominator strategy to order three fractions with explanations, and her reflective interview after the WCAO. The data has been video-recorded and transcribed for analysis (see Figure 3, for the sake of space, here, only show the beginning part of the transcript).

Pre-test	Post-test
	
Hope said while she gave the above solution in the pre-test, "I want to find a common denominator. So, this is, so between 3 and 4, the obvious one is, 2, is 12 to me. But then I also have to include eight, So, instead I'm going to use 24, and I should use a factor tree, but in my mind, the numbers just kind of pop out. And so, I'm going to set up my little tables to make these all out of 24. 24 is, yeah, it is divisible by three. So, to get to 24 from three, you have to multiply	Hope talked while she gave the above solution in the post-test, "I believe next I'm going to do the common denominator strategy, which this was the strategy that I was the most comfortable with when we started this project and like, I would just kind of default to it. Yes. Okay, So I'll start by listing out my fractions two thirds, three fourths, and three-eighths. So, the goal is to find a number where each denominator is a factor of it. And we want to find the least common multiple of these three fractions. So, like off the top my head obviously, I know what it is, but what younger

it by eight. So, I'm going to do the same thing on the top. So, two times eight becomes 16 and 24. And to get from four to 24, you have to multiply by six. So again, do the same thing on the top. And to get from eight to 24, you have to multiply by three. So, I'm going to multiply the top by three. And so, these two, this one [3/8] and this one [2/3] are kind of like the denominators are, I always think of them as like inverses of each other, because 8 times 3 is 24 and then 3 times 8 is 24. But now we can just compare the numerators. So, we have 16, 18, and 9…"	students will likely be taught to do is list out multiples of these denominators until they find one in common. So, 3,6, 9, 12, 15, 18, 21,24,27 And then we'll move on to 4, 8, 12, 16, 20, 24. So now I see that two of these have 24 in common. So, then I can check and see if eight would also go into 24, which evenly in which it will. So, 8, 16, 24. So that means that my least common multiple is 24. So now I'm going to setup fractions to convert them. So, 3 over 4 to something over 24, two thirds to something over 24. And what was last three eighths, something over 24. So, 3 over 4 to something… over 24, So, we want to multiply the denominators of each fraction by whatever factor it takes to make them into 24. But we also have to multiply that same number to the top in order to keep the fractions proportional. Because if we only multiply it a four by six to get it to 24. And we said it was 3 out of 24. That's not going to be the same ratio as three-fourths. Three-fourths isn't equivalent to this number. And so, what we're really doing is multiplying each number by one but by different versions of one to increase that number. So now I'm going to setup fractions to convert them... So, 3 over 4 to something over 24... So, to get four to 24, we have to multiply by six. So, six times three is 18. So, three-fourths becomes 18 over 24. So, we're really multiplying this fraction by six over six, which is the same as multiplying by one, but this just scales it up …"

Figure 3. Hope's Pre-and Post- Written Response and Partial of the Transcript of Her Verbal Response

We applied adapted Toulmin's model to explore Hope's understanding of the common denominator strategy in the pre-and post-tests through her participation in WCAO. Any differences in diagramming were discussed among researchers until a consensus was reached. We identified her talking about the common denominator strategy in her reflective interview after the WCAO to triangulate and contextualize our results.

Results

In this section, we report our results of Hope's understanding of the common dominator strategy in terms of structure and content changes from pre-test to post-test. The following Figure 4 shows the basic structure of Hope's argument from the pre-test and post-test. We found that she made two typical changes regarding an argument's structure and content. From the structure perspective, we saw Hope made two warrants from implicit to explicit, and added more warrants to justify her claims post-WCAO. Additionally, from the content perspective, Hope transferred from talking about the specific task only to talking about the common denominator strategy in general, especially the fundamental ideas. She used more than one warrant to support a claim after the WCAO. These changes suggest Hope has developed a more relational understanding of the specific strategy after engaging in the WCAO session.

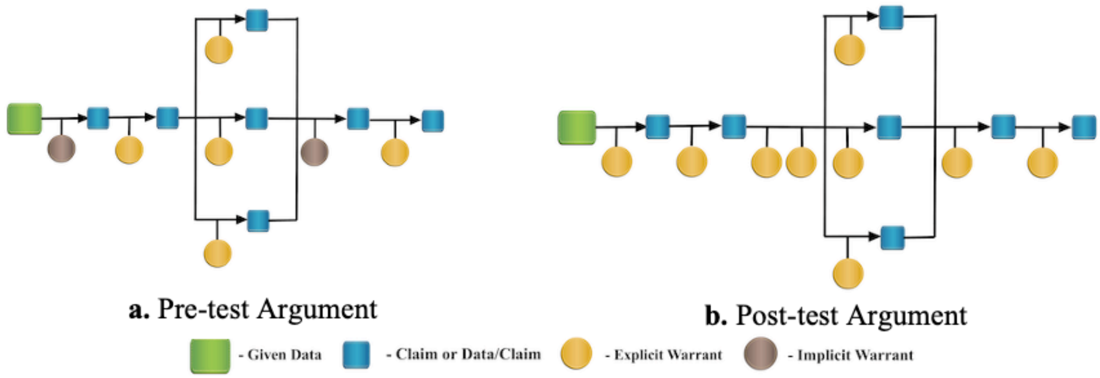


Figure 4. Toulmin's Model of Hope's Argument in the Pre-and Post-Test

Structure Changes

As Figure 4a and Table 1 show, Hope made seven claims and supported these claims with five explicit warrants and two implicit warrants in the pre-test. Hope made eight claims and supported the claims with nine explicit warrants in the post-test, as Figure 4b shows. Thus, Hope made one more claim backed up with two warrants and transferred two warrants from implicit to explicit after the WACO engagement. These results indicate that Hope increased the number of connections after engaging in WCAO.

Table 1. Summary of the Structure Changes

Argumentation Component	Pre-test	Post-test	Changes
Claims	7	8	**+1**
Explicit warrants	5	9	+2
Implicit warrants	2	N/A	Transferred two warrants from *implicit* to *explicit*

Content Changes

From comparing the content of specific argument components, we found that Hope made two essential changes: a) understanding the fundamental idea of the common denominator strategy and b) understanding the rationale of the procedure for finding equivalent fractions.

Understand the Fundamental Idea of the Common Denominator Strategy. One fundamental idea of the common denominator strategy is to understand why we want to apply it to compare 2/3, 3/4, and 3/8 rather than compare them directly. Hope developed her answers after WCAO. As we mentioned, the common denominator strategy was the fraction comparison strategy that Hope was "most comfortable with" and "just kind of default to" use. But she did not think about the why questions before WCAO. As Hope expressed in her reflective interview, "For common denominators, and I just thought of it as, oh, you know, we have the same denominator, so we can just add [compare] them." This expression suggests an implicit connection between the "same denominator" and "just [compare] them." However, Hope could articulate this connection explicitly after WCAO. She stated in the reflective interview, "It as we have a common denominator so that we only have to look at the numerator because before we were comparing two variables [denominator and numerator] and [now] we are only comparing one [numerator]." This statement shows Hope made connections between the number of variables, the affordance of the common denominator strategy, and the goal of fraction comparison, suggesting progress in understanding the fundamental idea of the common denominator strategy after WCAO.

The connections Hope made between the number of variables and the affordance of the common denominator strategy supported her in providing more warrants to her claim in the post-test. For instance, Hope claimed in the pre-test that "now we can just compare the numerators" without providing an explicit warrant of why we can only compare the numerator after she transferred all the fractions into the fractions with a common denominator 24. In contrast, in the post-test, Hope made a more precise claim that "By finding the common denominator we've made, so that we only need to look at the numerator," and provided an explicit warrant by stating that "Because every denominator is the same, so we know that all of the pieces are the exact same size... So, we only order the numerators from least to greatest." This example shows progress in Hope's relational understanding of the common denominator strategy, in which she transformed an implicit warrant (previous knowledge about only comparing the numerator) into an explicit warrant (using the common denominator strategy to make "all of the pieces are the exact same size," that is, to control one variable).

Understand the Rationale of the Procedures for Finding Equivalent Fractions. After WCAO, Hope also understood why we must multiply the same non-zero whole number by the numerator and denominator when finding equivalent fractions. Hope said in the reflective interview that she "knew you multiply by the numbers" but "wasn't understanding exactly how we create those equivalent fractions [works]" before WCAO. Nevertheless, she learned that "it [transferring to equivalent fractions] stays the same because you're really just multiplying by one and scaling it up by doing, like, four over four" from engaging in WCAO.

The new insight Hope gained that "you're really just multiplying by one" seemed to support Hope in providing two new explicit warrants to justify why the finding equivalent fraction procedure works in the post-test. Her first warrant used indirect reasoning, in which Hope stated in the post-test, "[For 3/4], because if we only multiply a four by six to get it to 24 [only multiply the denominator]. And we said it was 3 out of 24. That's not going to be the same ratio as three-fourths. Three-fourths isn't equivalent to this number." Here, Hope justified the rationale of multiplying the same number by the numerator by showing the mathematical conflict created by not doing so (3/24 "is not going to be the same ratio as" and "isn't equivalent" to 3/4).

The second warrant justified why the procedure of multiplying the same non-zero whole number to the numerator and denominator can maintain equivalence by connecting to the multiplicative identity property one that "a x 1 = 1 x a = a" and the fact that 1= n/n ($n \neq 0$). Hope stated this idea specifically in the post-test as follows,

What we're really doing [multiply the same number to the numerator or denominator] is multiplying each number [each fraction] by one but by different versions of one to increase that number [the denominator]. So [for fraction ¾] to get four to 24, we have to multiply by six. So, we're really multiplying this fraction by six over six, which is the same as multiplying by one, but this just scales it up. So, six times three is 18. So, three-fourths becomes 18 over 24.

Hope's above understanding could be expressed in an algebraical form that,

$$\frac{a}{b} = \frac{a \cdot n}{b \cdot n} = \frac{a}{b} \cdot \frac{n}{n} = \frac{a}{b} \cdot 1 = \frac{a}{b} \quad = \frac{c}{d} \ (n \neq 0)$$

For the case of ¾, this reasoning process could be expressed as,

$$\frac{3}{4}=\frac{3\cdot 6}{4\cdot 6}=\frac{3}{4}\cdot\frac{6}{6}=\frac{3}{4}\cdot 1=\frac{3}{4}=\frac{18}{24}\ (n\ \neq 0)$$

The above results showed that Hope had improved her relational understanding of the common denominator strategy by making connections between the procedure of finding equivalent fractions, indirect reasoning, ratio, different versions of one, and the property after engaging in WCAO.

Conclusions and Implications

We identified what relational understandings Hope developed from both the structural and content perspectives by examining Hope's use of the common denominator strategy in the pre-and-post-tests through Toulmin's (1958/2003) model. These findings exemplified that Toulmin's model could be a useful analytic tool to characterize the progress of a person's relational understandings by concerning both the content and structure lens. These two lenses provide a more comprehensive picture of understanding a person's development of relational understanding of mathematical concepts. Hope's case also shows that although a PST has mastered a strategy for solving tasks correctly, plentiful spaces for expanding their relational understandings exist. Collective argumentation (e.g., WCAO) could be an effective pedagogical approach because it allows PSTs to address their learning needs using their strength in mathematical procedures. Through engaging in collective argumentation, furthermore, WCAO provides a context for preservice teachers to understand collective argumentation better, which may support them in learning to implement collective argumentation in their future teaching. This study urges us to explore further how engaging in the WCAO session impacted Hope and other participants' development of rationale understanding of mathematical concepts.

We also notice this study's limitation in that we only examined one critical case study of Hope. We studied how Hope developed her conceptual understanding of fractions before and after participating in WCAO. Although we cannot claim that the findings will generalize to other learning of mathematical concepts with the WCAO approach, the findings may be generative for teaching and learning mathematics through an online collaborative learning community.

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