Proceedings of the Forty-Fifth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education

Engaging *All* Learners

VOLUME 1: TEACHING

Reno, Nevada

Oct 1-4, 2023

Editors:

Teruni Lamberg
University of Nevada, Reno
terunil@unr.edu

Diana Moss
University of Nevada, Reno
dmoss@unr.edu

PME-NA 2023

Oct. 1-4, 2023 in Reno, Nevada

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1). University of Nevada, Reno.

Citation:

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol 1). University of Nevada, Reno.

ISBN: 978-1-7348057-2-7

DOI: 10.51272/pmena.45.2023

PME-NA HISTORY AND GOALS

PME-NA History and Goals

PME came into existence at the Third International Congress on Mathematical Education (ICME-3) in Karlsrühe, Germany, in 1976. It is affiliated with the International Commission for Mathematical Instruction. PME-NA is the North American Chapter of PME. The first PME-NA conference was held in Evanston, Illinois in 1979. Since their origins, PME and PME-NA have expanded and continue to expand beyond their psychologically oriented foundations. The major goals of the International Group and the North American Chapter are:

1. To promote international contacts and the exchange of scientific information in the psychology of mathematics education.

2. To promote and stimulate interdisciplinary research in the aforesaid area, with the cooperation of psychologists, mathematicians, and mathematics teachers; and

3. To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

PME-NA Membership

Membership is open to people who are involved in active research consistent with PME-NA's aims or who are professionally interested in the results of such research. Membership is open on an annual basis and depends on payment of dues for the current year. Membership fees for PME-NA (but not PME International) are included in the conference fee each year. If you are unable to attend the conference but want to join or renew your membership, go to the PME-NA website at http://pmena.org. For information about membership in PME, go to http://www.igpme.org and visit the "Membership" page.

PME-NA Contributors

PME-NA Steering Committee

Elected Members

Doris Jeannotte (Chair) – Université du Québec à Montréal
Ayman Aljarrah – Acadia University
Leslie Dietiker – Boston University
Christopher Kurz – Rochester Institute of Technology
Zareen Rahman – James Madison University
Angeles Dominguez – Tecnológico de Monterrey
Xiangquan (James) Yao – Pennsylvania State University
María S. García González – Universidad Autónoma de Guerrero
Oyemolade (Molade) Osibodu – York University
Casey Griffin (Graduate Student Rep.) – University of Delaware
Tabatha Rainwater (Graduate Student Rep.) – University of Tennessee, Knoxville

Appointed Members

Aaron Brakoniecki (Webmaster) – Boston University
Winnie Ko (Treasurer) – Indiana State University

Conference Organizing Committees

Current Conference (2022-2023)

Teruni Lamberg
Chair
University of Nevada, Reno
terunil@unr.edu

Shera Alberti-Annunzio
Conference Coordinator
University of Nevada, Reno
shera@unr.edu

Local Organizing Committee

Diana Moss*
University of Nevada, Reno
dmoss@unr.edu

Lynda Wiest
University of Nevada, Reno
wiest@unr.edu

Glenn Waddell*
University of Nevada, Reno
gwaddell@unr.edu

Rachael Welder*
University of Nevada, Reno
rwelder@unr.edu

Alysia Goyer*
Stockton University
alysia.goyer@stockton.edu

Joseph Antonides*
Virginia Tech
jantonides@vt.edu

*Served as strand leaders

Future Conference Chair (2023-2024)

Karl W. Kosko
Kent State University
kkosko1@kent.edu

Past Conference Co-Chairs (2021-2022)

Alyson Lischka
Middle Tennessee State University
alyson.lischka@mtsu.edu

Elizabeth B. Dyer
University of Tennessee Knoxville
edyer8@utk.edu

Ryan Seth Jones
Middle Tennessee State University
ryan.Jones@mtsu.edu

Jennifer N. Lovett
Middle Tennessee State University
jennifer.lovett@mtsu.edu

Jeremy Strayer
Middle Tennessee State University
jeremy.strayer@mtsu.edu

Sponsor
University of Nevada, Reno

Reviewers

Abbaspour Tazehkand, Shahabeddin
Acevedo, Carlos Ivan
Adeolu, Adewale
Aguilar, Jair
Akgul, Enisa
Akuom, Denish Ogweno
Aljarrah, Ayman
Altshuler, Mari
Alyami, Hanan
Alzayyat, Ahmad
Ambrose, Rebecca C
Anthony, Monica
Antonides, Joseph
Aqazade, Mahtob
Aryal, Harman Prasad
Asempapa, Reuben
Ataide Pinheiro, Weverton
Austin, Christine Kathryn
Austin, Christopher
Azimi Asmaroud, Seyedehkhadijeh
Bailey, Nina Gabrielle
Balady, Steve
Baniahmadi, Mona
Bennett, Amy Been
Bermudez, Hillary
Bertolone-Smith, Claudia Marie
Bharaj, Pavneet Kaur
Bieda, Kristen N
Bofferding, Laura
Boncoddo, Rebecca
Bondurant, Liza
Bostic, Jonathan David
Boyce, Steven
Bozzano, Patricia Eva
Brakoniecki, Aaron
Brass, Amy
Brown, Yuriko Hoshiya
Bui, Mai
Butler, Rebecca
Cabañas-Sánchez, Guadalupe
Cannon, Susan
Cardetti, Fabiana
Castanheira, Brittney
Cavey, Laurie Overman
Caviness, Stephen
Chandler, Kayla
Chavez, Oscar
Chen, Lizhen
Chicalote Jiménez, Tania Azucena
Choi, Kyong Mi
Cirillo, Michelle
Complete Name
Cook, Candies
Cordero Osorio, Francisco
Corey, Douglas
Corven, Julien
Czocher, Jennifer A
Daniel, Amy L
Davis, Jon D.
De Alba, Carlos Alejandro
Dietiker, Leslie
Dinapoli, Joseph
Dobie, Tracy Elyse
Doherty, Kristin
Dominguez, Angeles
Dubbs, Christopher
Ducloux, Kanita
Duni, Dhimitraq
Edwards, Belinda
Egbedeyi, Temitope
Elizondo, Andrea Martina
Erwin, Alicia
Escobar Durán, Uriel
Fajardo, Maria Del Carmen
Fan, Yiyun
Feikes, David
Fink, Heather
Forde, Elizabeth
Franks, Asia
Frazee, Leah Michelle
Gantt, Allison L.
García González, María S.
Gargroetzi, Emma Carene
Gebremichael, Andualem Tamiru
Ghousseini, Hala
Gilbertson, Nicholas J
Gómez-Árciga, Adrián
Goodson-Espy, Tracy
Goyer, Alysia
Graysay, Duane
Greenstein, Steven
Grewall, Tejvir Kaur
Griffin, Camille
Griffin, Casey
Gualdron Pinto, Elgar
Hackenberg, Amy J
Haiduc, Ana-Maria
Hall, Jennifer
Han, Simon Byeonguk
Hand, Victoria
Hartmann, Christine
He, Jia
Hidayat, Angga
Hohensee, Charles
Hong, Dae S.
Hunt, Jessica H.
Izard, Blair
Jarry-Shore, Michael
Jeannotte, Doris
Kacerja, Suela
Kamaldar, Azar
Kamlue, Nitchada
Karanevich, Peter
Kaufman-Ridout, Bailey Anne
Kebreab, Lybrya
Kent, Laura Brinker
Kercher, Andrew
Kerrigan, Sarah
Kim, Hee-Jeong
Kim, Youngjun
Kirkland, Patrick
Kirwan, J Vince
Kitchen, Richard
Ko, Winnie
Ko, Yi-Yin
Kochmanski, Nicholas

Kokushkin, Vladislav
Krell, Michael W
Kurz, Christopher A
Kwon, Oh Hoon
Lamberg, Teruni
Lassak, Marshall
Leatham, Keith R.
Lee, Boram
Lee, Carrie W
Lee, Hwa Young
Leonard, Helene
Lim, Dexter
Lloyd, Mary Elizabeth Riley
López-Iñesta, Emilia
Love, Candice
Lovin, Louann
Macdonald, Beth L.
Magiera, Marta T.
Mainzer, Emily
Maldonado Reynoso, Norma Patricia
Males, Lorraine M
Margolis, Claudine
Marsh, Dalton Dayne
Martin, Tami S.
Martínez Uribe, Alfredo
Martinez-Soto, Eduan
Mask, Walker
Matranga, Anthony
Mauntel, Matthew
Max, Brooke
Meagher, Michael S
Méndez Huerta, Dinorah
Moldavan, Alesia Mickle
Montero-Moguel, Luis E
Moss, Diana L.
Nagle, Courtney
Naresh, Nirmala
Nitta, Kathleen
Norton, Anderson
O'Dell, Jenna R
Odiwuor, Brian
Orozco, Claudia
Ortiz Galarza, Mayra Lizeth
Ortiz, Daniel
Oslund, Joy Ann
Osibodu, Oyemolade
Otten, Samuel
Pacheco, Mike
Paliwal, Veena
Panorkou, Nicole
Paoletti, Teo
Park, Hyejin
Park, Jungeun
Partridge, Eric
Patterson, Cody L
Phelps-Gregory, Christine M
Piatek-Jimenez, Katrina
Poling, Lisa L
Polly, Drew
Provost, Amanda
Pujiyanto, Fnu
Pynes, Kristen D'Anna
Quansah, Abigail Lois
Rahman, Zareen Gul
Rainwater, Tabatha
Raja, Waleed Ashraf
Rathouz, Margaret
Ricks, Thomas
Roberts, Sarah A.
Robinson, Molly L
Roman, Christopher Orlando
Roman, Kathryn E.
Roscioli, Kate
Ruiz, Steven L
Rupnow, Rachel
Rygaard Gaspard, Brandi
Saldaña, Mike
Sanchez Wall, Lina
Sankaranarayanan, Ananthi
Sayavedra, Alyssa
Schwarts, Gil
Sepulveda, Francisco
Sevier, John
Sianturi, Iwan Andi Jonri
Siebert, Daniel
Simon, Martin
Simpson, Amber
Smith, Amy
Smith, Bernard Ronald
Smith, Shawnda
Smithey, Montana
Smucker, Karoline
Son, Kyunghoon
Sourwine, Jasmine
Stoddard, Elyssa
Suárez, Mayra Zulay
Suazo-Flores, Elizabeth
Sung, Yewon
Swart, Michael I.
Tague, Jenna
Talbot, Jennifer
Thrasher, Emily
Towers, Jo
Tucci, Anthony
Turner, Blake O'Neal
Turner, Erin
Tyburski, Brady A
Varghese, Sijo
Voyias, Karley Violet
Waddell, Jr., Glenn
Warshauer, Hiroko Kawaguchi
Watkins, Jonathan D
Webb, David C
Webel, Corey
Wei, Xinyu
Weiland, Travis
Welder, Rachael Mae
Wessman-Enzinger, Nicole Marie
Westby, Kathryn R.
Wheeler, Ann
Willett, Brooklynn
Winsor, Matthew
Witt, Nicholas
Wynn, Lynda
Yao, Xiangquan
Yavuz, Selim
Yilmaz, Zuhal
Zaldívar, José David
Zhou, Lili
Zhuang, Yuling

Zimmerman, Stacey C.	Zolt, Holly	Zwanch, Karen

Preface

The Forty Fifth Annual meeting of the North American chapter of the International Group for the Psychology of Mathematics Education was held PME-NA 45 in Reno, Nevada, Oct. 1-4, 2023. The conference theme is listed below:

Engaging All Learners

Math learning should be a joyful experience for all students. When students are engaged and inspired, they are motivated to learn. Instruction that targets the learning needs and interests of our students makes it possible for students to excel in learning math. Participants in the conference explored how to create conditions to support learning that build on student engagement and interest in addition to other research engaged by the PME-NA community. The specific conference theme questions explored as part of the conference was:

- How can we engage all students to learn math content by building on their interest and motivation to learn?
- How do we design learning environments that take students and learning into account?
- What are the design features of tools and curricula design features considering student engagement and interest in supporting learning?
- How do we build partnerships with schools and the community to support student engagement and math learning?
- What research agendas should we pursue to ensure that all students reach their potential by paying attention to engagement and learning needs?

The acceptance rate for Research Report was 45%, the acceptance rate for brief research reports was 70 %. The acceptance rate for posters was 90%. Note: some papers were accepted in alternate format than originally proposed. The total number of participants who submitted proposals as co-authors was 1083.

Plenary Speakers

Motivation and Embodied Cognition

- Mitchell J. Nathan, Ph.D., University of Wisconsin at Madison
- James Middleton, Ph.D., Arizona State University

Connecting Math to Real-world Experiences, Culture and Technology

- Lisa Lunney Borden, Ph.D., St. Francis Xavier University, Canada
- Jose Luis Cortina, Ph.D., National Pedagogical University, Mexico City
- Theodore Chao, Ph.D., Ohio State University

Play Experiences and Math Learning Panel Presentation, *"What Do You See in Mathematical Play?"*

- Nathaniel Bryan, Ph.D., Ed.D., The University of Texas at Austin
- Melissa Gresalfi, Ph.D., Vanderbilt University
- Naomi Jessup, Ph.D., Georgia State University
- Amy Parks, Ph.D. Michigan State University
- Tran Templeton, Ed.D., Teachers College Columbia University
- Anita Wager, Ph.D. Vanderbilt University

Preparing Teachers to Engage Students (closing the plenary sessions)

- Robert Berry III, Ph.D., University of Arizona

The local organizing committee would like to thank the steering committee for all their support and everyone who helped make this conference a success.

Table of Contents

PLENARY PAPERS

AN ARGUMENT FOR ENGAGEMENT AS A FUNDAMENTAL CONSTRUCT FOR UNDERSTANDING MATHEMATICS LEARNING

UN ARGUMENTO A FAVOR DEL COMPROMISO COMO CONSTRUCTO FUNDAMENTAL PARA COMPRENDER EL APRENDIZAJE DE LAS MATEMÁTICAS

James A. Middleton
Arizona State University
jimbo@asu.edu

An extended (and probably unnecessary) parallel is drawn between engagement in mathematics and engagement in musical performance. Key facets of engagement are described and a model of how mathematics engagement plays out in task-level activities is discussed in light of new findings related to its social and emotional facets. Implications for instructional practices that flow from this research are presented. The article concludes with suggestions for future research that incorporates understandings of identity and emotional object as promising directions.

Keywords: Engagement, motivation, emotion, affect, identity

My old mentor, Tom Romberg, he of *Curriculum and Evaluation Standards for School Mathematics* fame, once reflected to me that mathematics is like music. "You don't become a musician by playing scales. You become a musician by playing music." In my other life, I am a musician, and I love creating music, writing, and composing songs that try to say something about life and its meaning, and playing these songs to entertain and uplift people. When Teruni asked me to perform a couple of my songs prior to this address, I thought about the parallels between my relationship with music and my relationship to mathematics, and crafted this essay using some of these parallels as examples that might resonate to those of us interested in mathematical engagement—the sense of connection people have with mathematics, and their interaction with it and others as they experience it.

I recently finished a marathon 37-hour studio session with my band. During those 37 hours, our team of 6 band members, one sound engineer, backup singers and "support team" (family) worked collaboratively to develop the structure of sixteen songs, overlaying different instrumental and vocal parts, and troubleshooting problem areas of tempo, pitch, orchestration, and voicing—all areas that together, make a song potentially compelling to the listener. At the end of those 37 hours, we had rough cuts of each of our songs that served as sort of "existence proofs" that our creative ideas were correct. But even at the end of the session, the songs are not polished, the final beauty yet unrevealed.

The process of developing this body of work echoes the general research on engagement in mathematics. Members of our community of practice displayed various facets of engagement: Cognitive, Behavioral, Affective/Emotional, and Social, at different intensities at different times in the process of solving the many problems we encountered when creating something that for us was new and pleasing. One of my musical mentors, Igor Glenn, who played with Glen Campbell, Bobby Weir, The New Christy Minstrels, and John Denver told me, "Jim, being a musician is spending your life collecting licks and then putting them together in new ways." Much like music, when learners are "engaged" in mathematics, they employ deep processing, instantiating, or developing procedures for combining smaller units of thinking into routines that, when compiled, constitute a problem solution (Cognitive Engagement). In music we can liken this to composing a song such that the finished product has intellectual and aesthetic rigor.

In mathematics, learners must also perform tasks, using appropriate tools for thinking and communication (Behavioral Engagement). In music, one must be able to play the notes, the licks and riffs that make up the pieces of the composition. Notice that the procedures enacted—the notes, licks and riffs are not the music. They are elements of understanding that are combined to create the music. The tools musicians use—instruments, click-tracks, microphones, and effects—determine how a group of musicians can enact the composition, affording and constraining the possible ways in which a composition can be played.

In mathematics, learners use emotional feedback to gauge task efficacy and to monitor engagement while learning (Affective/Emotional Engagement). Musically, the "feel"—the sense of emotion, whether it is happiness, sadness, frustration, or anger—one experiences while playing is transferred to the finished piece. Sometimes these emotions get in the way of quality performance—inhibiting the singer, for example, from adequately forming the right tone, as their throat closes with sadness. Sometimes emotions related to poor performance engender a negative feedback cycle that prevents the musician from enacting skills that they had previously been able to perform. But sometimes, emotion empowers a player to try something new that elevates the playing of herself and her bandmates to a new level of understanding of not just the song, but of musicianship itself.

Lastly, in mathematics, learners play off their peers, reacting to and modifying behavior to support each other and to get value added from the collaboration (Social Engagement). Sometimes this type of engagement is productive, leading to improved learning, and sometimes it is inhibitive, leading to a breakdown of learning and its associated cognitive and social relationships. In recent work, my colleague Mandy Jansen and our team listened to students reflect on their relationship with their peers as they learned mathematics together. Their responses were laden with references to practices that impacted their own engagement. Such practices included social loafing, disruptive behaviors, disrespecting others, including the teacher, and domineering conversation, as examples of social interactions that inhibited individual engagement. But they also referred to practices such as helping one another, providing encouragement, and explanation as reasons why they remained engaged (Riske, et al., 2021). Likewise in music, members of a team must listen to each other, playing with their eyes and ears, such that each person's contribution is unique, but coordinated with those of their bandmates. Respect, attention, support, and encouragement are all critical for one's musical development.

Much of the dynamics of engagement witnessed by teachers is due to the complexity of social acts in service to mathematical discourse. I have used the metaphor of improvisational jazz in previous articles, making the case that members of a community of practice play much the same roles as musicians in a band, each contributing some element to the larger discourse that corresponds to their expertise, confidence in their expertise, and their personal interpretation of the mathematics as well as the roles others take on. Curriculum serves as a basic composition by which the discourse has structure, and where, given the sociomathematical norms established in the classroom, individuals can "riff" off each other, providing creative input to the discourse, such that what ultimately results is a performance unlike any other. Yet, it still maintains the structure of the composition—the curriculum—to ensure coherence and aesthetic wholeness. Also like many bands, the "conductor"—the teacher—listens to the composition as it is being played, correcting mistakes, interpreting, and orchestrating the performance to display rigor and aesthetic quality.

When each of these facets of engagement are entered into in productive ways, students create new mathematical knowledge, feel interested and efficacious, value mathematical thinking as useful and important, and ultimately tend to pursue mathematically intensive work more readily in the future (Middleton, Jansen, & Goldin, 2017). Just like music, a performance becomes an episode that may be called up from memory later and replayed in the head or used as the basis for future performances. With the use of recording devices such as whiteboards, flip charts, graphing utilities and collaborative software, these memories don't have to stay in one's head. They can be made public—replayed—such that others may appreciate the thinking, borrow ideas, and make them their own, critique them and add new contributions (Lamberg & Middleton, 2009).

Now, there is only so far that analogies can take us before they become tiresome. But I hope that those of you who enjoy music and especially those who have played or sang with others resonate (see what I did there?) with this description.

When it comes to research on mathematics engagement, new advances in our understanding are being developed every day. I have boiled these findings down into four fuzzy take-aways that reflect the larger body of research on engagement, but that highlight new and important avenues that may benefit scholars who do not focus on engagement, per se, but who must take it into account when researching learning, teaching, and curriculum. These take-aways are:

Mathematics engagement is embodied—When we engage in mathematical thinking, our motor cortex, endocrine system, attention and arousal centers of the brain, cortical processing, sensory and effector systems are activated—always. This, of which my colleague and friend, Mitch Resnick will speak at this conference, is some of the most exciting work in our field, indeed in the entire field of cognition. For instance, we have evidence that, when we observe others doing a task, the same centers in our brain light up as if we ourselves were doing the task. When we observe others emote, we likewise tend to interpret the experience in ways that evince the same emotion as the observed. When we rotate objects mentally, our heads tilt and our hands wave, mimicking the transformation. Emotional mirroring of this sort is often likened to contagion, wherein one person may begin to laugh triumphantly, and others follow suit (Iaocoboni, 2009; Mafessoni & Lachmann, 2019).

Mathematics Engagement is emotional—Emotion is implicated into every facet of engagement. It is not separate from cognition, behavior, or social functioning (Meyer & Turner, 2006). Without taking emotion into account, we cannot understand the reasons students persist in the face of difficulty, nor when they engage in practices that may be maladaptive from a mathematical perspective. It is also becoming more evident that mathematics itself—its doing and the resulting products of that doing—is emotional. Our feelings color the directions we take, the problem-solving processes and tools we employ, and whether potentially productive ways of thinking are followed up upon (Goldin, 2007). Lastly emotion directly impacts the depth of cognitive processing one can employ when engaged in challenging mathematics. Students who interpret their experience as interesting and enjoyable tend to persist in the face of failure versus students who are frustrated and anxious (Tulis & Fulmer, 2006).

Mathematics Engagement is multifaceted—when I say that engagement is multifaceted, I do not mean in a trivial way—that there are many variables that contribute to engagement. What I mean is that, at any time in a learner's experience, they engage in *many* things: Mathematics problem solving, negotiating social relationships, general mood relating to life events, specific state characteristics, and physiological dynamics that each color their behavior in the moment of learning. This multidimensional situatedness implies that not all students will be optimally engaged at any one moment in time. While this is obvious to all of us who have taught before, it has not always been taken as a design imperative for curriculum and instruction. In the past, this fact has been overlooked in an attempt to create teacher-proof, general materials that turn out to have little meaning for learners (Thompson, 2013). Recent attempts to address this fact have led to true innovations including contextualized curriculum, positioning and other affirming teaching practices, and adaptive curriculum technologies (e.g., Gresalfi & Hand, 2019). Each of these attempts has been successful, such that curriculum, at least, and some instruction is quite different now than it was 40 years ago.

But it begs the question, "Is it possible, or even desirable for all learners to be optimally engaged in all tasks at all times?" My own answer to this question is tied to my belief (and extensive evidence) that each student has different gifts, different skills, preferences, and proclivities they have developed because of their past experiences in and out of school. To address each facet of engagement, cognitive, behavioral, social, and emotional at optimal levels, we must take the long-term approach, while designing tasks for the short term. What I mean by this is that social engagement is the critical variable that will determine the optimization. For any task, if we optimize cognitive engagement, there must be multiple levels of challenge and multiple means by which students are able to enter the task. Similarly with behavioral engagement, some students will be more adept at different times with different tools for different purposes. Emotional engagement will be dependent upon the varied histories of students and their past relationship with mathematics, their peers, and schooling in general. No single task will optimize all these facets for all students all the time.

Social engagement is the lever by which, even though at the task level engagement is not always optimal, over time, the support of the group, if orchestrated with care and inclusion as key principles can bolster lagging engagement, reward exceptional engagement, and offload sub-tasks to different members of the group to account for each learner's personal preferences and proclivities. Every task may not be optimal, but over time, all students may have the opportunity and the disposition to engage productively.

Mathematics Engagement is the result of feedback—Following up on this, one of the key revelations in the past 20 years in the fields of affect, motivation, engagement, and identity, is the notion that what happens in the short term impacts the long term. This may seem to be a kind of no-brainer—of course what we learn in the moment impacts our longer-term understandings. That is why we teach, to create moments that provide the opportunity for people to learn and grow mathematically. But in the field of educational psychology the focus on general processes—traits to be more precise—has overshadowed the importance of state processes to the extent that the unique nature of mathematics concepts, pedagogical practices, and social environment have been largely ignored, and their respective and collective impact, unexplored (Middleton, Jansen, & Goldin, 2017). But work in situational interest (Renninger & Hidi, 2019), achievement-related emotions (Forsblom, et al., 2022), task-based efficacy (Midgely et al., 1989), and self-regulated learning (Bell & Pape, 2014) show how dramatically what happens in the moment of learning becomes consolidated into longer-term trait-like habits and preferences.

A Model of Task-Level Engagement

This work is being combined with work on social facets of engagement to reveal a rather complicated, but still relatively simple system of influences where feedback from the moment reinforces the long-term attitude of students, which in turn, impacts the student's initial model of the learning situation, thereby partly determining their patterns of engagement (Middleton et al., 2023).

Together, these newer empirical results can provide some direction for the design of mathematics curriculum and teaching practices. The syllogism I am drawing here is complex. The web of implications from object to predicate flows to areas yet unknown, but here are my understandings of the field of engagement-related research currently, and their implications for the improvement of the mathematical well-being of students (See Figure 1).

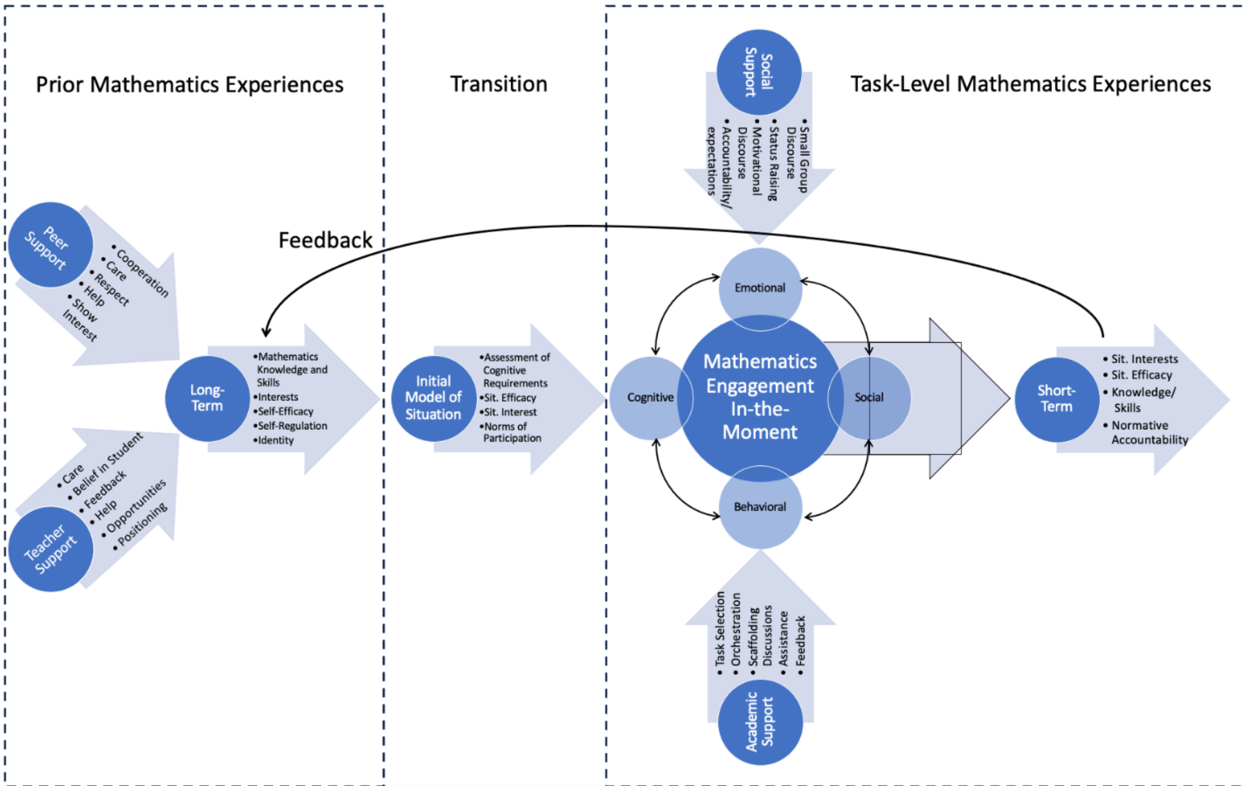


Figure 1. Major interacting facets impacting mathematics engagement.

When we think of prior mathematics experiences, the residue of these experiences makes up a person's memories. In those memories, students recall support they have received by their peers (or not), as well as that provided by their teachers (or not). Other factors from the student's culture and out-of-mathematics class experiences are also implicated here, but I have not included them in Figure 1 for clarity's sake. These memories of mathematics past are used as evaluative templates to help the student determine their role, mode, requisite knowledge, and feelings about the task set before them.

There is a transition point between the introduction of a task, and its doing. In this transition, a student may assess their own ability with respect to the task requirements (task efficacy), their interest in the mathematics, and in the context (situational interest), and how they will fit in with their group mates (norms of participation). These things determined, even if tentatively, will color the student's engagement-in-the-moment.

Then, at the task level, while engaged, there is constant monitoring of the cognitive, behavioral, social, and emotional characteristics of the task. As one is engaged, the initial model of the situation may be altered as one recognizes that they have knowledge to bring to bear on the problem, or that their group mate brings up an important point. So, the situational interest displayed, the situational efficacy attributed, the behaviors employed, and the person's social role play off each other dynamically. Peers and teacher provide potential support in the form of scaffolding, motivational talk, helping and feedback. The social acts teachers and peers proffer in carrying out the task can improve aspects of engagement (for some students) or inhibit them (for other).

Together, the initial model, its dynamic interaction with situational variables, and the support interjections of teacher and peers provide information for the learner to update portions of their long-term engagement attitudes. This feedback reinforces already developed notions of mathematics and the learner's ability and role within it and provides another episodic memory that can be called upon to direct future mathematics engagement in similar situations.

Over time, the stereotypical mathematics problems given to students, the routines of mathematics classes, and the kind of feedback typically provided students create robust, long-term attitudes towards mathematics that color students' future expectations of mathematical engagement, including their tendency to select mathematically intensive coursework and future occupational aspirations (Sullivan, 2013; Betz, 2023). We are all familiar with reports showing the steady decline of positive mathematics motivation from the middle grades onward. Most students leave compulsory education with less-than-positive long-term engagement patterns. So, what are the key levers we can employ to reduce the slope of this decline? Might we even be able to reverse it for more students? It is to this question I devote the remainder of this essay.

Implications for Practice

It has always baffled me why our field tends to focus its attention on the long-term manifestations of mathematics engagement instead of the short-term. It is only in the moment of learning where we can make an impact. Luckily there are a handful of practices that have been shown to improve patterns of engagement, particularly those associated with social support--the degree that students have opportunities to interact in a social environment such that their ideas are valued, and they feel welcome to be there. In other words, when students can feel like they matter, both their mathematical thinking and they as members of the learning community, they will tend to become more engaged:

Small Group Discourse. Discourse for engagement encourages nearly all participants in a group to share their thinking. Moreover, students listen to try to understand each other's thinking to build consensus around the mathematical idea.

Status Raising. The teacher *and students* regularly use language that magnifies *specific student's strengths* with respect to knowing and doing mathematics and assumes capability instead of shortcomings. This kind of feedback serves a reward purpose, enhancing the sense of task efficacy in the learner, and as information for others regarding what good mathematical thinking looks like. It is important that such status raising is distributed authentically to good ideas across each member of the classroom.

Motivational Discourse. When the teacher provides explicit language that supports student motivation using warm and welcoming language. These efforts can take several forms: 1) Focusing on the process of learning, challenging students, viewing errors as constructive, or supporting persistence; 2) Modeling positive affect to reduce anxiety and address emotional needs; and 3) Encouraging peer support and collaboration emphasizing joint goals & shared responsibilities.

Accountability/high expectations. When the teacher holds students accountable for engaging with learning of content such that all students are expected to participate in class (work hard and put forth effort, communicate their thinking, and listen to the teacher and each other). A number of strategies here have been shown to be productive including: 1) Cold calling or using random calling to lower incidence of loafing; 2) circulating the room and, while doing so, pushing students to keep working on the math (not letting students get away with not trying hard in their groups); 3) Using a timer to let students know how much time they have; 4) Enabling students to share their thinking without being called upon; 5) Using structured protocols to help students learn to talk with one another – here, group work roles are explicitly referred to; and 6) ANY methods that don't allow students to hide—making explicit efforts to get more students involved in learning content.

Together, these practices have been shown to be related to increased situational interest and task efficacy in students and reduced negative emotions towards mathematics. Each of these outcomes is important, but the reduction of negative emotions directed towards mathematics has a multiplier effect. That is, negative emotions in mathematics are strongly, negatively associated with the following aspects of engagement: situational interest in the task, social engagement, task efficacy, and beliefs that the mathematics in the task is useful and important (Middleton, et al., 2023). So not only do these practices directly support the development of interest and mathematics self-efficacy, they also may indirectly reduce the impact of the negative emotions students exhibit in relation to challenging and difficult content.

In a kind of shocking finding, my colleagues and I found that academic support practices like selecting challenging tasks, scaffolding discussions, assisting, and providing feedback can actually reduce some important aspects of affective/emotional and cognitive engagement (Middleton, et al., 2023). When tasks are highly challenging, for example, students necessarily make more mistakes, become confused and potentially frustrated. Teachers' attempts to maintain a rigorous level in mathematics learning experiences can reduce students' sense of efficacy and interest, which, unchecked, may lead to less productive engagement, or engagement in other activities that are unrelated to learning mathematics. In our conversations with students, and in large scale quantitative assessment, students in our studies felt that challenge was negative, and that pressing them to explain was also detrimental in relation to their engagement.

The implications of these two seemingly contradictory findings, to me are this: It is the social support system in the classroom that undergirds rigorous work. The teacher, like the conductor of an orchestra, must always be pushing to improve the knowledge students are displaying. This can be difficult and disconcerting and may cause negative feelings such as frustration and embarrassment. However, with teacher and peers providing social support—i.e., status raising, motivational discourse, encouragement to keep level of effort high and others, the negative impacts, motivationally, of rigor may be overcome such that the value of the knowledge gained in pursuing and tackling hard problems takes a prominent role in promoting a strong sense of efficacy and interest (Ahmed, et al., 2010). My friend Gerry Goldin called this complex interplay between emotional content "affective structures" that can take negative emotional content and re-interpret it as positive when barriers are overcome, and when a person's contribution to the collective understanding is respected and valued (Goldin, et al., 2011).

Next Steps in Research

We have an opportunity at this time, to greatly enhance our understanding of mathematics learning by taking engagement seriously. In the past we have focused too exclusively on the cognitive or behavioral aspects of engagement to the detriment of its emotional and social counterparts. These two facets are pregnant with possibility, and I would like to end this essay with some directions I see as fruitful.

Emotional Object. The latest research on emotion shows quite convincingly that we experience multiple emotions simultaneously. Work done by Schukalow et al., (2023), and my colleagues (Middleton et al., 2023) explain this phenomenon in part by the fact that emotions are not just positive or negative (valence), they also have intensity (e.g., very angry versus ticked off), they have activation (e.g., active such as enjoyment, or passive such as boredom), and they have object (e.g., I am ashamed of myself, or I am ashamed for my classmates). The combinations of these features create the vast majority of affective patterns we see in the classroom. Furthermore, they can be shown to associate with different manifestation of engagement. Mapping these emotional patterns to develop an understanding of productive emotional engagement and its role in interpreting and evaluating learning experience is a new and promising area of research.

Social Support. Our understanding of social support is being sped along by researchers focusing on culturally appropriate practices and identity, as well as others focusing on inclusion. It is time for the main body of mathematics education research to embrace this activity and try to incorporate their ideas into our body of work in engagement. As an example, I would like to point out Melissa Gresalfi and Vicki Hand's (2019) lovely model of identity construction that accounts for identity to be seen as building from norms of practice which exist within the frames and storylines within which students are positioned. These frames and storylines exist within larger cultural narratives that guide one's mathematics identity within one's larger set of identities related to race, class, gender, and intersectionality. Questioning dominant racial and economic narratives by which mathematics and its place in society have been positioned, and proposing intervening narratives has the potential for personal frames and storylines to become more supportive of diverse identities. But also, in the classroom, positioning, and redefining what it means to be mathematically competent, corresponds to the very social support practices I highlight in this article. They have the potential to scaffold students' playing a new kind of music in their mathematical experiences. Perhaps mathematics learning will become, like jazz, a beacon of inclusive artists and aficionados learning from each other and growing as new, diverse perspectives on mathematics and its role in our lives are shared.

Conclusion

The time is ripe for our community to step back and look at mathematics learning from a broader perspective: A perspective that accounts for the experience of learning—the engagement one has with the content, one's self, and one's community—as the fundamental unit of change in mathematics education. Those moments in which students persist in challenging tasks, supported by peers and teachers to showcase their ability, and where that ability is valued and respected will improve, ultimately, the affective responses learners have towards mathematics, and more importantly, change their identity with respect to mathematics in ways far more profound than focusing on a particular task sequence, or on teachers' practices isolated from how they impact engagement.

References

Ahmed, W., Minnaert, A., van der Werf, G., & Kuyper, H. (2010). Perceived social support and early adolescents' achievement: The mediational roles of motivational beliefs and emotions. *Journal of youth and adolescence*, *39*, 36-46.

Bell, C. V., & Pape, S. J. (2014). Scaffolding the development of self-regulated learning in mathematics classrooms: Results from an action research study examine the development of self-regulated learning behaviors in a seventh grade mathematics class. *Middle School Journal*, *45*(4), 23-32.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1). University of Nevada, Reno.

Betz, N. E. (2023). Integrating 50 Years of Theory and Research: Suggestions for Directions on Women's Career Development. The Counseling Psychologist, 51(6), 849–861. https://doi.org/10.1177/00110000231176246
Forsblom, L., Pekrun, R., Loderer, K., & Peixoto, F. (2022). Cognitive appraisals, achievement emotions, and students' math achievement: A longitudinal analysis. Journal of Educational Psychology, 114(2), 346.
Gresalfi, M., & Hand, V. M. (2019). Coordinating situated identities in mathematics classrooms with sociohistorical narratives: A consideration for design. ZDM, 51, 493-504.
Goldin, G. A. (2020). Aspects of affect and mathematical modeling processes. In R. E. Lesh, E. Hamilton, & J. Kaput (Eds.), Foundations for the future in mathematics education (pp. 281-296). New York: Routledge.
Goldin, G. A., Epstein, Y. M., Schorr, R. Y., & Warner, L. B. (2011). Beliefs and engagement structures: Behind the affective dimension of mathematical learning. ZDM, 43, 547-560.
Iacoboni, M. (2009). Imitation, empathy, and mirror neurons. Annual review of psychology, 60, 653-670.
Lamberg, T. d., & Middleton, J. A. (2009). Design Research Perspectives on Transitioning From Individual Microgenetic Interviews to a Whole-Class Teaching Experiment. Educational Researcher, 38(4), 233–245. https://doi.org/10.3102/0013189X09334206
Mafessoni, F., & Lachmann, M. (2019). The complexity of understanding others as the evolutionary origin of empathy and emotional contagion. Scientific reports, 9(1), 5794.
Meyer, D. K., & Turner, J. C. (2006). Re-conceptualizing emotion and motivation to learn in classroom contexts. Educational Psychology Review, 18, 377-390.
Middleton, J.A., Wiezel, A., Jansen, A. et al. Tracing mathematics engagement in the first year of high school: relationships between prior experience, observed support, and task-level emotion and motivation. ZDM Mathematics Education 55, 427–445 (2023). https://doi.org/10.1007/s11858-022-01432-9
Renninger, K. A., & Hidi, S. E. (2019). Interest development and learning. In K. A. Renninger & S. E. Hidi (Eds.), *The Cambridge handbook of motivation and learning* (pp. 265–290). Cambridge University Press.
Riske, A. K., Cullicott, C. E., Mirzaei, A. M., Jansen, A., & Middleton, J. (2021). Student Engagement with the "Into Math Graph" Tool. Mathematics Teacher: Learning and Teaching PK-12, 114(9), 677-684.
Schukajlow, S., Rakoczy, K., & Pekrun, R. (2023). Emotions and motivation in mathematics education: Where we are today and where we need to go. ZDM–Mathematics Education, 55(2), 249-267.
Sullivan, P., Aulert, A., Lehmann, A., Hislop, B., Shepherd, O., & Stubbs, A. (2013). Classroom Culture, Challenging Mathematical Tasks and Student Persistence. Mathematics education research group of Australasia.
Thompson, P. W. (2013). In the absence of meaning... . In Leatham, K. (Ed.), Vital directions for research in mathematics education (pp. 57-93). New York, NY: Springer.
Tulis, M., & Fulmer, S. M. (2013). Students' motivational and emotional experiences and their relationship to persistence during academic challenge in mathematics and reading. Learning and Individual Differences, 27, 35-46.

Acknowledgments

I would like to thank all those people who have inspired my rather chaotic work over the years. In particular, I would like to thank my students for continually pushing me into new and ever more exciting directions in mathematics, psychology, teacher education and their unholy union.

DESIGNING INSTRUCTIONAL RESOURCES TO SUPPORT TEACHING

El Diseño de Recursos Educativos para Apoyar a la Enseñanza

José Luis Cortina
Universidad Pedagógica Nacional, México
jcortina@upn.mx

Jana Višňovská
The University of Queensland, Australia
j.visnovska@uq.edu.au

We discuss the importance of bringing teaching to the forefront of instructional design. We do so by describing the process of developing an instructional sequence for early number, using design research. The instructional sequence was developed with the specific aim of supporting teaching, conceived as a complex and demanding job, not reducible to predictable routines. The sequence has caught the interest of an unexpected number of teachers in Mexico. We have followed up with some of them and have documented that the resource has benefited their practice significantly. In our account of the design process, we highlight what—from a theoretical point of view—we have come to regard as three guiding ideas that are central to designing for supporting teaching: (1) designing for a resource to be viable in teachers' classrooms, (2) designing for a resource to be regarded by teachers as relevant to their practice and (3) designing so that a teacher who has just taken an interest in a resource might fruitfully engage with it in her practice.

Keywords: Number Concepts and Operations, Instructional Activities and Practices, Teacher Knowledge.

In the development of resources to support mathematics instruction in schools, instructional designers must necessarily take a position regarding both learning and teaching. Whether implicitly or explicitly, multiple considerations come into play not only about how students are to learn mathematics, but also about the roles that teachers are to play in instruction. A strong inclination has been to prioritize in the designs issues related to student learning. This, of course, is an easily justifiable position since the improvement of students' mathematics learning is central to our aims as mathematics educators. However, this inclination carries the risk of overlooking teaching and even misconstruing it, jeopardizing the suitability of instructional design products to help improve mathematics education.

Commonly, resources are designed with the expectation of them being capable of adequately supporting students' learning, if administered appropriately. Many are developed through research and are based on rather complex and robust conceptualizations of mathematics learning. In contrast, not much attention is typically given in the design process to how teaching is conceptualized. Often, it ends up being framed as a practice in which compliance and adherence are considered to be essential. At the least, it is commonly expected that, for the use of the resource to render positive results, teachers would have to adhere to the indications of the developers. Hence, in instructional design, teaching is frequently conceptualized as a practice of a predominantly administrative and organizational nature, where teachers are seen to be operating within what Biesta (2007) refers to as "a causal model of professional action" (p. 7).

The framing of teaching as a mainly administrative and organizational occupation stands in sharp contrast with how research in the field has come to understand it. It has been shown that teachers necessarily shape how instructional activities play out in classrooms (Brown, 2009; Gueudet & Trouche, 2012; Pepin, 2018), and decide how student learning ultimately gets to be supported (Biesta, 2007; Dewey, 1997; Lampert, 2001). It has been also shown that good mathematics teaching is demanding, uncertain, and not reducible to predictable routines (Ball & Cohen, 1999; Lampert, 2001; Schifter, 1995). This is due largely to the relational and adaptive nature of teaching. Teaching mathematics well "necessarily requires that teachers teach in response to what students do as they engage in solving mathematical tasks"(Jackson et al., 2013, p. 647).

The aforementioned research makes it reasonable to question the value of instructional resources that are designed under the assumption that teaching is a predominantly administrative and organizational occupation. The use of resources that are based on an erroneous conceptualization of teaching might not lead to the improvement of mathematics instruction. But would it even be possible to design resources in which it is assumed that teachers are the ultimate producers of instruction? If so, could the design of such resources contribute to the substantial improvement of the teaching and learning of mathematics in classrooms?

For several years, our work as researchers and instructional designers has focused on exploring what the development of resources for supporting mathematics teaching entails. These are resources to be used by teachers to support their students' learning, developed with the understanding that good mathematics teaching requires functioning effectively in uncertain and indeterminate situations, where it is necessary to constantly make autonomous judgments, based on a pertinent rationale (see Hoyle, 2008).

In the paper, we first explain the research approach we have taken in developing the instructional sequence, which is based on the methodological principles of design research. Next, we give an account of the process of developing of the instructional sequence, which involved three research cycles, each of which led us to develop theoretical insights into the design of instructional resources to support teaching, and to modify the instructional sequence. At the end of paper, we discuss how, of the three guiding ideas, we came to regard the second as the leading one: designing for a resource to be regarded by teachers as relevant.

Research Approach

Our research has been conducted following the general guidelines of design research (e.g., Bakker, 2018; Cobb et al., 2003), where we have pursed both, the crafting of an educational innovation and the development of theory. It has consisted, so far, of three research cycles. Each cycle has been conducted with the purpose of both improving an instructional sequence on early number and of developing and refining theoretical ideas. In the following three sections of this paper, we explain what each of these cycles has entailed.

First Research Cycle, a Dual Classroom Design Study

The first research cycle in the development of the instructional sequence consisted of conducting a *dual* classroom design study (Gravemeijer & van Eerde, 2009), in a third-grade preschool classroom (equivalent to kindergarten in the USA). The design study was conducted in collaboration with Jesica, a preschool teacher who at the time was enrolled in a Master program and was being supervised by the first author. The decision to conduct the design study was prompted by Jesica's concerns about the lack of success she had had in teaching her preschool students to solve additive word problems, required by the Mexican curriculum.

The first two authors were familiar with the resource known as the *Patterns and Partitioning* instructional sequence (P&P; Cobb, Boufi, et al., 1997; Cobb, Gravemeijer, et al., 1997; McClain & Cobb, 1999), which had been demonstrated to be viable in supporting students to develop sophisticated early number ideas. These include developing notions that allow them to solve simple additive problems by reasoning about composing or decomposing quantities, instead of by counting by ones (Steffe, 1992; Steffe et al., 1988). Although the P&P was developed as part of the initial phase of a classroom design study conducted in a first-grade classroom, its designers considered that it could be a valuable instructional resource in kindergarten (McClain & Cobb, 1999).

The P&P was designed to support the collective development of early number ideas by providing opportunities for students to reason about patterns and partitions of collections of up to ten items, leveraging whole-class discussions. For example, in one of the initial classroom activities, students are supported to develop familiarity with pairs of numbers that add up to five, by discussing the different ways in which five monkeys could be in two trees .

The dual classroom design study was conducted with two chief goals. The first entailed investigating the practicality of using the P&P in a classroom like Jesica's. This goal included inquiring about the adaptations that would be necessary to make to the instructional sequence to increase its possibilities of becoming a viable resource in this type of classroom.

The second goal was to inquire about Jesica's educational practice in relation to the use of the P&P. Among other things, we planned to investigate how she would make sense of the instructional sequence, whether she would regard it as a useful resource and, if so, how, and why. In addition, we wanted to find out if the use of the sequence would lead to positive changes in her practice. If so, what would those changes be, and which elements of the instructional sequence would have favored them.

In preparing for the classroom design study, we assessed the students' elementary understandings of the number, that were prerequisites for children to be able to productively engage with the instructional tasks of the P&P sequence. It was found that the vast majority of Jesica's students had not yet developed those elementary understanding. Some children in Jesica's classroom were only successful with the word number sequence up to three and could correctly identify the names of only one or two single digit numerals (Cortina & Peña, 2018; Peña, 2018).

The results of the assessment presented the research team with two problems directly related to developing instructional resources for supporting teaching. The first one concerned the viability of the resource. The activities proposed at the starting point of the P&P sequence did not connect to what students in a classroom like Jesica's already understood about numbers and could do. Clearly, the P&P sequence could not readily be used in a classroom like Jesica's, if expected to be a resource that would help a teacher in supporting the development of numerical notions in their students.

In addition, the resource offered no guidance as to how it would be viable to start to work with children in such a classroom, so that they could be supported to eventually become readily capable of participating in the instructional activities of the P&P sequence. This meant that the P&P sequence not only lacked prompt viability in Jesica's classroom but also relevance for a teacher like her. A teacher in Jesica's position could reasonably—and correctly—consider the P&P sequence to be unsuited to her teaching practice, given the educational profile of the students with whom she worked.

The results of the assessment led us to significantly modify the instructional sequence, in an attempt to make it viable in Jesica's classroom and relevant for her. The starting point was changed and a whole new phase was included at the start, aimed at supporting children's development of essential number notions. The version of the P&P Instructional Sequence that was used in the classroom design study, with the modification we included, is presented in Table 1.

In an intensive out-of-classroom collaboration with the first author, Jesica conducted the classroom design study in her preschool (Peña, 2018; Peña et al., 2018). The first author and Jesica met after each lesson to analyze classroom events and co-design the following classroom activities. During these interactions, the first author both supplied elaborations of the rationale that justified the instructional sequence and kept a record of what clarifications and elaborations were needed.

Table 1. The First Version of the Modified P&P Instructional Sequence

Phase	Overarching teaching goal	Specific learning goals
1	*Support the development of the essential number understandings up to five*	*Master the number word sequence* *Enumerate with one-to-one correspondence* *Use fingers to represent numbers* *Identify the names of written numerals*
2	Support students' reasoning about patterns and partitioning with numbers up to five	Reason about (and subitise) spatial patterns Reason about (and subitise) finger patterns Reason about number partitions in the 10-frame Subitise and reason about spatial patterns in the 10-frame Reason about how to solve arithmetic problems by composing or decomposing quantities, instead of by counting by ones
3	Support the development of the essential number understandings up to ten	Master the number word sequence Enumerate with one-to-one correspondence Use fingers to represent numbers Identify the names of written numerals
4	Support students' reasoning about patterns and partitioning with numbers up to ten	Reason about (and subitise) finger patterns Subitise and reason about spatial patterns in the 10-frame Reason about number partitions in the rekenrek Reason about how to solve arithmetic problems by composing or decomposing quantities, instead of by counting by ones

In an intensive out-of-classroom collaboration with the first author, Jesica conducted the classroom design study in her preschool (Peña, 2018; Peña et al., 2018). The first author and Jesica met after each lesson to analyze classroom events and co-design the following classroom activities. During these interactions, the first author both supplied elaborations of the rationale that justified the instructional sequence, and kept a record of what clarifications and elaborations were needed.

The classroom design study consisted of 21 instructional sessions that were taught over a 5-month period. In Phase 1, the teacher supported collective engagement in repeated counting with words and symbols through activities such as singing number songs, playing number-word games and board games. In Phase 2, for example, students reasoned about number partitions up to five in the 10-frame with a narrative involving a watermelon stall with two decks (see Figure 1, left). They were asked to advise a teacher's friend on arranging a number of watermelons on her market stall. While students proposed different arrangements, the teacher kept a record of these on the board, specifying how many watermelons would be in the top and bottom decks (see Figure 1, right).

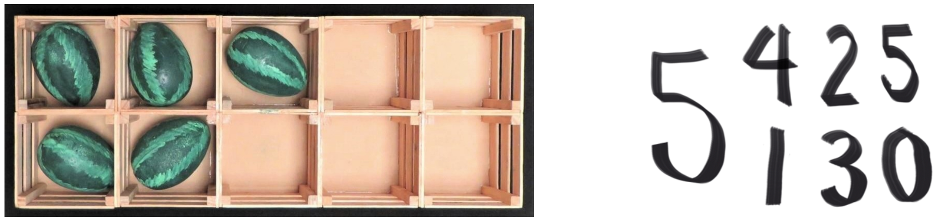


Figure 1. A watermelon stall and the capture of the record the teacher kept.

Phase 3 included similar activities to Phase 1, but with number words, collections, and written numerals up to 10. Phase 4 culminated in students reasoning about composing and decomposing quantities when solving problems about passengers getting on and off a Tour Bus.

> The bus left a park with 4 tourists, made a stop at a museum, and arrived at the destination with 10 tourists onboard.

When the teacher first asked the whole class to explain what happened at the museum, it was considered obvious that more tourists had boarded the bus. Lupe explained: "Six got on because six are missing for ten". Hernan, when asked whether he understood Lupe's response, said: "Yes! Six are missing because there were four, and six are ten". Both Lupe and Hernan were amongst the children who showed the least understanding of early number at the beginning of the classroom design study. Their responses to the problem illustrate how the great majority of the students not only came to solve rather advanced additive problems correctly, but how they did so by reasoning composing or decomposing quantities, not by counting by ones.

Our analysis indicated that the modified instructional sequence was viable for the targeted preschool classrooms (Peña, 2018; Peña et al., 2018). At the completion of the classroom design study, the team members had a strong practice-embedded understanding of how the designed resource could be used to support the reasoning about patterns and partitions up to 10, in educational settings like the one in which Jesica taught.

Second Research Cycle, Developing an Online Resource

The second research cycle was the result of the two authors remaining in communication with Jesica and maintaining a collaboration. After concluding her master's studies, Jesica started to teach in a new school. There, she was questioned by her concerned principal and colleagues about why she did not follow the recommendations of the Ministry of Education regarding how to teach mathematics in preschool. She was using whole class activities rather than small group work, and was focusing on small numbers instead of "maintaining the challenge" by teaching tasks with larger numbers. Instead of responding to pressure by reverting to institutionally legitimate forms of teaching, Jesica defended her teaching decisions by referring to her students' initial assessments, the results of the classroom design study in which she had participated, and the research literature.

Her teaching soon became of interest to her principal and supervisor when they realized how Jesica's students became much more eager to participate in mathematics than was typical in the classrooms of the school zone. Jesica was asked to give short workshops during the staff meetings, both at the school and the school-cluster levels. Several of her colleagues started to approach her for advice. We understood that Jesica alone could not provide support to her colleagues comparable to that she had received during her Master study. Hence, the aim of the second research cycle was to develop resources that could help Jesica in her efforts to support her colleagues in improving their early number instruction.

A website was created (www.sentidonumerico.com) with online resources intended to aid Jesica in leading the workshops and helping the teachers in their planning and decision making. We understood that if the website was to become a resource capable of *participating with teachers* on their selection of goals and tasks that were suitable for their students, our first job was to facilitate its plentiful, independent, and confident use by teachers. We assured her that the website was designed well for Jessica's colleagues most accessible, and often the only, computer: a smartphone. It included descriptions and short videos of classroom activities, and printable tools that teachers could adapt and use in their classrooms (e.g., game boards). In addition to the immediate physical accessibility of the tasks and tools, our aim was to make the conceptual resources accessible. Each activity was thus labelled to identify a specific learning goal and a phase of the sequence (see Table 1).

Based on access data and teachers' questions, we noticed that Jesica's colleagues were much more interested in the classroom activities aimed at supporting essential number understandings (Table 1, Phases 1, 3), than in the rest of the online resources. They readily recognized the importance of their students developing basic counting skills, despite this not been a priority explicitly stated in the Mexican curriculum. In contrast, they did not recognize the relevance of the learning goals within P&P phases of the sequence and found it difficult to justify the considerable investment of time and effort that pursuing these goals required.

The contribution of P&P activities to supporting early number learning was the key innovation of the resource. Like any substantial teaching innovation, this presented teachers with what research on implementation refers to as an ecological disruption (Koichu et al., 2021). Unsurprisingly in hindsight, the teachers initially responded to this by non-participation with the innovative parts of the resource.

The research team construed this situation as a design problem that entailed both the relevance of the resource and its clarity, for the teachers. The resolution required finding ways of communicating effectively that the innovation addressed problems that the teachers already considered relevant to their teaching, and how it did so.

It is worth clarifying here that, over several decades, "problem solving" has been a key goal in the Mexican mathematics curricula (Secretaría de Educación Pública, 2011), including at the preschool level. Not surprisingly, the concern for making students problem solvers has entered the prevailing teacher culture. We knew that there was much frustration amongst Jesica's colleagues, because only a few of their students ever became proficient in solving word problems. In addition, it was apparent to us that if the teachers were to focus solely on supporting the development of essential number understandings, their students would not come to solve the Tour Bus problems with flexibility when larger numbers were involved, as the only number patterns at their disposal would be those of sequential order (i.e., counting up and perhaps down by ones, cf. Graven, 2016).

Our instructional design challenge thus became to find ways of adjusting the instructional sequence so that the teachers could more easily recognize the close relation between the concerns they already had regarding their students' difficulties with problem solving and the pursue of the P&P objectives. We made three modifications to the instructional sequence with the expectation that it would help the teachers to recognize that (a) by supporting their students to reach the key learning goals of the instructional sequence, they would be providing them with valuable means for problem solving, and (b) while the essential number understandings were necessary for becoming a proficient problem solver, they were not sufficient.

The first modification involved renaming "number understandings" to "number skills", to align with the language in which teachers made connections to their practice. The second modification involved renaming the teaching goals as addressing "basic" vs. "advanced" number skills (see new Phases 2 and 3 in Table 2). This removed the non-transparent "patterns and partitioning" language and presented the advanced number skills as a continuation of the basic skills, aimed at enhancing students' problem solving beyond the activity of counting. The third modification involved downgrading the Phase 3 (Table 2) into a transition stage to further support teachers in recognizing the advanced number skills goals as the key ones they needed to support.

Table 2. The Second Modification of the P&P Instructional Sequence

Phase	Overarching teaching goal	Specific learning goals
1	Support the development of the essential number understandings up to five	Master the number word sequence Enumerate with one-to-one correspondence Use fingers to represent numbers Identify the names of written numerals
2	Support students' reasoning about patterns and partitioning with numbers up to five	Reason about (and subitise) spatial patterns Reason about (and subitise) finger patterns Reason about number partitions in the 10-frame Subitise and reason about spatial patterns in the 10-frame Reason about how to solve arithmetic problems by composing or decomposing quantities, instead of by counting by ones
3	Support students' reasoning about patterns and partitioning with numbers up to ten	Reason about (and subitise) finger patterns Subitise and reason about spatial patterns in the 10-frame Reason about number partitions in the rekenrek Reason about how to solve arithmetic problems by composing or decomposing quantities, instead of by counting by ones

Formal and informal data indicated to us that the modifications encouraged considerable increase in teachers' engagement with both the advanced number skills activities and the rationale of the instructional sequence. However, our aim here is to illustrate the type of criteria that were followed when trying to design a resource to support teaching, when teaching is understood to be demanding, complex and independent, and teachers are considered to be the ultimate producers of instruction. Our design failure, which manifested as teachers' lack of use of the key parts of the resource, shows where the designers' work was unfinished. As we have already illustrated, the resource was not ready, for teachers who had not participated directly in its development, to be easily recognized by them as worthwhile of fully engaging with it.

Third Research Cycle, Adjusting the Instructional Sequence and the Online Resources

In facilitating teacher workshops with Jesica, we became aware that there was one issue underlying the design of the sequence that was far from being clear to the teachers, namely, that it was critical for children to experience fun, joy, belonging and success as they got involved in the instructional activities, particularly at the beginning of the instructional sequence (Phase 1, see Table 2). During the classroom design study, we had attempted to ensure that the children had these kinds of experiences, regardless of how competent they were.

The modifications that we made to the instructional sequence during the second research cycle, seemed to make it easier for teachers to recognize the importance of supporting their students' development of relatively complex number skills with numbers up to ten. However, they seemed to think that the pursue of such a goal would require a kind of instruction in which they would have to be instructing and correcting the children, constantly. The instructional activities of Phase 1 (see Table 2) were based on stories and games, and when Jesica used them, she focused on conveying to her students that they were good at what she was asking them to do. In the case of Jesica's colleagues, it seemed that engaging students in mathematical activities expecting the children to enjoy them, and without correcting their mistakes, presented a significant pedagogical innovation.

From a learning perspective, our consideration about trying to ensure that children experienced fun and joy, when engaging in mathematics instruction, was justified in the literature (Boaler, 2019; Parks, 2020). In addition, we viewed teaching reliant on correction of errors as a case in point of what Rancière (1991) refers to as stultification in schooling: a mechanism through which students are subtly but persistently positioned as incompetent and lacking. At this point in the development of the instructional sequence, we became aware that supporting students to experience fun, joy, belonging and success was also of critical importance for teaching. As we explain next, it was directly related to the possibility of the instructional sequence being viable in the teachers' classrooms and, also, relevant to them.

Students' willingness to engage in mathematical activities makes teaching more manageable for the teacher, and the teacher more likely to participate with the resource. Conversely, if students come to view mathematical activities as tiresome, and themselves as incompetent and lacking, they will probably become reluctant to participate. This could easily create classroom management complications and lead a teacher to consider an instructional resource as unfitting for her class.

Once again, we construed the situation as being an instructional design problem, one that concerned the clarity of the instructional sequence and also, as we already explained, its viability and its relevance. We realized that the sequence of goals (Table 2) that described the instructional sequence addressed only mathematical content, keeping the forms of classroom engagement hidden. We then decided to include a new phase at the start of the in the instructional sequence (Phase 0; see Table 3), where the main goal is to support children's willingness to engage in early number activities, and enjoyment of doing so. We developed activities and teaching routines aimed at supporting teachers in coming to value and pursue this initial teaching goal based on noticing their students' enthusiasm.

Encouraging Results

At the current stage of our research project, we feel that there is still much that we need to understand in relation to designing resources that support teaching. However, there have been promising developments that suggest that we are on a proper path. Some of them are related to the interest that Jesica's colleagues have shown in the instructional sequence, which we have already mentioned. Others are related to later developments that have taken place on a larger scale.

Table 3. The Final Modification of the P&P Instructional Sequence

Phase	Overarching teaching goal	Specific learning goals
0	Support the development of an interest in and a taste for counting and numbers	Become interested and show joy when engaging in activities that involve counting or working with numbers

The last version of the instructional sequence has caught the interest of an unexpected number of teachers. In 2020, after Mexican schools closed for COVID-19, we started to collaborate with several teaching organizations and offered intensive online workshops, organized in 2-hour increments over three consecutive days. Although we do not know how the attending teachers adapted the designed resource in their teaching, they valued the experience positively, to a surprising degree. One of the workshops was attended by 850 teachers who were present during all three days, even though no external incentives were provided to participate. The Facebook community that we created to keep in touch with the teachers has reached 7000 members.

In the summer of 2022, we offered a 27-hour online teacher professional development course, to be delivered in eleven Saturday sessions, throughout the 2022-2023 school year. The course was advertised in the Facebook community. We designed the course contemplating that we would have between ten and fifteen participants. However, the number of applications exceeded one hundred. We decided to select thirty participants, giving priority to the teachers who worked in public school and, amongst them, those who worked in schools located in rural areas, where professional development opportunities are scarce.

At the time of writing this paper, nine of the eleven sessions have been held. There are twelve teachers who have participated in all the sessions. Overall, the course has helped us identify issues worth looking at closely, to improve the way in which the instructional sequence is formulated, and also the suitability of the resources we provide teachers online. It has also helped us to identify clearly that the instructional sequence can become a valuable resource for teachers who work in significantly different kinds of schools, one that helps them make judgments about how their groups are progressing, provides them with a pertinent rationale for deciding how to continue the educational work, and offers them clear suggestions about the activities, manipulatives and other means that they can employ in pursing specific learning goals.

Discussion

Our work as researchers and instructional designers has focused on exploring what the development of resources for supporting mathematics teaching entails, when good mathematics teaching is understood as being independent, intellectually demanding, and complex. The resources we seek to develop would be for teachers to use, with their full consent, and they would be regarded by teachers as useful and beneficial to achieve their educational endeavor.

Our research has led us to propose three theoretically oriented ideas to guide the design, refinement, and improvement of these types of resources: (1) designing for a resource to be viable in teachers' classrooms, (2) designing for a resource to be regarded by teachers as relevant to their practice and (3) designing so that a teacher who has just taken an interest in a resource might fruitfully engage with it in her practice. In retrospect, we recognize the second as being the leading idea. In fact, the importance of the other two can be understood in terms of how they can make a resource come to be regarded by teachers as relevant.

Concerning the first guiding idea, teachers would not be likely to consider a resource as relevant if when trying to engage with it, they find that it is incompatible with their students' current mathematical abilities, or with their real possibilities of progress. Teachers will also not likely consider a resource as relevant if they find that the proposed means of support are not well suited to achieve the specified learning objectives. In other words, a resource that is not viable in a teacher's classroom, is unlikely to be regarded as relevant, at least in the long run.

In developing resources to support teaching, it is thus important that they be crafted through a careful and rigorous instructional design process, such as that involved in conducting a classroom design study. The reader will recall that the P&P sequence was developed as part of a classroom design study, and the adaptations we made to it during the first research cycle were the result of another study of the same type.

But the viability of an instructional resource might be necessary condition for it to be relevant for teachers, but not sufficient. A resource might not be regarded by teachers as worthwhile of expending time and effort to engage with it and learn how to use it, if they do not see it as useful in pursuing goals they already consider important—regardless of how rigorous the research process of developing it was, how innovative it is, and that it has been used successfully in other classrooms. Hence, as we did during the second research cycle, the development of a resource to support teaching may require extra design efforts, tailoring it so that teachers can clearly recognize it as worthwhile. To be sure, this redesign process should not harm the viability of the resource, or else its relevance would be compromised.

The importance of the third guiding idea can also be seen in terms of relevance to the teacher. From the perspective of developing resources to support teaching, the proposed starting point of a resource, crafted in the form of an instructional sequence, must not only be viable in terms of students' current capacities to participate in the proposed activities. The activities must also be viable in terms of teachers' current capacities to engage with them productively. If a teacher can not readily engage in the use of an instructional sequence in a way that is productive and meaningful to her, it is likely that she will not regard it as worthwhile.

Conclusion

Instructional design has long been seen as a good way to seek improvements in mathematics education. For the most part, the focus has been placed on students' learning. Resources have been developed with the intention of making it possible to recreate in classrooms what research has found to be important in learning mathematics, both in general and of specific fields and ideas. Teachers, for their part, have been commonly seen as those responsible for implementing the innovations. They have been expected to have faith in them and to be keen to assume the expenditures of time, effort and learning that incorporating the innovations into their teaching will involve.

In this paper we have introduce a different approach to instructional design, one that entails considering that the successful support of mathematical learning requires good teaching, and that this kind of teaching is achieved by teacher who act with conviction, will, judgment, and autonomy. This generates new challenges for instructional design. As we have explained here, these challenges include ensuring that the designed resources can be recognized by teachers as relevant to their teaching, that they are viable in teachers' classrooms, and that they allow teachers to engage fruitfully with the resources, from the start.

References

Bakker, A. (2018). Design research in education: A practical guide for early career researchers Routledge. https://doi.org/10.4324/9780203701010

Ball, D. L., & Cohen, D. K. (1999). Developing practice, developing practitioners: Towards a practice-based theory of professional education. In G. Sykes & L. Darling-Hammond (Eds.), Teaching as the learning profession: Handbook of policy and practice (pp. 3-32). Jossey-Bass.

Biesta, G. J. J. (2007). Why "what works" won't work: Evidence-based practice and the democratic deficit in educational research. Educational Theory, 57(1), 1-22. https://doi.org/10.1111/j.1741-5446.2006.00241.x

Boaler, J. (2019). Developing mathematical mindsets: the need to interact with numbers flexibly and conceptually. American Educator, 42(4), 8-33.

Brown, M. W. (2009). The teacher–tool relationship: Theorizing the design and use of curriculum materials. In J. T. Remillard, B. A. Herbel-Eisenmann, & G. M. Lloyd (Eds.), Mathematics teachers at work (pp. 17-36). Routledge. https://doi.org/10.4324/9780203884645

Cobb, P., Boufi, A., McClain, K., & Whitenack, J. (1997). Reflective discourse and collective reflection. Journal for Research in Mathematics Education, 28, 258-277. https://doi.org/10.2307/749781

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in education research. Educational Researcher, 32(1), 9-13. https://doi.org/10.3102/0013189X032001009

Cobb, P., Gravemeijer, K., Yackel, E., McClain, K., & Whitenack, J. (1997). Mathematizing and symbolizing: The emergence of chains of signification in one first-grade classroom. In D. Kirshner & J. A. Whitson (Eds.), Situated cognition: Social, semiotic, and psychological perspectives (pp. 151-232). Lawrence Erlbaum.

Cortina, J. L., & Peña, J. (2018). Nociones numéricas de alumnos mexicanos de tercero de preescolar (Numerical notions of Mexican third-year preschool students). Educación Matemática, 30(3), 103-123. https://doi.org/10.24844/EM3003.05

Dewey, J. (1997). Experience and education. Free Press.

Gravemeijer, K., & van Eerde, D. (2009). Design research as a means for building a knowledge base for teachers and teaching in mathematics education. The Elementary School Journal, 109(5), 510-524. https://doi.org/10.1086/596999

Graven, M. (2016). When systemic interventions get in the way of localized mathematics reform For the Learning of Mathematics, 36(1), 8-13.

Gueudet, G., & Trouche, L. (2012). Teachers' work with resources: Documentational geneses and professional geneses. In G. Gueudet, B. Pepin, & L. Trouche (Eds.), From text to 'lived' resources: Mathematics curriculum materials and teacher development. Springer. https://doi.org/10.1007/978-94-007-1966-8

Hoyle, E. (2008). Changing conceptions of teaching as a profession: Personal reflections. In D. Johnson & R. Maclean (Eds.), Teaching: Professionalization, Development and Leadership (pp. 285–304). Springer Dordrecht. https://doi.org/10.1007/978-1-4020-8186-6_19

Jackson, K., Garrison, A., Wilson, J., Gibbons, L., & Shahan, E. (2013). Exploring relationships between setting up complex tasks and opportunities to learn in concluding whole-class discussions in middle-grades mathematics instruction. Journal for Research in Mathematics Education, 44(4), 646-682 https://doi.org/10.5951/jresematheduc.44.4.0646

Koichu, B., Sánchez Aguilar, M., & Misfeldt, M. (2021). Implementation-related research in mathematics education: The search for identity. ZDM Mathematics Education, 53, 975–989. https://doi.org/10.1007/s11858-021-01302-w
Lampert, M. (2001). Teaching problems and the problems of teaching. Yale University Press.
McClain, K., & Cobb, P. (1999). Supporting students' ways of reasoning about patterns and partitions. In J. V. Copley (Ed.), Mathematics in the early years. National Council of Teachers of Mathematics.
Parks, A. N. (2020). Creating joy in PK–grade 2 mathematics classrooms. The Mathematics Teacher, 113(1), 61–64. https://doi.org/10.5951/mtlt.2019.0250
Peña, J. (2018). El sentido numérico en preescolar (Number sense in preeschool) [Unpublished Masters Thesis]. Universidad Pedagógica Nacional]. Mexico City.
Peña, J., Cortina, J. L., & Visnovska, J. (2018). What happened at Frida's museum? Teaching Children Mathematics, 25(3), 175-179. https://doi.org/10.5951/teacchilmath.25.3.0174
Pepin, B. (2018). Enhancing teacher learning with curriculum resources. In L. Fan, L. Trouche, C. Qi, S. Rezat, & J. Visnovska (Eds.), Research on mathematics textbooks and teachers' resources: advances and issues (pp. 359-374). Springer. https://doi.org/10.1007/978-3-319-73253-4
Rancière, J. (1991). The ignorant schoolmaster. Stanford University.
Schifter, D. (1995). Teachers' changing conceptions of the nature of mathematics: Enactment in the classroom. In B. S. Nelson (Ed.), Inquiry and the development of teaching: Issues in the transformation of mathematics teaching. Education Development Center.
Secretaría de Educación Pública. (2011). Programa de estudio 2011. Guía para la educadora. Educación básica. Preescolar (Program of Studies 2011. Teacher guide. Preschool). Author.
Steffe, L. P. (1992). Learning stages in the construction of the number sequence. In J. Bideaud, C. Meljac, & J.-P. Fischer (Eds.), Pathways to number: Children's developing numerical abilities (pp. 83-88). Lawrence Erlbaum. https://doi.org/10.4324/9780203772492
Steffe, L. P., Cobb, P., & von Glasersfeld, E. (1988). Construction of arithmetic meanings and strategies. Springer. https://doi.org/10.1007/978-1-4612-3844-7

STORYTELLING, MATHEMATICS, AND COMMUNITY

Theodore Chao
The Ohio State University
chao.160@osu.edu

In this plenary discussion, Dr. Chao presents his research framework and reflections from engaging in Digital Mathematics Storytelling within Black, Asian American, and Asian American communities in multiple countries. The framework, based heavily around storytelling, counter-storytelling, and Critical Race Theory, has been employed as a workshop to elicit mathematics video stories from youth and mathematics teachers. Here, Dr. Chao reflects on what he's learned from these workshops and how he's started to recognize not only the power of storytelling for forging mathematics and community identities, but the dangers to our society because of social media and weaponized uses of mobile video everywhere. He ends by calling for a new critical digital media literacy within our field of mathematics education.

Keywords: Equity, Inclusion, and Diversity; Social Justice; Technology

Objectives

Storytelling, a fundamental thread in the fabric of human culture, serves as the original medium through which knowledge, traditions, and values get passed down from generation from generation (de Jager et al., 2017; Prusak et al., 2012). Storytelling can be seen as the original form of culturally sustainable pedagogy, embodying the essence of shared human experiences across diverse cultures (Paris & Alim, 2017). Storytelling also entices young minds, particularly opening up space for voice and agency from young peoples from oppressed communities to express themselves (Love, 2014; Nunez-Janes & Cruz, 2013). Furthermore, storytelling can serve as an inclusive tool for acknowledging and respecting the heterogeneity of all cultures and communities (Solórzano & Yosso, 2002).

Furthermore, storytelling plays a significant role in decolonizing educational practices (San Pedro & Kinloch, 2017). Focusing on storytelling challenges the hegemony of Western epistemologies by recognizing and centering knowledge as emerging from the people and the community, rather than imposing knowledge bases from outside the community as absolute truth (Matias & Grosland, 2016). This is particularly crucial in indigenous and marginalized communities, where traditional knowledge systems have often been undervalued or entirely erased.

In the context of mathematics education, storytelling offers an innovative approach to learning. Storytelling enables youth to not only showcase the ways they engage with mathematics within their communities but also facilitates their self-positioning in relation to mathematics (Chao et al., 2022). Through stories, learners can relate mathematical concepts to their lived experiences, thus making learning more relatable and meaningful (Zazkis & Liljedahl, 2009).

The objective of this plenary discussion is to weave together the intricate threads of storytelling, community, and mathematics, and envision how they are intricately intertwined. This is a snapshot of my thinking in progress, as I engage in the work of how storytelling and mathematics connect through my own lenses as a Chinese American cisgender male living in the United States. I have engaged in this research work with Black, Asian American, and Asian communities in various parts of the United States, Vietnam, and Indonesia. This plenary discussion serves as a place to collect my reflections and insight on what I think I'm learning. And what I am finding is that storytelling in mathematics, in this moment, is significant not only for educational researchers but also for individuals who are listeners, amplifiers, and storytellers in the modern digital landscape. Through our conceptual and community imagination, I hope this plenary discussion details how storytelling can be harnessed as a transformative tool in mathematics education and community-building, especially within our modern technology-driven, hyper-connected, and polarized society.

Theoretical Perspectives

Storytelling: The Heart of Community

Storytelling serves as the backbone of human civilization. In an era where technology is rapidly evolving, storytelling remains a steadfast medium through which traditions, values, and knowledge are continually passed down (Prusak et al., 2012). For centuries, communities have relied on stories to fortify their cultural heritage and impart wisdom to subsequent generations. Furthermore, storytelling can foster inclusive environments in which youth develop strong senses of identity and agency, often based upon their community, family, and heritage.

However, with the advent of the digital age, mainstream media has (in)advertently created a disconnect between individuals and their traditional storytelling roots. The dominant narrative structures, often aligned with Western storytelling formats, overshadow the rich diversity of storytelling that defines various cultures. This creates a single format for how stories are shared, creating a true single-story experience, one in which all stories have a traditional protagonist who must go through Acts 1, 2, and 3 to culminate in a nice, tidy ending. This mono-cultural storytelling dominance robs children of the variance of storytelling types and formats from diverse cultures and narratives. Commercialized mainstream media not only shows the same types of stories over and over again, be it in books, movies, and YouTube videos, but encourages passive viewership of youth to consume, not just large amounts of media, but also goods and services (Hill, 2011). *So while we live in a world surrounded by stories and technology to create our own stories, I see that many of the stories we see are still only the stories from the dominant groups, and they are not used to share important knowledge or culture, but rather to sell, to influence, and to propagate, rather than helping all of us develop our own storytelling skills or learn to grow closer to our communities.*

We are Inherent Storytellers

Human beings have been guardians and purveyors of stories since the dawn of consciousness. The rich tapestry of our shared heritage is painstakingly embroidered with narratives passed down through millennia. In the Black and Asian American communities I have been working within, storytelling ascends beyond tradition; it becomes the lifeblood that courses through generations, linking the past to the present and foreshadowing the future. From the profound ancestral lore among Asian American families to the sagas of defiance and resilience among Black American families, our stories are interwoven into our very fabrics of existence.

And so, our children are natural-born storytellers. Their vines of imagination and their buds of narration germinate early. However, traditional education structures often weed out these budding storytelling skills, especially when connected to mathematics. A folk science myth that continually lives within our schools is the archaic dichotomy of a "left brain" that focuses on logic versus a "right brain" that focuses on creativity, wrongfully separating mathematics from creativity and completely dissociating from storytelling (Geake, 2008). In truth, we know that mathematics is an art, intrinsically tied to our storytelling ability (Zazkis & Liljedahl, 2009). But the straitjackets of standardized mathematics teaching continue to suffocate this creative link between mathematics and storytelling. *So, this is not a mere call to arms; it is a clarion call for an intellectual insurgency to reclaim spaces within mathematics education that allow children to unfurl their stories and, through this, bolster their cultural identities and critical thinking: All of us and our children are mathematical storytellers.*

Narrative Identity and Counter-Storytelling

Identity is not only embodied within the stories a person tells about themselves, but also encompasses the actual act of narrating or storytelling (Sfard & Prusak, 2005). Identity is a verb, made and remade through the act of storytelling. Our stories are not merely descriptions of a static reality, but rather dynamic constructs that can change over time and context. Our narratives serve as constructs that embody our range of experiences, characteristics, and expectations, thereby defining the creation and evolution of our personal and social identities. Even more important than telling a story to explore our identity is the way that identities are reified and endorsed through the acceptance, validation, and re-telling of our narratives. Simply put, our stories are our identities.

Counter-storytelling, therefore, involves sharing stories and experiences that challenge existing dominant (and oppressive) narratives and stereotypes (Solórzano & Yosso, 2002). Counter-storytelling is a tool for individuals in marginalized communities to highlight their experiences and perspectives, and challenge destructive narratives that perpetuate harmful stereotypes. Through counter-storytelling, individuals and communities reclaim their own narratives and thereby their own identities (Chao et al., 2021). *And we only learn how to tell counter-stories if we know how to tell stories first.*

Need for Creating Safe Spaces for Sharing Stories

The transformative kiln of storytelling happens during the moment of collective sharing - around campfires, across kitchen tables, or over steamers of dim sum. Magic happens when individual narratives, through collective listening and feedback, metamorphize into a story for the community during the sharing that happens in a *storycircle*. I build on this concept of storycircles, using Lambert's StoryCenter model (2013), as a safe space for a small group of storytellers to share their stories in progress, not just to elicit feedback and commentary from others, but to also feel out various parts of themselves as they take on and inhabit their own stories.

In Black and Asian American communities, our youth already navigate an intricate labyrinth of identity and marginalization based on the many shifting ways they are positioned and how they position themselves. Engaging in a storycircle, then, in which a young person shares a story in progress, a story they are still feeling out, to others who are listening and not judging, serves as a crucible for self-realization, self-actualization, and agency. Here, sharing is not just cathartic, but an act of defiance and self-assertion. I am no longer just the way you see me; I am telling my story and together, we are remaking who I am through this story.

I see a parallel the storycircle in the realms of ciphers, written about extensively in the hip-hop education world (Alim et al., 2023; Emdin, 2016; Love, 2014). As a fan of hip-hop, I am enthralled at the magic that happens when artists encircle and express their truths, building off each other to create rhymes and stories. These ciphers become sanctuaries of unrestrained creativity, solidarity, and expression in which artists embody their full selves. However, the main barrier for entering a hip-hop cipher is the ability to rhyme or spit. The cost for entry can be perceived to be quite high. In contrast, my conceptualization of storycircles is that they are a "soft" option for anyone to be able to share their story in whatever manner they want, be it verbally, through text, or image. I position the storycircle as an inviting space of collaboration, to collectively generate, define, and revise our community mathematics stories. *My emphasis on the safe space of a storycircle is to refocus our gaze from the polished, finished story and instead on the intricate, delicate process of how we collectively weave our mathematics stories together.*

Grounding Principles of Black Lives Matter

While the Black Lives Matter movement started as a collective call for systemic justice against police murder of Black people in the United States, the movement itself has become a global call for justice for all oppressed peoples, particular the violence endured under the vestiges of imperialism and white supremacy (*What We Believe*, 2018). The Black Lives Matter movement has significantly influenced my scholarship, my activism, and my thinking, particularly the movement's guiding principles (Chao & Marlowe, 2019). These thirteen guiding principles are: (1) Restorative Justice, (2) Empathy, (3) Loving Engagement, (4) Diversity, (5) Globalism, (6) Queer Affirming, (7) Trans Affirming, (8) Collective Value, (9) Intergenerational, (10) Black Families, (11) Black Villages, (12) Unapologetically Black, and (13) Black Women (Mathews & Jones, 2022). While each of these guiding principles is important, in this storytelling work, I find connections to the guiding principles of (2) Empathy, (3) Loving Engagement, (4) Diversity, (5) Globalism, (8) Collective Value, and (9) Intergenerational.

A focus on storytelling emphasizes listening to each other, creating opportunities to hear each other's stories, and practicing *empathy*. And through hearing each other's stories and sharing our stories with each other, we engage in *loving engagement*, seeing, listening, and recognizing each other as fellow humans. When we hear more and more of each other's stories, we gain an appreciation for the *diversity* of all of us, collectively, and start to see that our own life histories are *global*, are intertwined with a much deeper, much more ancient narrative than what we have been taught at school. And it is these stories, often from voices unheard, that help define *collective value*, allowing every single individual a space to share their story, to share their voice, to share their identity. And the power of stories is that, when they are shared, they often do not focus just on one perspective or one generation, they connect generations, they connect us to our ancestors, biological or chosen, so that we see ourselves as part of a much deeper *intergenerational* dynasty. So, while I do not position my work as an active part of the Black Lives Matter movement, I do align heavily with these guiding principles and hope that my framing of storytelling can also build on what I see as important guiding principles for all our youth from oppressed communities.

AsianCrit

Asian American identity is a tapestry in itself, forged through political solidarity against the onslaught of imperial forces that wrought havoc across Asia over the past four centuries - tearing asunder histories, languages, and communities (Lee, 2015; Takaki, 1998). This identity, tethered not only to ancestral bonds, but to the shared tales of survival, coalition-building, and triumph, is nurtured and perpetuated through storytelling. Even the very term *Asian American* is a political one, created as an act of collation building during the Third World Liberation movement to fight for Asian American studies and acknowledge the solidarity between the Asian American community and the Black Power movement (Takaki, 1998).

To fully examine and position the Asian American experience through a critical lens, I use AsianCrit, a subbranch of Critical Race Theory (Iftikar & Museus, 2018; Museus & Iftikar, 2013). AsianCrit offers a lens through which the experiences and identities of Asian Americans can be analyzed and understood, going beyond surface-level perceptions, and delving into the complexities that define the Asian American experience. The seven tenets of AsianCrit are: (1)

Asianization, (2) Transnational Contexts, (3) (Re)Constructive History, (4) Strategic (Anti)Essentialism, (5) Intersectionality, (6) Story, Theory, and Praxis, and (7) Commitment to Social Justice.

While each of these tenets is deeply connected to my own history and voice as an Asian American scholar, I focus heavily on the sixth tenet: Story, Theory, and Praxis. I recognize the power of storytelling as a mechanism of voice and communication within the Asian American community and how the multiple and complex stories of Asian Americans that counter racist stereotypes and destructive narratives have been rare in our history (Chen & Buell, 2017). An AsianCrit perspective centers the ways storytelling is a powerful tool in unraveling the rich tapestry of experiences, while not only challenging stereotypes and promoting social justice, but also engaging in the complicated journey of both building a collective Asian American community. This is complex, as this work must both recognize the many and multiple groups that too often are essentialized under the banner of Asian American while promoting community solidarity, which is basically the fourth tenet, Strategic (Anti)Essentialism. This complexity is one of the paradoxes of Asian America, to both celebrate one's unique ethnic and ancestral heritage, while also acknowledging the necessary ways that collective solidarity in the United States is built under the general category of Asian American. *And to me, there is no better way to engage with these paradoxes than through telling the stories of people who live in and around these paradoxes while creating their own counter-stories of what it means to be Asian American.*

Our Technology Won't Save Us, But We Can Save It

I want to address the role of video storytelling in today's age of *TikTok*, *YouTube*, and *Instagram*. When I first started my work in digital media, back as an undergraduate in the late 1990s, I was fascinated with the ways I could capture so much video data onto a single MiniDV tape and then edit that footage in a lossless format. A future in which everyone had access to a virtual television studio seemed on the horizon. Now, more than twenty years later, that future has come alive in a way few of us could imagine. Yes, we live in world in which almost all of us can make engaging, polished videos seemingly instantly on our mobile devices, capturing all aspects of our lives to share with the public. Yet, other developments in the evolution of digital video have happened too.

Mobile Video as Democracy. First, the ubiquity of live streams have allowed for documenting police brutality as it happens, effectively allowing the world to bear witness to the inhumane treatment that Black people face in their everyday lives (Hockin & Brunson, 2018). This use of video is not only powerful, but it has also transformed the conversation around racism in the United States through video evidence that systemic violence, so brutal and jarring that many of those who live within unaffected communities had trouble believing that this type of violence could still existed. Videos of Black individuals being murdered during routine traffic stops and videos of elderly Asian Americans being attacked on the street continually pop up on my own social media feeds. I'm often not in the mental space to watch them, because they are triggering. Yet I understand the power these images have in sparking anger, in creating action, and demanding change.

Mobile Video Spreading Extremism. Second, however, the salaciousness of these violent videos brings up another side effect of the "video everywhere" era of today: clickability. Because so many videos are accessed through social media sites focused on generating views, engaging viewers to spend more and more time on their sites, and on collecting user data, the extremeness of videos that are immediately clickable or enticing has created an increase amount of videos that push on extreme viewpoints (Crain & Nadler, 2019). The amount of young people who claim to have been radicalized to a particular cause because of being served a more extreme YouTube video after a more extreme YouTube video is alarming, in which viewers are recommended videos with extreme or dark views after watching videos with relatively mainstream viewpoints (Ribeiro et al., 2020). This radicalization is scary and has major implications in the ways our young people are engaging with the world today. I certainly did not imagine that a byproduct of the democratization of video production would be that our world views would become extremely fractured, that our societies would become so polarized, or that once dead and buried philosophies such as fascism, eugenics, or race-based mathematics intelligence would find new communities of supporters.

The Need for Digital Media Literacy. Third, the ever presence of mobile video media has pawned significant dangers surrounding our own privacy, mental well-being, and safety. Students deal with bullying at school but on their social media feeds, feelings of isolation or depression can be magnified when staring at seemingly perfect photographs of one's peers, and issues of privacy abound as students and teachers can be captured on video at any time. *Digital media literacy* is a general term that encompasses the ways that students can be literate not just about the digital media they are surrounded with, but also be aware of how to safeguard their privacy, their mental sanctity, and be conscious of the psychological warfare being waged on them through social media (Park, 2012). A focus on digital media literacy allows young minds to critically engage with digital content and learn how to leverage digital media not just in a safe way, but to actually effect change within their communities (Yue et al., 2019).

And so, it is within this world that I am hoping to explore how to bring things back (while looking forward) by using storytelling. The digital video revolution brings both opportunities and challenges. While it offers unprecedented avenues for sharing and amplifying stories across the globe, potentially enriching educational experiences and fostering a sense of community, the unregulated nature of social media and video streaming apps pose terrifying dangers, such as misinformation, cyberbullying, extremism, the destruction of democracy, and global genocide. Our students and us need to walk into this technological landscape warily, with tools for our own protection, and safeguards so that our children can navigate through it safely.

This is not just a call for revolutionizing mathematics education through storytelling but my urgent plea for a larger systemic metamorphosis. The same colonial forces that thrived on both a "divide and conquer" strategy to create infighting rather than solidarity and "bread and circuses" strategy to create distracting entertainment to dissuade revolution are mirrored in the divisive nature of social media. Our youth must be shielded and ready to fight.

I believe that as our children learn about mathematics in the world to better navigate it, they should also learn about how to navigate the digital landscape carefully and conscientiously, learn about their own power as digital storytellers while being vigilant of its potential, extremism-inducing pitfalls. We can weave together a safe space for mathematics storytelling, digital media literacy, and community building for our young minds to make their world a better place.

The Digital Mathematics Storytelling Workshop

The centerpiece of this research is the *Digital Mathematics Storytelling Workshop*. This workshop is structured around a few key modules or exercises, designed to engage young individuals or mathematics educators in the art of storytelling and its connection to mathematics. These workshops can last for as little as three days to as long as six weeks. In each of these workshops, I enact a participant action research framework, in which the participants themselves decide how the research should be enacted, what sorts of data they interested in, and the relationship between the workshop and the research study (Bang & Vossoughi, 2016; Kindon et al., 2007; Mirra et al., 2015). This aspect, of asking participants to decide what the research should look like, does not always go smoothly. Overall, the workshop involves these modules:

1. **Mathematics in Daily Life**: In the initial stage, participants create brief videos that capture the essence of mathematics as it manifests in their homes, families, and communities. This stage is critical as it helps participants see the ubiquity and relevance of mathematics in their own everyday life.
2. **Drafting and Sharing**: Then, participants dedicate time to crafting a rudimentary story centered around the mathematics they saw or their personal relationship with it. This story is then shared within small groups through a structured format known as a Storycircle. In a Storycircle, each participant narrates their evolving story to others who listen intently without interruption.
3. **Reflection and Refinement**: After the Storycircle, participants reflect upon the feedback received and consider the emotional experience of narrating their story to an audience. Armed with this insight, they meticulously refine their stories.
4. **Video Creation**: In this stage, participants adapt their revised stories into video format. This phase allows them to experiment with different multimedia elements to further enhance the expressiveness of their narratives.
5. **Community Screening**: In the final stage, we organize a community screening party where all participants share their stories to friends, family, and community members. Each storyteller presents their stories; the screening becomes a platform for dialogue as the audience provides feedback and engages in discussions inspired by the stories.

I am continually revising these modules, hoping to attain the research goals of fostering and eliciting participants' mathematics and storyteller identities. I have found that this process starts the process of bridging the gap between mathematics and storytelling, encouraging participants to embrace their mathematics selves as an integral and meaningful part of their narratives and communities. Yet, I still struggle to foster stories that reflect the breadth of the incredible stories and ideas that participants share during their storycircles.

Reflections on My Research Findings

In this section, while I do not report direct research results, I offer some broader findings and reflections, based on my undertaking on digital mathematics storytelling in various contexts.

Storycircle is Life: We Must Have Space to Tell Mathematics Stories

Storytelling is a quintessential human tradition. And yet, so many of the youth and the teachers I have worked with have very little confidence in their abilities as a storyteller. I think this is due to two things. First, in the vast amount of media we consume, often we see the same Eurocentric story structures again and again. So, when participants have a template in their mind of what a story should look and feel like, they often have a traditional three act story structure in their mind. This might feel disconnected with the ways that they might be thinking about or perceiving stories in their mind, so it makes them hesitant to share their story. Not all stories have to follow the same format as a blockbuster Marvel or Pixar film. Second, many of them have never had the opportunity to think about their own mathematics-connected stories, outside of devising a story problem. For them, a mathematics story must involve some sort of mathematics operation or number sentence, which limits the types of stories they can tell.

A poignant instance of this is a simple observation made during one of my first digital storytelling studies: a 7-year-old Black girl was intricately arranging beads in rows and telling me a story about how she was counting the beads in each row. As a mathematics educator, I am always mesmerized when listening to a child show me their counting strategies. However, one of the adult educators from the community was nearby and astutely recognized what the child was doing and asked the child if she was counting up how many beads she wanted in each braid of her hair. Immediately, the child excitedly said yes, and the story morphed into one about the mathematics behind the beading patterns she envisioned for her hair. Something I, as someone who has less familiarity with Black hair culture and braiding, did not pick up on when listening to the story. What this anecdote showed me is why we need to share our stories in progress to others in our community, who are there not just to offer feedback, but to help us better connect to what these stories even mean to ourselves.

I have found that when my participants engage in their first storycircle, it is a difficult and vulnerable space. But it is these difficult spaces that are crucial in helping them start to see that they have the power to tell stories using their own voice. While our students might have multiple opportunities in their daily school experiences to engage in some form of mathematics and opportunities to periodically engage in story creation in their literacy or arts classes, they have probably never had the opportunity to engage in any form of mathematics storytelling. All of us, not just our students, need these conducive spaces to inspire us to perceive and weave our mathematics stories divergently, to tell our mathematics narratives in our own voice. And this development is not instantaneous; it requires time, nurturing, and exposure to an array of storytelling forms. So, we need spaces like the storycircles, so that, in the end, all of us can be the storytellers we are meant to be.

Screenings Matters: The Collective Sharing of Stories

Public sharing and collective feedback imbue storytelling with a transformative energy that fosters community building, social change, and identity affirmation. I have always loved the celebration that occurs when young people can share their stories publicly, to showcase all the hard work they put into making their story happen. Often, this screening is a catered event, with family, friends, and the community invited to view the stories.

However, I have since learned that this screening is an important also a part of the process. Just because a video is "finished" for a screening, does not necessarily mean it is completed. During the screening, particularly when the screening is attended by the storyteller's peers, the discussion that erupts around the story helps better connect the mathematics to the story. Or the discussion helps the storyteller better understand what the story means to them. For example, during one screening, a 7th-grade Asian American girl made a story about how she makes earrings and is trying to figure out how much to charge for those earrings. In the discussion, students brought up the concept of a wage, how the idea of trying to figure out how much money made from the profit of making and selling one earing might be too volatile, but as a worker, you are deserving of a steady wage for your labor. I was astounded at this insight and realized that this discussion, during the final screening, was an extremely mathematical conversation that also led us to connect to ideas of the role of capitalism in our society. What this showed me is that the final screening opens many opportunities for important discussions that can extend the mathematics in the original video.

Beware The Video Everywhere Era

While platforms like *TikTok* and *YouTube* have opened a whole new and exciting space for video storytelling for today's youth, I have begun to feel, from my participants, how wary they are about the fact that they are continually surrounded by peers with phones that could be recording them at any time. I want to be clear here. Yes, while I use digital video heavily in this work, I also want us, as mathematics education researchers, to advocate for our youth as they navigate environments where the ubiquity of constant video recording is becoming a source of anxiety. In almost every single digital mathematics storytelling workshop I have done recently, I have heard stories of people being filmed without their consent and ridiculed on social media. I have also heard from teachers who are constantly vigilant about phones in their classrooms, wary that they are being filmed and can be vilified on social media.

So, while I see endless possibilities for community engagement and education through video media, so are the dangers. With the omnipresence of phones everywhere, we must be aware that many of our youth and teachers are extremely apprehensive about invasions of privacy and the potential misuse of their images. We must also be aware of the ways online video tools can be weaponized against vulnerable individuals or communities, often bullying or shaming people based on the same categories of gender, race, ability, or language that we see attacked again and again. In recent work, I have completely rethought my use of public community screenings, instead offering opportunities for participants to have smaller, more intimate screenings of their stories, so that they do not feel as vulnerable or exposed.

Critical Digital Media Literacy

Finally, I have come to realize that we live in an exciting, but fragile time, an era beset by a deluge of information and misinformation, with no real way for any of us to discern what is real and what is spin. The construct of digital media literacy might not be enough, as it focuses heavily on the curation and sharing of digital media in and of itself. What we need for our students and ourselves are stronger skills to better critique and harness the swirling currents of the digital age, skills that help us create systems of basic safety, skills that help us recognize how to consume thoughtful media versus extreme media.

I believe our youth can be equipped not only with the skills to navigate the digital world but also with the knowledge and conviction to utilize these platforms in ways that uplift themselves and their communities. I am generally referring to these skills as *Critical Digital Media Literacy*, a loose grouping of the skills I see as necessary for youth to critically analyze the content they watch, understanding the psychological impact of the social media the engage with, and learn how to utilize digital media for positive change. So, as we embrace storytelling and its evolution through digital media, let us not lose sight of what we are trying to do. We are at a crossroads, where our digital spaces have largely been overtaken by traditional capitalist notions focused on generating the largest number of users to sell things to or control. But we can envision a better digital space, one based on the community knowledges and ways of knowing that come about in the stories we tell. It is our responsibility to cultivate spaces where diverse stories flourish, where communities find solace and power, and where the next generation are not merely consumers but conscientious creators and guardians of narratives that shape their world.

Discussion

My journey using Digital Mathematics Storytelling has taught me so much, not only about how to get people to make and share mathematics story videos. What I have learned is where the edges of our world of mathematics education and our digital society are not clear, where our models break, where there needs to be more definition, and where our role comes in. What I have also learned is that this intersection between storytelling, community, and mathematics education is much more difficult to navigate than I anticipated. Below are spaces where I am still struggling in this work.

Reciprocity between Storytelling and Mathematics. What I still do not have a strong handle on is the reciprocal relationship between storytelling and mathematics, how stories illuminate mathematical concepts and, in turn, how mathematics enriches the fabric of the stories. I know that it can happen, but I am still not exactly sure what it looks like and how to elicit it. In theory, I know that when children associate mathematical theories with their daily lives through storytelling, they internalize concepts with an intensity that rote learning may never achieve. But what does this look like? The stories that my participants tell still dance around this deeper connection to mathematics.

Engaging in Counter-Storytelling. Engaging marginalized communities through counter-storytelling is where I thought this work would take me. And yet, even with setting up nurturing environments where individuals can engage in counter-storytelling to challenge dominant narratives and stereotypes, this is not what they choose to do. The prospect of storytelling is so visceral and so personal, I feel somewhat inauthentic suggesting that their stories need to have a political purpose, that their stories must also fight stereotypes while talking about mathematics.

Sustainable Community Building and Identity Formation through Storytelling. We know that communities are formed through storytelling. Yet, in the storytelling workshops I have helped create, I cannot help feeling that the whole community falls apart once the workshops are over. How can we authentically create communities that are based on the identities that come out during the storytelling, that become true communities about individuals and not just a storytelling workshop?

I end with these wonderings and hope that sharing my story with you has helped you in your own journey. I hope to hear your story too.

References

Aguirre, J. M., Turner, E., Bartell, T. G., Kalinec-Craig, C., Foote, M. Q., McDuffie, A. R., & Drake, C. (2012). Making connections in practice how prospective elementary teachers connect to children's mathematical thinking and community funds of knowledge in mathematics instruction. Journal of Teacher Education, 0022487112466900.

Alim, H. S., Chang, J., & Wong, C. (Eds.). (2023). Freedom Moves: Hip Hop Knowledges, Pedagogies, and Futures.

Bang, M., & Vossoughi, S. (2016). Participatory Design Research and Educational Justice: Studying Learning and Relations Within Social Change Making. Cognition and Instruction, 34(3), 173–193. https://doi.org/10.1080/07370008.2016.1181879

Chao, T., Adams Corral, M., Ozturk, A., Lin, H.-C., & Hidayat, A. (2022). Eliciting youth mathematics stories: The impact of a digital mathematics storytelling summer camp experience. In A. E. Lischka, E. B. Dyer, R. S. Jones, J. Lovett, J. Strayer, & S. Drown (Eds.), Proceedings of the forty-fourth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Middle Tennessee State University.

Chao, T., Adams-Corral, M., Ozturk, A., Lin, H.-C., & Li, Y. (2021). Community math stories: Informal adult educators exploring mathematics identity through digital mathematics storytelling. In D. Olanoff, K. Johnson, & S. M. Spitzer (Eds.), Productive Struggle: Persevering Through Challenges: Proceedings of the 43rd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. (pp. 143–152). Widener University, Towson University, and West Chester University. http://www.pmena.org/pmenaproceedings/PMENA%2043%202021%20Proceedings.pdf

Chao, T., & Marlowe, M. (2019). Elementary Mathematics and #BlackLivesMatter. Bank Street Occasional Paper Series, 2019(41). https://www.bankstreet.edu/research-publications-policy/occasional-paper-series/occasional-paper-series-41/elementary-mathematics-and-blacklivesmatter/

Chen, G. A., & Buell, J. Y. (2017). Of models and myths: Asian(Americans) in STEM and the neoliberal racial project. Race Ethnicity and Education, 0(0), 1–19. https://doi.org/10.1080/13613324.2017.1377170

Crain, M., & Nadler, A. (2019). Political Manipulation and Internet Advertising Infrastructure. Journal of Information Policy, 9, 370–410. https://doi.org/10.5325/jinfopoli.9.2019.0370

de Jager, A., Fogarty, A., Tewson, A., Lenette, C., & Boydell, K. (2017). Digital Storytelling in Research: A Systematic Review. The Qualitative Report, 22(10), 2548–2582.

Emdin, C. (2016). For White folks who teach in the Hood... and the rest of y'all too: Reality pedagogy and urban education. Beacon Press.

Geake, J. (2008). Neuromythologies in education. Educational Research, 50(2), 123–133. https://doi.org/10.1080/00131880802082518

González, N., Andrade, R., Civil, M., & Moll, L. (2001). Bridging Funds of Distributed Knowledge: Creating Zones of Practices in Mathematics. Journal of Education for Students Placed at Risk (JESPAR), 6(1–2), 115–132. https://doi.org/10.1207/S15327671ESPR0601-2_7

Hill, J. A. (2011). Endangered childhoods: How consumerism is impacting child and youth identity. Media, Culture & Society, 33(3), 347–362. https://doi.org/10.1177/0163443710393387

Hockin, S. M., & Brunson, R. K. (2018). The Revolution Might Not Be Televised (But It Will Be Lived Streamed): Future Directions for Research on Police–Minority Relations. Race and Justice, 8(3), 199–215. https://doi.org/10.1177/2153368716676320

Iftikar, J. S., & Museus, S. D. (2018). On the utility of Asian critical (AsianCrit) theory in the field of education. International Journal of Qualitative Studies in Education, 31(10), 935–949.

Kindon, S., Pain, R., & Kesby, M. (2007). Participatory action research approaches and methods. Connecting People, Participation and Place. Abingdon: Routledge, 260.

Lambert, J. (2013). Digital storytelling: Capturing lives, creating community. Routledge.

Lee, E. (2015). The Making of Asian America: A History. Simon and Schuster.

Love, B. (2014). Urban storytelling: How storyboarding, moviemaking, and Hip-Hop-based education can promote students' critical voice. English Journal, 103(5), 53–58.

Mathews, S. A., & Jones, D. (2022). Black lives matter at school: Using the 13 guiding principles as critical race pedagogies for Black citizenship education. The Journal of Social Studies Research.

Matias, C. E., & Grosland, T. J. (2016). Digital Storytelling as Racial Justice Digital Hopes for Deconstructing Whiteness in Teacher Education. Journal of Teacher Education, 67(2), 152–164. https://doi.org/10.1177/0022487115624493

Mirra, N., Garcia, A., & Morrell, E. (2015). Doing youth participatory action research: Transforming inquiry with researchers, educators, and students. Routledge.

Moll, L. C., Amanti, C., Neff, D., & Gonzalez, N. (1992). Funds of knowledge for teaching: Using a qualitative approach to connect homes and classrooms. Theory into Practice, 31(2), 132–141.

Museus, S. D., & Iftikar, J. (2013). An Asian Critical Theory (AsianCrit) Framework. In M. Y. Danico & J. G. Golson (Eds.), Asian American Students in Higher Education (pp. 18–29). Routledge.

Nunez-Janes, M., & Cruz, A. (2013). Latino/a students and the power of digital storytelling. Radical Pedagogy, 10(2), 1–16.

Paris, D., & Alim, H. S. (2017). Culturally sustaining pedagogies: Teaching and learning for justice in a changing world. Teachers College Press.

Park, S. (2012). Dimensions of Digital Media Literacy and the Relationship with Social Exclusion. Media International Australia, 142(1), 87–100. https://doi.org/10.1177/1329878X1214200111

Prusak, L., Groh, K., Denning, S., & Brown, J. S. (2012). Storytelling in organizations. Routledge.

Ribeiro, M. H., Ottoni, R., West, R., Almeida, V. A. F., & Meira, W. (2020). Auditing radicalization pathways on YouTube. Proceedings of the 2020 Conference on Fairness, Accountability, and Transparency, 131–141. https://doi.org/10.1145/3351095.3372879

San Pedro, T., & Kinloch, V. (2017). Toward Projects in Humanization: Research on Co-Creating and Sustaining Dialogic Relationships. American Educational Research Journal, 54(1_suppl), 373S-394S. https://doi.org/10.3102/0002831216671210

Sfard, A., & Prusak, A. (2005). Telling identities: In search of an analytic tool for investigating learning as a culturally shaped activity. Educational Researcher, 34(4), 14–22.

Solórzano, D. G., & Yosso, T. J. (2002). Critical race methodology: Counter-storytelling as an analytical framework for education research. Qualitative Inquiry, 8(1), 23–44.

Takaki, R. (1998). Strangers from a Different Shore: A History of Asian Americans (Updated and Revised). Little, Brown and Company.

Turner, E., Foote, M. Q., Stoehr, K. J., McDuffie, A. R., Aguirre, J. M., Bartell, T. G., & Drake, C. (2016). Learning to Leverage Children's Multiple Mathematical Knowledge Bases in Mathematics Instruction. Journal of Urban Mathematics Education, 9(1). http://ed-osprey.gsu.edu/ojs/index.php/JUME/article/view/279

What We Believe. (2018). Black Lives Matter. https://blacklivesmatter.com/about/what-we-believe/

Yue, A., Nekmat, E., & Beta, A. R. (2019). Digital literacy through digital citizenship: Online civic participation and public opinion evaluation of youth minorities in Southeast Asia. Media and Communication, 7(2), 100.

Zazkis, R., & Liljedahl, P. (2009). Teaching mathematics as storytelling. In Teaching Mathematics as Storytelling. Brill.

WHAT DO YOU SEE IN MATHEMATICAL PLAY?

Melissa Gresalfi
Vanderbilt University
melissa.gresalfi@vanderbilt.edu

Amy Noelle Parks
Michigan State University
parksamy@msu.edu

Anita A. Wager
Vanderbilt University
anita.wager@vanderbilt.edu

Nathaniel Bryan
The University of Texas at Austin
nathaniel.bryan@austin.utexas.edu

Naomi Jessup
Georgia State University
njessup@gsu.edu

Tran Templeton
Columbia University
ttn2114@tc.columbia.edu

As part of a longitudinal study focused on mathematical play, we (Melissa, Amy, and Anita) are often faced with questions about what counts as play and what mathematics (and other learning) we see in play, and whose play is most likely to be seen or dismissed. Rather than discuss our findings from classroom videos of kindergarten children engaged in mathematical play, we asked scholars who bring different lenses to research on play, young children, and teaching and learning mathematics to look at some of our data and provide their perspectives. In this session, we will share video and discuss with our panel (Nathaniel, Naomi, and Tran) various ways to interpret that video. This paper provides background on the potential of mathematical play and the details of the study that generated data for analysis. We conclude with a copy of a transcript that is associated with a video we will watch during the plenary with hopes that participants will watch prior to the session and come with their own questions/perspectives.

Keywords: early childhood, identity, mathematics, play

Introduction

As content standards become more rigorous and demanding, and high stakes assessment becomes the norm even early grades, the time for exploration and play is growing increasingly scarce (Miller & Almon, 2009). Even in the earliest grades, curriculum can be tightly scripted, recess strictly timed, and toys are often absent or hidden. While prescriptive activities are efficient solutions to the now-defined work of schooling--moving large numbers of students through a large numbers of topics—this approach rarely supports students in developing robust number sense. Further, the practices that such approaches to teaching mathematics require, such as a focus on efficiency, speed, and memorization, are known to undermine students' enjoyment and deep understanding (Boaler, 2002; Boaler & Staples, 2008). In contrast, classrooms that offer time for exploration, that emphasize reasoning and understanding over accuracy and speed, and which place student identity at the center of instructional design, have been found to support a more productive relationship with the domain of mathematics (Cobb et al., 2009; Gresalfi, 2009). And yet, for a variety of reasons, such classroom practices have been challenging to develop at scale.

One aspect of mathematical exploration that is a topic of great interest, but has received very little empirical focus, is mathematical play. While the role of play in supporting student learning is well-understood and often advocated for very young students, we have little understanding about mathematical play for older children. Indeed, once children enter formal schooling, discussions of learning rarely reference play, and work investigating the design of mathematical learning environments has seldom explores whether or how play might be involved.

Researchers who study play across contexts define play as “spontaneous, naturally occurring activities with objects that engage attention and interest” (Lifter & Bloom, 1998)p. 164). Burghardt (2010) argued that play is spontaneous or pleasurable, functional, different from similar serious behaviors, repeated, and initiated in the absence of stress. Increasingly, educators have recognized that play provides both social and cognitive benefits, such as increasing creativity, reducing stress, and promoting problem solving (Elkind, 2007). In this work, we define play as pleasurable activities that allow children to explore, to engage with interesting materials, to make choices, and to possibly engage in social interactions. The play is *mathematical* if the play context can promote a mathematical learning goal.

For many, “mathematics” and “play” are terms that anchor two opposite extremes, with things that look like “math” resembling nothing that looks like “play.” But for mathematicians, play is an acknowledged element of engaging the discipline, in part because mathematicians are offered a different version of mathematics with which to engage. Rather than taking as their task the acquisition of other people’s knowledge, mathematicians are afforded the opportunity to think with and about mathematics, to inquire into its structure, its limits, and its possibilities. Many scholars have highlighted how play allows one to think beyond oneself, to test and explore the limits of ideas (Gadamer, 1975; Vygotsky, 1978), and such ideas are expressed by mathematicians. For example, as Mathematician Sharon Whitton states:

> Play has a role both in the work of mathematics and in the evolution of mathematics. Although play is not often acknowledged as a major contributing factor in mathematicians’ work, their methods of inquiry resemble many of the behaviors of children involved in meaningful play (cited in Bergen (2009), p.421).

We argue that play should not be reserved only for those who are tasked with exploring the frontiers of mathematics, but rather, that play is itself an important vehicle for exploration, understanding, enjoyment, and learning. In play children can encounter and explore mathematical concepts and relationships through their engagement with carefully chosen materials (Ginsburg, 2006; Tudge & Doucet, 2004), such as when they compare quantities of toys, compose and decompose shapes of wooden blocks, and count forward and backward on a gameboard (Seo & Ginsburg, 2004). In lessons where children engage in mathematical play, they have opportunities to solve problems and explore concepts in low-stakes contexts that encourage social engagement, experimentation, and the use of interesting materials (Parks, 2015).

Prior research suggests that designing spaces that allow students to explore mathematical ideas, encounter parameters and structures, develop a sense of mathematical aesthetic, and engage in cycles of revision and justification would be a productive use of class time (Sinclair, 2004, 2026). Likewise, research has documented what young children learn in mathematics through play, such as through developing spatial reasoning through block building (Casey et al., 2008) or magnitude through linear board games (Siegler & Ramani, 2008). However, little research has sought to link mathematical learning in schools to playful experiences in mathematics for children in the primary grades (Wager & Parks, 2014), or has addressed how such play could be supported.

Why Study Play

Bringing play into the context of mathematics requires recognizing the mathematical content embedded in spontaneous play and the possibilities for designing mathematical engagement that embody a spirit of playfulness. For example, work on embodied cognition by Lakoff and Nunez (2000) has demonstrated how spontaneous play with containers can build concepts that support set thinking about numbers while play on balance beams can create an embodied sense of the meaning of equality. At the same time, research on mathematical play that has been orchestrated by adults, such as work with linear board games or puzzles, has been shown to develop mathematical understandings while still retaining many of the features of play (Kamii et al., 2004; Siegler & Ramani, 2008).

The Role of Play in Learning and Identity

There is growing evidence that play can support students' mathematical competence and the development of productive mathematical identities (Wager & Parks, 2014). Mathematical play offers opportunities for children to engage in all of the standards for mathematical practice, for example by making sense of problems and persevering in solving them, or exploring the structure of tools, symbols, and numbers. Play with particular materials, such as blocks (Wolfgang et al., 2003), linear games (Siegler & Ramani, 2008), and puzzles (Clements et al., 2012) has been shown in a range of empirical research to impact young children's mathematical learning of both number and geometry.

With respect to identity, bringing play into mathematics also provides opportunities for children to come to see themselves as "mathematics people" (Parks, 2015, p. 83). When teachers create opportunities for children to engage in mathematical play, and when they label children's play with mathematics vocabulary, they help children see themselves as people who enjoy mathematics (Esmonde, 2009). In addition, unlike formal lessons, play contexts often allow for children and teachers to use language more fluidly and to enact different kinds of social relationships, which may be more likely to mirror the ways that children talk and play at home. This can be particularly meaningful for children from low-income or marginalized families who may experience conflicts between the performance of school mathematics in formal lessons and the ways they engage at home ((Enyedy & Mukhopadhyay, 2007). Reshaping of mathematics norms and practices serves to position students differently with respect to the discipline, broadening not only the kinds of mathematical experiences that students might have, but also, how those experiences lead them to reach different conclusions about their abilities and preferences (Gresalfi & Hand, 2019).

Supporting Play in the Mathematics Classroom

Our approach to design begins with the starting point that the classroom is an ecosystem (Greeno, 1991; Gresalfi et al., 2012): changing one element of the system can influence others, but not always directly in a simple causal relationship. Play is an excellent example of an activity that cuts across multiple elements of a system, as it involves attending to the tools and physical resources available in a classroom, but also the concomitant norms and practices that make space for such toys and tools to become objects of inquiry. Likewise, while students' beliefs about themselves and about mathematics influence how they might play in math class, those beliefs develop through their participation with these classroom norms (Gresalfi, 2009; Hand, 2010). Thus, in thinking about how to support mathematical play in elementary school classrooms, we articulate the different elements of the system, and how those elements of this ecosystem interact.

The Role of Tools in Attuning Students to Mathematical Ideas

Within early childhood, Montessori (Montessori, 1917) was one of the earliest advocates for supporting children's mathematical learning through their independent engagement with material objects. Current studies demonstrate that young children who experience Montessori programs (even when randomly assigned) perform more strongly on mathematics assessments than children who do not (Laski et al., 2015; Lillard & Else-Quest, 2006). Research suggests that a wide variety of toys and tools can be productive in promoting mathematics learning, but that these tools are most productive when children have multiple opportunities to explore the same materials over time in a variety of contexts, when adults support children in assigning mathematical meaning to the materials, and when materials make mathematics (rather than the everyday world) salient (Laski et al, 2015). Likewise, research on children's mathematical engagement in museums has demonstrated that allowing children agency in their interactions with well-designed materials supports their engagement with mathematical concepts (Kelton et al, 2018) and that observing children's physical engagements can provide insight into their developing understandings (Nemirovsky et al., 2012).

The Role of Classroom Culture

Classrooms that support learning through mathematical play offer a different experience for students than traditional math classrooms. This shift requires an environment in which students have agency to explore, make their own decisions, and feel comfortable making mistakes - an environment that invites students to participate (Gutiérrez, 2012) and develop their identities as learners and doers of mathematics. Agency, the opportunity to take action with regard to one's own learning (Gresalfi, Martin, Hand, & Greeno, 2009), enables students to participate in mathematics in ways that are meaningful and sensible to them (Gresalfi & Cobb, 2006; Nasir & Hand, 2006). Classroom environments that provide opportunities to participate exercise agency can "transform how students see themselves as mathematical thinkers, how they see the discipline, and ultimately, the mathematics they learn" (Turner et al., 2013), p. 229).

The Role of Teachers in Supporting Mathematical Play

The potential of play to support students' mathematical engagement depends largely on the teacher. Although students are naturally experts at play, they are less likely, on their own, to play in ways that easily translate to rich mathematical thinking. When adults make conscious choices in constructing play environments and choose to intentionally engage with children during play and other informal activities, student mathematical engagement is enhanced (Trawick-Smith et al., 2016; Van Oers, 2010) In particular, teachers' use of 'math talk' during formal and informal lessons has been shown in a variety of studies to have a significant impact on children's later learning (Levine et al., 2010; Wiebe Berry & Kim, 2008). More broadly, there is growing evidence that providing materials that support mathematical play and intervening in play to deepen thinking and extend vocabulary can promote more significant early mathematics learning (e.g., van Oers, 2010; Trawick-Smith, Swaminathan & Liu, 2015; Wager, 2013). Trawick-Smith, Swaminathan, and Liu (2015) found that teachers' asking of open-ended questions during play and providing of appropriate levels of guidance during play (not too much or too little) had positive relationships with mathematics learning in a pretest/posttest study. However, despite the benefits of appropriate adult interactions in mathematical play, research has found that teachers sometimes fail to lift up the mathematics in children's play because they are not skilled at recognizing the mathematics (Moseley, 2005), lack the time to observe play (Seo & Ginsburg, 2004), or are constrained by curricula and other instructional demands (Parks & Bridges-Rhoads, 2012).

The Playful Learning Project and the Challenges of Analysis

This four-year longitudinal study was designed to investigate play in early elementary math education by developing in-depth accounts of: 1) how kindergarten teachers learn to integrate play into their instruction over multiple years, and how their teaching changes over time; 2) how the task of integrating play changes with different demands of mathematics curricula over the grades (kindergarten through second grade); and 3) how the relationship that students develop with mathematics might be transformed by experiencing playful mathematics learning over their early elementary careers (kindergarten through second grade).These fairly straightforward goals have become much more complex as we have embarked upon the study, and new challenges have emerged. Specifically, and the he focus of this paper and our plenary, is to explore and problematize mathematical play by considering what mathematical thinking looks like when it is transformed through play, what counts as play, and how some children's play is visible, suppressed, or otherwise rendered invisible by broader structures and biases, from researchers, teachers, and other children.

Over the first year of the project, the team worked with six kindergarten teachers who taught in teams in three classrooms at a racially and economically diverse public school in the Southern United States. The teachers and the project team co-designed four weeks of instruction with the goal of introducing playful tasks into the lessons of the mandated curriculum. During the weeks these lessons were taught, the project team video recorded children's engagement in small groups using GoPro 360 cameras, which captured all four students per table simultaneously. We also videoed the teacher during whole-class and small group instruction. From the three classrooms across the four units, we captured 580 videos of student mathematical play, ranging from two to twelve minutes in duration.

Because the GoPro cameras record in 360 degrees, analyzing the videos presents several challenges in terms of attention. Often four children were engaged in four separate activities, although sometimes the whole group would come together or partners would team up. The video could be "unwrapped" to see all four children at once or could be watched so the viewer could control the focus of attention. As a research group, we found that where we directed attention in the videos and the sense we made of interactions during play about mathematics was shaped by on our professional backgrounds (e.g., former early childhood teachers and former high school mathematics teachers often attended to different aspects of the mathematics), our identities (e.g., our racial identities shaped our focus and interpretations) and our academic histories (e.g., being immersed in learning sciences, early childhood education, or teacher education framed interactions in different ways). This became even more complex when teachers were included in watching video, as their perspectives brought in additional differences.

Conversations across the group about these differences reminded us of a decades old special issue in the *Journal of Learning Sciences* (Sfard & McClain, 2002) where a group of researchers analyzed the same video through a variety of lenses to provide insight on the role of symbolic tools in shaping mathematical thinking. At the time, we remembered being struck by the different understandings of the same video that each theoretical perspective allowed, but in looking back we noticed instead how similar each of the socio-cultural perspectives taken really were. This raised some questions for us. We wondered about what perspectives we were missing even within the diversity of our research group. We were especially concerned with perspectives that were not represented in our research group and that we would be unlikely to encounter at mathematics education research conferences, but yet felt highly relevant to play.

We were interested in what scholars who centered young children's perspectives and the social context of schooling might see in the children's play. This wondering led to this plenary. We choose a video from the first year of our study, which took place in November, during a unit on counting to twenty. The video shows four children playing at a center invited children to design a zoo with plastic animals by putting a specified number of animals in each "pen." We chose this video because it revealed rich and diverse social interactions among the children and because we felt that all the mathematics was not immediately apparent. The points of ambiguity, both social and mathematical in nature, makes it a good candidate for rich analysis, particularly analysis that draws from different perspectives.

The children pictured in this video are called Quentin, King, and Kiera (pseudonyms). Quentin is a black boy who was moved into the classroom about two weeks prior to the episode, in an effort to balance the children with "challenging behaviors" across the three kindergarten classes. In his previous classroom, Quentin was frequently removed from activities due to the teacher's perception that he was being disruptive to the class. King is a black boy who tended to be quiet in class but paid close attention to instructions in general and math in particular and was almost always on task. Kiera is a white girl who also tended to be quiet in class and was reluctant to offer answers to questions in whole-class settings. There is a fourth child in the video from whose parents did not give consent to participate, and therefore we do not include his data (he spoke very little in this interaction).

We invited three scholars who center young children in their work to share their insights as part of the plenary. They are Dr. Tran Templeton, whose work draws on critical childhood studies to explore how young children make sense of their own lives, Dr. Nathaniel Bryan, whose work draws on Critical Race Theory and PlayCrit theories to understand the experiences of young Black boys and their teachers, and Dr. Naomi Jessup, whose work draws on Critical Race Theory to reimagine mathematics education in humanizing ways, with particular attention to Black children.

Rather than share these scholars' perspectives in this paper, we invite readers to prepare for the plenary by engaging on their own with the transcript of the video and to form their own questions and (admittedly) partial understandings to bring to the conversation. The episode shown in the transcript below shows the children engaging in the zoo activity described above, along with occasional interruptions from the kindergarten teacher, Ms. Lane.

In reviewing the transcript, we encourage readers to think about the following prompts:

- What questions about mathematics and play do the interactions in the video raise for you?
- What theoretical frames do you think would illuminate the analysis?
- Which moments in the video do you find most engaging? Why?

Table 1: Transcript of Zoo Video

Time	Quentin	King	Kiera	Ms. Lane
0:00	[Sticks tongue out at camera and giggles]			
0:15	King you have two yellow ones. I have two orange ones. I know.. [unclear]			

	Two yellow ones.			
0:26		A lion. That's what I want.		
0:28	Can I have that orange one cause I have this orange one right here and I want this?			
0:35	Can I have that orange one? I just want the orange one.			
0:45			Look these look the exact same. [showing animal to Quentin]	
0:50	Yeah. I can find them. ROAR!!			
0:55		Stop		
0:56	Let's make this a shoe store			
1:00		Hey		
1:09	[Gasp!] I found another one. Let's put them all together.			
1:15	These together [pairs up matching animals]. I got to get another giraffe.			
1:29		No. We can't put that together.		
1:38		What's that?		
1:40	[Gasp] Why do you put the stuff up?			
1:42		Oh my goodness. This is something like a.. that's a dinosaur, but I don't even know what it's called.		
1:47	You're messing up our game. We're trying to put two of each on the board.			
1:53				xxx, are you playing?
1:56				Okay, try to do that one right there. [pointing to the box labeled 7 in the zoo in front of xxx]
1:59	We're doing this one.			
2:03				After you finish you have to make sure that you count to make sure that the amount of animals matches this number

				[points to the “13” on the stickynote in the box that Quentin and King are working on].
2:11	We’re not done. We got to put a ri... [Adds elephant to animals that King is counting]	[pointing and counting each animal] 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, thir...		
2:25	Well, we can knock all of them down [knocks about half of the animals down and gathers them together] and count to..Let put. Let’s grab. I’m going to grab a different animal [grabs animal from bin] So yeah….	[knocks down and grabs other half of the animals to put back into the bin]		
2:37		I’m going to count while I put them on there.		
2:41	[place hippo in 13 square] Oooooneeee			
2:42		[places elephant in 13 square] One		
2:43	[touches elephant] No that will be two. [picks up and places hippo again] One			
2:45		One [touches elephant]		
2:46	[touches elephant] Two			
2:47	[adds dinosaur] Three	No. No. [picks up elephant. One. [picks up hippo] Two. [adds giraffe] Three.		
2:53	[adds gorilla] Four	[points to hippo, giraffe, elephant, dinosaur, then gorilla each in turn counting] One, two, three, four, five.		
2:56	[adds duck] Six			
3:00		[adds rhinoceros] Seven.		
3:04	[adds squirrel] Eight. Hey you’re copying me.	[adds chetah] Eight.		
3:08		[adds deer] Nine.		
3:10	[adds lion] Ten.			
3:12		[adds giraffe] Ten		
2:13	[adds hippo] Eleven. [adds buffalo] Twelve.[adds kangaroo] Thirteen.			

3:23		No that's thirteen there. We're done.		
3:25		Now we need to do...		
3:26	16. Let me scoot this over there a little bit [pulls zoo toward him so that the 16 box lies flat on the table.			
3:32		These are a lot of animals		
3:35				
3:39	I'm gonna put a fox on our [unclear]. I want a fox. [add fox to 16 box]. One. King can I go - Can I help you with 16? I'm trying to do 16. [moves around to King's side of the table]. I want to help with 16.			
3:57	[takes a handful of animals from bin]			
4:08		One. [unclear]		
4:12	We doing sixteen. [unclear] Okay.			
4:14		18. Stop! [places lizard, fox, cheetah on 18 box each in turn while counting] One. Two.		
4:18	Okay. I'm gonna do sixteen.	Three.		
4:22	[places panda, tapir, koala, lizard each in turn while counting] One	[places giraffe on 18 box] Four		
4:27	two, three			
4:31		Five		
3:33	Four. Uh, will you [unclear]. Excuse me King! [frustrated]. I'm trying to put this on the table to make sure it works. King, can I get one of these animals?			
4:47	[pointing to each animal in the 16 box] One two, three, four, five.			Y'all don't have to sit down if you don't want to. You can stand up if it's easier. Is it easier?
4:53	Yes			Like what Quentin is doing?
5:03				Okay, we will scoot your chair in. That

				way you can walk around the table and look at [unclear].
5:08	[scoots his chair in and then goes to scoot King’s chair in] King. King. Excuse me. We can push our chair up the teacher said and walk around.			
5:20	[touching each animal in 16 box and counting them in turn] One, two, three, four, five, six.	[touching each animal in 18 box and counting them in turn] One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen.		
5:29		[puts hippo in 18 box] fourteen.		
5:32		Sixteen. Eighteen.	[put 4 new animals in 18 box]. Fifteen, sixteen, seventeen, eighteen.	
5:35	One, two, three, four, five, six. [adds new animal to 16 box] Seven.	What’s this? A bull! That’s a bull!		
5:44	Kiera, could you give me one of the animals? I need an animal.			
6:02	I know what these are. Are we? Uh, Kiera – there's three of us over here.			
6:07			I know.	
6:10	You’re smushing me.			
6:12			[moves around the table to 7 block]	
6:13	What about this guys? We can scoot your chair down here and make it easier.		[moves back to side with King and Quentin]	
6:24		What is this? A monkey?		
6:28	Yeah! Uh, I don’t know. Gasp! I know what these are! Um. They’re twins. Who wants to match the twin with m–? Let’s match the the animals! So animal...			
6:46			Let’s match the animals.	
6:49	So this one goes with...			
6:52		Oh, it’s not matched yet.		
6:54	I’m matching			
6:55			Me too!	

6:56	So them go together. The ones that don't go together let's put them over here [touches 16 box]			
7:03	These go together [picking up a pair of matched animals]			
7:04			Here, these go together. [hands Quentin a cheetah that matches a cheetah in his hand]	
7:06	Oh yeah. Uh, it's right...yeah, uh it's another one. Here. [hands her the cheetah]			
7:13			Here. Ha! [hands Quentin a giraffe]	
7:15	No. I don't need that one. These two match. And these match.These two match. I've got one over here that matches. [handing Kiera pairs of animals each time]			
7:34	Is this an elephant?			
7:35			Yep	
7:36	Okay elephant			
7:45	[continues pairing up animals and putting them in front of Kiera]. We're trying to match them!			
7:50			That's you. I'm just standing right here.	
7:54			That's him that's not me. That's him.	
7:57	It's you too. I'm giving them to you and then you're putting them on the thing.		Nuh-uh!	
8:00	We're not doing – We're not doing anything BAD!			
8:05			We just match them.	
8:08	We're just trying to take care of the animals [unclear]			
8:12		We don't need to do that		
8:15			Here, here's this.	
8:22		Look at the timer [to Quentin]		

8:25			Oh, it's a 6! Clean up. Let's clean up. [all students frantically pick up animals and put them back in bin]	
8:35	[takes bin and put it in front of himself] Fast! Faster! Faster than a - Let's do it faster than a giraffe.			
8:45				Calm it down okay. Go ahead and have a seat.

Acknowledgements

This material is based upon work supported by the National Science Foundation under grant number DRL- 2101356. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

References

Bergen, D. (2009). Play as the Learning Medium for Future Scientists, Mathematicians, and Engineers. American journal of play, 1(4), 413-428.

Boaler, J. (2002). Learning from teaching: Exploring the relationship between reform curriculum and equity. Journal for Research in Mathematics Education, 33, 239-258.

Boaler, J., & Staples, M. (2008). Creating mathematical futures through an equitable teaching approach: The case of Railside school. Teachers College Record, 110, 608-645.

Burghardt, G. M. (2010). Defining and Recognizing Play. In P. Nathan & A. D. Pellegrini (Eds.), The Oxford Handbook of the Development of Play (pp. 0). Oxford University Press. https://doi.org/10.1093/oxfordhb/9780195393002.013.0002

Casey, B. M., Andrews, N., Schindler, H., Kersh, J. E., Samper, A., & Copley, J. (2008). The development of spatial skills through interventions involving block building activities. Cognition and Instruction, 26(3), 269-309.

Clements, D. H., Wilson, D. C., & Sarama, J. (2012). Young children's composition of geometric figures: A learning trajectory. In Hypothetical Learning Trajectories (pp. 163-184). Routledge.

Cobb, P., Gresalfi, M. S., & Hodge, L. (2009). An interpretive scheme for analyzing the identities that students develop in mathematics classrooms. Journal for Research in Mathematics Education, 40(1), 40-68.

Elkind, D. (2007). The power of play: Learning what comes naturally. Da Capo Lifelong Books.

Enyedy, N., & Mukhopadhyay, S. (2007). They don't show nothing I didn't know: Emergent tensions between culturally relevant pedagogy and mathematics pedagogy. The Journal of the Learning Sciences, 16(2), 139-174.

Esmonde, I. (2009). Mathematics learning in groups: Analyzing equity in two cooperative activity structures. The Journal of the Learning Sciences, 18(2), 247-284.

Gadamer, H.-G. (1975). Truth and method. Seabury Press.

Ginsburg, H. P. (2006). Mathematical play and playful mathematics: A guide for early education. Play= learning: How play motivates and enhances children's cognitive and social-emotional growth, 145-165.

Greeno, J. G. (1991). Number sense as situated knowing in a conceptual domain. Journal for Research in Mathematics Education, 22, 170-218.

Gresalfi, M., & Hand, V. M. (2019). Coordinating situated identities in mathematics classrooms with sociohistorical narratives: A consideration for design. ZDM, 51(3), 493-504.

Gresalfi, M. S. (2009). Taking up opportunities to learn: Constructing dispositions in mathematics classrooms. Journal of the Learning Sciences, 18, 327-369.

Gresalfi, M. S., Barnes, J., & Cross, D. (2012). When does an opportunity become an opportunity? Unpacking classroom practice through the lens of ecological psychology. Educational Studies in Mathematics, 80(1-2), 249-267.
Gresalfi, M. S., & Cobb, P. (2006). Cultivating students' discipline-specific dispositions as a critical goal for pedagogy and equity. Pedagogies: An International Journal, 1, 49-58.
Hand, V. (2010). The co-construction of opposition within a low-track mathematics classroom. American Educational Research Journal, 47, 97-132.
Kamii, C., Miyakawa, Y., & Kato, Y. (2004). The development of logico-mathematical knowledge in a block-building activity at ages 1–4. Journal of Research in Childhood Education, 19(1), 44-57.
Laski, E. V., Jor'dan, J. R., Daoust, C., & Murray, A. K. (2015). What makes mathematics manipulatives effective? Lessons from cognitive science and Montessori education. SAGE Open, 5(2), 2158244015589588.
Levine, S. C., Suriyakham, L. W., Rowe, M. L., Huttenlocher, J., & Gunderson, E. A. (2010). What counts in the development of young children's number knowledge? Developmental Psychology, 46(5), 1309.
Lifter, K., & Bloom, L. (1998). Intentionality and the role of play in the transition to language. Transitions in prelinguistic communication, 7, 161-196.
Lillard, A., & Else-Quest, N. (2006). Evaluating montessori education. Science, 313(5795), 1893-1894.
Miller, E., & Almon, J. (2009). Crisis in the kindergarten: Why children need to play in school. Alliance for Childhood (NJ3a).
Montessori, M. (1917). The Advanced Montessori Method: Spontaneous activity in education (F. Simmonds, Trans.; Vol. 1). Frederick A. Stokes Company.
Moseley, B. (2005). Pre-service early childhood educators' perceptions of math-mediated language. Early Education and Development, 16(3), 385-398.
Nasir, N. i. S., & Hand, V. M. (2006). Exploring sociocultural perspectives on race, culture, and learning. Review of educational research, 76(4), 449-475.
Nemirovsky, R., Kelton, M. L., & Rhodehamel, B. (2012). Gesture and imagination: On the constitution and uses of phantasms. Gesture, 12(2), 130-165.
Parks, A. N. (2015). Exploring mathematics through play in the early childhood classroom. Teachers College Press.
Parks, A. N., & Bridges-Rhoads, S. (2012). Overly scripted: Exploring the impact of a scripted literacy curriculum on a preschool teacher's instructional practices in mathematics. Journal of Research in Childhood Education, 26(3), 308-324.
Seo, K.-H., & Ginsburg, H. P. (2004). What is developmentally appropriate in early childhood mathematics education? Lessons from new research. In J. Sarama, A.-M. DiBiase, & D. H. Clements (Eds.), Engaging young children in mathematics: Standards for early childhood mathematics education (pp. 91-104). Lawrence Erlbaum Associates.
Sfard, A., & McClain, K. (2002). Analyzing tools: Perspectives on the role of designed artifacts in mathematics learning. Journal for the Learning Sciences, 11, 153-161.
Siegler, R. S., & Ramani, G. B. (2008). Playing linear numerical board games promotes low-income children's numerical development. Developmental science, 11(5), 655-661.
Sinclair, N. (2004). The roles of the aesthetic in mathematical inquiry. Mathematical Thinking and Learning, 6(3), 261-284.
Sinclair, N. (2026). Mathematics and beauty: Aesthetic approaches to teaching children. Teachers College Press.
Trawick-Smith, J., Swaminathan, S., & Liu, X. (2016). The relationship of teacher–child play interactions to mathematics learning in preschool. Early Child Development and Care, 186(5), 716-733.
Tudge, J. R., & Doucet, F. (2004). Early mathematical experiences: Observing young Black and White children's everyday activities. Early Childhood Research Quarterly, 19(1), 21-39.
Turner, E., Dominguez, H., Maldonado, L., & Empson, S. (2013). English learners' participation in mathematical discussion: Shifting positionings and dynamic identities. Journal for Research in Mathematics Education, 44(1), 199-234.
Van Oers, B. (2010). Emergent mathematical thinking in the context of play. Educational Studies in Mathematics, 74, 23-37.
Vygotsky, L. S. (1978). Mind and society: The development of higher psychological processes. Harvard University Press.
Wager, A., & Parks, A. N. (2014). Learning mathematics through play. Handbook of play and learning in early childhood, 216-227.

Wiebe Berry, R. A., & Kim, N. (2008). Exploring teacher talk during mathematics instruction in an inclusion classroom. The Journal of Educational Research, 101(6), 363-378.

Wolfgang, C., Stannard, L., & Jones, I. (2003). Advanced constructional play with LEGOs among preschoolers as a predictor of later school achievement in mathematics. Early Child Development and Care, 173(5), 467-475.

PREPARING TEACHERS TO ENGAGE STUDENTS FOR EQUITABLE MATHEMATICS EDUCATION

Robert Q. Berry, III
University of Arizona
rqberry@arizona.edu

Preparing teachers to teach mathematics is at the intersection of the three areas where cultural and racial knowledge intersects with content and pedagogical content knowledge. Consequently, preparing teachers to teach mathematics must consider all peoples' practices. This highlights two Black girls marginalized by their teacher, which provides the space for discussing teacher discretion and systemic violence. A significant takeaway when preparing teachers is to get them to think about how they can lead with mathematics rather than violence.

Keywords: Equity, Inclusion, and Diversity; Classroom Discourse; Teacher Educators

Preparing Teachers: Content, Pedagogy, & Cultural and Racial Knowledge

Mathematics teacher education has often concentrated on equipping teachers with the necessary skills to address three key areas: (a) the content of mathematics they should teach, (b) knowledge of pedagogy for teaching mathematics, and (c) the qualifications required for teaching mathematics (Berry et al., 2014). Teachers should have deep knowledge of mathematics content, which correlates positively to students' achievement (Howard & Milner, 2021). Additionally, teachers should know the pedagogy for teaching mathematics or mathematical knowledge for teaching, which is positively related to teachers' use of effective mathematics teaching practices (Ottmar et al., 2015). By focusing on these three areas, too often, preparing teachers to engage in equitable mathematics teaching receives minimal attention. There is widespread agreement among professional organizations in the field of mathematics education that preparing teachers to develop equitable frameworks in their teaching is necessary for addressing inequities in students' experiences (Association of Mathematics Teacher Educators, 2020; National Council of Teachers of Mathematics, 2014, 2018, 2020A, 2020B). Martin (2019) argued that there is still a considerable amount of effort required to establish equitable and inclusive access to mathematics education for those who have been historically excluded and disenfranchised.[1]

Teachers who teach students from historically marginalized and disenfranchised populations should not only be prepared to understand mathematics content and pedagogy but should also be prepared to understand ways of using their students' racial and cultural backgrounds in their teaching. Howard and Milner (2021) highlight three areas of knowledge for preparing teachers for equitable education: (a) content knowledge, (b) pedagogical content knowledge, and (c) cultural and racial knowledge. Preparing teachers to teach mathematics is at the intersection of the three areas where cultural and racial knowledge intersects with content and pedagogical

[1] When I use historically excluded and disenfranchised, I am not ascribing a sweeping set of attributes to Black, Latinx, Indigenous, and poor peoples. I understand that collapsing these groups into one group does not acknowledge the intersectionality within these groups. However, there are shared histories and experiences among historically excluded and disenfranchised people.

content knowledge. Consequently, preparing teachers to teach mathematics must consider all peoples' practices, including African, Indigenous, and non-Westernized perspectives (Mukhopadhyay et al., 2009). Building relationships and considering the practices of all peoples are described by many researchers as building on students' "funds of knowledge." Funds of knowledge encompass various aspects of people's lives, such as cultural experiences, artifacts, values, emotions, language, and identity (Moll & Gonzalez, 2004).

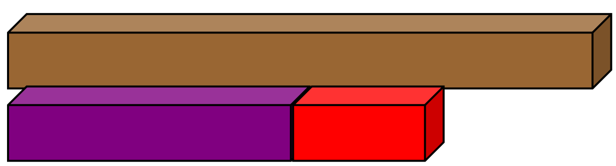
An example is provided by Civil and Khan (2001), who delved into teaching practices to establish a connection between students' familial and cultural experiences and mathematics content. These works all question the belief that teaching mathematics is culturally neutral and that there is a set of universally accepted teaching practices. Consequently, preparing teachers to teach mathematics must elicit frameworks for making meaningful connections between teaching with students' cultures, lives, and experiences.

The vignette below highlights the importance of making meaningful connections between teaching with students' cultures, lives, and experiences. The vignette focuses on the role of mathematical discourse as a pedagogical practice intersecting with Black girls' experiences and ways of knowing. Specifically, the Black girls, Alexa and Jasmine, are marginalized by their teacher, which provides the space for discussion of teacher discretion and systemic violence due to the girls being marginalized.

Vignette: Alexa and Jasmine

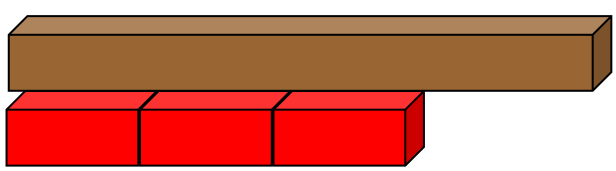
Ms. Lewis, a fourth-grade teacher, uses Cuisenaire Rods to make sure her students understand the concepts of unitizing (treating an object as a unit or whole), partitioning (separating a unit/whole into equal parts), and iterating (repeating a part to produce identical copies of it) (McCloskey & Norton, 2009). In previous grades and lessons, students used Cuisenaire Rods to compare fractions, add fractions with like denominators, and subtract fractions with like denominators. In this lesson, Ms. Lewis builds on the previously learned concepts to introduce adding fractions with unlike denominators. She starts by saying, "If the brown rod is the whole, how might we solve $\frac{1}{2} + \frac{1}{4}$?"

The students worked on the problem individually and then in pairs. Alexa and Jasmine, two Black girls, paired up to share their work and thinking with one another. Alexa's representation of the problem is below.



Alexa stated, "since brown is the whole, purple is one-half because two purples make a brown, and red is one-fourth because four reds make a brown."

Simultaneously, Jasmine explains her representation below.



Jasmine stated, “the purple is the same as two reds, so two reds are one-half. That is the same as two-fourths. So, you put another red, and the answer is three-fourths.” The discussion between the girls continued, with both working to make sense of the representations and convince each other why their representation made more sense.

Alexa and Jasmine were talking and referencing their representations simultaneously in a dynamic rather than a linear pattern. Their voices were elevated, and they expressed themselves nonverbally using hand gestures. While making her rounds to the pairs in the classroom, Ms. Lewis said to Alexa and Jasmine, “Ladies, one at a time.” There was no acknowledgment of their mathematical representation or the content of what was said.

When it was time for the whole group class discussion, Alexa and Jasmine enthusiastically raised their hands to share but were not acknowledged. After three students shared, Ms. Lewis presented a second task. “If the blue rod is the whole, how might you solve $\frac{2}{3}+\frac{2}{9}$?”

Alexa decided not to engage with the task, and Jasmine spent much time trying to get Alexa to do the task. Alexa asked, “Why should I do it? . . . she not gonna call on us.” Jasmine responded, “that don’t matter . . . come on, just do it.” After a few moments, Ms. Lewis made her rounds to Alexa and Jasmine and noted that neither had started the second task. Ms. Lewis stated, “I need you, ladies, to get started and follow directions.” Jasmine started representing the task with the Cuisenaire Rods, while Alexa decided not to engage.

As I reflect on Alexa and Jasmine’s positioning in Ms. Lewis’ classroom, I wonder if the girls’ discourse pattern was perceived as not competent. As a result, Alexa felt marginalized and disconnected. What if Ms. Lewis had led by getting a sense of the girls’ mathematics understandings, rather than focusing on the ways the girls were engaging? If Ms. Lewis had taken this approach, would Alexa have continued to stay engaged throughout the lesson? Alexa and Jasmine’s discourse patterns are informed by their experiences within the Black community. This chapter challenges readers to examine the intersections between discourse patterns informed by communal experiences, teaching practices, and systemic violence.

Mathematical Discourse, Systemic Violence, and Discretionary Spaces

There is widespread agreement on the role of mathematical discourse in positively impacting students’ mathematical experience. Promoting and valuing students’ participation in mathematical discourse is a way of positioning them as mathematically competent (Berry, 2018). Mathematical discourse involves several practices, such as asking and answering questions, exchanging mathematical representations and ideas, and constructing, evaluating, and refining arguments, among other methods. The use of mathematical discourse practices allows students to demonstrate their mathematical understanding and to connect with other students. However, mathematics teachers must be mindful of how focusing on discourse practices to the exclusion of mathematical and cultural understanding and experiences can position students as incompetent. In the vignette, Ms. Lewis focused on discourse without acknowledging Alexa and Jasmine’s mathematical understanding. She responded to how the girls engaged in discourse rather than their mathematical understanding reflected in their discourse.

Even when focusing on mathematical understanding for discourse, students may be positioned as “at-risk.” For example, constructing viable arguments and critiquing others’ reasoning are practices used in mathematics teaching that might have risk-taking implications for students. Constructing viable arguments and critiquing others’ reasoning are practices that, when performed outside the mathematics classroom, may put historically excluded students, specifically Black students, at risk of negative consequences. When Black students construct arguments and critique the reasoning of those in authority or engage in this practice outside of mathematics classrooms, educators must consider the potential consequences. Argumentation may involve the projection of voice, tone, positioning of bodies, and proximity. These forms of engagement may be considered deficit if there is little understanding of social and cultural contexts for argumentation. In preparing teachers it is important to convey that there may be teaching practices that position students as “at-risk.”

The potential risk for Black students engaging in argumentation occurs when their ways of engagement do not align with expected norms for participating in this type of discourse, creating the potential for negative consequences. Alexa and Jasmine’s discourse patterns were not aligned with Ms. Lewis’ expectations for students’ participation in discourse. Many Black students come from communities where argumentation is non-linear and can be perceived as loud and aggressive, with several people talking simultaneously while engaging with one another. The assumption may be that no listening is occurring if multiple people are talking simultaneously. Such an assumption ignores the experiences and resources students bring from their communities. This argumentation pattern is inconsistent with turn-taking, where one person speaks while others listen, with everyone using “inside voices.” Policing argumentation based on assumptions and a specific pattern of discourse ignores that students may be highly engaged in mathematics. Understanding the assumptions and argumentation patterns helps teachers recognize that discourse in some communities is dynamic and multi-tiered.

Research in mathematics education reveals that Black students often face devaluation, inequity, exclusion, and violence (Battey, 2013; Berry, 2008; Joseph et al., 2019; Martin, 2015; McGee & Martin, 2011). In everyday language, “violence” is primarily defined as physical aggression against a person or group (Leonardo, 2013). In this context, violence is used to describe the emotional and psychological acts occurring in spaces where students feel disconnected and marginalized. Although this type of violence may not result in physical harm, it is still considered as such due to its detrimental effect on an students emotional and psychological well-being. The experiential aspects of violence in mathematics for historically excluded students can result in teaching and learning that lead to little or no engagement in activities that promote reasoning. In the second task in the vignette, Alexa chose not to participate because she felt she had not been acknowledged for her mathematical thinking, resulting in her feeling disconnected. An argument can be made that in this scenario, Alexa experienced violence.

Many Black students come to schools and classrooms with contexts rooted in the culture and traditions of the Black experience (Berry, 2020). Consequently, educators must know and understand Black students’ resources and find ways to incorporate these into mathematics teaching and learning. Considering Alexa and Jasmine’s perspectives in the vignette, it is plausible that their discourse pattern might be rooted in their cultural context and familiar traditions. Unfortunately, classroom teaching practices too often ignore context and traditions familiar to students and default to normalizing whiteness as the source of standard practices. This does not imply that all such practices are detrimental to Black students; however, it is necessary to understand how classroom practices differentially impact students’ experiences.

Teachers make daily decisions in classrooms that have a significant impact on their students' experiences. According to Ball (2018), these decisions are made in what he refers to as "discretionary spaces." In mathematics, discretionary spaces include but are not limited to task selection, means of engagement, positioning of students, and decisions about discourse. Ball

(2018), argues that discretionary spaces are driven by both policies and the autonomous discretion of teachers. For example, policy-driven practices might require specific teaching practices and resources, test preparation strategies for standardized tests, and standardizing content coverage through the narrow use of a pacing guide. The tensions between policy-driven teaching practices and teacher autonomy create what Ball (2018) described as a paradox of constraint and discretion in teaching.

Constraint teaching while positioned to disrupt inequities, can also restrict teachers' use of their professional judgment through standardization, and impede efforts to make schools contextually and culturally responsive. Discretion in teaching can make it possible to teach in contextually sensitive and culturally responsive ways and enable teachers to connect the school to the world, but it can also allow marginalization and other forms of oppression to flow into schools. Thus, Ball's (2018) paradox suggests that there are nuanced spaces necessary for both constraint and discretion in teaching to address systemic violence.

Ms. Lewis used a district-level mandated curriculum in her classroom that incorporated many teaching strategies supported by the National Council of Teachers of Mathematics (NCTM, 2020). For example, the curriculum incorporated mathematical discourse strategies that encourage students to provide reasoning and justifications for their mathematical thinking. While one can argue that using the curriculum and its materials are policy-driven, implementing those practices was at Ms. Lewis' discretion. It was clear that Ms. Lewis followed the basic outline of the lesson as prescribed in the curriculum materials. However, her decision not to acknowledge Alexa and Jasmine was at her discretion. The paradox is that the curriculum was designed to provide a high-quality mathematics experience for students, but the implementation can lead to disconnection and marginalization.

Every classroom is fertile with opportunities to elevate or stifle students through various modes of participation, valuing their thinking, and building community. Ball (2018) unpacks micro-moments in which teachers send implicit and explicit messages to students through decisions made in discretionary spaces. For some students, discretionary spaces serve as a source of systemic violence. Systemic violence can address behavior, position students' mathematical competency, and relate to culture and language in mathematics teaching. One way to disrupt systemic violence is to focus on students' humanity, cultural ways of knowing, and experiences that can create opportunities for engagement through mathematical discourse.

Leading with Mathematics

Let's ponder how the lesson would have been different if Ms. Lewis had focused on mathematics instead of how Alexa and Jasmine engaged with one another. Instead of saying, "Ladies, one at a time," what if Ms. Lewis had said, "Tell me about your representation" or "Describe your Cuisenaire Rods"? These questions suggest that leading with mathematics creates opportunities for both the teacher and the learner. One major takeaway from the vignette when preparing teachers is to get them to think about how they can lead with mathematics rather than violence. Leading with mathematics focuses on pedagogical choices that position students as competent, experts, and safe. Teachers use questions to elevate students' mathematical thinking and representations rather than focusing solely on behaviors and how students are engaging in discourse while excluding mathematics. Positioning students as competent, experts, and safe demonstrates the discretionary spaces teachers use as part of their professional judgment to diminish and eliminate systemic violence and psychological, cultural, and spiritual harm to students. When teachers engage in discourse practices to support positive identity development, build community, and give students autonomy to engage in mathematics using the resources they bring to the classroom, they can reflect on the impact of their practices on students' well-being. Reflecting on this vignette should push mathematics teacher educators to consider how to prepare teachers to use spaces of discretion.

References

Association of Mathematics Teacher Educators (AMTE). (2020). Standards for Preparing Teachers of Mathematics:(black+ White Version). Information Age Publishing, Incorporated.

Ball, D. L. (2018, April). Just dreams and imperatives: The power of teaching in the struggle for public education. In Presidential Address to the American Educational Research Association Annual Meeting, New York (Vol. 19).

Berry, R. Q., III. (2008). Access to upper-level mathematics: The stories of successful African American middle school boys. Journal for Research in Mathematics Education, 39(5), 464‒488.

Berry, R. Q., III. (2018, July). Positioning students as mathematically competent [NCTM blog post]. https://www.nctm.org/News-and-Calendar/Messages-from-the-President/Archive/Robert-Q_-Berry-III/Positioning-Students-as-Mathematically-Competent/

Berry, R. Q., III. (2020, February). How do we help teachers teach math to Black kids? My response [NCTM blog post]. https://www.nctm.org/News-and-Calendar/Messages-from-the-President/Archive/Robert-Q_-Berry-III/February-2020/

Berry, R. Q., III, Ellis, M. & Hughes, S. (2014). Examining a history of failed reforms and recent stories of success: Mathematics education and Black learners of mathematics in the United States. Race Ethnicity and Education, 17(4), 540–568. doi:10.1080/13613324.2013.818534.

Civil, M., & Khan, L. H. (2001). Mathematics instruction developed from a garden theme. Teaching Children Mathematics, 7(7), 400 – 405.

Howard, T. C., & Milner, H. R. (2021). Teacher preparation for urban Schools. In H. R. Milner & K. Lomotey (Eds.), Handbook of urban education (2nd ed., pp. 221–237). Routledge.

Joseph, N. M., Hailu, M. F., & Matthews, J. S. (2019). Normalizing Black girls' humanity in mathematics classrooms. Harvard Educational Review, 89(1), 132–155.

Leonardo, Z. (2013). The story of schooling: Critical race theory and the educational racial contract. Discourse: Studies in the Cultural Politics of Education, 34(4), 599–610.

Martin, D. B. (2015). The collective Black and Principles to Actions. Journal of Urban
Mathematics Education 8(1), 17–23.

Martin, D. B. (2019). Equity, inclusion, and antiblackness in mathematics education. Race Ethnicity and Education, 22(4), 459–478. https://doi.org/10.1080/13613324.2019.1592833

McCloskey, A. V., & Norton, A. H. (2009). Using Steffe's advanced fraction schemes. Mathematics Teaching in the Middle School, 15(1), 44–50.

McGee, E. O. & Martin, D. B. (2011). "You would not believe what I have to go through to prove my intellectual value!" Stereotype management among academically successful Black mathematics and engineering students. American Educational Research Journal, 48(6), 1347–1389. doi:10.3102/0002831211423972.

Moll, L. C., & González, N. (2004). Engaging life: A funds of knowledge approach to multicultural education. In J. Banks & McGee Banks (Eds.), Handbook of research on multicultural education (Second edition) (pp. 699 – 715). NY: Jossey-Bass.

Mukhopadhyay, S., Powell, A. B., & Frankenstein, M. (2009). An ethnomathematical perspective on culturally responsive mathematics education. In B. Greer, S. Mukhopadhyay, A. B. Powell, & S. Nelson-Barber (Eds.), Culturally responsive mathematics education (pp. 65 – 84). New York, NY: Routledge.

National Council of Teachers of Mathematics. (2014). Principles to actions: Ensuring mathematical success for all. Reston, VA: NCTM.

National Council of Teachers of Mathematics (2018). *Catalyzing Change in High School Mathematics: Initiating Critical Conversations*. Reston, VA: NCTM.

National Council of Teachers of Mathematics (2020A). *Catalyzing Change in Early Childhood and Elementary Mathematics: Initiating Critical Conversations*. Reston, VA: NCTM.

National Council of Teachers of Mathematics (2020B). *Catalyzing Change in Middle School Mathematics: Initiating Critical Conversations*. Reston, VA: NCTM.

Ottmar, E. R., Rimm-Kaufman, S. E., Larsen, R. A., & Berry, R. Q., III (2015). Mathematical knowledge for teaching, standards-based mathematics teaching practices, and student achievement in the context of the responsive classroom approach. American Educational Research Journal, 52(4), 787-821.

WELCOME TO THE WORLD OF MATHEMATICS -- WHERE ANYTHING IS POSSIBLE!

¡Bienvenido al Mundo de las Matemáticas, Donde Todo es Posible!

Mitchell J. Nathan
University of Wisconsin–Madison
MNathan@wisc.edu

In my Keynote Address to PME-NA 45, I offer an embodied framework for naming what makes mathematics powerful for mathematicians and scientists, yet intractable for many learners. The essential claim is this: Students reside in the Real World, where math is grounded, embodied and meaningful, while mathematics resides in the ungrounded, disembodied realm of the UnReal World. To make <u>all</u> educational experiences <u>meaningful</u>, I consider ways to prepare students to be tourists to the UnReal World, such as progressive formalization and immersion in eXtended Reality (XR). Even so, educators must remember that learners remain citizens of the Real World even when visiting the UnReal World. I share examples of how embodied learners make sense of UnReal things, method of making bridges between these worlds, and concerns that entrenched assessment practices neglect the nonverbal ways of knowing expressed by embodied learners.

Welcome to the world of mathematics -- where anything is possible! This is both wonderful (for mathematicians) and terrible (for students). *Wonderful*, in that mathematics delivers a set of formal systems with precise and powerful rules for generating (and verifying) mathematically correct statements to model patterns, both real and imagined *Terrible*, because the truths that are generated are detached from the Real World and frequently defy the intuitions and expectations students have developed from their lived experiences. I offer this frame because my scholarship is positioned with a guiding value: **All educational experiences should be *meaningful* to students.** I arrive at this based on ethical considerations that educational institutions must be committed to improving students' lives and futures, and based cognitive considerations that meaningful information is more readily accessed, retained, and applied (Bransford et al., 2000). I refer to this focus on meaningful learning as **achieving a grounded understanding** (Nathan, 2014). Ideas and symbols that are abstract and unfamiliar become grounded for learners when they are connected to one's lived and felt experiences, including those ideas and symbols that are embodied through one's perceptions, actions, and social and cultural practices.

1. Students Reside in the Real World; Mathematics in the UnReal World

Mathematics education encounters an apparent enigma: If people naturally do math as they go about their lives, what makes learning math so difficult? The quandary falls away when we recognize that the math people do is what I call **Grounded and Embodied Math**, or **GEM**. It is the math that people *can* do because as embodied beings, people are grounded in their lived, contextually bound experiences. This is the mathematics of the **Real World (RW)** where people -- myself included! -- actually reside. Here in the RW, things have mass, and extent; operations take time and energy; capacities are limited, space confined. Hardly anything (perhaps <u>not</u> anything) is truly linear, exact, discrete, or known with certainty. In the RW where GEM occurs, all things are *not* possible. Herein lies a central challenge for math education.

One can also imagine a different world, **the UnReal World (unRW)**, where **<u>UnG</u>rounded and dis<u>E</u>mbodied <u>M</u>ath (unGEM)** is relevant and has meaning. This is a realm filled with dimensionless points, lines of infinite length, perfect correlations, and certainty.

I use these terms, ***RW, unRW, GEM, and unGEM*** *-- cumbersome as they may be -- to help firm up my thoughts about the current challenges facing math education and some of the promising pathways forward.*

When we teach math, we are really asking *two* things of students:

I. Master the formal systems of notation that describes the behavior of objects that adhere to the rules of unGEM; and

II. Become a resident of unRW, so that unGEM becomes meaningful.

Our educational institutions predominantly support students only with the first of these, and largely ignores the second. I believe it is imperative that we consider both propositions because there are many documented occasions where students can reliably apply the rules of unGEM (consistent with Proposition I), even as the formal statements of unGEM remain meaningless with respect to the world in which they reside (counter to Proposition II; e.g., Koedinger & Nathan, 2004; Landy & Goldstone, 2007). More commonly, students adhere to Proposition I by mastering unGEM even when they can demonstrate their understanding of the mathematical ideas to which the unGEM symbols refer. This is observed, for example, when students who speak different languages exhibit nonverbal ways of effectively expressing their mathematical reasoning during collaboration, thus circumventing unfamiliar terms (Swart et al., 2021).

2. Preparing Tourists to the UnReal World

One reason for this disconnect is a lack of *common ground* between phenomena in the RW and the unRW (Alibali et al., 2019). Naming the dichotomy between the grounded and embodied nature of the RW and the ungrounded and disembodied nature of the unRW helped me frame my goals in terms of an educational question: *How can we support students to become competent tourists in unRW*? I am aware of three promising ways (which need not be mutually exclusive):

1. *Progressive formalization*. This builds on the natural ability of humans to incrementally ground from concrete experiences to idealized descriptions (i.e., abstractions) of those experiences by establishing a new grounding of previously unfamiliar ideas in successive iterations. For example, manipulating physical objects (i.e., performing operations on RW objects) often provides the necessary grounding when first encountering numbers, while numbers later ground arithmetic operations, arithmetic comes to ground algebraic expressions, and so on. Realistic Mathematics Education (Gravemeijer & Doorman, 1999; Van den Heuvel-Panhuizen & Drijvers, 2020) and Concreteness Fading (Bruner, 1966; Fyfe et al., 2014; Ottmar & Landy, 2017) are two approaches to progressive formalization in math education with substantial empirical support. These approaches incrementally shifts RW experiences towards unGEM of unRW by successively stripping away the perceptual richness of RW actions and objects, so they come to serve as increasingly idealized representations of the original experience.

2. *Professional Vision*. Changes in one's perceptions and language through instruction enables one to see and describe RW phenomena through a lens of unGEM (Goodwin, 1994; Stevens & Hall, 1998). Powerful examples of this are seen in Bob Moses' Algebra Project (Moses & Cobb, 2002), Learning Along Lines (Ma, 2016; Taylor, 2017), and Dallas Math Walks (Wang & Walkington, 2023). These approaches brings unGEM into RW by grounding unRW concepts in students' lived perceived and embodied experiences.

3. *Immersion in Microworlds*. This invites learners into spaces where objects only behave according to rules of unGEM. The includes approaches with varying degrees of immersion, from tablets that present manipulable mathematical objects for number (TouchMath; Sinclair & Heyd-Metzuyanim, 2014), algebraic expressions (Graspable Math; Ottmar et al., 2015), and geometric objects (Carbri, GeoGebra), to eXtended Reality (XR) environments (including VR and AR) that sensorially immerse one in unRW spaces (Dimmel et al., 2023; Johnson-Glenberg & Megowan-Romanowicz, 2017; Walkington et al., 2022). These approaches puts residents of RW in unRW to adopt the practices of unGEM; and, ideally, reflect upon those practices when they return to RW.

Each of these approaches rests on the amazing plasticity of the human neural system, and a realization that with the appropriate sensorimotor experiences, we are able to (temporarily) realign our expectations for how things ought to behave in the world. Examples such as the rubber hand illusion (National Geographic, 2015), illustrate the range of situations to which humans are able to adjust.

3. We Remain Citizens of the Real World Even When we Visit the UnReal World

As educators and educational researchers, it is important to keep in mind that people are inclined to apply the rules and expectations that have successfully guided their RW experiences even when visiting the unRW. To illustrate, humans -- mathematicians included, LOL! -- are inclined to interpret phenomena of the unRW through a GEM lens. Thus, numbers *are* locations in space, operations *are* physical actions on mathematical "objects," limits *are* fictive motion (Lakoff & Núñez, 2000), logic *is* a sequence of cause and effect, and set theoretic statements *are* idealizations of containers for collecting, categorizing and classifying objects (Johnson, 1980). In visual art, M. C. Escher's *Ascending and Descending,* intrigues us because it offers a twist (literally and actually) on our experiences of gravity the RW. Even the *Theory of Relativity,* arguably the paradigmatic case of thinking "outside the box," is said to have its origin in the mundane nature of Einstein's contemplations of the variations in train schedules across Europe as he commuted to and from his job at the patent office in Bern (Galison, 2004). It is not that people are unable to perceive unGEM objects, but that they will project their RW interpretations onto objects from unGEM. We see this abound from work analyzing students' math errors (e.g., Landy & Goldstone, 2007; Koedinger & Nathan, 2004; VanLehn, 1990) demonstrating that these errors often arise when people try to make unGEM objects conform to *their* RW. Thus, it is important to acknowledge -- and accommodate -- learner's natural tendencies.

This should not be taken to suggest that people are basically literal thinkers, or lack imagination. Contrarily, people exhibit great imagination in their contributions to science, technological innovation, and the arts. Rather, these realizations highlights how people are naturally inclined to use their experiences to rationalize problem spaces, and, further, to point out that even people's imagination is not limitless; it is tethered to one's RW, embodied experiences. Artists such as Escher (1972; see note 1) in the visual realm, and Pink Floyd in music (Waters, Wright, Mason, & Gilmour, 1970; see note 2) demonstrate that even when people imagine behaviors that violate the laws of the RW, they do so while standing on *terra firma*.

While we all live in the RW, there are actual residents of the unRW: ChatGPT*n* (currently at version 4) is one example, and so is your smartphone. These residents are comfortable with living in ungrounded, formal spaces. Borges' (1962/ 2007) *Library of Babel* is one such space. Borges imagined an institution that contains every conceivable book by generating every orthographic variations of the alphabet, regardless of whether it is readable or accurate (e.g., the entire contents of my future talk). Personally, as a trained mathematician, I like to visit unRW and partake in unGEM, but as a trained engineer, I do not wish to reside there permanently.

One of our superpowers as human beings is our ability to process *all* events from a grounded and embodied perspective. Just because humans have developed strings of symbols to represent systems of language, culture, art, and mathematics, does not mean that we take comfort in their arbitrary associations to ideas and events. Unlike AI, we do not simply store these formalisms verbatim or process them blindly. Rather, people project meaning onto these symbol systems because of their cultural associations, embodied affordances, and the mental simulations they invoke (Bransford et al., 1972; Gallese & Sinigaglia, 2011; Glenberg & Robertson 2000). This tethering to worldly experiences is one of the advantages people have over AI systems and it is one reason why humans excel in expressing themselves through art and mathematics. Practice and feedback from the RW affords humans opportunities to get better at accurately recognizing and generating images that represent RW objects. Ironically, generative AIs trained on synthetic data produced by other generative AIs rapidly degrade with practice and feedback, and the images they produce become further removed from real-looking objects (Alemohammad et al., 2023). unGEM, operating in its own self-referential world, becomes self-consuming and increasingly absurd (Eisenstein, 2023).

4. Final Thoughts: Assessing the Understanding of Grounded and Embodied Learners

Since its inception, the PME community has acknowledged a key insight by focusing on the *Psychology* of Mathematics Education. Educational practitioners and leaders, working with members of the education research community, must bridge GEM and unGEM. At its core, these bridging activities necessitate a continued understanding of the psychological needs of students and teachers to adopt the reasoning and practices of an unReal World that has tremendous utility for modeling the RW and gifting it with innovations that further the Public Good. Existing and emerging educational practices and technologies offer promising inroads, but ultimately depend upon an understanding of human thinking, human development and human behavior.

This is made imminently clear by examining a persistent misalignment between our emerging conceptualizations of student knowledge and current knowledge assessment practices. Once we recognize the grounded and embodied nature of people's mathematical thinking and learning, including students' nonverbal and nonsymbolic ways of knowing and expressing their thinking, it is clear that formative and summative knowledge assessment practices must follow suit. Fortunately, teachers who themselves engage in and reflect upon their own embodied mathematical behaviors seem to become more inclined to notice students' nonverbal ways of expressing their mathematical reasoning and to address students' embodied behaviors in their formative assessment practices (Sung, Swart & Nathan, 2021). Considerations of both formative and summative assessment practices designed with grounded and embodied learners in mind offer some of the most promising next steps for advancing mathematics education for all learners.

Author Notes

Note 1. M. C. Escher's lithograph *Waterfall* (Dutch: *Waterval*; Escher, 1972) gives the appearance of a normal flow of water that endlessly cycles upward or downward. The watercourse uses two Penrose triangles to create the illusion. The Penrose triangle was designed by Oscar Reutersvärd in 1934, and independently discovered by Roger Penrose in 1958 (Penrose & Penrose, 1958).

Note 2. Pink Floyd's (1970) "Echoes" is a composition on the *Meddle* album that runs over 23 minutes. The composers used the Shepard-Risset Glissando, a variant of the Shepard Tone created by psychologist Roger Shepard (1964), which can be heard at the end of composition, beginning just after the 22-minute mark in the Meddle version, https://www.youtube.com/watch?v=KBca3xf-j3o.

Acknowledgments

I am grateful for comments on an earlier draft from my colleague and dear friend, Dr. Michael Swart.

References

Abrahamson, D., Nathan, M. J., Williams–Pierce, C., Walkington, C., Ottmar, E. R., Soto, H., & Alibali, M. W. (2020). The future of embodied design for mathematics teaching and learning. *Frontiers in Education, 5*(147). Doi.org/10.3389/feduc.2020.00147

Alemohammad, S., Casco-Rodriguez, J., Luzi, L., Humayun, A. I., Babaei, H., LeJeune, D., … & Baraniuk, R. G. (2023). Self-consuming generative models go mad. *arXiv preprint arXiv:2307.01850*.

Alibali, M. W., Nathan, M. J., Boncoddo, R., & Pier, E. (2019). Managing common ground in the classroom: teachers use gestures to support students' contributions to classroom discourse. *ZDM*, *51*, 347-360.

Bransford, J. D., Barclay, J. R., & Franks, J. J. (1972). Sentence memory: A constructive versus interpretive approach. *Cognitive Psychology, 3*(2), 193– 209.

Bransford, J. D., Brown, A. L., & Cocking, R. R. (2000). *How people learn*. Washington, DC: National Academy Press.

Bruner, J. S. (1966). *Toward a theory of instruction*. Harvard University Press.

Dimmel, J., Pandiscio, E., & Bock, C. (2021). The geometry of movement: encounters with spatial inscriptions for making and exploring mathematical figures. *Digital Experiences in Mathematics Education*, *7*(1), 122-148.

Eisenstein, C. (2023). From homogeneity and bedlam to sense and sensibility, Aug 7, 2023, https://charleseisenstein.substack.com/p/from-homogeneity-and-bedlam-to-sense?r=98ztt&utm_campaign=post&utm_medium=email

Escher, M. C. (*1972*) *The Graphic Works of M. C. Escher*. London: Pan Books.

Fyfe, E. R., McNeil, N. M., Son, J. Y., & Goldstone, R. L. (2014). Concreteness fading in mathematics and science instruction: A systematic review. *Educational Psychology Review, 26*, 9-25.

Galison, P. (2004). *Einstein's clocks and Poincare's maps: Empires of time.* WW Norton & Company.

Gallese, V., & Sinigaglia, C. (2011). What is so special about embodied simulation?. *Trends in Cognitive Sciences, 15*(11), 512-519.

Glenberg, A. M., & Robertson, D. A. (2000). Symbol grounding and meaning: A comparison of high-dimensional and embodied theories of meaning. *Journal of Memory and Language*, *43*(3), 379-401.

Goodwin, C. (1994). Professional vision. *American Anthropologist, 96*, 606– 633.

Gravemeijer, K., & Doorman, M. (1999). Context problems in realistic mathematics education: A calculus course as an example. *Educational Studies in Mathematics, 39*(1-3), 111-129.

Johnson-Glenberg, M. C., & Megowan-Romanowicz, C. (2017). Embodied science and mixed reality: How gesture and motion capture affect physics education. *Cognitive Research: Principles and Implications*, *2*(1), 1-28.

Koedinger, K. R., & Nathan, M. J. (2004). The real story behind story problems: Effects of representations on quantitative reasoning. *The Journal of the Learning Sciences*, *13*(2), 129-164.
Lakoff, G., & Núñez, R. (2000). *Where mathematics comes from*. New York: Basic Books.
Landy, D., & Goldstone, R. L. (2007). How abstract is symbolic thought?. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, *33*(4), 720.
Ma, J. Y. (2016). Designing disruptions for productive hybridity: The case of walking scale geometry. *Journal of the Learning Sciences*, *25*(3), 335-371.
Moses, R., & Cobb, C. E. (2002). *Radical equations: Civil rights from Mississippi to the Algebra Project*. Beacon Press.
Nathan, M. J. (2014). Grounded mathematical reasoning. In L. Shapiro (Ed.) *The Routledge handbook of embodied cognition* (pp. 171-183). Routledge.
Nathan, M. J., Soto, H., Closser, A. H., Ottmar, E., Abrahamson, D., & Walkoe, J. (2021, October). Embodied mathematical imagination and cognition (EMIC) Research Colloquium. In Olanoff, D., Johnson, K., & Spitzer, S. M. (2021). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1934-1937). Philadelphia, PA.
National Geographic (2015). Is That My Real Hand? *Breakthrough*, *National Geographic* (Air date Nov 4, 2015). https://www.youtube.com/watch?v=DphlhmtGRqI Accessed on August 23, 2023.
Ottmar, E. R., Landy, D., Goldstone, R. L., & Weitnauer, E. (2015). Getting from here to there: Testing the effectiveness of an interactive mathematics intervention embedding perceptual learning. In D. C. Noelle, R. Dale, A. S. Warlaumont, J. Yoshimi, T. Matlock, C. D. Jennings, et al. (Eds.), *Proceedings of the Thirty-Seventh Annual Conference of the Cognitive Science Society* (pp. 1793–1798). Cognitive Science Society.
Ottmar, E., & Landy, D. (2017). Concreteness fading of algebraic instruction: Effects on learning. *Journal of the Learning Sciences, 26*(1), 51-78.
Penrose, L. S. & Penrose, R. (1958). Impossible objects: A special type of visual illusion. *British Journal of Psychology, 49*(1), 31–33. doi:10.1111/j.2044-8295.1958.tb00634.x
Shepard, R. N. (1964). Circularity in judgments of relative pitch. *The Journal of the Acoustical Society of America*, *36*(12), 2346-2353. doi: 10.1121/1.1919362
Sinclair, N., & Heyd-Metzuyanim, E. (2014). Learning Number with TouchCounts: The role of emotions and the body in mathematical communication. *Technology, Knowledge and Learning*, *19*(1), 81-99.
Stevens, R., & Hall, R. (1998). Disciplined perception: Learning to see in technoscience. In M. Lampert & M. L. Blunk (Eds.) *Talking mathematics in school: Studies of teaching and learning*, (pp. 107-149). Chicago: Cambridge University Press.
Sung, H., Swart, M. I., & Nathan, M. J. (2021, October). Enhancing K-12 pre-service teachers' embodied understanding of the geometry knowledge through online collaborative design. In Olanoff, D., Johnson, K., & Spitzer, S. M. (2021). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 909-917). Philadelphia, PA.
Swart, M. I., Schenck, K. E., Xia, F., Kim, D., Kwon, O. H., Walkington, C., & Nathan, M. J. (2020, October). Embodiment as a Rosetta Stone: Collective conjecturing in a multilingual classroom using a motion capture geometry game. In Ana Isabel Sacristán & José Carlos Cortés (Eds.) (pp. 2206-2214), *Proceedings of the 42nd annual conference of the North-American chapter of the International Group for the Psychology of Mathematics Education (PME-NA 42)*. Mazatlán, Sinaloa, Mexico. doi.org/10.51272/pmena.42.2020-382
Taylor, K. H. (2017). Learning along lines: Locative literacies for reading and writing the city. *Journal of the Learning Sciences*, *26*(4), 533-574.
Van den Heuvel-Panhuizen, M., & Drijvers, P. (2020). Realistic mathematics education. In. S. Lerman (Ed.) *Encyclopedia of mathematics education*, (pp. 713-717). Springer.
VanLehn, K. (1990). *Mind bugs: The origins of procedural misconceptions*. Cambridge, MA: MIT press.
Walkington, C., Nathan, M. J., Hunnicutt, J., Washington, J., & Holcomb-Webb, K. (2022). Learning geometry through collaborative, embodied explorations with augmented reality holograms. In *Proceedings of the 16th International Conference of the Learning Sciences-ICLS 2022, pp. 1992-1993*. International Society of the Learning Sciences.
Wang, M., & Walkington, C. (2023). Investigating problem-posing during math walks in informal learning spaces. *Frontiers in Psychology*, *14*, 1106676.
Waters, R., Wright, R., Mason, N., & Gilmour, D. (1970). Echoes [Recorded by Pink Floyd]. On *Meddle*. London, UK: Harvest Records.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1). University of Nevada, Reno.

MATHEMATICAL KNOWLEDGE FOR TEACHING

A FRAMEWORK FOR PEDAGOGICAL CONTENT KNOWLEDGE IN TEACHING MEASURES OF CENTRAL TENDENCY

Dhimitraq Duni
Eastern New Mexico University
dhimitraq.duni@enmu.edu

Keywords: Teacher knowledge, Data analysis and statistics

Pedagogical content knowledge (PCK) has been a topic of much interest for the last 37 years (Duni, 2019). When trying to understand teacher knowledge, in introducing measures of central tendency (MCTs) in middle school, finding a framework to adapt to this work was difficult. The problem is that most frameworks are too general and do not lend themselves to the questions being posed. So, the lack of specificity in the existing frameworks forced me to the create a new conceptual framework to analyze teacher knowledge.

As established by previous research (Ball, Thames, &Phelps, 2008; Hill et al., 2008; Groth, 2013), I conceptualized PCK as the knowledge of curriculum (KC), knowledge of content and students (KCS), and knowledge of content and teaching (KCT). What I contributed and elaborated on, was the specific components of each of knowledge variate as it pertains to teaching MCTs in middle school statistics. I analysed curriculum documents (NCTM, 2000; CCSSO, 2010), policy documents (Franklin et al., 2007; Franklin et al., 2015), and research (Mokros & Russel, 1995; Strauss and Bichler, 1998; Tarr & Shaughnessy, 2003; Zawojewski & Shaughnessy, 2000) pertaining to how students learn, and how teachers teach MCTs in middle school statistics (Duni, 2018). From that literature review I compiled the different components of the framework (that I will present here) that I used to analyze teachers' pedagogical content knowledge.

Components

KC emphasizes the need for students to understand the differences and choose between the different MCTs, and that students need to have a conception of the mean that does nor revolve around the algorithm of add-and-divide. Kader et al. (2013) and NCTM (2000) recommended the concept of the fair share and balance point as good introductions to the mean. NGA & CCSSO (2010) recommended that students should also understand that all three MCTs are used to represent the data with a single number, and that students should make connections between MCTs and variability.

KCS incudes mostly conceptions and misconceptions students have about finding MCTs. Students tend to: think of minimums and maximums as outliers, ignore 0 in the data set when computing measures of center, not order data before finding the median, use the mode in the beginning to represent typicality, ignore mean when it is not part of the data, have difficulty calculating weighted means, only know the mean procedurally.

Lastly, KCT included the investigative cycle at the center as well as recommendations to connect statistics to other areas of mathematics, relying on real data, and making effective use of technology (Franklin et al., 2007; NCTM, 2000).

This research shows that the existing overarching frameworks on PCK are useful, but they need to be adapted to the specific work done. Having this framework allows faculty to be specific on recommendations for teacher preparation and professional development.

References

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389–407. doi:10.1177/0022487108324554

Franklin, C., Kader, G., Mewborn, D., Moreno, J., Peck, R., Perry, M., & Scheaffer, R. (2007). *Guidelines for assessment and instruction in statistics education (GAISE) report: A pre-K-12 curriculum framework.* Alexandria, VA: American Statistical Association.

Franklin, C. A., Bargagliotti, A. E., Case, C. A., Kader, G. D., Scheaffer, R. L., & Spangler, D. A. (2015). *The statistical education of teachers*. Alexandria, VA: American Statistical Association. Retrieved from www.amstat.org/education/SET/SET.pdf

Duni. D. (2018). *The look of pedagogical content knowledge in middle school statistics* (Doctoral dissertation). Retrieved from https://ir.library.illinoisstate.edu/etd/952

Duni, D. (2019). Using Computer Generated Animations to Examine Pedagogical Content Knowledge in Statistics Teaching. In *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 568-573).

Groth, R. E. (2013). Characterizing key developmental understandings and pedagogically powerful ideas within a statistical knowledge for teaching framework. *Mathematical Thinking and Learning*, *15*(2), 121–145. doi:10.1080/10986065.2013.770718

Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education*, *39*(4), 372–400.

Kader, G. K., Jacobbe, T., Wilson, P. S., & Zbiek, R. M. (2013). *Developing essential understanding of statistics for teaching mathematics: Grades 6-8*. Reston, VA: National Council of Teachers of Mathematics.

Mokros, J., & Russell, S. J. (1995). Children's concepts of average and representativeness. *Journal for Research in Mathematics Education*, *26*(1), 20–39. doi:10.2307/749226National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author. National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common core state standards for mathematics*. Washington, DC: Author. Retrieved from http://www.corestandards.org/assets/CCSSI_Math Standards.pdf

Strauss, S., & Bichler, E. (1988). The development of children's concepts of the arithmetic average. *Journal for Research in Mathematics Education*, *19*(1), 64–80. doi:10.2307/749111

Tarr, J. E., & Shaughnessy, J. M. (2007). Student performance in data analysis, statistics, and probability. In P. Kloosterman & F. K. Lester. Jr. (Eds.), *Results and interpretations of the 2003 mathematics assessment of the National Assessment of Educational Progress* (pp. 139–168). Reston, VA: National Council of Teachers of Mathematics.

Zawojewski, J. S., & Shaughnessy, J. M. (2000). Mean and median: Are they really so easy? *Mathematics Teaching in the Middle School*, *5*(7), 436–440.

IMPACT OF STORYTELLING ON COGNITIVE, BEHAVIORAL AND EMOTIONAL ENGAGEMENT IN PRIMARY MATHEMATICS EDUCATION

Salman Biland
National University of Sciences and Technology
salmanbiland@gmail.com

Teruni D. Lamberg
University of Nevada, Reno
terunil@unr.edu

Student's engagement in learning impacts student performance in math. Therefore, a quasi-experimental study was conducted to investigate the impact of using story telling as a context to teach math verses using a traditional lecture approach in a fifth-grade classroom. The control group received traditional instruction while the experimental group received a digital intervention that incorporated story telling. The findings revealed that using a ST approach of teaching was more effective in enhancing students' engagement at grade 5 in mathematics classroom.

Keywords: Mathematical Knowledge for Teaching; Storytelling; Student's Engagement.

The objective of this study is to investigate how storytelling could enhance behavioral, emotional, and cognitive engagement in the math-classroom. Storytelling (ST) incorporates both logical (i.e., what makes sense) and aesthetic (i.e., how it feels to the reader) aspects. It addresses what make sense and how it feels. It ignites both cognitive and emotional engagement in learning. This study investigated: How storytelling support 5th grade students' emotional, behavioral, and cognitive engagement in math classroom.

Integrating Storytelling with math, engages students because it provides students a meaningful context that makes learning interesting (Lemonidis & Kaiafa, 2019; Zazkis & Liljedahl, 2009). Therefore, students are more engaged, and they perform better in math (Moyer, 2000). There are three kinds of engagement: Cognitive (Lemonidis & Kaiafa, 2019) social (Altieri, 2009), and emotional engagement (Goral & Gnadinger, 2006; Toh et al., 2016). Storytelling approach in classroom have incorporate all these kinds of engagements (Marsico, Molo, Albano, & Perin, 2019; Kim & Li, 2021). Behavioral engagement (BE) means the observable acts of students being involve in learning (Fredricks et al., 2004; Griffin et al., 2008). Students' participation and efforts to perform tasks and interact appropriately are categorize as behavioral engagement. In our framework,

A quasi-experimental design research was used to compare traditional method and storytelling-based approach (Gribbons & Herman, 1997). Sixty fifth grade students took part in this study and filled out a questionnaire related to behavioral engagement, cognitive engagement, and emotional engagement. The control group participated in a traditional lecture where they listened to a lecture, took notes, and did sample problems and the experimental group participated in a technology-based instruction that incorporated story telling. The findings revealed that the technology-based lessons that incorporated story telling significantly increased student engagement.

References

Altieri, J. L. (2009). Strengthening connections between elementary classroom mathematics and literacy. Teaching Children Mathematics, 15(6), 346–351.

BaLemonidis, C., & Kaiafa, I. (2019). The Effect of Using Storytelling Strategy on Students' Performance in Fractions. *Journal of Education and Learning*, *8*(2), 165-175.
Corno, L., & Mandinach, E. B. (1983). The role of cognitive engagement in classroom learning and motivation. *Educational psychologist*, *18*(2), 88-108.
Fredricks, J. A., Blumenfeld, P. C., & Paris, A. H. (2004). School engagement: Potential of the concept, state of the evidence. *Review of educational research*, *74*(1), 59-109.
Goral, M. B., & Gnadinger, C. M. (2006). Using storytelling to teach mathematics concepts. *Australian primary mathematics classroom*, *11*(1), 4-8.
Griffin, M. A., Parker, S. K., & Neal, A. (2008). Is behavioral engagement a distinct and useful construct?. *Industrial and Organizational Psychology*, *1*(1), 48-51.
Kim, D., & Li, M. (2021). Digital storytelling: Facilitating learning and identity development. *Journal of Computers in Education*, *8*, 33-61.
Marsico, G., Mollo, M., Albano, G., & Pierri, A. (2019). Digital Storytelling and Mathematical Thinking: An Educational Psychology Embrace. *International Journal of Innovation in Science and Mathematics Education*, *27*(6).
Moyer, P.S. (2000). A remainder of one: Exploring partitive division. Teaching Children Mathematics, 6(8), 517.
Rotgans, J. I., & Schmidt, H. G. (2011). Cognitive engagement in the problem-based learning classroom. *Advances in health sciences education*, *16*, 465-479.
Sagayadevan, V., & Jeyaraj, S. (2012). The role of Emotional Engagement in Lecturer-Student interaction and the Impact on Academic Outcomes of Student Achievement and Learning. *Journal of the Scholarship of Teaching and Learning*, *12*(3), 1-30.
Toh, T.L, Cheng, L. P., Jiang, H., & Lim, K. M. (2016). Use of comics and storytelling in teaching mathematics. In *Developing 21st Century Competencies In The Mathematics Classroom: Yearbook 2016, Association Of Mathematics Educators* (pp. 241-259).
Zazkis, R., & Liljedahl, P. (2009). Teaching Mathematics as Storytelling. Rotterdam, The Netherlands: Sense Publishers.

MATHEMATICS AND VISUAL IMPAIRMENTS: STUDY OF TEACHERS OF STUDENTS WITH VISUAL IMPAIRMENTS MATHEMATICAL PEDAGOGICAL CONTENT KNOWLEDGE

Tasnim Alshuli
University of Arizona
talshuli@arizona.edu

Keywords: Mathematical Knowledge for Teaching; Cognition; Students with Disabilities; Teacher Knowledge; Equity, Inclusion, and Diversity

Objective

Mathematics is considered to be a "gatekeeper" (e.g., Douglas & Attewell, 2017), a subject that allows some students to move forward while holding others back (Moses, Kamii, Swap, and Howard, 1989). Students with visual impairments (SVI) are among those who are excluded from mathematics education due to the visual nature of mathematics which can limit their access (Bell & Silverman, 2019; Jitngernmadan, Stöger, Petz, & Miesenberger, 2017).

Teachers for students with visual impairments (TSVIs) are one source of academic support for SVIs. However, TSVIs struggle to support SVIs in mathematics education because they often lack mathematics education in their academic and professional development (McBride, 2020). This poster considers what mathematical pedagogical content knowledge (MPCK) TSVIs bring to their work with SVIs in mathematics education.

Theoretical Framework

Chick's (2006) MPCK framework was used to analyze the data in three categories: (1) Clearly PCK, (2) Content Knowledge in a Pedagogical Context, and (3) Pedagogical Knowledge in a Content Context.

Methods

I conducted a 60-minute semi-structured interview with a math teacher working in a school for SVIs. The teacher also had 15 years teaching mathematics in mainstream schools.

Results and Implications

The teacher was very purposeful in his use of Clearly PCK and of Content Knowledge in a Pedagogical Context. For example, when working with fractions, the teacher considered whether the student was sighted or not. For a sighted student, the teacher would use visual symbolic representation to explain fractions. However, for a SVI, the teacher would use verbal sense-making and specific wording to support the SVI to do fractions in their head. After the verbal sense-making stage, the teacher would then work with the SVI on symbolic representation in braille or large print. The teacher used Pedagogical Knowledge in a Content Context when he encouraged SVIs to use their tools, engage with others, develop a growth mindset and positive attitude towards mathematics, and have better access and inclusion in mathematics fields.

The study showed the differences between one teachers work with SVIs and sighted students. The study also showed that TSVIs might focus on the visual aspects of math and the necessary adaptations rather than focusing on SVIs' sensemaking with or without those adaptations and accommodations. Future work might investigate how to best support TVIs' MPCK development to effectively support SVIs.

References

Bell, E. C., & Silverman, A. M. (2019). Access to math and science content for youth who are blind or visually impaired.

Chick, H. L., Baker, M., Pham, T., & Cheng, H. (2006, July). Aspects of teachers' pedagogical content knowledge for decimals. Proceedings of the 30th annual conference of the International Group for the Psychology of Mathematics Education (Vol. 2, pp. 297-304).

Douglas, D., & Attewell, P. (2017). School mathematics as gatekeeper. The Sociological Quarterly, 58(4), 648-669. Jitngernmadan, P., Stöger, B., Petz, A., & Miesenberger, K. (2017). Extraction Methodology of Implicit Didactics in Math Schoolbooks for the Blind. Harnessing the Power of Technology to Improve Lives (pp. 843-850). IOS Press.

McBride, C. R. (2020). Critical issues in education for students with visual impairments: Access to mathematics and the impact of the Covid-19 pandemic (Doctoral dissertation, University of Georgia).

Moses, R., Kamii, M., Swap, S. M., & Howard, J. (1989). The algebra project: Organizing in the spirit of Ella. Harvard Educational Review, 59(4), 423-444.

RECOGNIZING REFERENT UNIT IN FRACTION MULTIPLICATION PROBLEMS: IS THE WHOLE ALWAYS THE SAME?

Kun Wang	Chandra Hawley Orrill	Rachael Eriksen Brown
UMass Dartmouth	UMass Dartmouth	Penn State Abington
kwang1@umassd.edu	corrill@umassd.edu	reb37@psu.edu

Teachers' attention and flexibility on referent unit is important to better understand fractions and fractions operations while it is documented that teachers struggle with it. In this study, we explored teachers' different levels of identifying referent unit in a fraction multiplication problem involving a drawn representation. By analyzing data from five pilot interviews with five middle school mathematics teachers, we found out that teachers attended referent unit differently. Moreover, their different levels of mastering referent unit related to what they view as a whole through their thinking.

Keywords: Referent Unit, Fraction Multiplication/Division, Visual representations, Teacher Knowledge

Backgrounds

Numerous studies have investigated the understanding of fractions among both in-service and pre-service teachers. Given that fraction plays an important role for upper elementary and middle school grades (e.g., CCSSM, 2010), including topics such as using visual models to understand fraction multiplication and division, a robust teaching of this content is necessary. Recent studies have reported that teachers having constrains on comprehension of fraction operations and on facilitating fractions across different representations (Copur-Gencturk & Ölmez, 2022; Izsák et al., 2019; Lee, 2017; Lee et al., 2011; Lo & Luo, 2012; Philipp & Hawthorne, 2015).

Prior Research

Recent research noted about teachers' struggle with referent unit. For example, teachers' attention to referent unit and their flexibility with the referent unit are considered as indicators of understanding of fractions as well as fraction operations. Referent unit is different from unit in a way that referent unit may change according to the situational needs, as Lee (2017) defined: "*referent units* are units that are needed when numbers are embedded in problem situations." (p. 329). Lee et al. (2011) investigated on how twelve teachers' reasoning and understanding of mathematical visual representations with referent unit, and they found out that the participants lack flexibility of keeping track of "unit to which a fraction refers" (p. 204). Consistent with them, Lee (2017) studied 111 pre-service teachers, only 12% of whom showed flexibility of referent unit by providing "appropriate representations" (p. 345). To examine teachers' attention to referent unit, Copur-Gencturk and Ölmez (2022) reported approximately half of their in-service teacher participants (N=246) attended referent unit, by referring "different wholes or unit" where 1/3 could be greater than ½.

While there is limited research particularly examining teacher' understanding of referent unit, making sense of visual representations is often adopted as a way of such kind of exploration, such as length representations or area model representations (Izsák et al., 2019; Lee et al., 2011;

Orrill et al., 2008; Orrill & Brown, 2012). Those studies documented variety level of teachers' struggle with coordinating fractional reasoning in the context of both representations.

Perspectives

Majority research demonstrated that teachers could solve fraction multiplication/division problems by algorithm (e.g., Lo & Luo, 2012), but more importantly, they need to make sense of it both for themselves and thus for their students. Specifically, in the case of referent unit, they need to know how to facilitate the fractions operations in different visual representations, so that they can help with future students' conceptual understanding. According to Shulman (1986), it requires not only teachers' content knowledge (CK) but also their pedagogical content knowledge (PCK), since PCK includes knowledge of students and making sense of different representations to students. To explore teachers' knowledge through this lens, we are driven by this research question: to what extend teachers attend to referent unit when interpreting drawn representation?

Methods

The present study involved conducting interviews with a convenience sample of five middle school mathematics teachers, consisting of two currently practicing teachers (Kevin and Hunter) and three recently retired teachers (Laura, Beth, and Wendy). They are all pseudonymous. Kevin taught at a private school, while the rest of the participants were public school teachers. For two of the retired teachers, they are considered within the context of school settings, because Beth teaches as a long-term substitute and Wendy last taught in the 2019-2020 school year. Among the sample, Beth was the only Black teacher, while the other four participants were white.

Each interview, conducted over Zoom, lasted approximately an hour, during which the teachers were asked to work on tasks on Jamboard. Meanwhile, they were asked to share their screen for both parties to show their pointers moving around or their notes or scratches while they are thinking. Jamboard containing 19 pages of questions related to nine different situations, with one to two questions per screen. The interviews were recorded and transcribed by Zoom, and one of the authors reviewed and edited the transcript to ensure the accuracy of transcribed conversation.

This work focuses on only one area model representation (figure 1) form a situation where there are four different students' work. We first had the teachers looked at four students' work separately, with a purpose of revealing teachers' knowledge about how to make sense of each area model representation from students' work. Before they responded, they were informed to *consider the fraction multiplication problem ¾ × ⅔, and to solve it using an area model, students colored in the model as shown below*. Following the students' drawing, we asked them to solve task one with two separate questions: *Where do you see ¾ in Donald's drawing? How about 2/3?*

Our intention for task one is to capture whether teachers can identify there are two different *wholes* in this representation. Specifically, the first *whole* is the entire rectangular shape, we can consider it as ¾ [of the whole rectangle], or three blue of the whole shape, which represents the blue-shaded parts. However, the second *whole* as we intended now is supposed to be the blue-shaded parts, through which it reflected a thinking of 2/3 [of the blue shaded parts], or 2 parts of 3 blue parts represents the two orange-shaded parts.

Data was analyzed qualitatively to determine whether each teacher successfully identified the two different *wholes* as we indented to, if not, what *whole* did they refer to when they responded to the two separate interview questions.

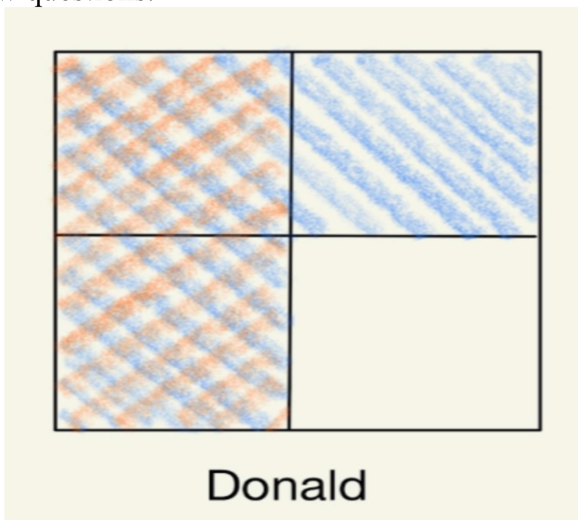


Figure 1: A student's work of modeling $^3/_4 \times {}^2/_3$

Results

Where Did the Teachers See 3/4

All five teachers identified ¾ as the blue-shaded parts quickly. They either marked the three blue shaded parts on Jamboard or mentioned it as the "blue lines." Interestingly, when they talked, they did not explicitly refer to the relationship of the blue shaded parts to the whole rectangle. Only one teacher, Kevin, explicitly attended to the relationship of the blue section to the whole by typing a note to himself on Jamboard saying, "Blue=3/4 of whole."

Regardless of their ways of indication (verbal or written), we agreed that all of these five teachers were aware of where the 3/4 could be seen in this representation, with recognizing the whole as the entire rectangular shape.

Where Did the Teachers See 2/3

Among five teachers, two teachers (Kevin and Laura) successfully identified 2/3 as two orange parts of three blue parts. Kevin typed his notes on Jamboard saying, "Red [Orange] = 2/3 of blue shaded". Laura explained, "…so the two-thirds are the orange lines going this way, and the two-thirds doesn't include this one [the top right blue part], so it is 2/3 of the blue lines."

The other three teachers explicitly expressed confusion when they trying to identify 2/3. For example, Wendy mentioned that the denominator "is confusing," because she thought the student did not realize "[the whole rectangular shape] is not partitioned into thirds." Similarly, Hunter said, "I honestly don't see the two-thirds… I would like to cut it into three pieces, but I don't see where the two-thirds are in his drawing." Further extending the idea of partitioning that Hunter introduced, Beth said, "…where is [2/3]? Um…In order to multiply this [2/3], I would have done it [the whole rectangle] into twelfths, because I could see it a little better: 6/12."

All three of these teachers (Wendy, Hunter, and Beth) were unable to identify where the 2/3 appears in the representation and they also all seemed to be searching for 2/3 of the original rectangle, rather than 2/3 of the 3/4. Wendy and Hunter wanted to partition the entire rectangle into fourths or into thirds. Thus, we interpreted that they were not shifting to think about a different whole when they were looking for 2/3. Even though Beth gave an alternative way of partitioning the entire rectangle into a common denominator (twelfths), she was trying to find both ¾ and 2/3 in the original rectangle, rather than attending to the whole to which each fraction referred. In another words, she was thinking of the same whole.

In contrast, Kevin and Laura were smoothly and explicitly, transitioning from the-entire-rectangle-as-a-whole into the blue-shaded-part-as-a-whole. Thus, we concluded that, two of the five teachers showed some flexibility of referent unit as they successfully identified what the unit is based on the situational needs.

Discussion

The present study is consistent with prior research about teachers struggle with referent unit (e.g., Izsák, 2008) and provide more evidence on the difficulties teachers have on identifying referent unit when they deal with the area model for fraction multiplication situations. Our findings revealed that it was not anomalous that the given area (rectangle) was conceived as the whole by teachers even when the discussion should have been about the blue section of the rectangle. This is the same phenomenon described by Izsák (2008) who drew from Steffe's (1994; 2003) work with children to explain the issue as the teacher not attending to two levels of units: the whole rectangle and the new "whole" comprised of the three blue boxes.

Interestingly, as one of four different student representations in this pilot task, the researchers had anticipated that Donald's drawing would be relatively clear for teachers. However, it seemed that the other representations, which were all area models partitioned into more equal pieces were easier overall. Additional analysis needs to be completed to determine the extent to which the teachers attended to the shifting unit in the other drawn representations. We ponder whether this task, with its four unique area models, might differentiate teachers' understanding of referent units.

By illustrating the importance of looking at whether teachers attended to referent units in a drawn representation situated in fraction multiplication problem, we hope to contribute on the existing research about teachers' attention as well as flexibility of referent unit.

Acknowledgments

This material is based upon work supported by the National Science Foundation under Grant No. 2201125. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

References

Common Core State Standards Mathematics. (2010). *Common core state standards for mathematics.* http://www.corestandards.org/Math/

Copur-Gencturk, Y., & Ölmez, İ. B. (2022). Teachers' attention to and flexibility with referent units. International *Journal of Science and Mathematics Education, 20*(6), 1123-1139.

Izsák, A., Kulow, T., Beckmann, S., Stevenson, D., & Ölmez, I. B. (2019). Using coordinated measurement with future teachers to connect multiplication, division, and proportional relationships. *Mathematics Teacher Educator, 8*(1), 49-75.

Lee, M.Y. Pre-service teachers' flexibility with referent units in solving a fraction division problem. *Educ Stud Math 96*, 327–348 (2017). https://doi.org/10.1007/s10649-017-9771-6

Lee, S. J., Brown, R. E., & Orrill, C. H. (2011). Mathematics teachers' reasoning about fractions and decimals using drawn representations. *Mathematical Thinking and Learning, 13*, 198–220.

Lo, J.-J., & Luo, F. (2012). Prospective elementary teachers' knowledge of fraction division. *Journal of Mathematics Teacher Education, 15*(6), 481–500. doi:10.1007/s10857-012-9221-4

Orrill, C. H., & Brown, R. E. (2012). Making sense of double number lines in professional development: Exploring teachers' understandings of proportional relationships. *Journal of Mathematics Teacher Education, 15*(5), 381–403. https://doi.org/10.1007/s10857-012-9218-z

Orrill, C.H., Sexton, S., Lee, S.-J., & Gerde, C. (2008). Mathematics teachers' abilities to use and make sense of drawn representations. *In The International Conference of the Learning Sciences 2008: Proceedings of ICLS 2008.* Mahwah, NJ: International Society of the Learning Sciences.

Philipp, R. A., & Hawthorne, C. (2015). Unpacking the referent units in fraction operations. *Teaching Children Mathematics, 22*(4), 240–247.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational researcher, 15*(2), 4-14.

TAKING STRUCTURE SERIOUSLY: WHAT DO TEACHERS NOTICE ABOUT INVARIANCE IN FRACTIONS

Chandra Hawley Orrill
UMass Dartmouth
corrill@umassd.edu

Rachael Eriksen Brown
Penn State Abington
reb37@psu.edu

Kun Wang
UMass Dartmouth
kwang1@umassd.edu

Keywords: Rational Numbers & Proportional Reasoning, Teacher Knowledge, Mathematical Knowledge for Teaching

Objective

For decades, there has been a call in mathematics teaching and learning to focus on mathematical structures. This was a pillar of "new math" (Phillips, 2015) and part of the way the National Council of Teachers of Mathematics (NCTM) defined mathematics in both sets of standards (NCTM, 1989, 2000). It is also a Standard for Mathematical Practice in the Common Core (CCSSI, 2010). However, little research considers teachers' knowledge of structural relationships. Certainly, it has been posited that teachers who understand mathematical structure will be more capable of supporting students in learning about them (Mason et al., 2009). This conjecture has neither been tested nor explored in terms of the extent to which teachers are aware of structural relationships.

Here, we report a pilot study aimed at understanding how practicing teachers understand invariance as it relates to fractions. While invariance is often considered for proportional reasoning situations, it is rarely a focus of discussion or instruction for fractions. Thus, we wondered how teachers might make sense of novel tasks that asked them to attend to aspects of invariance with fractions.

Response to Issue

This poster reports an exploratory study undertaken as a pilot of an interview instrument. We interviewed a convenience sample with five current (Kevin and Hunter) or recently retired (Laura, Beth, and Wendy) middle school mathematics teachers. (All names are pseudonyms)

We considered how the teachers responded to a single item contrasting fractions and proportions by considering how two different drawings (one area model and one with shaded dots) might show that 2/3 is equivalent to 8/12. One of the main take-aways was that these participants did not attend to the relationship of the numerator to the denominator in their sensemaking about equivalence. Interestingly, the teachers also had different conceptions of the referent unit as they interpreted the drawings, as well.

Given that this was only one task considered by five participants, we are careful not to overstep our claims. More questions need to be asked of more teachers to understand this issue. However, the work reported here supports our assertion that teachers may not attend to structural aspects of fraction situations when solving for themselves. Further, it suggests that research is warranted in this area.

Acknowledgments

This material is based upon work supported by the National Science Foundation under Grant No. 2201125. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

References

Common Core State Standards Initiative (2010). *Common core standards for mathematics.* https://learning.ccsso.org/wp-content/uploads/2022/11/Math_Standards1.pdf

Mason, J., Stephens, M., & Watson, A. (2009). Appreciating mathematical structure for all. *Mathematics Education Research Journal, 21*(2), 10-32.

National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics.* National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. National Council of Teachers of Mathematics.

Phillips, C. J. (2015). *The new math: A political history.* The University of Chicago Press.

TEACHERS' KNOWLEDGE AND USE OF VISUAL FRACTION REPRESENTATIONS

Karl W. Kosko
Kent State University
kkosko1@kent.edu

Christine K. Austin
Kent State University
causti17@kent.edu

Maryam Zolfaghari
Kent State University
mzolfagh@kent.edu

The purpose of this study was to examine the relationship between in-service teachers (ISTs') reported use of manipulatives and their pedagogical content knowledge for teaching fractions (PCK-Fractions). The study's results indicated no significant relationship between ISTs' reported use of visual representations and their PCK fractions. However, trends were observed across ISTs' education, taught grade levels, PCK, and use of visual representation. The implications and future needs for the study are discussed in the paper.

Keywords: Mathematical representations; instructional activities and practices; teacher knowledge.

Mathematical Knowledge for Teaching (MKT) involves the content and pedagogical content knowledge used by teachers to engage in pedagogy in its various forms (Ball et al., 2008; Izsák, 2008). Over the past two decades, research on MKT has shown various factors are associated with lower and higher MKT. For example, while years of experience alone are positively associated with higher MKT scores (Copur-Gencturk & Li, 2023; Hill, 2010), a stronger association is typically found when focusing on experience in contexts where specific concepts are taught (Herbst & Kosko, 2014; Hill, 2010; Zolfaghari et al., 2022). Various other factors have been explored including those associated with indicators of content knowledge (Copur-Gencturk & Li, 2023; Hill, 2010; Ko & Herbst, 2020) and pedagogy used (Hill et al., 2008; Jacobson et al., 2021; Morin, 2013). This paper focuses on one particular pedagogical approach often posed as associated with higher MKT: use of visual representations and manipulatives.

Manipulative use has long been tied with definitions of MKT. For example, Hill et al. (2005) describe developing items of specialized content knowledge so that teachers could "show or represent numbers or operations using pictures or manipulatives…" (p. 388). Later validating the Mathematical Quality of Instruction rubric, Hill et al. (2008) included use of representations, including manipulatives, as a category that aligned with MKT scores. Examining teachers' MKT for fraction multiplication, Izsák (2008) noted a relationship between teachers' skill and frequency in using visual representations for fraction arithmetic and their demonstrated MKT. Morin (2013) found that the relationship was particularly evident in one teacher's incorporation of a concrete to figurative to abstract progression for students' meaning-making. This included knowledge of varying manipulatives that allowed for adjusting activities should one representation not facilitate the connections to underlying concepts the teacher sought. Examining the topic more explicitly, Jacobson et al. (2021) found that teachers' evaluation of visual representations corresponded with their demonstrated pedagogical content knowledge. Despite the common assumption of manipulative use being tied to higher MKT scores, the bulk of such scholarship is qualitative. Thus, the purpose of this exploratory study is to examine whether and to what degree Inservice teachers' pedagogical content knowledge for teaching fractions (PCK-Fractions) is associated with their reported use of manipulatives for teaching fractions.

Pedagogical Content Knowledge for Teaching Fractions

Research on MKT for fractions is common, with studies providing evidence for a lack of content knowledge amongst preservice and inservice teachers (Erdam, 2016; Huang et al., 2009; Izsák et al., 2019). For example, Izsák et al. (2019) reported many teachers were unable to coordinate three levels of units – an important conceptual level for understanding fraction multiplication and division. Such reasoning is also important as Izsák (2008) observed teachers' ability to coordinate such units corresponded with how they used visual representations. Similarly, Thurtell (2019) found that preservice teachers who had less robust content knowledge of fractions were limited in their use of visual representations. Thurtell (2019) suggested that an overreliance on symbolic representation as an ideal for representing mathematics was at fault. Rather, "the undercurrent of calculational views evident in the preservice teachers' learning approaches were reflected clearly in their teaching approaches" (p. 305) despite dispositions generally supportive of using visual representations. More recently, Zolfaghari et al. (2022) found that preservice teachers who have field experiences in upper elementary grades (3-5) demonstrated higher PCK for fractions. Key in this finding is that Zolfaghari et al. (2022) defined higher PCK as an ability to assess more sophisticated unit coordination in students' reasoning. Taken altogether, scholarship on teachers' MKT for fractions suggest content knowledge is generally weaker than it should be. However, preservice teachers would benefit from explicit experience in upper elementary classrooms (Zolfaghari et al., 2022) and by interrogating an overreliance on symbolic representations for teaching fractions (Thurtell, 2019).

Visual Representations and Manipulatives

Visual representations are commonly advocated for the teaching and learning mathematics (Bolden et al., 2015). Visual representations are defined as both concrete and pictorial, with common mathematical visuals including groups of and array models for multiplication (Kosko, 2018) and fraction strips, number lines, and pie charts for fractions (Cramer et al., 2008; Tunç-Pekkan, 2015). Different studies have reported teachers' usage of visual representations at various grade levels. For instance, surveying 603 primary and 336 secondary teachers, Howard et al. (1997) found that most teachers felt confident in using visual representations. However, the use of visual representations was significantly lower in secondary mathematics classrooms compared to elementary classrooms. Similarly, Gilbert and Bush (1988) surveyed 220 elementary teachers and found that as teachers' grade level increased, their use of visual representations decreased. Examining 820 teachers from K–10, Swan and Marshall (2010) found teachers' reduced use of visual representations was due to the complexity of topics taught. Rather, "teachers associate the use of mathematics manipulatives with concept formation and hence to be abandoned when the mathematics becomes more complex." (p.17). These findings were corroborated by similar surveys (O'Meara et al., 2020; Uribe-Flórez & Wilkins, 2010). Additionally, O'Meara et al. (2020) found that teachers' use of manipulatives was negatively associated with a lack of training available in how to use them.

Method

Participants & Measures (1 paragraph)

The sample consisted of 47 in-service teachers (ISTs) who taught in Midwestern schools districts. Much of this sample identified as white (n = 45), female (n = 36), and had an average of 18.23 years of teaching experience. Our sample consisted of 12 third grade, 18 fourth grade, 6 fifth grade, and 11 sixth grade teachers. Of the 47 participants, 37 reported to have a master's degree which includes a master's in general education/curriculum (n = 10), elementary education

(n = 3), secondary mathematics (n = 2), literacy/reading (n = 10), administration/leadership (n = 6), education psychology (n = 2), and education technology (n = 1).

Participants were recruited via email from nearby school districts that taught grades third through sixth and were asked to participate in a survey to be completed on Qualtrics. The survey consisted of 14 pictures of fraction manipulatives that the ISTS had to indicate if they were aware of (see Table 1). If they indicated that they knew the manipulative they were then asked how often they used the manipulative (physical and virtual) on a Likert scale (0 = Never, 1 = Rarely, 2 = Sometimes, 3 = Often, 4 = Always). Then, the ISTs were given a Pedagogical Content Knowledge of Fractions instrument (PCK-Fractions) that was designed to measure a teacher's level of PCK for the teaching and learning of fractions of upper elementary mathematics (Zolfaghari et al., 2021; 2022). The instrument consisted of 17 multiple choice items that have the teachers look at a student's mathematical work and assess their reasoning.

Analysis & Results

ISTs demonstrated higher PCK-Fraction scores (M = 0.35, SD = 0.96), which is to be expected when considering the measure assesses teachers from preservice to inservice (see Zolfaghari et al., 2022). On average, most of the sampled ISTs are aware of various fraction manipulatives (Table 1). However, fewer participants knew of using Geoboards (n = 24) or playdoh (n = 5) for teaching fractions. Spearman Rho correlation coefficients were calculated to examine the relationship between ISTs' PCK-Fractions, their knowledge of various fraction manipulatives and their reported use. To facilitate this, we created composite variables for all visual representations (excluding symbolic numeral and symbolic number lines) for knowledge of (α=.82, M=.72, SD=.23) and use of (α=.83, M=1.33, SD=.61) various visual representations of fractions. The correlation analysis indicates a strong significant relationship between the number of manipulatives known and the average reported use (ρ = 0.580, p < .001). However, no statistically significant relationship was observed between PCK-Fractions with either knowledge (ρ = -0.230, p = .879) or reported use (ρ = -0.097, p = .516) of fractions visual representations.

Table 1: Percent of Participants who Know of Manipulatives and Average Reported Use

	Know	Use		Know	Use
Symbolic Numerals w/ Pictures	95.75%	M=2.93	Counters	80.85%	M=1.36
Symbolic Numerals	100%	M=2.60	Fraction Tiles	74.47%	M=1.30
Symbolic Number Line	97.87%	M=2.50	Fraction Squares	74.47%	M=1.11
Fraction Strips	89.36%	M=2.11	Linking Cubes	70.21%	M=1.02
Fraction Circles	93.62%	M=1.89	Cuisenaire Rods	68.10%	M=0.77
Tactile Number Line	80.85%	M=1.51	Geoboards	51.06%	M=0.53
Pattern Blocks	74.47%	M=1.41	Play-Doh	10.64%	M=0.06

Likert Use Scale: 0 = Never, 1 = Rarely, 2 = Sometimes, 3 = Often, 4 = Always

To better understand these results, we examined the potential effect of various demographic variables that may influence teachers' use of manipulatives. Of particular interest, we found that the highest degree earned (bachelors or masters) had a statistically significant and negative relationship with the use of manipulatives (ρ = -0.397, p = .006). Table 2 reports participants by their highest degree earned and the grade level taught along with the average years of experience and average reported use (0 = Never, 1 = Rarely, 2 = Sometimes, 3 = Often, 4 = Always). As seen in Table 2, other than fifth grade, ISTs who have a master's degree, on average, reported using manipulatives less frequently. Another interesting trend evident in Table 2 is that the specific grade level taught appears to influence reported manipulative use, with grade 6 teachers reporting the lowest use of visual representations and grade 5 the highest. Unfortunately, the relatively small sample size per subgroup in Table 2 prevented further statistical analysis.

Table 2: Average Reported Years' Experience by Highest Degree Earned and Grade Level Taught

Highest Degree	Grade	PCK	Experience	Use
Bachelors (n = 10)	Third (n = 3)	M = -0.03 SD = 0.66	M = 7.00 SD = 6.56	M = 1.73 SD = 1.00
	Fourth (n = 5)	M = 0.22 SD = 1.19	M = 15.60 SD = 11.87	M = 2.08 SD = 0.90
	Fifth (n = 2)	M = -1.23 SD = 1.82	M = 6.50 SD = 2.12	M = 1.45 SD = 1.06
Masters (n = 37)	Third (n = 9)	M = 0.43 SD = 0.72	M = 13.11 SD = 8.45	M = 1.07 SD = 0.46
	Fourth (n = 13)	M = 0.49 SD = 1.00	M = 20.46 SD = 7.68	M = 1.28 SD = 0.72
	Fifth (n = 4)	M = 1.58 SD = 0.39	M = 25.00 SD = 4.69	M = 1.68 SD = 0.19
	Sixth (n = 11)	M = 0.15 SD = 0.58	M = 23.73 SD = 6.97	M = 0.84 SD = 0.49

Discussion

Results presented here are from an exploratory study examining the relationship between PCK for fractions and ISTs' reported knowledge and use of visual representations for fractions. Despite significant scholarship positing a relationship between use of visual representations and MKT (Hill et al., 2008; Izsák, 2008; Jacobson et al., 2021), we found no statistically significant relationships. However, trends in the descriptive data suggest that grade level and graduate coursework may have varying effects on visual representation use and the role of PCK. Our study did not examine the role of content knowledge, and future work should consider this given prior research suggesting its influence on how teachers use visual representations (Izsák, 2008; Thurtell, 2019). Additional research is also needed to better understand how graduate work may detract from use of manipulatives. More focus may be drawn to the concentration of such degrees (i.e., the majority in this study were in literacy due partly to a state mandate), the quality of degrees themselves, and so forth. One final implication from this study is that despite knowledge of various visual representations, most teachers use such representations less than "often" when teaching fractions. This finding is alarming, given that more effective use of manipulatives occurs at least on a weekly basis (Uribe-Flórez & Wilkins, 2010).

Acknowledgments

Research reported here received support from the National Science Foundation (NSF) through DRK-12 Grant #1908159. Any opinions, findings, and conclusions or recommendations expressed in this paper are those of the authors and do not necessarily reflect the views of NSF.

References

Bolden, D., Barmby, P., Raine, S., & Gardner, M. (2015). How young children view mathematical representations: a study using eye-tracking technology. Educational research, 57(1), 59-79. https://doi.org/10.1080/00131881.2014.983718

Cramer, K., Wyberg, T., & Leavitt, S. (2008). The role of representations in fraction addition and subtraction. Mathematics teaching in the middle school, 13(8), 490-496.

Copur-Gencturk, Y., & Li, J. (2023). Teaching matters: A longitudinal study of mathematics teachers' knowledge growth. Teaching and Teacher Education, 121, 103949.

Erdem, E. (2016). Prospective middle school mathematics teachers'(PMTs) content knowledge about concepts 'fraction' and 'rational numbers.' *Journal of Education and Training Studies*, *4*(6), 80–92. https://doi.org/10.11114/jets.v4i6.1431

Gilbert, R. K., & Bush, W. S. (1988). Familiarity, Availability, and Use of Manipulative Devices in Mathematics at the Primary Level. School Science and Mathematics, 88(6), 459-69.

Herbst, P., & Kosko, K. W. (2014). Mathematical knowledge for teaching and its specificity to high school geometry instruction. In J. Lo, K. R. Leatham, & L. R. Van Zoest (Eds.), *Research trends in mathematics teacher education* (pp. 23-46). New York: Springer. https://doi.org/10.1007/978-3-319-02562-9_2

Hill, H. C. (2010). The nature and predictors of elementary teachers' mathematical knowledge for teaching. Journal for Research in Mathematics Education, 41(5), 513-545.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. American educational research journal, 42(2), 371-406.

Hill, H. C., Blunk, M. L., Charalambous, C. Y., Lewis, J. M., Phelps, G. C., Sleep, L., & Ball, D. L. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. Cognition and instruction, 26(4), 430-511.

Howard, P., Perry, B., & Tracey, D. (1997). Mathematics and Manipulatives: Comparing Primary and Secondary Mathematics Teachers' Views.

Huang, T. W., Liu, S. T., & Lin, C. Y (2009) Preservice teachers' mathematical knowledge of fractions. *Research in Higher Education Journal*, *5*, 1.

Izsák, A. (2008). Mathematical knowledge for teaching fraction multiplication. Cognition and instruction, 26(1), 95-143.

Izsak, A., Jacobson, E., & Bradshaw, L. (2019). Surveying middle-grades teachers' reasoning about fraction arithmetic in terms of measured quantities. *Journal for Research in Mathematics Education*, *50*(2), 156–209. https://doi.org/10.5951/jresematheduc.50.2.0156

Jacobson, E., Liu, J., Bharaj, P. K., & Savich, T. (2021). Teachers' Representational and Contextual Justifications for Selecting Pedagogical Representations. In Olanoff, D., Johnson, K., & Spitzer, S.(2021). Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (pp. 447-455).

Ko, I., & Herbst, P. (2020). Subject matter knowledge of geometry needed in tasks of teaching: relationship to prior geometry teaching experience. *Journal for Research in Mathematics Education*, *51*(5), 600-630.

Kosko, K. W. (2019). Third-grade teachers' self-reported use of multiplication and division models. School Science and Mathematics, 119(5), 262-274. https://doi.org/10.1111/ssm.12337

Morin, M. L. (2021). *The Role of Mathematical Knowledge for Teaching in Instruction for Multiplicative Reasoning: Unpacking a Teacher's Decision-Making Process* (doctoral dissertation). University of Colorado at Denver.

O'Meara, N., Johnson, P., & Leavy, A. (2020). A comparative study investigating the use of manipulatives at the transition from primary to post-primary education. International Journal of Mathematical Education in Science and Technology, 51(6), 835-857. https://doi.org/10.1080/0020739X.2019.1634842

Swan, P., & Marshall, L. (2010). Revisiting mathematics manipulative materials. Australian Primary Mathematics Classroom, 15(2), 13-19.

Thurtell, E. (2019). *Fractions speak louder than words: Investigating preservice primary teachers' knowledge and understanding for teaching fractions with representations* (doctoral dissertation). University of Wollongong.

Tunç-Pekkan, Z. (2015). An analysis of elementary school children's fractional knowledge depicted with circle, rectangle, and number line representations. Educational Studies in Mathematics, 89(3), 419-441. https://doi.org/10.1007/s10649-015-9606-2

Uribe-Flórez, L. J., & Wilkins, J. L. (2010). Elementary school teachers' manipulative use. School Science and Mathematics, 110(7), 363-371. https://doi.org/10.1111/j.1949-8594.2010.00046.x

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1). University of Nevada, Reno.

Zolfaghari, M., Austin, C. K., & Kosko, K. W. (2021). Exploring teachers' pedagogical content knowledge of teaching fractions. *Investigations in Mathematics Learning, 13*(3), 230-248. https://doi.org/10.1080/19477503.2021.1963145

Zolfaghari, M., Kosko, K. W., & Austin, C. K. (2022). Examining the nature of pedagogical content knowledge (PCK) with a validation argument for the PCK-Fractions measure. In A. E. Lischka, E. B., Dyer, R. S. Jones, J. N. Lovett, J. Strayer, & S. Drown (Eds.), *Proceeding of the 44rd annual meeting of the North American Chapter for the Psychology of Mathematics Education* (pp. 730-738).

ALIGNING TEACHERS' MOVEMENTS AND IDENTITIES

Maria Eloisa Nuguid
Purdue University
mnuguid@purdue.edu

Keywords: Teacher Educators, Informal Education, Cognition.

Objectives

To address the need of making mathematics meaningful to all learners, I explore the research question: How do teachers' identities align with their movements in a small group mathematics station? I argue that teachers communicate parts of their identities through movement. Directly aligning with the theme, body-based movement such as gesturing facilitates learning, meaning-making, rapport-building, and enhances students' engagement.

Theoretical Framework

I use frameworks of embodied cognition and identity as performance. Embodied cognition is a process through which an organism navigates its surroundings using its body and connects new ideas and representations to prior experiences, facilitating meaning-making (Nathan, 2021; Núñez et al., 1999). Examples include gesturing such as pointing or tracing a triangle in the air (Alibali & Nathan, 2012). I focus on a sociocultural lens of embodied cognition to account for both individual and collaborative interactions (Danish et al., 2020). Next, identity as performance is "the performance and the recognition of the self…by telling stories, joining groups, [and] acting in a particular way at a particular time" (Butler, 1988; Darragh, 2016, p. 29). Both the teacher's performance and input from students inform her identity.

Methods

I apply a qualitative case study methodology (Merriam & Tisdell, 2016) to provide intensive and holistic description and characterize movement and identity. After listening and participating in a read-aloud in a kindergarten classroom, kindergarteners separated into integrated stations around the classroom including a mathematics station. Using qualitative methods, I found through transcription, coding, and thematic analysis of video and interview data that I collected that participants made both conscious and subconscious choices with actions. Through an interpretivist perspective, I aligned parts of my participants' semi-structured interviews about their identities with the movements that they performed during facilitation to extract meaning.

Results and Implications

My first participant identified as a teacher and mother. Through past experiences of teaching elementary students, some of her conscious actions aligned with her identity as a teacher like lightly touching a students' forearm to redirect attention, forming shapes with her hands, and folding her hands in her lap to model respect. My second participant identified as both a teacher and teacher educator, and similarly she used conscious movements to connect the activity to real life contexts while also noting in an interview that some movements were subconscious. For future directions, I plan to study facilitation in informal science, technology, engineering, and mathematics (STEM) learning spaces.

References

Alibali, M. W., & Nathan, M. J. (2012). Embodiment in Mathematics Teaching and Learning: Evidence From Learners' and Teachers' Gestures. *The Journal of the Learning Sciences*, *21*(2), 247–286. https://www.jstor.org/stable/23266406

Butler, J. (1988). Performative Acts and Gender Constitution: An Essay in Phenomenology and Feminist Theory. *Theatre Journal*, *40*(4), 519. https://doi.org/10.2307/3207893

Danish, J. A., Enyedy, N., Saleh, A., & Humburg, M. (2020). Learning in embodied activity framework: a sociocultural framework for embodied cognition. *International Journal of Computer-Supported Collaborative Learning*, *15*(1), 49–87. https://doi.org/10.1007/s11412-020-09317-3

Darragh, L. (2016). Identity research in mathematics education. *Educational Studies in Mathematics*, *93*(1), 19–33. https://doi.org/10.1007/s10649-016-9696-5

Merriam, S. B., & Tisdell, E. J. (2015). *Qualitative research: a guide to design and implementation* (Fourth edition). John Wiley & Sons.

Nathan, M. J. (2021). *Foundations of embodied learning: a paradigm for education*. Routledge.

Núñez, R. E., Edwards, L. D., & Matos, J. F. (1999). Embodied Cognition as Grounding for Situatedness and Context in Mathematics Education. *Educational Studies in Mathematics*, *39*(1/3), 45–65. https://www.jstor.org/stable/3483160

MATHEMATICAL PROCESSES AND PRACTICES

A FRAMEWORK FOR DESIGNING GRAPHING TASKS FROM THE GROUND UP

Hwa Young Lee
Texas State University
hylee@txstate.edu

Teo Paoletti
University of Delaware
teop@udel.edu

Holly Zolt
Texas State University
hmz10@txstate.edu

Mai Bui
Texas State University
mtb104@txstate.edu

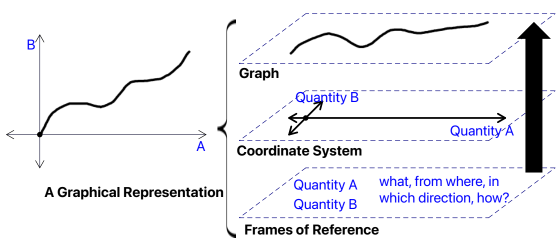
Hamilton Hardison
Texas State University
hhardison@txstate.edu

Allison L. Gantt
University of Delaware
agantt@udel.edu

Brandi Rygaard Gaspard
Texas State University
brr102@txstate.edu

Keywords: Middle School Education, Instructional Activities and Practices, Cognition

Graphical representations are common in STEM fields for modeling, communicating, and analyzing phenomena. Results from research (e.g., Lai et al., 2016), including large-scale national assessments (e.g., Nation's Report Card, 2005), indicate that U.S. students have not been provided sufficient opportunities for developing rich graphing understandings. However, much of the research and tasks used to examine students' graphing understandings have assumed that students have established understandings of the Cartesian plane needed to construct and interpret graphs. In our work, we address this issue by designing tasks that attend to students' understandings of three layers constituting a graphical representation: frames of reference, coordinate systems, and graph (Joshua et al., 2015; Lee et al., 2020) (see Figure 1a).



(a)

	Task Design/Re-Design Framework Guiding Questions (TDGQs)
1. Frames of Reference (FoR)	**TDGQ1.1.** Is the task situated in (what we perceive to be) a spatial or quantitative context? **TDGQ1.2.** What features of the task will/did support students in identifying (a) a reference point, (b) a directionality, and (c) a unit of measure?
2. Coordinate Systems (CS)	*(All of the questions above and)* **TDGQ2.1.** Is the task situated in (what we consider) a spatial or quantitative context? **TDGQ2.2.** What features of the task will/did support students in coordinating two or more FoR to construct a spatial or quantitative coordinate system?
3. Graphs	*(All of the questions above and)* **TDGQ3.1.** Is the graph situated in (what we perceive to be) a spatial or quantitative CS? **TDGQ3.2.** What features of the task will/did support students in constructing or interpreting graphs in a spatial or quantitative CS?

(b)

Figure 1: (a) Three layers of a graphical representation and (b) task design framework

In this poster, we leverage conceptual analysis (Thompson, 2008) to unpack the three layers of a graphical representation and our associated task design framework (see Figure 1b). We illustrate how this novel framework was used to design tasks for middle-grades students and the role the framework played when analyzing students' engagement with the tasks.

We implemented the tasks in one-on-one clinical interviews (Clement, 2000; Goldin, 2000) with 15 sixth-grade students to gain insights into their existing graphing understandings at the onset of a teaching experiment (Steffe & Thompson, 2000) aimed at building upon these students' graphing understandings. We discuss affordances and limitations of our approach, as well as potential task revisions we are considering as a result of our analyses of students' activities.

Acknowledgments

This work was made possible by funding from the National Science Foundation, Grants #2200777 and #2200778.

References

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In A. E. Kelly & R. A. Lesh (Eds.), Handbook of Research Design in Mathematics and Science Education (pp. 547–589). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A. E. Kelly & R. A. Lesh (Eds.), Handbook of research design in mathematics and science education (pp. 517–545). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Joshua, S., Musgrave, S., Hatfield, N., & Thompson, P. W. (2015). Conceptualizing and reasoning with frames of reference. In T. Fukawa-Connelly, N. Infante, E., K. Keene, & M. Zandieh (Eds.), Proceedings of the 18th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education (pp. 31–44). Pittsburgh, PA: RUME.

Lai, K., Cabrera, J., Vitale, J. M., Madhok, J., Tinker, R., & Linn, M. C. (2016). Measuring graph comprehension, critique, and construction in science. Journal of Science Education and Technology, 25(4), 665-681.

Lee, H. Y., Hardison, H., & Paoletti, T. (2020) Foregrounding the background: Two uses of coordinate systems. For the Learning of Mathematics, 40(1), 32–37. Nation's Report Card (2005) Item Maps. Retrieved from https://www.nationsreportcard.gov/itemmaps/?subj=MAT&grade=8&year=2005.

Steffe, L., & Thompson, P. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), Research design in mathematics and science education (pp. 267-307).

Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano & A. Sépulveda (Eds.), Plenary Paper presented at the Annual Meeting of the International Group for the Psychology of Mathematics Education, (Vol 1, pp. 31–49). Morélia, Mexico: PME.

A LOCAL CAUSAL EXPLANATION FOR WHY STUDENTS MIGHT CONFLATE RATES OF CHANGE AND AMOUNTS OF CHANGE

Elizabeth Roan
Texas State University
Eaw109@txstate.edu

Jennifer A. Czocher
Texas State University
Czocher.1@txstate.edu

Previous research on differential equations notes that students tend to conflate rate of change and amounts of change. In this study, we create a local causal explanation (Maxwell, 2004) for why one participant did distinguish between the rate of change in a population and change in a population, while the other did not. We conjecture that students may only be able to construct (or talk about) rates of change if they have the appropriate quantities available to them in which to coordinate.

Keywords: Modeling, Undergraduate Education, Advanced Mathematical Thinking.

It has been well documented that students tend to conflate the rates of change with change in the amounts (Mkhatshwa & Doerr, 2018; Rasmussen & King, 2000; Rasmussen & Marrongelle, 2006; Rowland & Jovanoski, 2004). For example, Rowland and Jovanoski (2004) found that students consistently did not distinguish between "amounts" and "rates of change of amounts" throughout different contexts (adding and removing fish in a pond farm, drugs being administered to a patient, and a car slowing down). This aligns with Rasmussen and King's (2000) study on finding starting points to teach differential equations via guided reinvention. In their study, they noted that one student distinguished between the rate of change of the fish population and the change in the fish population, while two students did not. They inferred that students did not make the distinction when they did not "conceptualize the situation in a way that involved a rate of change". We aimed to investigate why participants might use language that describes change in amounts when talking about the rate of change of amounts. We address the question: What conditions led participants to (not) distinguish between the rate of change of a population and the change in a population when constructing a model of how quickly a disease spreads through a community?

Theoretical perspective

To describe the conditions under which participants distinguished the rate of change of a population and change in a population, we will first explain the theoretical perspective we take on how one may construct a differential equation (a type of mathematical model) to represent real-world situations. We operationalize mathematical modeling using theories of quantitative reasoning. With this lens, quantities are seen as the building blocks of the model (Larson, 2013). A quantity is different from a variable; it is a mental construct that is characterized by conceptualizing an object that has an attribute that an individual can imagine measuring (Thompson & Carlson, 2017). For example, when modeling disease transmission, one could conceptualize a quantity like the number of sick people at a specific time. The object would be the sick population, the attribute would be the amount, and one could envision counting the number of sick people on a given day even if it would not be possible to do. One constructs a mathematical model by operating on the quantities they impose onto the task scenario. Operating on quantities has been described in terms of quantitative operations and covariational reasoning (Kularajan & Czocher, 2022). A quantitative operation is a conceptual operation where an individual creates a new quantity in relation to one (or more) already created quantities (Ellis, 2007; Thompson, 2011). For example, one could additively compare the number of sick individuals on day one with the number of sick individuals on day three to create the change in the amount of sick population during two days. Covariational reasoning occurs when one envisions two quantities that can vary and thinks about the ways they change in relation to each other, and how they vary simultaneously. For example, an individual could envision the number of sick people at an instance varying with time. Covariational reasoning has been described through increasing levels of more complex ways of coordinating pairs of quantities (Thompson & Carlson, 2017). An important theoretical implication of taking this lens is that the models one constructs are dependent upon the quantities they impose onto the task scenario (Czocher et al., 2022).

Methods

This study was part of a larger study of the efficacy of facilitator scaffolding moves during modeling. Participants in the larger study worked on several modeling tasks over the course of 10 1-hour long sessions. Data were collected via individual clinical task-based interviews held over Zoom. We report on 2 STEM undergraduates' work on the disease transmission task, which was the last modeling problem in the sessions. In the disease transmission task, participants were asked to imagine a disease spreading through a community of sick and well members and asked to model the rate of spread of the disease through the community. The canonical differential equations associated with this task are: $\frac{dS}{dt} = \alpha \times S(t) \times W(t); \frac{dW}{dt} = -\alpha \times S(t) \times W(t)$,where $S(t)$ represents the number of sick people at time t, $W(t)$ represents the number of well people at time t, and α represents the probability of transmitting the disease between the sick and well individuals. We report on work from Roion, a computer science major, who built a model for the amount of change of the sick population in an instant, and Niali, an electrical engineering major, who built a model for the rate of change of the sick population. Both participants self-reported typically receiving As and Bs in their math classes. We chose to report on these two students to contrast one participant who did make the distinction between the rate of change of an amount and the change in amount and one who did not. The contrasting cases offer a chance to examine why a participant may or may not distinguish between the rate of change of a population and the change in a population. Data were analyzed by taking stock of the quantities students imposed onto the task scenario by noting the object, attribute, and evidence of quantification (Czocher et al., 2022). We then documented how and why (or inferred the reason why) participants chose to combine quantities to construct the rate of change of the infected population. This was done by attending to participants' answers when asked "why did you multiply/divide/subtract/add quantity X with/by quantity Y." When participants did not explicitly state their reason, we inferred why they combined quantities by appealing to the dimensional units participants ascribed to their quantities.

Results

Given our theoretical perspective, the conditions under which participants did/did not distinguish the rate of change of a population and the change in a population are determined by the quantities they imposed onto the task scenario and how the participants decided to operate on those quantities.

Roion's final equation modeling how quickly a disease spreads through a community is shown in Figure 1. Roion built his equation as follows: First, Roion constructed two quantities: the number of infected people at time t, represented by $I(t)$, and the number of well people at time t, represented by $W(t)$. Roion then multiplicatively combined the number infected people at time t and the number of well people at time t to create a new quantity called the maximum interactions between infected people and well people at time t. Roion then took a subset of the maximum number of interactions to get the number of interactions that happen (which he denoted $c \times W(t) \times I(t)$). He then took a subsequent proportion of the interactions that actually happen to get the number of interactions that result in infection (which he denoted $a \cdot (c \cdot W(t) \cdot I(t))$. Roion did not ascribe any dimensional units to a and c. To him, they were unitless

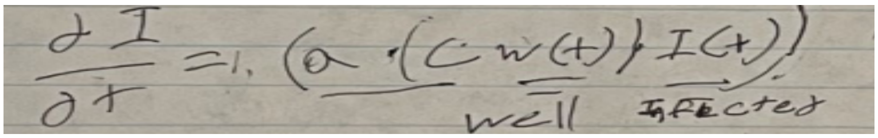


Figure 1 Roion's equation amount of change of the sick population in an instant

proportions, and so $a \cdot \left(c \cdot W(t) \cdot I(t)\right)$ represented an amount of interactions at time t that result in infection. We infer that Roion calculated the amount of change in an instant.

Niali's final equation that modeling quickly a disease spreads through a community is shown in Figure 2 (left).

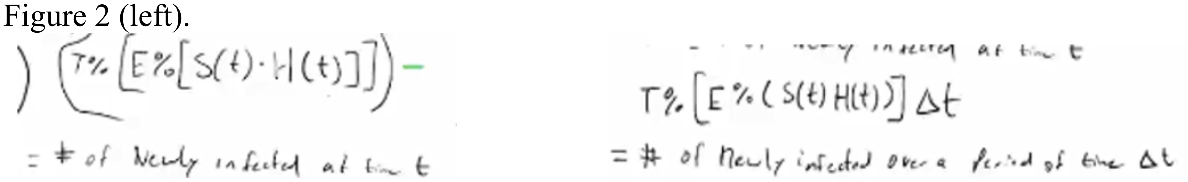


Figure 2 Niali's equations for newly infected at time t (left), and newly infected during an arbitrary duration of time (right)

Niali followed a similar path to Roion at first. For Niali, $S(t)$ denoted the number of sick people at time t, and $H(t)$ denoted the number of healthy people at time t. Niali combined these two quantities multiplicatively to get the maximum number of interactions between the two populations at time t. Niali then multiplied by two percentages to get the number of interactions that resulted in infection at time t (E represented the percentage of interactions that actually happened and T represented the percent of actual interactions where disease was transmitted). Niali noted that the number of interactions that resulted in infection at time t is the same calculation as newly infected at time t. It is at this point that Niali and Roion diverged. Niali coordinated the number of newly infected with an arbitrary duration of time to construct the number of newly infected during a duration of time Δt as seen in Figure 2 (right). By doing this, Niali is coordinating the multiplicative change in the infected population with additive change in time. This coordination results allowed Niali to write an equation for the average rate of change of the infected population.

Discussion

In this paper, set out to describe the conditions that led participants to (not) distinguish between the rate of change of a population and the change in a population when constructing a differential equation. By attending to the quantities participants imposed onto the scenario and how they operated on those quantities, we were able to detect differences in how our participants constructed their models for disease spread. These differences may explain why one may/may not distinguish between rates and amounts. In our examples, Niali distinguished the amount of change in the sick population from the rate of change in the sick population because the quantities he imposed onto the task scenario could be operated on to create a rate of change. Niali created quantities called newly infected during Δt and called duration of time (Δt) and operated on those quantities to create the average rate of change of the sick population (a coordination of the newly infected and duration of time). Roion, however, did not impose quantities onto the scenario that *could* be operated on to create a rate of change. We found no evidence that Roion imposed a quantity that described the duration of time in this task, thus we hypothesize that Roion did not have the "building block" necessary to create a rate of change of the sick population, but he did have the "building blocks" necessary to build a model for the amount of change in an instant. Our theoretical perspective and methodological approach allowed us to create a local causal explanation for why Niali did distinguish between the rate of change in the sick population and change in the sick population, while Roion did not. Our interpretation is consistent with Ellis et al. (2012)'s work on building exponential growth with 8th-grade students. In her study, the students were able to conceptualize exponential growth as constant multiplicative rates of change by coordinating multiplicative ratios of height (multiplicative change in the plant) with additive differences for the time (additive change in time). Importantly, the students first had to quantify the multiplicative change in the plant and change in time as quantities themselves that could be coordinated. That is, students may only be able to construct (or talk about) rates of change if they have the appropriate quantities available to them in which to coordinate.

A goal of this paper was to document contrasting cases of students thinking about differential equations to inform future research and instruction about how students think about differential equations in a canonical context. Rasmussen & King (2000) originally gave students a SIR disease model and then moved on to an unbounded exponential growth model when students had difficulty articulating the meaning of the differential equations. Researchers and teachers might consider helping the student quantify change in amount and change in time such that the student has the opportunity to coordinate the two to construct a rate of change. However, given that both participants in our study came to a model that described the spread of the disease through a community that looked normatively correct and was satisfactory for them, we suggest that researchers and teachers consider whether it might be beneficial to the student to background the distinction between rate of change of an amount and the change in an amount in an instant when working with students at this level. That is, if we want students to develop mathematical models for real-world contexts that are meaningful for them that work mathematically the same as the normatively correct model, then it might be worth allowing students to use amount languages when discussing rates of change. We do not assert that our results would generalize across people. The strength of the relationship between the quantities a student has available and their use of amount language to talk about rates needs further exploration.

Acknowledgments

Research reported in this paper is supported by National Science Foundation Grant No. 1750813 with Dr. Jennifer Czocher as principal investigator, and by Doctoral Research Support Fellowship given by Texas State University.

References

Czocher, J., Hardison, H. L., & Kularajan, S. S. (2022). A bridging study analyzing mathematical model construction through a quantities-oriented lens. *Educational Studies in Mathematics*. https://doi.org/10.1007/s10649-022-10163-3

Ellis, A. (2007). The Influence of Reasoning with Emergent Quantities on Students' Generalizations. *Cognition and Instruction*, *25*(4), 439-478.

Ellis, A., Ozgur, Z., Kulow, T., Williams, C., & Amidon, J. (2012). Quantifying exponential growth: The case of the Jactus. *Quantitative reasoning and mathematical modeling: A driver for STEM integrated education and teaching in context*, *2*, 93-112.

Kularajan, S. S., & Czocher, J. A. (2022). *STEM Undergraduates' Structural Conception of Rate of Change* forty-fourth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Nashville, Tennessee.

Larson, C. (2013). Modeling and quantitative reasoning: the summer jobs problem. In R. Lesh, P. L. Galbraith, C. Haines, & A. Hurford (Eds.), *Modeling Students' Mathematical Modeling Competencies* (pp. 111-118).

Maxwell, J. (2004). Causal Explanation, Qualitative Research, and Scientific Inquiry in Education. *Educational researcher*, *33*, 3-11. https://doi.org/10.3102/0013189X033002003

Mkhatshwa, T. P., & Doerr, H. M. (2018). Undergraduate Students' Quantitative Reasoning in Economic Contexts. *Mathematical thinking and learning*, *20*(2), 142-161.

Rasmussen, C., & King, K. (2000). Locating starting points in differential equations: a realistic mathematics education approach. *International Journal of Mathematical Education in Science and Technology*, *31*(1), 161-172.

Rasmussen, C., & Marrongelle, K. (2006). Pedagogical Content Tools: Integrating Student Reasoning and Mathematics in Instruction. *Journal for Research in Mathematics Education*, *37*(5), 388-420.

Rowland, D. R., & Jovanoski, Z. (2004). Student interpretations of the terms in first-order ordinary differential equations in modelling contexts. *International Journal of Mathematical Education in Science and Technology*, *35*(4), 503-516.

Thompson, P. W. (2011). Quantitative Reasoning and Mathematical Modeling. In L. L. Hatfield, S. Chamberlain, & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education* (Vol. 1, pp. 3357). University of Wyoming.

Thompson, P. W., & Carlson, M. (2017). *Variation, Covariation, and Functions: Foundational Ways of Thinking Mathematically*. National Council of Teachers of Mathematics.

AN APPLICATION OF HABERMAS CONSTRUCT OF RATIONALITY TO SUPPORT STUDENTS' PROOF VALIDATION SKILLS

Yuling Zhuang
Emporia State University
yzhuang@emporia.edu

Proof validation plays a significant role in students' understanding and learning of mathematical proofs. Recent studies have shown that university students were lacking skills in proof validation and were challenged by the implementation of the appropriate acceptance criteria when validating proofs. Drawing on Habermas' construct and rational questioning, the present study develops a proof validation framework that seeks to improve students' proof validation skills. The written responses of students' proof validation were analyzed based on their use of the proof validation framework in the context of a transition-to-proof course. The results showed that students primarily focused on the epistemic rationality component when they accept or reject the purported proof and that the students experienced difficulties with meeting the requirements of rationality components. Some educational implications are provided.

Keywords: Reasoning and Proof, Research Method, Design Experiment.

Mathematical proof is an essential component of undergraduate mathematics courses and serves as a prerequisite for students' advanced mathematical learning. Research in mathematics education indicates that most students have experienced substantial difficulty with proofs at the undergraduate level or beyond (Selden, 2012; Sommerhoff & Ufer, 2019; Weber, 2010). Many universities have instituted *transition-to-proof courses* (cf. Moore, 1994) to help students to review methods of mathematical proof (e.g., direct proof, proof by contradiction, proof by induction) to equip them for advanced mathematics courses that require mathematical proving skills. Students are expected to be able to read, evaluate, and write proofs through such a transition-to-proof course. The research studies on these courses have primarily concentrated on investigating students' beliefs regarding proofs, as well as their approaches to and challenges with proof comprehension, construction, and validation (e.g., Ko & Knuth, 2013; Mejia-Ramos et al., 2012; Segal, 2000; Selden & Selden, 2015; Weber, 2010). A few studies on proof validation have reported that university students often encounter substantial obstacles in accurately validating proofs and offering solid justifications for their judgments (see Alcock & Weber, 2005; Bleiler et al., 2014; Ko & Knuth, 2013; Kirsten & Greefrath, 2023; Selden & Selden, 2003, 2015; Sommerhoff & Ufer, 2019). However, little empirical research exists on how to improve students' skills of *validation of proofs* as in Selden and Selden, 1995 which focused on "proofs as texts that establish the truth of theorems and on readings of, and reflections on, proofs to determine their correctness" (Selden & Selden, 2003, p. 5). Proof validation plays a significant role at university level mathematics because university students who take advanced mathematics courses are expected to spend substantial study time reading and evaluating proofs and arguments that are presented by lectures or textbooks (Inglis & Alcock, 2012; Selden & Selden, 2003; Weber, 2004). Researchers (e.g., Alcock & Weber, 2005; Kirsten & Greefrath, 2023; Selden & Selden, 2003; Sommerhoff & Ufer, 2019) have suggested that validation of proofs should be taught explicitly in the university.

Studies that have focused on proof validation have typically explored the ability of students to differentiate between valid and invalid arguments (e.g., Alcock & Weber, 2005; Bleiler et al., 2014; Ko & Knuth, 2013; Selden & Selden, 2015; Weber, 2010). In his study of 28 undergraduate mathematics majors who had completed a transition to proof course, Weber (2010) found that the students judged invalid proofs to be valid proofs 60% of the time (64 out of 106). Furthermore, researchers have sought to gain insights into the cognitive processes involved in students' proof validation process by employing assessment models to better understand how students make these judgments. For instance, Selden and Selden (2003) found that undergraduates focused too much on *the surface features* of an argument, in which students tend to check proofs step by step rather than by the logical structure of proofs. Harel and Sowder (2007) proposed the idea of using "proof schemes" to describe and evaluate student performance in proving. The results of these studies revealed that many students judged invalid empirical arguments to be valid proofs because their ability to determine whether arguments were proofs was very limited, and they had problems with implementing appropriate acceptance criteria for validating proofs. However, these assessment models or proof schemes mainly focus on documentation of students' current situations or difficulties they experience while validating proofs, so these models cannot address the distance between the students' proof validation performance and the teacher's expectations. Consequently, several researchers (e.g., Selden & Selden, 2013, 2015; Sommerhoff & Ufer, 2019; Weber & Alcock, 2005) have advocated for further research aimed at developing instructional tools to address students' challenges with proof validation. They have argued that instruction in transition-to-proof courses should incorporate the introduction of these tools with the goal of enhancing students' proof validation skills.

Proof by contradiction (PBC) is an essential proof method across all mathematical content areas and is often viewed as more difficult than direct proof (Quarfoot & Robin, 2022). According to Robin and Quarfoot (2022), students need opportunities for validation of proofs using the PBC method before being able to link these experiences in their written work. The present study examined how a cohort of mathematics students used a proof validation framework that was adapted from Habermas' (1998) construct of rationality in the context of a transition-to-proof course to validate proofs that focused on PBC.

Theoretical Framework

In the field of mathematics education, Boero (2006) started to use Habermas' (1998) construct of rationality as a theoretical framework to study various mathematical discursive activities (e.g., proving, argumentation, problem-solving) based on the three components of rationality: epistemic (inherent in the control of validation of statements), teleological (inherent in the strategic choice of tools to achieve the goal of the activity), and communicative (inherent in the conscious choice of suitable means to communicate understandably within a given community). In recent years, following Boero (2006), several researchers (e.g., Boero et al., 2010; Boero & Planas, 2014; Morselli & Boero, 2011; Urhan & Bülbül, 2022a, 2022b; Zhuang, 2020; Zhuang & Conner, 2018, 2020, 2022a) have used Habermas' construct of rationality to analyze students' behavior in proving and problem-solving processes and developed instructional tools to help teachers plan and manage argumentative discourse according to rationality components. Moreover, considering the gap exists between the rationality of teachers and students, some researchers (e.g., Boero & Planas, 2014; Boero et al., 2018; Urhan & Bülbül, 2022; Zhuang & Conner, 2022a) have suggested the explicit introduction of Habermas' construct of rationality to students for pedagogical purposes to facilitate students' awareness of rationality requirements in proving and argumentation activities according to the teacher's expectations.

Drawing on Habermas' (1998) construct of rational behavior, Zhuang and Conner (2018, 2022a) developed a *rational questioning framework* for teachers to support students in meeting the requirements of rationality in argumentation. The present study views rational questioning as a didactical tool that can be introduced to students in order to guide them in the validation of proofs as shaped by rationality requirements; because rational questioning guide students to concern the choice of appropriate and efficient methods (teleological rationality) on the basis of mathematical knowledge (epistemic rationality), such as rules, theorems, axioms, and principles and communicate in a precise way with the choice of understandable means of communication within the shared mathematical community (communicative rationality), which corresponds to what policy documents and mathematics educators suggest as important characteristics of proof in mathematics (Boero et al., 2010; Boero & Planas, 2014; NCTM, 2000; Stylianides, 2007).

By adapting Habermas' (1998) construct of rationality and Zhuang and Conner's (2018, 2022a) rational questioning framework, this study develops a *proof validation framework* (see Table 1) to support students' proof validation skills via the use of rational questioning. The purpose of this study is to explore how students validate the p*urported proofs* (cf. Selden & Selden, 2003) through the use of the developed proof validation framework. More specifically, this study addresses the following research questions:

1. Which rationality component is privileged when students accept or reject the purported proof?
2. Which rationality components students are competent when they validate the purported proof?

Table 1: Proof Validation Framework

Components of Habermas' Rationality	Rational questioning to consider when validating purported proofs
Epistemic Rationality (*ER*): Uses valid definitions, axioms, and theorems shared by the mathematical community.	E1. What are the mathematical definitions, axioms, or theorems stated in the proof? Are they true? E2. Are there any other definitions, axioms, or theorems that would more reliably account for the stated mathematical claims? E3. What are the warrants or reasons used to support the stated mathematical claims and mathematical arguments? E4. Does the proof provide correct warrants or reasons to justify the stated mathematical claims and mathematical arguments? E5. Are the warrants or reasons convincing enough to help someone understand why the stated mathematical claims are true?
Teleological Rationality (*TR*): Employs efficient proof strategies (e.g., direct proofs, proof by	T1. What the proof methods (e.g., proof by contradiction, direct proof) are used to prove the stated mathematical claims and

contradiction, proof by induction, etc.) to achieve the goal of proof.	mathematical arguments? Are the proof methods used correctly and logically to achieve the goal of the proof? T2. How efficient is the applied proof method? Could any other methods be taken into account? T3. Could these proof methods be used to solve a similar problem?
Communicative Rationality (*CR*): Writes with forms of expression (e.g., mathematical language, visual representations, symbolic notation, etc.) that are understandable in the mathematical community.	C1. Are the proofs or arguments represented clearly? C2. Do the proofs or arguments contain mathematically correct language, visual representations, or symbolic notations? C3. Are there any ways to organize or write the proofs or arguments more clearly? Are there any irrelevant or distracting points?

Methodology

By following an instrumental case study (Stake, 1995), this study analyzed a cohort of mathematics graduate students' written responses from one of their course assignments in an online transition-to-proof course. The course serves as an introductory course for a master's mathematics program at a comprehensive state university in the United States to help students to review methods of mathematical proofs before they take advanced graduate-level mathematics courses. Because it is an online course, the participants of this study come from all over the world. Most participating students hold a bachelor's degree related to mathematics (i.e., pure mathematics, applied mathematics, mathematics education) or have studied a STEM-related major (e.g., computer science) for their undergraduate degrees. A few students have some experience in proof-related mathematics courses, such as advanced calculus and abstract algebra, but none of them have taken a transition-to-proof course that specifically focuses on proof validation and written techniques. Many students in this class hope to teach mathematics at a community college after completing the master's program in mathematics.

During the first couple of weeks of the course, the instructor introduced the proof validation framework (see Table 1) to the class and guided students to use the framework to validate the proofs that focused on direct proof methods through an online interactive learning community. Next, the students were expected to use the proof validation framework to complete the weekly Proof Writer's Workshop (PWW) assignment in which the students needed to read, evaluate, and critique at least one purported proof that is common to the new proof methods that are covered in the course. Students had to decide whether the purported proof that was given by the instructor should receive a pass (solid) proof, a revised (developing proof), or a failed grade (flawed proof), and the students were asked to justify their validations. In the assignment, the students were given a reflection question about the use of the proof validation framework: "Which components of rationality help you analyze this proof/argument? Explain." after they evaluated the purported proof.

The data in this study include 16 participating students' written responses on a purported proof that focused on PBC. The purported proof and the reflection question from their PWW assignments are shown in Figure 1. Before working on the PWW assignment, students were assigned to read the textbook *How To Prove It* (Velleman, 2019) in regard to PBC content. For this purported proof, the instructor expects the students to determine whether a fundamental flaw exists in reasoning about the definition of set difference (epistemic rationality). Although the PBC was used correctly, it may not be the most efficient proof method to apply, proof by contrapositive can be considered as a more efficient strategy (teleological rationality). This purported proof is well-written in text, but it could be more consistent with symbolic notations, such as the use of the set difference signs as "\" or "-" (communicative rationality).

This study conducted a microanalysis of each student's written responses for one of their weekly PWW assignment (Corbin & Strauss, 2015). The responses provided by the students for the first question were utilized to assess the accuracy of their proof validation, as well as to analyze the rationality components in which students demonstrated competence during the validation of the proposed proof (*RQ2*). Through the use of the constant comparative method (Glaser & Strauss, 1967), the students' answers to the reflection question were used to investigate which rationality component is privileged when they make judgments (*RQ1*).

The goal of the proof writer's workshop assignment is to help you get a better sense of what counts as a "solid proof", a "developing proof" and a "flawed proof". Each assignment will have at least one proof/argument for you to read, evaluate and critique. You will take the role of the grader: your job is to decide whether each proof/argument should receive a pass (solid proof), a revise (developing proof) or a failed grade (flawed proof).

Theorem 1: Suppose $A \subseteq B$ and C is any set. If $x \in A - C$ then $x \in B - C$.

Proof. We will give a proof by contradiction. Suppose for contradiction that $x \in A \setminus C$ but $x \notin B \setminus C$. From the definition of set difference, this tells us that $x \notin B$ and $x \in C$. But $x \in A \setminus C$ tells us that $x \in A$ and $x \notin C$. It is impossible to have $x \in C$ and $x \notin C$, so we have a contradiction. Therefore the theorem statement is true. □

Q1: Overall, would you rate this proof/argument as a pass or a revise? Or do you think there is a fundamental flaw in this proof/argument? Please justify your decision. You may consider evaluation proof/argument via the use of the Proof Validation Framework.

Reflection Question: Which component(s) of rationality help you analyze this proof/argument? Explain.

Figure 1: Proof Writer's Workshop (PWW) Assignment Used in this Study

Results

Through the analysis of the students' PWW assignment written responses, 14 out of 16 students gave the purported proof (see Figure 1) either a revised or a failed grade. Interestingly, the two students who made the wrong evaluation, in which they gave a grade of pass, did not validate the purported proof based on rationality components. This section reports how these 14 students validated the purported proof through the use of the proof validation framework (see Table 1).

Privileged Rationality Component

Based on students' written responses to the reflection question, all 14 students stated that they evaluated the given purported proof through epistemic rationality. Most of the students drew on epistemic rationality to analyze the correctness of the definition of set difference (under the guidance of E1 rational questioning, see Table 1). For instance, one student responded, "The epistemic rational element was the primary element that helped me analyze this proof because not stating a valid definition is what leads to the flaw in reasoning. You could also say that a valid definition (set difference) was incorrectly applied." A few students pointed out that the purported proof had a lack of warrants or reasons to support the stated theorem because the definition of the subset (given $A \subseteq B$) should be employed (under the guidance of E4 rational questioning).

Eight students mentioned that they applied teleological rationality when they validated the purported proof. One student highlighted that "The main rationality element that helped the most with this proof is teleological because the use of PBC seemed odd to me." The students generally drew on teleological rationality to evaluate the correctness of the proof method (under the guidance of T2 rational questioning, see Table 1). However, students rarely focused on the effectiveness of the proof method (under the guidance of T3 and T4 rational questioning). Only one student suggested that the proof could be shorter if a direct proof method were employed.

Eight students concentered on the communicative rationality of the purported proof. Most of them were satisfied with the language and notation that was used in the given purported proof (under the guidance of C1 and C2 rational questioning). Some students have different opinions as to whether the proof method should be stated in the beginning. For instance, one student stated that "It was also nice to have them state the proof method in the first sentence, so the reader knows what assumptions should be expected." Another student commented, "Maybe a little redundant to say proving by contradiction and then stating again that they were supposing for contradiction."

Overall, when validating the given purported proof, 7 students articulated that all three components of rationality helped them to validate the purported proof. Five students merely mentioned epistemic rationality. One student focused on both epistemic and teleological components of rationality and another focused on both epistemic and communicative components of rationality. In this sense, epistemic rationality appears to be the privileged rationality component when students accept or reject the given purported proof.

Competent Rationality Components

Among students who focused on the epistemic rationality component, 11 students were able to determine the correctness of the definition of set difference as stated in the purported proof, "From the definition of set difference, this tells us that $x \notin B$ and $x \in C$. " In addition, five of them corrected the definition in their written responses either by stating the correct definition or providing a counterexample. On the other hand, three students thought that the writer stated the definition of set difference correctly.

Of eight students who attended the teleological rationality component, four thought that the PBC method was applied correctly. Another four students articulated that the writer did not follow the PBC strategy due to incorrect assumptions. For instance, one student stated that "The use of contradiction is wrong since the contradiction used, $x \notin B - C$ which contradicts the primary given assumption." The rest of the three students argued that the writer should assume $x \notin A - C$ because according to their understanding of PBC, the writer should assume the negation of the *if* portion of the statement.

The students who have paid attention to the communicative rationality all agreed that the purported proof was written clearly in the text. As one student commented, "The proof was written clearly and used correct notations." Several students recommended improving the written work by ensuring consistency in the usage of notations, such as the set difference sign.

The results of this study also showed that 3 out of 14 students provided both invalid interpretations of epistemic and teleological rationality even though they rated the purported proof as revised. In addition, the two students who gave the grade of pass did not apply the proof validation framework. Without focusing on any rationality component, one student simply commented that "This proof looks to be sufficient to support the conclusion." Another student applied the direct proof method to show that the given purported proof was true in terms of the theorem and therefore concluded that the presented purported proof should pass with a lack of concentration on the purported proof itself.

Discussion

Researchers have used Habermas' (1998) construct to identify and interpret the challenges that are experienced by university students in mathematical proving and problem-solving activities (e.g., Boero & Morselli, 2009; Urhan & Bülbül, 2022b). This study responds to the call within the field to conduct research that develops approaches to teaching proof validations explicitly. The results of this study indicate that the proof validation framework (see Table 1) based on Habermas' (1998) construct of rationality and guided by rational questioning (Zhuang, 2020; Zhuang & Conner, 2018, 2022a) provides students with a tool to validate mathematical proofs in terms of rationality components rather than simply focused on what Selden and Selden (2003) called the surface features of proofs and arguments. According to Boero et al. (2010), the goal of developing rational behavior in proving must be guided and promoted by teachers. The introduction of the proof validation framework to students scaffolds them to be aware of the rationality requirements inherent in proving and to facilitate students' maturation of acting rationality in proving activities in a long-term teaching intervention. As one student replied in the reflection question, "All three of the elements of rationality helped me to analyze this proof. I found that the rational questions provided helped me the most. The questions allowed me to look for specific reasoning in the proof. This allowed me to break apart the proof, sentence by sentence, and apply the rational elements."

Previous research that examined students' abilities in proof validation has primarily concentrated on assessing the accuracy of their judgments regarding the warrants or reasons presented in the purported proofs (Alcock & Weber, 2005; Selden & Selden, 2003; Sommerhoff & Ufer, 2019; Weber & Alcock, 2005). Habermas' construct provides us with a more comprehensive frame in which to understand varying aspects of students' acceptance criteria for validating mathematical proofs, especially for teleological rationality and communicative rationality. The construct enables teachers to identify students' competence in rationality components and embeds students' competence in rationality components into their classroom instruction.

The result of this study showed that the epistemic rationality was represented as the privileged rationality component when students validated the given purported proof. This finding is unsurprising given that the principal flaw inherent in the purported proof lies in the incorrect definition of the stated set difference. This finding is also consistent with previous studies (see Alcock & Weber, 2005; Inglis & Alcock, 2012; Ko & Knuth, 2013; Kirsten & Greefrath, 2023; Selden & Selden, 2003; Sommerhoff & Ufer, 2019) in which students tended to check proofs line by line to make sure that each statement in the argument is mathematically correct. In addition, this study found that only 8 out of 14 students considered either the teleological or the communicative rationality component when they made judgments. This result implies that students prioritized the evaluation of the truthfulness of the stated assertions, rather than critically assessing the appropriateness and effectiveness of the employed proof method, as well as the clarity of the proof presentation. What is most disappointing is that only a single student acknowledged the efficiency of utilizing PBC in the purported proof. Hence, it is imperative for teachers to provide guidance to students regarding the teleological and communicative rationality during the process of proof validation, in order to direct the students' attention toward the overall strategy employed and the written structure of the proof.

Although the students who applied the proof validation framework (see Table 1) correctly rated the purported proof as either revised or failed under the guidance of rational questioning, it was noticed that some students' interpretations of rationality components were incompetent. The failures of interpretations include inadequate knowledge of the definition of set difference (epistemic rationality) and the incorrect assumption of negating *if* statement regards PBC (teleological rationality). These results demonstrate the importance to ask students to provide justifications for their validation according to the three rationality components because it enables teachers to determine the rationality components at which the students are competent. In this way, teachers can plan and design the follow-up process of teaching not only by focusing on students' gaps in mathematical knowledge but also by taking the rationality components into account based on the rationality requirements that teachers expect students to achieve in proving activities.

Implications and Future Directions

A number of studies (e.g., Kirsten & Greefrath, 2023; Selden & Selden, 2003, 2015; Sommerhoff & Ufer, 2019) have consistently revealed the challenges encountered by university students in the process of validating proofs. However, the skills critical for proof validation often do not receive adequate emphasis within the mathematics classroom.

This study proposes the idea of introducing the proof validation framework (see Table 1) to students in a transition-to-proof course. This framework provides a way for students to implement rationality components when they validate proofs, so they may benefit from interventions that focus on the requirements of rationality behavior in proving activities. Moreover, the utilization of the proof validation framework enables teachers to evaluate students' proof validation performance at a meta-level, as described by Boero et al. (2010), encompassing an awareness of the constraints associated with the three components of rationality as well as the content of the proof. For pedagogical purposes, the proof validation framework scaffolds students to realize proving as a rational process and guides them to gradually move to an awareness of epistemic, teleological, and communicative requirements of rationality that is inherent in proving as a long-term teaching intervention.

The present study proposes one potential way to improve students' proof validation skills, and other possible ways to improve students' skills in proof validation are worthy of investigation. Zhuang and Conner (2022b) discussed the use of students' incorrect answers through classroom-based argumentation, so it may be beneficial to examine errors in proofs that are constructed by students. Additionally, it would be interesting to investigate whether students are able to apply the skills of proof validation in terms of rationality components to their construction of proofs.

References

Alcock, L., Hodds, M., Roy, S., & Inglis, M. (2015). Investigating and improving undergraduate proof comprehension. Notices of the AMS, 62(7), 742–752.

Alcock, L., & Weber, K. (2005). Proof validation in real analysis: Inferring and checking warrants. The Journal of Mathematical Behavior, 24(2), 125-134. https://doi.org/10.1016/j.jmathb.2005.03.003

Bleiler, S. K., Thompson, D. R., & Krajčevski, M. (2014). Providing written feedback on students' mathematical arguments: Proof validations of prospective secondary mathematics teachers. Journal of Mathematics Teacher Education, 17(2), 105-127. https://doi.org/10.1007/s10857-013-9248-1

Boero, P. (2006). Habermas' theory of rationality as a comprehensive frame for conjecturing and proving in school. In J. Novotná, H. Moraová, M. Krátká & N. Stehlíková (Eds.), Proceedings of the 30th conference of the International Group for the Psychology of Mathematics Education (Vol. 2, pp. 185–192). Prague: PME.

Boero, P., Douek, N., Morselli, F., & Pedemonte, B. (2010). Argumentation and proof: A contribution to theoretical perspectives and their classroom implementation. In M. F. F. Pinto & T. F. Kawasaki (Eds.), *Proceedings of the 34th conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 179–205). Belo Horizonte: PME.

Boero, P., Fenaroli, G., & Guala, E. (2018). Mathematical argumentation in elementary teacher education: The Key Role of the Cultural Analysis of the Content. In A. Stylianides & G. Harel (Eds.), Advances in mathematics education research on proof and proving (pp. 49-67). Springer.

Boero, P., & Morselli, F. (2009). The use of algebraic language in mathematical modelling and proving in the perspective of Habermas' theory of rationality. In V. Durand-Guerrier, S. Soury-Lavergne & F. Arzarello (Eds.), Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education (pp. 964–973). INRP.

Boero, P. & Planas, N. (2014). Habermas' construct of rational behavior in mathematics education: New advances and research questions. In Liljedahl, P., Nicol, C., Oesterle, S., & Allan, D. (Eds.), Proceedings of the Joint Meeting of PME 38 and PME-NA 36 (Vol. 1, pp. 205–235). Vancouver: PME.

Corbin, J. M., & Strauss, A. L. (2015). Basics of qualitative research: Techniques and procedures for developing grounded theory (Fourth edition). SAGE.

Glaser, B. G., & Strauss, A. L. (1967). The discovery of grounded theory: Strategies for qualitative research. Aldine.

Habermas, J. (1998). On the pragmatics of communication (M. Cooke Ed.). MIT Press.

Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. In F. Lester (Ed.), Second handbook of research on mathematics teaching and learning, Vol. 2, pp. 805–842. Information Age Publishing.

Hodds, M., Alcock, L., & Inglis, M. (2014). Self-explanation training improves proof comprehension. Journal for Research in Mathematics Education, 45(1), 62-101. https://doi.org/10.5951/jresematheduc.45.1.0062

Inglis, M., & Alcock, L. (2012). Expert and novice approaches to reading mathematical proofs. Journal for Research in Mathematics Education, 43(4), 358-390. https://doi.org/10.5951/jresematheduc.43.4.0358

Ko, Y. Y., & Knuth, E. J. (2013). Validating proofs and counterexamples across content domains: Practices of importance for mathematics majors. The Journal of Mathematical Behavior, 32(1), 20-35. https://doi.org/10.1016/j.jmathb.2012.09.003

Kirsten, K., & Greefrath, G. (2023). Proof construction and in-process validation–Validation activities of undergraduates in constructing mathematical proofs. The Journal of Mathematical Behavior, 70, 101064. https://doi.org/10.1016/j.jmathb.2023.101064

Mejia-Ramos, J. P., Fuller, E., Weber, K., Rhoads, K., & Samkoff, A. (2012). An assessment model for proof comprehension in undergraduate mathematics. Educational Studies in Mathematics, 79(1), 3-18. https://doi.org/10.1007/s10649-011-9349-7

Moore, R. C. (1994). Making the transition to formal proof. Educational Studies in mathematics, 27(3), 249-266. https://doi.org/10.1007/BF01273731

Morselli, F., & Boero, P. (2009). *Proving as a rational behaviour: Habermas' construct of rationality as a comprehensive frame for research on the teaching and learning of proof.* In V. Durand-Guerrier, S. Soury-

Lavergne & F. Arzarello (Eds.), Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education (pp. 211-220). Lyon: CERME.
Morselli, F., & Boero, P. (2011). Using Habermas' theory of rationality to gain insight into students' understanding of algebraic language. In Early algebraization (pp. 453-481). Springer, Berlin, Heidelberg.
National Council of Teachers of Mathematics (2000). Principles and standards for school mathematics. https://www.nctm.org/Standards-and-Positions/Principles-and-Standards/
Quarfoot, D., & Rabin, J. M. (2022). A hypothesis framework for students' difficulties with proof by contradiction. International Journal of Research in Undergraduate Mathematics Education, 8(3), 490-520. https://doi.org/10.1007/s40753-021-00150-z
Rabin, J. M., & Quarfoot, D. (2022). Sources of students' difficulties with proof by contradiction. International Journal of Research in Undergraduate Mathematics Education, 8(3), 521-549. https://doi.org/10.1007/s40753-021-00152-x
Segal, J. (2000). Learning about mathematical proof: Conviction and validity. Journal of Mathematical Behavior, 18, 191–210. https://doi.org/10.1016/S0732-3123(99)00028-0
Selden, A. (2012). Transitions and proof and proving at the tertiary level. In G. Hanna & M. de Villiers (Eds.), Proof and proving in mathematics education: The 19th ICMI study (pp. 391–420). Dordrecht: Springer.
Selden, J., & Selden, A. (1995). Unpacking the logic of mathematical statements. Educational studies in mathematics, 29(2), 123-151. https://doi.org/10.1007/BF01274210
Selden, A., & Selden, J. (2003). Validations of proofs considered as texts: can undergraduates tell whether an argument proves a theorem? Journal for Research in Mathematics Education, 34(1), 4–36. https://doi.org/10.2307/30034698
Selden, A., & Selden, J. (2015). Validation of proofs as a type of reading and sense-making. In K. Beswick, T. Muir, & J. Wells (Vol. Eds.), Proceedings of the 39th conference of the international group for the psychology of mathematics education: Vol 4, (pp. 145–152). Hobart, Australia: PME.
Sommerhoff, D., & Ufer, S. (2019). Acceptance criteria for validating mathematical proofs used by school students, university students, and mathematicians in the context of teaching. ZDM Mathematics Education, 51(5), 717-730. https://doi.org/10.1007/s11858-019-01039-7
Stake, R. E. (1995). The art of case study research. SAGE Publications.
Stylianides, A. J. (2007). Proof and proving in school mathematics. Journal for Research in Mathematics Education, 38(3), 289–321. https://doi.org/10.2307/30034869
Velleman, D. J. (2019). How to prove it: A structured approach. Cambridge University Press.
Weber, K. (2004). Traditional instruction in advanced mathematics courses: A case study of one professor's lectures and proofs in an introductory real analysis course. Journal of Mathematical Behavior, 23, 115–133. https://doi.org/10.1016/j.jmathb.2004.03.001
Weber, K. (2010). Mathematics majors' perceptions of conviction, validity, and proof. Mathematical thinking and learning, 12(4), 306-336. https://doi.org/10.1080/10986065.2010.495468.
Weber, K., & Alcock, L. (2005). Using Warranted Implications to Understand and Validate Proofs. For the Learning of Mathematics, 25(1), 34–51. http://www.jstor.org/stable/40248484
Urhan, S., & Bülbül, A. (2022a). Analysis of mathematical proving in geometry based on Habermas' construct of rationality. Mathematics Education Research Journal, 1-31. https://doi.org/10.1007/s13394-022-00420-2
Urhan, S., & Bülbül, A. (2022b). Habermas' construct of rationality in the analysis of the mathematical problem-solving process. Educational Studies in Mathematics, 112, 175-197. https://doi.org/10.1007/s10649-022-10188-8
Zhuang, Y. (2020). Using Habermas' Construct of Rational Behavior to Gain Insights into Teachers' Use of Questioning to Support Collective Argumentation (Publication No. 28024092) [Doctoral dissertation, University of Georgia]. ProQuest Dissertations and Theses Global.
Zhuang, Y. & Conner, A. (2018). Analysis of teachers' questioning in supporting mathematical argumentation by integrating Habermas' rationality and Toulmin's model. In T. Hodges, G. Roy, & A. Tyminski (Eds.), Proceedings of the 40th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (pp. 1323–1330). University of South Carolina & Clemson University.
Zhuang, Y., Conner, A. (2020). Teacher questioning strategies in supporting validity of collective argumentation: explanation adapted from Habermas' communicative theory. In A.I. Sacristán, J.C. Cortés-Zavala & P.M. Ruiz-Arias, (Eds.). Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Mexico (pp. 2288-2296). Cinvestav/AMIUTEM/PME-NA. https:/doi.org/10.51272/pmena.42.2020

Zhuang, Y., & Conner, A. (2022a). Teachers' use of rational questioning strategies to promote student participation in collective argumentation. Educational Studies in Mathematics, 111(2), 345-365. https://doi.org/10.1007/s10649-022-10160-6

Zhuang, Y. & Conner, A. (2022b). Secondary mathematics teachers' use of students' incorrect answers in supporting collective argumentation. Mathematical Thinking and Learning, 26, 1-24. https://doi.org/10.1080/10986065.2022.2067932

COMBINING COVARIATIONAL REASONING AND CAUSALITY TO CONCEPTUALIZE FEEDBACK LOOPS

COMBINANDO EL PENSAMIENTO COVARIACIONAL Y LA CAUSALIDAD PARA CONCEPTUALIZAR BUCLES DE RETROALIMENTACIÓN

Darío González
Universidad Autónoma de Chile
dario.gonzalez@uautonoma.cl

This paper introduces two theoretical constructs, open-loop covariation and closed-loop covariation, that combine covariational reasoning and causality to characterize the way that three preservice mathematics teachers conceptualize a feedback loop relationship in a mathematical task related to climate change. The study's results suggest that the preservice teachers' reasoning about feedback loop between quantities involved the ability to conceive closed-loop covariation, which in this study was characterized by two cognitive realizations: (i) the conception of simultaneous change and (ii) the recognition of circular causality. These realizations, at least for the participants, appeared to be independent from one another. The theoretical distinction between open- and closed-loop covariation could inform instructional strategies to develop students' ability to think about and model feedback loops.

Keywords: Cognition, Integrated STEM / STEAM, Modeling, Sustainability.

Purpose of The Study

Authors from disciplines as diverse as biology, chemistry, engineering, economy, and mathematics have proposed that current global, complex, politically charged, socio-scientific issues (universal income, evolution, pandemic and vaccines, climate change, etc.) require STEM professionals to understand them as complex systems (Ghosh, 2017; Orgill et al., 2019; Renert, 2011; Richmond, 1997; Roychoudhury et al., 2017; Schuler et al., 2018). This holistic perspective, known as systems thinking, focuses on understanding phenomena in terms of relationships, connectedness, and context. Systems thinking complements the analytic or reductionist perspective commonly used in STEM and STEM education fields. An important distinction between these perspectives involves causality; while the reductionist perspective focuses on linear, direct cause-and-effect relationships, systems thinking involves identifying complex causality relationships. An important type of these relationships are feedback loops, or a "succession of cause-effect relations that start and end with the same variable. It constitutes a circular causality, only meaningful dynamically, over time" (Barlas, 2002, p. 1147).

Mathematics represents a powerful way to make sense of relationships, which can be understood as two (or more) quantities changing together over time. Feedback loops, therefore, can be seen as two (or more) quantities changing simultaneously in a way such that, the first quantity *causes* the second quantity to change, and that change *causes* the first quantity to change again, and so on. In particular, I believe that combining covariational reasoning with the notion of causality can provide insights into how students can understand feedback loop relationships in mathematics.

In this paper, I combine covariational reasoning and causality to introduce two theoretical constructs, *open-loop covariation* and *closed-loop covariation*, which can characterize the way that three preservice teachers conceptualize a feedback loop relationship in a mathematical task related to climate change. The constructs also have the potential to address a gap in the literature focusing on covariational reasoning which have a tendency to consider only unidirectional implications of change and leave underexplored the role of real-world causality. I also discuss possible implications for mathematics learning and teaching.

Conceptual Framework

Covariational Reasoning and the Multiplicative Object

Covariational reasoning finds one of its earliest definitions in the work of Saldanha and Thompson (1998), who defined it as follows:

> Someone holding in mind a sustained image of two quantities' values (magnitudes) simultaneously. It entails coupling the two quantities, so that, in one's understanding, a multiplicative object is formed of the two. As a multiplicative object, one tracks either quantity's value with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value … An operative image of covariation is one in which a person imagines both quantities having been tracked for some duration, with the entailing correspondence being an emergent property of the image. (pp. 298-299)

The *multiplicative object* in their definition is analogous to the logical conjunction "and" that joins or units two propositions to produce one proposition that is true if and only if both of the constituent propositions are true. In the case of covariation, the multiplicative object joins the corresponding values of two covarying quantities so that the student "mentally unites their attributes to make a new attribute that is, simultaneously, one and the other" (Thompson et al., 2017, p. 96). This multiplicative object supports the student's ability to conceptualize two (or more) quantities changing simultaneously and interdependently.

Open- and Closed-Loop Covariation

Covariational reasoning support the conceptualization of two quantities changing simultaneously. However, simultaneity may not be enough to conceptualize feedback loop structures in a system. I propose that the way causality is conceived may also play an important role in conceptualizing a feedback loop relationship between two covarying quantities. To address this distinction, I introduced two constructs: *open-loop covariation* (OLC) and *closed-loop covariation* (CLC).

Let's consider the filling bottle problem where water is being pour into the bottle at a constant rate, increasing the volume of water, V, and the water height, h, in the bottle. The multiplicative object allows one to visualize V and h changing simultaneously over time so that there exist a pair $(V(t), h(t))$ for any t in some interval of conceptual time. One imagines that if V changes, so does h, and if h changes, so does V; they change together. The multiplicative object, thus, support a student's ability to conceptualize simultaneous change between two quantities.

However, the multiplicative object does not provide an answer to the questions of *how* and *why* a change in V results in a change in h, or vice versa. In this paper, I use the term causality as a way of describing how and why the state of a dynamic process changes as time goes on (Sauer, 2010). The volume of water V, or the space taken by the water in the bottle, is growing since more and more water is entering the bottle. One also observes (or imagines) that the water height, h, is increasing as water enters the bottle; that is, h grows as V grows. One could say that the growing volume of water in the bottle <u>causes</u> the water height to increase; the more water enters the bottle, the more the water height increases. This is a description of how and why a change in *V* causes a change in *h*. It is important to point out that thinking of causality in the opposite direction may not be as natural or intuitive; mathematically, changing the value of *h* would change the value of *V*, but it is hard to imagine that, in a real-world context, an increase in *h* would cause more water to enter the bottle. I use causality in that second "real-world" sense and consider the covariation of *h* and *V* as OLC, which shows *simultaneity* (*V* and *h* change together) and only *linear causality* (a change in *V* causes a change in *h*, and may not be as intuitive to say that a change in *h* causes a change in *V*).

The conceptualization of a feedback loop structure between covarying quantities is only possible when there exists circular causality between those quantities. For instance, consider a simple predator-prey model relationship in which *P* is the number of predators in a region and *N* is the number of preys in the same region. We assume that predation (magnitude of *P*) is the only factor affecting *N* and that prey availability (magnitude of *N*) is the only factor affecting *P*. With those assumptions, *P* increases when *N* increases because an increase in availability of prey produces prosperity for predators, who can reproduce more. However, when *P* increases passed certain value, *N* starts to decrease because an increase in predator means that more preys would die. When *N* decreases passed certain threshold value, *P* would start to decrease as well because predators do not have enough food to sustain their population. As *P* decreases passed certain value, *N* becomes to increase again because there are less predators and less preys get eaten. This is an example of CLC, which shows *simultaneity* (*P* and *N* change together) and *circular causality* (changes in *N* cause changes in *P*, which in turn cause changes in *N* again and so on). Conceptualizing circular causality, as illustrated by the predators-prey model, seems to be more cognitively demanding than conceptualizing linear causality (Ghosh, 2017; Hokayen et al., 2015; Roberts, 1978; Wellmanns & Schmiemann, 2022).

Methodology

This paper is part of a larger study that investigated how PSTs make sense of some elemental mathematics behind modeling climate change. Three secondary PSTs —hereafter Jodi, Pam, and Kris— enrolled in a mathematics education program at a large Southeastern university participated in the larger study. These PSTs had completed Calculus I and II and an Intro to Higher Mathematics course and were completing a Math Modeling for Teachers course by the time the larger study took place. The PSTs were asked to complete an original sequence of mathematical tasks while participating in individual, task-based interviews (Goldin, 2000). In this paper, I focus on the PSTs' responses to the Energy Balance task.

The Energy Balance (EB) Task

An *energy balance model* describes the continuous heat exchange between the sun, the planet's surface, and the atmosphere (Figure 1a). The planet's surface is warmed by a fraction of the sun's radiation (*S*). As the surface's temperature increases, it radiates heat towards the atmosphere (*R*). The majority of it (*B*) is absorbed by *greenhouse gases* (GHG), which rises the atmosphere's temperature. As it warms up, the atmosphere radiates a fraction of the absorbed heat back to the surface (*A*). The latter further increases the surface's temperature, which results in an increase of surface radiation towards the atmosphere and an increase in the atmosphere's temperature. The continuous heat exchange between the surface and the atmosphere is known as the *greenhouse effect* and has a key role in controlling the planet's mean surface temperature, *T*. The relative abundance of GHG in the atmosphere regulates the amount of heat it absorbs. Therefore, changes in the concentration of GHG are followed by changes in *T*.

The EB task (Figure 1b) describes a simplified situation that begins with an energy balance at the surface; that is, the surface absorbs heat at the same rate it releases it ($S + A = R$ in Figure 1a). Then, it is assumed that a unique and instantaneous pulse of carbon dioxide (CO_2) is released at $t = 0$ to increase its concentration in the atmosphere. The EB task focuses on CO_2 for it is one of the main drivers of global warming (IPCC, 2018). The instantaneous increase in CO_2 is followed by an instantaneous increase in A, the heat radiation from the atmosphere to the surface. Thus, at $t = 0$, the surface is absorbing heat at a higher rate than that at which it is releasing it ($S + A > R$ in Figure 1a). The surface then begins to warm up as time elapses (T increases as t increases) and to increase its heat radiation towards the atmosphere (R increases as t increases). The atmosphere also begins to warm up and to further increase its heat radiation back to the surface (A increases as t increases). This further warms the surface and further increases the surface heat radiation towards the atmosphere (T and R continue to increase as t increases). This energy feedback loop between the surface and the atmosphere allows for R to increase enough so that a new energy balance is reached ($S + A = R$). This balance is accompanied by a new (higher) value of T. Thus, the goal of the EB task was to engage PSTs into thinking about how the Earth's energy balance's response to an increase in CO_2 results in an increase in the mean surface temperature, T, thus connecting CO_2 pollution to global warming.

The current paper examines the PSTs' reasoning regarding the feedback loop between the surface and the atmosphere in terms of a covariation between R and A with respect to time. The results will mainly focus on the PSTs' responses to the second part of the EB task, in which they were asked to draw the graph of T as a function of time t (Figure 1b).

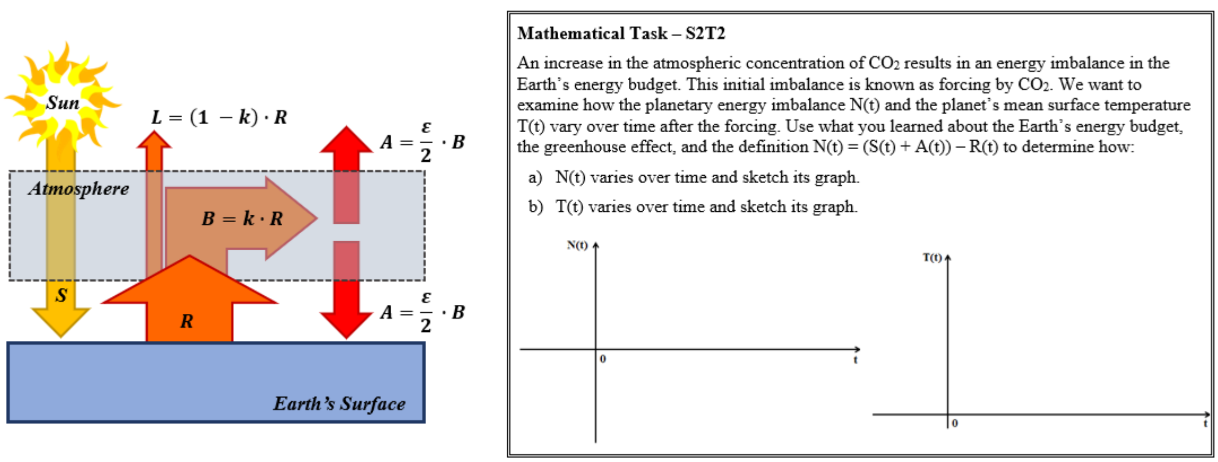


Figure 1: (a) An Earth's energy balance model (left) and (b) the EB task (right)

Data Collection

Before working on the EB task, the PSTs participated in a 32-minute-long, individual minilesson where some basic concepts related to the Earth's energy balance and the greenhouse effect were discussed. The goal was to provide PSTs with enough knowledge about those concepts so that they could start working on the EB task. The minilesson began with a 7-minute-long video retrieved from the NASA YouTube channel *NASAEarthObservatory* introducing the energy balance and the greenhouse effect. Then, each PST and I held a 5-minute-long Q&A session in which we clarified questions about the ideas discussed in the video. During the next 20 minutes, each PST and I worked with a diagram of the energy balance similar to the one in

Figure 1a. We talked about what each of the quantities *S*, *R*, *L*, *B*, and *A* represented and how they related to one another. Then, we talked about the energy balance at the surface as the equality $S + A = R$, and I illustrated it for some specific initial values of *S*, *A*, and *R*. We also discussed what inequalities such as $S + A > R$ or $S + A < R$ could mean in terms of temperature.

A week after the minilesson, each PST completed the EB task during a 60-minute-long, individual, task-based interview (Goldin, 2000). The interviews were semi-structured and had an interview protocol with pre-defined questions so that all participants received similar prompts during the interviews. To help PSTs understand how the energy balance responds to the increase in CO_2, the following recursive rules were made available to them: $B_t = k \cdot R_t$, $A_t = ½ \cdot B_t$, and $R_{t+1} = S + A_t$, for some $0 \leq k \leq 1$. By using these rules, they could find the values of *B*, *A*, and *R* for successive values of time, thus observing how these quantities change dynamically. This modelling strategy is known as *Discrete Event Simulation* and can be used to help students understand the dynamics of systems in particular situations (Hoad & Kunc, 2018).

Data Analysis

The interview videos and transcripts were analyzed through *thematic analysis* (Braun & Clarke, 2006; 2012). This qualitatively method of data analysis allows for systematically identifying, organizing, and offering insight into patterns of meaning across a data set though the development of codes and themes. Thematic analysis is a widely used analytic strategy to identify and make sense of collective or shared meanings and experiences. The method can be summarized into six phases: familiarizing yourself with your data, generating initial codes, searching for themes, reviewing themes, defining and naming, and producing the report.

I watched all interview videos and took notes while doing so. The videos were separated into shorter, more manageable *episodes*, each one covering a single topic or showing evidence of a particular way of reason covariationally. The notes informed my first round of coding for the interview transcripts. Then, the episode transcripts were sorted according to similar codes to look for patterns in participants' responses. This allowed me to revise and refine the initial codes, reducing them to two main themes. Thus, the five initial codes "asynchronous", "synchronous", "feedforward but not feedback", "verbalizing/indicating circularity", and "circular relationship" were collapsed into two themes: "Simultaneous Change", with "asynchronous" being the absence thereof and "synchronous" being the presence thereof, and "Circular causality", with "feedforward but not feedback" representing the absence thereof and "verbalizing/indicating circularity" and "circular relationship" representing the presence thereof.

Using the analytic framework previously described (Table 1), I indexed all episode transcripts into three analytic matrices, one per participant. These matrices allowed me to look for patterns in the distribution of themes, which provided the information needed to meet the research goals.

Results

The analysis revealed that, for this group of preservice teachers, two main cognitive realizations appear important to conceptualize the energy feedback loop between the surface and the atmosphere (the greenhouse effect) as a CLC between the quantities *R* and *A*: conceiving *simultaneity* of change and a *circular causality* relationship between those quantities. When one of these realizations was not supported, the PSTs developed inaccurate conceptualizations of the greenhouse effect and, by extension, this had an impact on their understanding of the link between CO_2 pollution and global warming.

Table 1: Analytic Framework

Theme	Descriptions
Simultaneity	<u>*Asynchronous*</u> The PST describes or represents changes in *A* and *R* as occurring asynchronously (*A* changes first, then *R* changes, then *A* again, and so on). <u>*Synchronous*</u> The PST describes or represents changes in *A* and *R* as occurring simultaneously as time elapses.
Causality	<u>*Linear*</u> The PST describes or represents causality in one direction between *A* and *R*. Either change in *A* causes change in *R* or change in *R* causes change in *A*. <u>*Circular*</u> The PST describes or represents a circular causality between *A* and *R* so that change in *A* causes change in *R*, which in turn causes change in *A* again.

The PSTs explored the greenhouse effect as an energy feedback loop while working on their graphs of $T = g(t)$. I start the discussion with Kris' work because she conceptualized the greenhouse effect in terms of CLC. More specifically, Kris made remarks about *R* and *A* increasing *simultaneously* as time, *t*, increased: "So, as *R* increases, *A* increases … the new *R* is affected by *S* plus *A* [*points at S and A*]. So, when *A* increases, *R* is going to be [bigger]. It can't just keep increasing". Kris also made remarks suggesting she conceived of a *circular causality* relationship between the quantities *R* and *A*.

> Well, [the surface] keeps in taking. I think it is warming up because once we added more CO_2, that is less of the emitted energy that is getting just like shut out passed the atmosphere, leaked from it. So then, more of it is going to be absorbed by the atmosphere … Whatever is absorbed by the atmosphere [*points at B*] is going to be absorbed back into the [*points at the surface*], well half of that plus the sun's energy [*points at S*] is going to be absorbed by the Earth, which is going to keep increasing, as we saw with like the 400 [*points at the R-value of "400"*]. Then, from the *A* value [*with a capped marker, traces the top half of a circle, going from R to A*], just with the *A*, [the surface] absorbs 160, and then we add a new *R*-value [*with the capped marker, traces the bottom half of the circle, going from A to R*], whatever that was, and then [the surface] absorbs 164 [*with the capped marker, re-traces the top half of the circle, going from R to A*]. So, I think it is going to keep increasing [*draws an increasing, concave-downward graph for T = g(t) that appears to have a horizontal asymptote that she labels as "new equilibrium temperature"*].

In the above excerpt, Kris repeatedly referred to the relationship between *R* and *A* as a "cycle", which suggests an awareness of circular causality between the quantities. Also, notice how she gestured both, the feedforward relationship from *A* to *R* and the feedback relationship from *R* back to *A*, further suggesting circular causality. Kris also drew an accurate graph of $T = g(t)$ showing asymptotic growth towards a new equilibrium value (Figure 2a); this suggests an awareness of the balancing quality of the feedback loop. Kris's CLC coincided with her demonstrating an accurate conception of the greenhouse effect, which guided her to conclude that an increase in CO_2 causes a warming effect over the planet's surface, correctly relating CO_2 pollution to global warming.

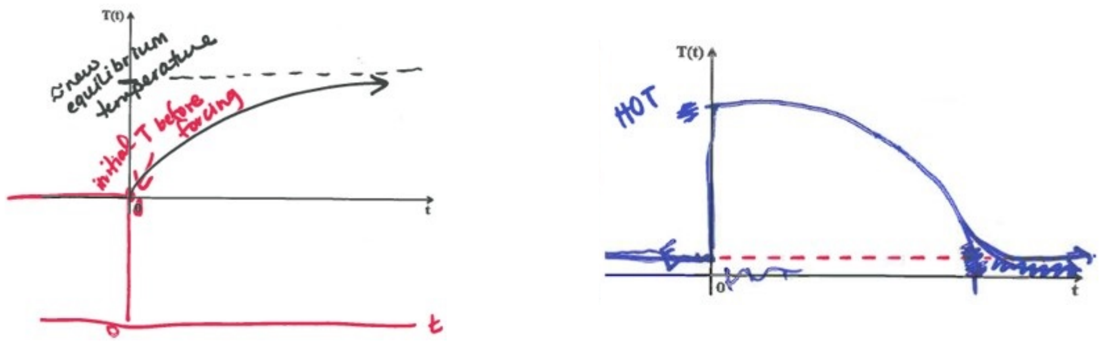


Figure 2: (a) Kris's graph of *T* = *g*(*t*) (left) and (b) Pam's graph of *T* = *g*(*t*) (right)

The case of Pam illustrates the conceptualization of OLC, where simultaneous change is conceived but not circular causality. She described *R* and *A* as two quantities increasing in tandem as *t* increased but only in the *A*-to-*R* direction ("*R* increases <u>*as*</u> [emphasis added] *A* increases. So, <u>*as*</u> [emphasis added] our *A* increases, *R* increases"). Pam also interpreted the increase in *R* and *A* as the planet's surface emitting more heat than the heat it was absorbing, which suggested additional evidence of only conceiving causality from *A* to *R*.

> A lot [of heat] is going in, but more is coming out, like *R* increases as *A* increases. So, as our *A* increases, *R* increases. But our *S* is staying the same. But our *A* is always less than *R*. So, more [heat] is coming out [*pauses to think*]. So, the Earth is trying to cool itself off, so the temperature is decreasing from here to here.

Pam again implied that an increase in *A* causes an increase in *R*, but did not appear aware that the increase in *R* also causes a new increase in *A*. She referred to *R* as heat leaving the surface and being larger than the heat absorbed by it. This claim overlooked that *A* is actually a fraction of *R* that is reabsorbed by the surface. This might have kept Pam from conceptualizing the feedback relationship from *R* back to *A*. Pam's OLC coincided with her demonstrating an inaccurate conception of the greenhouse effect, which may have led her to incorrectly conclude that the planet's surface cools down after an increase in CO_2, as her graph shows (Figure 2b).

Finally, Jodi's reasoning did not support simultaneous change but supported circular causality between *R* and *A*. In the following excerpt, Jodi appeared to imagine *R* and *A* as changing *asynchronously*—*A* changes first, then *R* changes, and then *A* again, and so on—but seemed aware of the circular causality between those two quantities.

> I am trying to look at the differences. So, like here [*points at the R-value "390" and the A-value "150"*]. Ok, so here the change was five [*points at the R-value "395" and the A-value "155"*]. The change was two [*points at the R-value "397" and the A-value "157"*]. Is it, I mean, is it not changing? … The flow of energy increased by five [*points at the A-value "155"*], but then it decreased by five [*points at the R-value "395"*]. Then it increased by two [*points at the A-value "157"*], and then it decreased by two [*points at the R-value "397"*]. So, it is almost as if there was no change in temperature because I associate like energy as kind of having a relationship with temperature. So, if the energy increases, then the temperature increases. But, in this scenario, an equal change in energy [*points at A*] was an equal change in output [*with her index finger, traces the bottom half of a circle from A to R*]

… Ok, cycle started here [*points at B*], and here the Earth's temperature would've been something. Equal input of energy, equal output of energy [*with her index finger, traces a circle connecting A, R and B*]. Ok. So, when the cycle started, there was an input of energy [*points at A*], and then it got released [*with her index finger, traces the bottom half of a circle from A to R*]. Another cycle starts [*points at B*], input of energy, release of energy [*with her index finger, traces a circle connecting B, A, and R*]. So, it would almost be like [*draws a periodic curve formed by identical arcs*].

Notice how Jodi described the changes in *R* and *A* as happening at different times rather than simultaneously. This suggests Jodi did not develop a multiplicative object and her reasoning may not be considered covariational. In contrast, she did allude to circular causality by tracing circles with her finger connecting *R* and *A*. I also interpreted the periodicity of her graphs (Figure 3) as additional evidence of awareness of circular causality. Jodi's way of reasoning coincided with her demonstrating an inaccurate conception of the greenhouse effect and arriving to an incorrect conclusion regarding the real impact of CO_2 pollution on the planet's surface temperature.

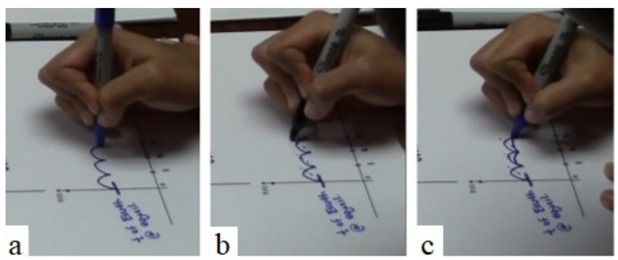


Figure 3: Jodi's periodic graphs of $T = g(t)$

Conclusion

Some situations that involve two quantities changing together over time also require the recognition of an underlying feedback loop structure between them. The study's results suggest that, for this group of preservice teachers, reasoning about feedback loop between quantities involved the ability to conceive *closed-loop covariation*, which in this study was characterized by two cognitive realizations: (i) the conception of *simultaneous change* and (ii) the recognition of *circular causality*. The first realization is based on the mental construction of a multiplicative object between those quantities (Saldanha & Thompson, 1998; Thompson et al., 2017), while the second realization involves noticing that changes in a quantity cause change in the second quantity, which in turn cause new changes in the first quantity and so on. The results also suggests that these two realizations, at least for these preservice teachers, appeared to be independent from each other. Kris demonstrated both realizations, while Pam and Jodi demonstrated one or the other but not both. An implication of this is that instructional strategies aiming to support students' ability to understand feedback loops mathematically should focus on developing both realizations at the same time.

It also important to point out that many school mathematical tasks involving change between quantities may only require *open-loop covariation*, where students oftentimes only need to attend to linear causality between the quantities. Examples of these are situations modeled by linear or quadratic functions. However, closed-loop covariation may be an interesting and novel way to explore situations involving exponential growth (e.g., compound interest or population growth), where the current value of the dependent quantity plays a role on how that quantity changes.

References

Barlas, Y. (2002). System dynamics: systemic feedback modeling for policy analysis. *Encyclopedia for life support systems*. UNESCO Publishing.

Braun, V., & Clarke, V. (2012). Thematic analysis. In H. Cooper, P. M. Camic, D. L. Long, A. T. Panter, D. Rindskopf, & K. J. Sher (Eds.), *APA handbook of research methods in psychology, Vol. 2. Research designs: Quantitative, qualitative, neuropsychological, and biological* (pp. 57–71). American Psychological Association. https://doi.org/10.1037/13620-004

Braun, V., & Clarke, V. (2006) Using thematic analysis in psychology. *Qualitative Research in Psychology, 3*(2), 77-101. https://doi.org/10.1191/1478088706qp063oa

Ghosh, A. (2017). *Dynamic systems for everyone*. Switzerland: Springer International Publishing. https://doi.org/10.1007/978-3-319-10735-6

Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A. Kelly & R. A. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 517–545). Lawrence Erlbaum Associates. https://doi.org/10.4324/9781410602725

Hoad, K., & Kunc, M. (2018). Teaching system dynamics and discrete event simulation together: A case study. *Journal of the Operational Research Society, 69*(4), 517-527. doi:https://doi.org/10.1057/s41274-017-0234-3

Hokayem, H., Ma, J., & Jin, H. (2015). A learning progression for feedback loop reasoning at lower elementary level. *Journal of Biological Education, 49*(3), 246-260. http://dx.doi.org/10.1080/00219266.2014.943789

Intergovernmental Panel on Climate Change (IPCC). (2018). *Global Warming of 1.5°C. An IPCC Special Report on the impacts of global warming of 1.5°C above pre-industrial levels and related global greenhouse gas emission pathways, in the context of strengthening the global response to the threat of climate change, sustainable development, and efforts to eradicate poverty.* Cambridge University Press. https://doi.org/10.1017/9781009157940

Orgill, M., York, S., & MacKellar, J. (2019). Introduction to systems thinking for the chemistry education community. *Journal of Chemical Education, 96*, 2720-2729. https://doi.org/10.1021/acs.jchemed.9b00169

Renert, M. (2011). Mathematics for life: Sustainable mathematics education. *For the Learning of Mathematics, 31*(1), 20-25. https://www.jstor.org/stable/41319547

Roberts, N. (1978). Teaching dynamic feedback systems thinking: An elementary view. *Management Science, 24*(8), 836-843. http://www.jstor.org/stable/2630381

Roychoudhury, A., Shepardson, D. P., Hirsch, A., Niyogi, D., Mehta, J., & Top, S. (2017). The need to introduce systems thinking in teaching climate change. *Science Educator, 25*(2), 73-81.

Richmond, B. (1997). The “thinking” in systems thinking: How can we make it easier to master? *The Systems Thinker, 8*(2), 1-5.

Saldanha, L. A., & Thompson, P. W. (1998). *Re-thinking covariation from a quantitative perspective: Simultaneous continuous variation*. Paper presented at the 20th annual meeting North American Chapter of the International Group for the Psychology of Mathematics Education, Raleigh, NC, United States.

Sauer, N. (2010) Causality and causation: What we learn from mathematical dynamic systems theory. *Transactions of the Royal Society of South Africa, 65*(1), 65-68. https://doi.org/10.1080/00359191003680091

Schuler, S., Fanta, D., Rosenkraenzer, F., & Riess, W. (2018). Systems thinking within the scope of education for sustainable development (ESD) ‑ a heuristic competence model as a basis for (science) teacher education. *Journal of Geography in Higher Education, 42*(2), 192-204. https://doi.org/10.1080/03098265.2017.1339264

Thompson, P. W., Hatfield, N. J., Yoon, H., Joshua, S., & Byerley, C. (2017). Covariational reasoning among U.S. and South Korean secondary mathematics teachers. *Journal of Mathematical Behavior, 48*, 95-111. http://dx.doi.org/10.1016/j.jmathb.2017.08.001

Wellmanns, A. & Schmiemann, P. (2022) Feedback loop reasoning in physiological contexts. *Journal of Biological Education, 56*(4), 465-485. https://doi.org/10.1080/00219266.2020.1858929

COMBINING COVARIATIONAL REASONING AND CAUSALITY TO CONCEPTUALIZE FEEDBACK LOOPS

COMBINANDO EL PENSAMIENTO COVARIACIONAL Y LA CAUSALIDAD PARA CONCEPTUALIZAR BUCLES DE RETROALIMENTACIÓN

Darío González
Universidad Autónoma de Chile
dario.gonzalez@uautonoma.cl

This paper introduces two theoretical constructs, open-loop covariation, and closed-loop covariation, that combine covariational reasoning and causality to characterize the way that three preservice mathematics teachers conceptualize a feedback loop relationship in a mathematical task related to climate change. The study's results suggest that the preservice teachers' reasoning about feedback loop between quantities involved the ability to conceive closed-loop covariation, which in this study was characterized by two cognitive realizations: (i) the conception of simultaneous change and (ii) the recognition of circular causality. These realizations, at least for the participants, appeared to be independent from one another. The theoretical distinction between open- and closed-loop covariation could inform instructional strategies to develop students' ability to think about and model feedback loops.

Keywords: Cognition, Integrated STEM / STEAM, Modeling, Sustainability.

Purpose of The Study

Authors from disciplines as diverse as biology, chemistry, engineering, economy, and mathematics have proposed that current global, complex, politically charged, socio-scientific issues (universal income, evolution, pandemic and vaccines, climate change, etc.) require STEM professionals to understand them as complex systems (Ghosh, 2017; Orgill et al., 2019; Renert, 2011; Richmond, 1997; Roychoudhury et al., 2017; Schuler et al., 2018). This holistic perspective, known as systems thinking, focuses on understanding phenomena in terms of relationships, connectedness, and context. Systems thinking complements the analytic or reductionist perspective commonly used in STEM and STEM education fields. An important distinction between these perspectives involves causality; while the reductionist perspective focuses on linear, direct cause-and-effect relationships, systems thinking involves identifying complex causality relationships. An important type of these relationships are feedback loops, or a "succession of cause-effect relations that start and end with the same variable. It constitutes a circular causality, only meaningful dynamically, over time" (Barlas, 2002, p. 1147).

Mathematics represents a powerful way to make sense of relationships, which can be understood as two (or more) quantities changing together over time. Feedback loops, therefore, can be seen as two (or more) quantities changing simultaneously in a way such that, the first quantity *causes* the second quantity to change, and that change *causes* the first quantity to change again, and so on. In particular, I believe that combining covariational reasoning with the notion of causality can provide insights into how students can understand feedback loop relationships in mathematics.

In this paper, I combine covariational reasoning and causality to introduce two theoretical constructs, *open-loop covariation* and *closed-loop covariation*, which can characterize the way that three preservice teachers conceptualize a feedback loop relationship in a mathematical task related to climate change. The constructs also have the potential to address a gap in the literature focusing on covariational reasoning which have a tendency to consider only unidirectional implications of change and leave underexplored the role of real-world causality. I also discuss possible implications for mathematics learning and teaching.

Conceptual Framework

Covariational Reasoning and the Multiplicative Object

Covariational reasoning finds one of its earliest definitions in the work of Saldanha and Thompson (1998), who defined it as follows:

> Someone holding in mind a sustained image of two quantities' values (magnitudes) simultaneously. It entails coupling the two quantities, so that, in one's understanding, a multiplicative object is formed of the two. As a multiplicative object, one tracks either quantity's value with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value … An operative image of covariation is one in which a person imagines both quantities having been tracked for some duration, with the entailing correspondence being an emergent property of the image. (pp. 298-299)

The *multiplicative object* in their definition is analogous to the logical conjunction "and" that joins or units two propositions to produce one proposition that is true if and only if both of the constituent propositions are true. In the case of covariation, the multiplicative object joins the corresponding values of two covarying quantities so that the student "mentally unites their attributes to make a new attribute that is, simultaneously, one and the other" (Thompson et al., 2017, p. 96). This multiplicative object supports the student's ability to conceptualize two (or more) quantities changing simultaneously and interdependently.

Open- and Closed-Loop Covariation

Covariational reasoning support the conceptualization of two quantities changing simultaneously. However, simultaneity may not be enough to conceptualize feedback loop structures in a system. I propose that the way causality is conceived may also play an important role in conceptualizing a feedback loop relationship between two covarying quantities. To address this distinction, I introduced two constructs: *open-loop covariation* (OLC) and *closed-loop covariation* (CLC).

Let's consider the filling bottle problem where water is being pour into the bottle at a constant rate, increasing the volume of water, V, and the water height, h, in the bottle. The multiplicative object allows one to visualize V and h changing simultaneously over time so that there exist a pair ($V(t)$, $h(t)$) for any t in some interval of conceptual time. One imagines that if V changes, so does h, and if h changes, so does V; they change together. The multiplicative object, thus, support a student's ability to conceptualize simultaneous change between two quantities.

However, the multiplicative object does not provide an answer to the questions of *how* and *why* a change in V results in a change in h, or vice versa. In this paper, I use the term causality as a way of describing how and why the state of a dynamic process changes as time goes on (Sauer, 2010). The volume of water V, or the space taken by the water in the bottle, is growing since more and more water is entering the bottle. One also observes (or imagines) that the water height, h, is increasing as water enters the bottle; that is, h grows as V grows. One could say that the growing volume of water in the bottle causes the water height to increase; the more water enters the bottle, the more the water height increases. This is a description of how and why a change in V causes a change in h. It is important to point out that thinking of causality in the opposite direction may not be as natural or intuitive; mathematically, changing the value of *h* would change the value of *V*, but it is hard to imagine that, in a real-world context, an increase in *h* would cause more water to enter the bottle. I use causality in that second "real-world" sense and consider the covariation of *h* and *V* as OLC, which shows *simultaneity* (*V* and *h* change together) and only *linear causality* (a change in *V* causes a change in *h*, and may not be as intuitive to say that a change in *h* causes a change in *V*).

The conceptualization of a feedback loop structure between covarying quantities is only possible when there exists circular causality between those quantities. For instance, consider a simple predator-prey model relationship in which *P* is the number of predators in a region and *N* is the number of preys in the same region. We assume that predation (magnitude of *P*) is the only factor affecting *N* and that prey availability (magnitude of *N*) is the only factor affecting *P*. With those assumptions, *P* increases when *N* increases because an increase in availability of prey produces prosperity for predators, who can reproduce more. However, when *P* increases passed certain value, *N* starts to decrease because an increase in predator means that more preys would die. When *N* decreases passed certain threshold value, *P* would start to decrease as well because predators do not have enough food to sustain their population. As *P* decreases passed certain value, *N* becomes to increase again because there are less predators and less preys get eaten. This is an example of CLC, which shows *simultaneity* (*P* and *N* change together) and *circular causality* (changes in *N* cause changes in *P*, which in turn cause changes in *N* again and so on). Conceptualizing circular causality, as illustrated by the predators-prey model, seems to be more cognitively demanding than conceptualizing linear causality (Ghosh, 2017; Hokayen et al., 2015; Roberts, 1978; Wellmanns & Schmiemann, 2022).

Methodology

This paper is part of a larger study that investigated how PSTs make sense of some elemental mathematics behind modeling climate change. Three secondary PSTs —hereafter Jodi, Pam, and Kris— enrolled in a mathematics education program at a large Southeastern university participated in the larger study. These PSTs had completed Calculus I and II and an Intro to Higher Mathematics course and were completing a Math Modeling for Teachers course by the time the larger study took place. The PSTs were asked to complete an original sequence of mathematical tasks while participating in individual, task-based interviews (Goldin, 2000). In this paper, I focus on the PSTs' responses to the Energy Balance task.

The Energy Balance (EB) Task

An *energy balance model* describes the continuous heat exchange between the sun, the planet's surface, and the atmosphere (Figure 1a). The planet's surface is warmed by a fraction of the sun's radiation (*S*). As the surface's temperature increases, it radiates heat towards the atmosphere (*R*). The majority of it (*B*) is absorbed by *greenhouse gases* (GHG), which rises the atmosphere's temperature. As it warms up, the atmosphere radiates a fraction of the absorbed heat back to the surface (*A*). The latter further increases the surface's temperature, which results in an increase of surface radiation towards the atmosphere and an increase in the atmosphere's temperature. The continuous heat exchange between the surface and the atmosphere is known as the *greenhouse effect* and has a key role in controlling the planet's mean surface temperature, *T*. The relative abundance of GHG in the atmosphere regulates the amount of heat it absorbs. Therefore, changes in the concentration of GHG are followed by changes in *T*.

The EB task (Figure 1b) describes a simplified situation that begins with an energy balance at the surface; that is, the surface absorbs heat at the same rate it releases it ($S + A = R$ in Figure

1a). Then, it is assumed that a unique and instantaneous pulse of carbon dioxide (CO_2) is released at $t = 0$ to increase its concentration in the atmosphere. The EB task focuses on CO_2 for it is one of the main drivers of global warming (IPCC, 2018). The instantaneous increase in CO_2 is followed by an instantaneous increase in A, the heat radiation from the atmosphere to the surface. Thus, at $t = 0$, the surface is absorbing heat at a higher rate than that at which it is releasing it ($S + A > R$ in Figure 1a). The surface then begins to warm up as time elapses (T increases as t increases) and to increase its heat radiation towards the atmosphere (R increases as t increases). The atmosphere also begins to warm up and to further increase its heat radiation back to the surface (A increases as t increases). This further warms the surface and further increases the surface heat radiation towards the atmosphere (T and R continue to increase as t increases). This energy feedback loop between the surface and the atmosphere allows for R to increase enough so that a new energy balance is reached ($S + A = R$). This balance is accompanied by a new (higher) value of T. Thus, the goal of the EB task was to engage PSTs into thinking about how the Earth's energy balance's response to an increase in CO_2 results in an increase in the mean surface temperature, T, thus connecting CO_2 pollution to global warming.

The current paper examines the PSTs' reasoning regarding the feedback loop between the surface and the atmosphere in terms of a covariation between R and A with respect to time. The results will mainly focus on the PSTs' responses to the second part of the EB task, in which they were asked to draw the graph of T as a function of time t (Figure 1b).

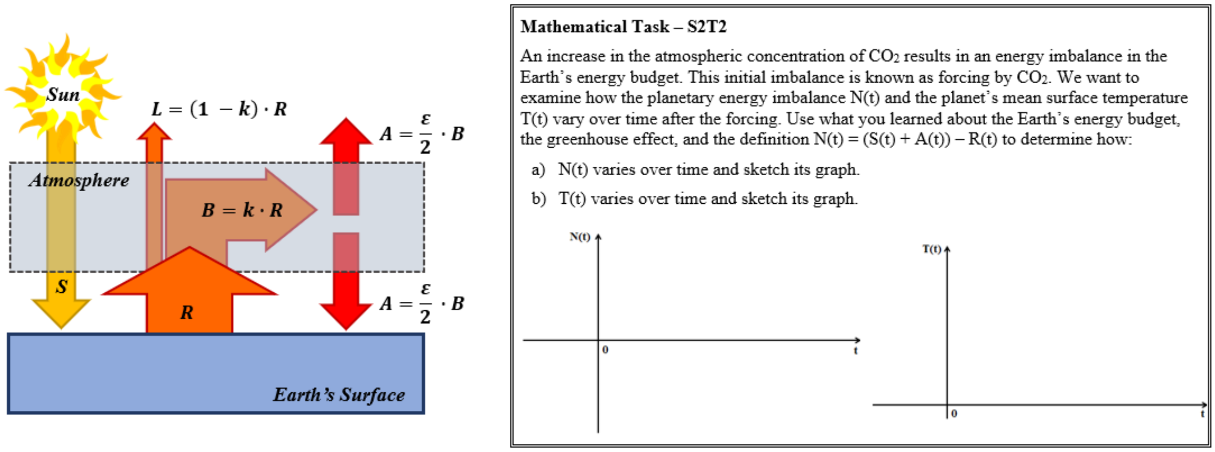


Figure 1: (a) An Earth's energy balance model (left) and (b) the EB task (right)

Data Collection

Before working on the EB task, the PSTs participated in a 32-minute-long, individual minilesson where some basic concepts related to the Earth's energy balance and the greenhouse effect were discussed. The goal was to provide PSTs with enough knowledge about those concepts so that they could start working on the EB task. The minilesson began with a 7-minute-long video retrieved from the NASA YouTube channel *NASAEarthObservatory* introducing the energy balance and the greenhouse effect. Then, each PST and I held a 5-minute-long Q&A session in which we clarified questions about the ideas discussed in the video. During the next 20 minutes, each PST and I worked with a diagram of the energy balance similar to the one in Figure 1a. We talked about what each of the quantities S, R, L, B, and A represented and how they related to one another. Then, we talked about the energy balance at the surface as the equality $S + A = R$, and I illustrated it for some specific initial values of S, A, and R. We also discussed what inequalities such as $S + A > R$ or $S + A < R$ could mean in terms of temperature.

A week after the minilesson, each PST completed the EB task during a 60-minute-long, individual, task-based interview (Goldin, 2000). The interviews were semi-structured and had an interview protocol with pre-defined questions so that all participants received similar prompts during the interviews. To help PSTs understand how the energy balance responds to the increase in CO_2, the following recursive rules were made available to them: $B_t = k \cdot R_t$, $A_t = ½ \cdot B_t$, and $R_{t+1} = S + A_t$, for some $0 \leq k \leq 1$. By using these rules, they could find the values of B, A, and R for successive values of time, thus observing how these quantities change dynamically. This modelling strategy is known as *Discrete Event Simulation* and can be used to help students understand the dynamics of systems in particular situations (Hoad & Kunc, 2018).

Data Analysis

The interview videos and transcripts were analyzed through *thematic analysis* (Braun & Clarke, 2006; 2012). This qualitatively method of data analysis allows for systematically identifying, organizing, and offering insight into patterns of meaning across a data set though the development of codes and themes. Thematic analysis is a widely used analytic strategy to identify and make sense of collective or shared meanings and experiences. The method can be summarized into six phases: familiarizing yourself with your data, generating initial codes, searching for themes, reviewing themes, defining and naming, and producing the report.

I watched all interview videos and took notes while doing so. The videos were separated into shorter, more manageable *episodes*, each one covering a single topic or showing evidence of a particular way of reason covariationally. The notes informed my first round of coding for the interview transcripts. Then, the episode transcripts were sorted according to similar codes to look for patterns in participants' responses. This allowed me to revise and refine the initial codes, reducing them to two main themes. Thus, the five initial codes "asynchronous", "synchronous", "feedforward but not feedback", "verbalizing/indicating circularity", and "circular relationship" were collapsed into two themes: "Simultaneous Change", with "asynchronous" being the absence thereof and "synchronous" being the presence thereof, and "Circular causality", with "feedforward but not feedback" representing the absence thereof and "verbalizing/indicating circularity" and "circular relationship" representing the presence thereof.

Using the analytic framework previously described (Table 1), I indexed all episode transcripts into three analytic matrices, one per participant. These matrices allowed me to look for patterns in the distribution of themes, which provided the information needed to meet the research goals.

Results

The analysis revealed that, for this group of preservice teachers, two main cognitive realizations appear important to conceptualize the energy feedback loop between the surface and the atmosphere (the greenhouse effect) as a CLC between the quantities R and A: conceiving *simultaneity* of change and a *circular causality* relationship between those quantities. When one of these realizations was not supported, the PSTs developed inaccurate conceptualizations of the greenhouse effect and, by extension, this had an impact on their understanding of the link between CO_2 pollution and global warming.

Table 1: Analytic Framework

Theme	Descriptions
Simultaneity	*Asynchronous* The PST describes or represents changes in A and R as occurring asynchronously (A changes first, then R changes, then A again, and so on). *Synchronous* The PST describes or represents changes in A and R as occurring simultaneously as time elapses.
Causality	*Linear* The PST describes or represents causality in one direction between A and R. Either change in A causes change in R or change in R causes change in A. *Circular* The PST describes or represents a circular causality between A and R so that change in A causes change in R, which in turn causes change in A again.

The PSTs explored the greenhouse effect as an energy feedback loop while working on their graphs of $T = g(t)$. I start the discussion with Kris' work because she conceptualized the greenhouse effect in terms of CLC. More specifically, Kris made remarks about R and A increasing *simultaneously* as time, t, increased: "So, as R increases, A increases … the new R is affected by S plus A [*points at S and A*]. So, when A increases, R is going to be [bigger]. It can't just keep increasing". Kris also made remarks suggesting she conceived of a *circular causality* relationship between the quantities R and A.

> Well, [the surface] keeps in taking. I think it is warming up because once we added more CO_2, that is less of the emitted energy that is getting just like shut out passed the atmosphere, leaked from it. So then, more of it is going to be absorbed by the atmosphere … Whatever is absorbed by the atmosphere [*points at B*] is going to be absorbed back into the [*points at the surface*], well half of that plus the sun's energy [*points at S*] is going to be absorbed by the Earth, which is going to keep increasing, as we saw with like the 400 [*points at the R-value of "400"*]. Then, from the A value [*with a capped marker, traces the top half of a circle, going from R to A*], just with the A, [the surface] absorbs 160, and then we add a new R-value [*with the capped marker, traces the bottom half of the circle, going from A to R*], whatever that was, and then [the surface] absorbs 164 [*with the capped marker, re-traces the top half of the circle, going from R to A*]. So, I think it is going to keep increasing [*draws an increasing, concave-downward graph for T = g(t) that appears to have a horizontal asymptote that she labels as "new equilibrium temperature"*].

In the above excerpt, Kris repeatedly referred to the relationship between R and A as a "cycle", which suggests an awareness of circular causality between the quantities. Also, notice how she gestured both, the feedforward relationship from A to R and the feedback relationship from R back to A, further suggesting circular causality. Kris also drew an accurate graph of $T = g(t)$ showing asymptotic growth towards a new equilibrium value (Figure 2a); this suggests an awareness of the balancing quality of the feedback loop. Kris's CLC coincided with her demonstrating an accurate conception of the greenhouse effect, which guided her to conclude that an increase in CO_2 causes a warming effect over the planet's surface, correctly relating CO_2 pollution to global warming.

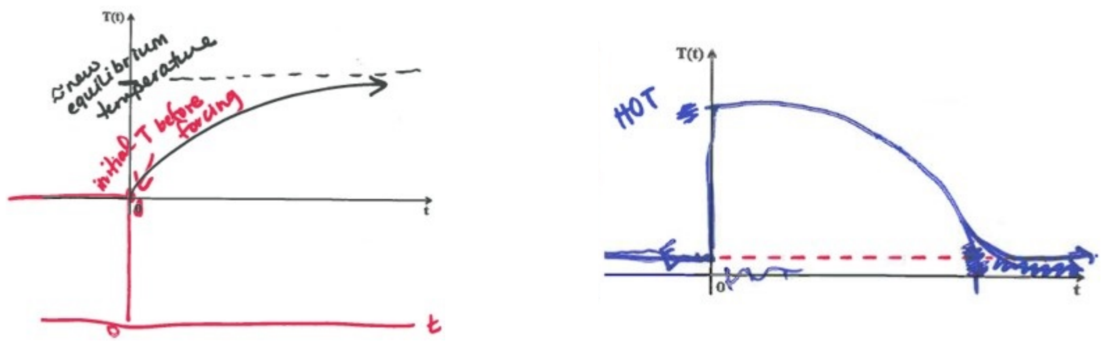


Figure 2: (a) Kris's graph of $T = g(t)$ (left) and (b) Pam's graph of $T = g(t)$ (right)

The case of Pam illustrates the conceptualization of OLC, where simultaneous change is conceived but not circular causality. She described *R* and *A* as two quantities increasing in tandem as *t* increased but only in the *A*-to-*R* direction ("*R* increases <u>*as*</u> [emphasis added] *A* increases. So, <u>*as*</u> [emphasis added] our *A* increases, *R* increases"). Pam also interpreted the increase in *R* and *A* as the planet's surface emitting more heat than the heat it was absorbing, which suggested additional evidence of only conceiving causality from *A* to *R*.

> A lot [of heat] is going in, but more is coming out, like *R* increases as *A* increases. So, as our *A* increases, *R* increases. But our *S* is staying the same. But our *A* is always less than *R*. So, more [heat] is coming out [*pauses to think*]. So, the Earth is trying to cool itself off, so the temperature is decreasing from here to here.

Pam again implied that an increase in *A* causes an increase in *R*, but did not appear aware that the increase in *R* also causes a new increase in *A*. She referred to *R* as heat leaving the surface and being larger than the heat absorbed by it. This claim overlooked that *A* is actually a fraction of *R* that is reabsorbed by the surface. This might have kept Pam from conceptualizing the feedback relationship from *R* back to *A*. Pam's OLC coincided with her demonstrating an inaccurate conception of the greenhouse effect, which may have led her to incorrectly conclude that the planet's surface cools down after an increase in CO_2, as her graph shows (Figure 2b).

Finally, Jodi's reasoning did not support simultaneous change but supported circular causality between *R* and *A*. In the following excerpt, Jodi appeared to imagine *R* and *A* as changing *asynchronously*—*A* changes first, then *R* changes, and then *A* again, and so on—but seemed aware of the circular causality between those two quantities.

> I am trying to look at the differences. So, like here [*points at the R-value "390" and the A-value "150"*]. Ok, so here the change was five [*points at the R-value "395" and the A-value "155"*]. The change was two [*points at the R-value "397" and the A-value "157"*]. Is it, I mean, is it not changing? … The flow of energy increased by five [*points at the A-value "155"*], but then it decreased by five [*points at the R-value "395"*]. Then it increased by two [*points at the A-value "157"*], and then it decreased by two [*points at the R-value "397"*]. So, it is almost as if there was no change in temperature because I associate like energy as kind of having a relationship with temperature. So, if the energy increases, then the temperature increases. But, in this scenario, an equal change in energy [*points at A*] was an equal change in output [*with her index finger, traces the bottom half of a circle from A to R*] … Ok, cycle started here [*points at B*], and here the Earth's temperature would've been

> something. Equal input of energy, equal output of energy [*with her index finger, traces a circle connecting A, R and B*]. Ok. So, when the cycle started, there was an input of energy [*points at A*], and then it got released [*with her index finger, traces the bottom half of a circle from A to R*]. Another cycle starts [*points at B*], input of energy, release of energy [*with her index finger, traces a circle connecting B, A, and R*]. So, it would almost be like [*draws a periodic curve formed by identical arcs*].

Notice how Jodi described the changes in *R* and *A* as happening at different times rather than simultaneously. This suggests Jodi did not develop a multiplicative object and her reasoning may not be considered covariational. In contrast, she did allude to circular causality by tracing circles with her finger connecting *R* and *A*. I also interpreted the periodicity of her graphs (Figure 3) as additional evidence of awareness of circular causality. Jodi's way of reasoning coincided with her demonstrating an inaccurate conception of the greenhouse effect and arriving to an incorrect conclusion regarding the real impact of CO_2 pollution on the planet's surface temperature.

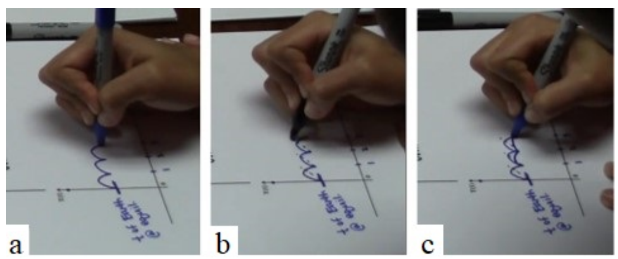


Figure 3: Jodi's periodic graphs of *T* = *g*(*t*)

Conclusion

Some situations that involve two quantities changing together over time also require the recognition of an underlying feedback loop structure between them. The study's results suggest that, for this group of preservice teachers, reasoning about feedback loop between quantities involved the ability to conceive *closed-loop covariation*, which in this study was characterized by two cognitive realizations: (i) the conception of *simultaneous change* and (ii) the recognition of *circular causality*. The first realization is based on the mental construction of a multiplicative object between those quantities (Saldanha & Thompson, 1998; Thompson et al., 2017), while the second realization involves noticing that changes in a quantity cause change in the second quantity, which in turn cause new changes in the first quantity and so on. The results also suggests that these two realizations, at least for these preservice teachers, appeared to be independent from each other. Kris demonstrated both realizations, while Pam and Jodi demonstrated one or the other but not both. An implication of this is that instructional strategies aiming to support students' ability to understand feedback loops mathematically should focus on developing both realizations at the same time.

It also important to point out that many school mathematical tasks involving change between quantities may only require *open-loop covariation*, where students oftentimes only need to attend to linear causality between the quantities. Examples of these are situations modeled by linear or quadratic functions. However, closed-loop covariation may be an interesting and novel way to explore situations involving exponential growth (e.g., compound interest or population growth), where the current value of the dependent quantity plays a role on how that quantity changes.

References

Barlas, Y. (2002). System dynamics: systemic feedback modeling for policy analysis. *Encyclopedia for life support systems*. UNESCO Publishing.

Braun, V., & Clarke, V. (2012). Thematic analysis. In H. Cooper, P. M. Camic, D. L. Long, A. T. Panter, D. Rindskopf, & K. J. Sher (Eds.), *APA handbook of research methods in psychology, Vol. 2. Research designs: Quantitative, qualitative, neuropsychological, and biological* (pp. 57–71). American Psychological Association. https://doi.org/10.1037/13620-004

Braun, V., & Clarke, V. (2006) Using thematic analysis in psychology. *Qualitative Research in Psychology, 3*(2), 77-101. https://doi.org/10.1191/1478088706qp063oa

Ghosh, A. (2017). *Dynamic systems for everyone*. Switzerland: Springer International Publishing. https://doi.org/10.1007/978-3-319-10735-6

Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A. Kelly & R. A. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 517–545). Lawrence Erlbaum Associates. https://doi.org/10.4324/9781410602725

Hoad, K., & Kunc, M. (2018). Teaching system dynamics and discrete event simulation together: A case study. *Journal of the Operational Research Society, 69*(4), 517-527. doi:https://doi.org/10.1057/s41274-017-0234-3

Hokayem, H., Ma, J., & Jin, H. (2015). A learning progression for feedback loop reasoning at lower elementary level. *Journal of Biological Education, 49*(3), 246-260. http://dx.doi.org/10.1080/00219266.2014.943789

Intergovernmental Panel on Climate Change (IPCC). (2018). *Global Warming of 1.5°C. An IPCC Special Report on the impacts of global warming of 1.5°C above pre-industrial levels and related global greenhouse gas emission pathways, in the context of strengthening the global response to the threat of climate change, sustainable development, and efforts to eradicate poverty.* Cambridge University Press. https://doi.org/10.1017/9781009157940

Orgill, M., York, S., & MacKellar, J. (2019). Introduction to systems thinking for the chemistry education community. *Journal of Chemical Education, 96*, 2720-2729. https://doi.org/10.1021/acs.jchemed.9b00169

Renert, M. (2011). Mathematics for life: Sustainable mathematics education. *For the Learning of Mathematics, 31*(1), 20-25. https://www.jstor.org/stable/41319547

Roberts, N. (1978). Teaching dynamic feedback systems thinking: An elementary view. *Management Science, 24*(8), 836-843. http://www.jstor.org/stable/2630381

Roychoudhury, A., Shepardson, D. P., Hirsch, A., Niyogi, D., Mehta, J., & Top, S. (2017). The need to introduce systems thinking in teaching climate change. *Science Educator, 25*(2), 73-81.

Richmond, B. (1997). The “thinking” in systems thinking: How can we make it easier to master? *The Systems Thinker, 8*(2), 1-5.

Saldanha, L. A., & Thompson, P. W. (1998). *Re-thinking covariation from a quantitative perspective: Simultaneous continuous variation*. Paper presented at the 20th annual meeting North American Chapter of the International Group for the Psychology of Mathematics Education, Raleigh, NC, United States.

Sauer, N. (2010) Causality and causation: What we learn from mathematical dynamic systems theory. *Transactions of the Royal Society of South Africa, 65*(1), 65-68. https://doi.org/10.1080/00359191003680091

Schuler, S., Fanta, D., Rosenkraenzer, F., & Riess, W. (2018). Systems thinking within the scope of education for sustainable development (ESD) – a heuristic competence model as a basis for (science) teacher education. *Journal of Geography in Higher Education, 42*(2), 192-204. https://doi.org/10.1080/03098265.2017.1339264

Thompson, P. W., Hatfield, N. J., Yoon, H., Joshua, S., & Byerley, C. (2017). Covariational reasoning among U.S. and South Korean secondary mathematics teachers. *Journal of Mathematical Behavior, 48*, 95-111. http://dx.doi.org/10.1016/j.jmathb.2017.08.001

Wellmanns, A. & Schmiemann, P. (2022) Feedback loop reasoning in physiological contexts. *Journal of Biological Education, 56*(4), 465-485. https://doi.org/10.1080/00219266.2020.1858929

BRIDGING SITUATIONAL AND GRAPHICAL REASONING TO SUPPORT EMERGENT GRAPHICAL SHAPE THINKING

Allison L. Gantt
University of Delaware
agantt@udel.edu

Teo Paoletti
University of Delaware
teop@udel.edu

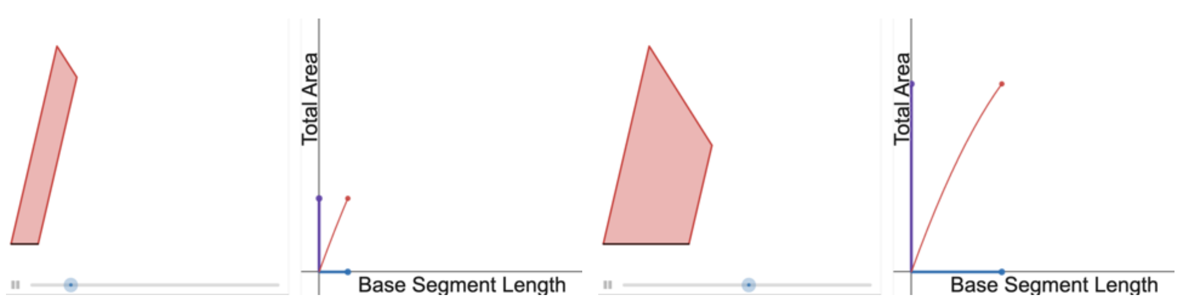
Srujana V. Acharya
University of Delaware
surjana@udel.edu

Claudine Margolis
University of Michigan
czmars@umich.edu

Emergent graphical shape thinking (EGST) entails conceiving a graph as being dynamically generated via the trace of a moving point constrained by two changing quantities. As such, Paoletti et al. (2023) argue that meanings for quantities within a situation and meanings for graphical representations must be connected, or bridged, to engage in EGST. In this report, we explore this bridging process through a case study investigating how two students made connections that bridge their situational and graphical meanings during their work on a mathematical task. We found that the pair's connections between situational and graphical meanings emerged most prominently only after recursive engagement with reasoning in both contexts. We discuss the implications of these findings for researchers and practitioners seeking to support students as they develop EGST.

Keywords: Algebra and Algebraic Thinking, Middle School Education, Learning Trajectories and Progressions; Reasoning

Students' graphical reasoning plays an important role in their learning across STEM fields and their participation as critical citizens (e.g., Glazer, 2011; Potgieter et al., 2008). In particular, emergent graphical shape thinking (EGST) can be useful, as it entails conceiving a graph as being dynamically generated via the trace of a moving point constrained by two changing quantities (Moore, 2021; Moore & Thompson, 2015). For instance, Figure 1 shows how a student can represent a conceived relationship between the dynamic quantities of the base segment length and area of a growing shape via a graph that is produced from the movement of a dynamic point. Paoletti et al. (2020) noted that EGST is important to interpret many graphs across STEM textbooks and practitioner journals in ways consistent with the authors' intentions. However, such reasoning is non-trivial. For example, less than 30% of the 121 U.S. teachers in Thompson et al.'s (2017) study provided evidence suggestive of EGST on a task that could potentially elicit such reasoning. Hence, there is a continued need to explore ways to support students' in developing EGST.



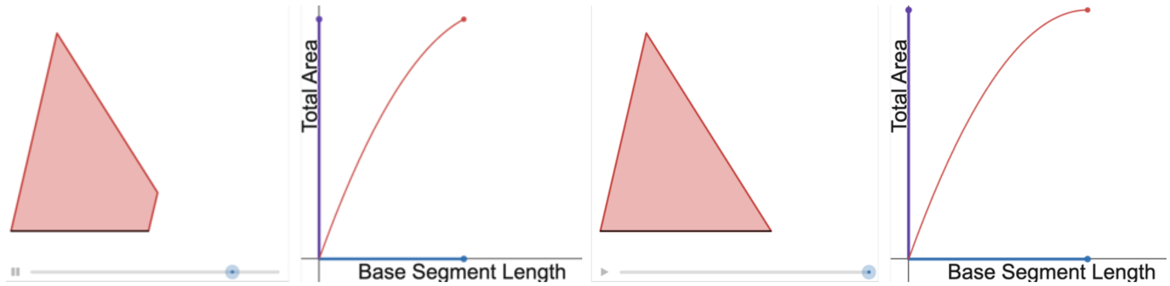


Figure 1: Four screenshots showing a point as a dynamic trace of the relationship between two situational quantities.

In order to engage meaningfully in EGST, Paoletti et al. (2023) argued that learners need to engage both in reasoning specific to the situation (real-world or abstract) and in reasoning about objects in graphical representations. However, the process by which students bridge their thinking in situational and graphical contexts to support EGST is still relatively unexplored.

In this paper, we present a case study (Yin, 2018) to explore learners' situational and graphical bridging process. We feature the activity of a pair of sixth-grade students as they engaged in a teaching experiment (Steffe & Thompson, 2000) to investigate how we might support students to engage in such bridging. To provide context for our case study, we first elaborate on the literature showing the relevance of situational and graphical representations to the development of EGST. Then, we present our methods and results, highlighting how students' iterative engagement with situational and graphical connections corresponded to greater specificity in both domains. Finally, we share implications for researchers and educators who wish to support their students in EGST.

Situations and Graphs in Emergent Graphical Shape Thinking

This case study builds upon the local instruction theory presented in Paoletti et al. (2023). The local instruction theory posited that students must engage in quantitative and covariational reasoning both with respect to situational quantities and with objects represented graphically prior to engaging in EGST. Reasoning quantitatively, both situationally and graphically, entails an individual constructing quantities to interpret their experiential worlds (Smith & Thompson, 2008; Steffe, 1991; von Glasersfeld, 1995). Covariational reasoning entails a learner mentally coordinating two varying quantities (Thompson & Carlson, 2017). Covariational reasoning often follows a developmental progression (Paoletti et al., 2023; Carlson et al., 2002) in which a student coordinates two quantities by thinking "of one, then the other, then the first, then the second, and so on" (Saldanha & Thompson, 1998, p. 299) until they have constructed a relationship that entails both quantities simultaneously being tracked for some duration. Researchers (Saldanha & Thompson, 1998; Thompson et al., 2017) refer to such a conception as a multiplicative object (i.e., a Cartesian product).

Although we often use mathematical representations (e.g., graphs, equations) to represent relationships between covarying quantities, covariational reasoning does not require such representations (e.g., Paoletti & Moore, 2018; Castillo-Garsow et al., 2013; Johnson, 2015). Learners often construct and coordinate covarying quantities to develop meanings for situational quantities and relationships between quantities. These *meanings for a situation* (M.S) can serve as the foundation for their mathematical activity (e.g., constructing a graph representing a relationship) or can result from their interpreting mathematical representations (e.g., developing a novel meaning for a situational relationship as they interpret a graph).

As a pre-requisite to engaging in EGST, learners must also engage in quantitative and covariational reasoning with respect to objects in a graphical representation (Moore, 2021; Paoletti et al., 2023). Paoletti et al. (2023) described a particular sequence of three *meanings in graphical representations* (M.R) that can support students' EGST. They described a learner must first consider how a segment length can represent a quantity's magnitude (M.R.1). Next, in a Cartesian coordinate system, a learner can consider changes in two orthogonal segment lengths in relation to two covarying quantities (M.R.2). Then, a learner can conceive of or anticipate a point in the coordinate system as a multiplicative object simultaneously representing the two segments' magnitudes (M.R.3).

Reflected in the above descriptions, to engage in EGST, learners must ultimately coordinate their situational (M.S) and graphical meanings (M.R) to conceive of a graph as being generated by the dynamic trace of a point. Paoletti et al. (2023) contended that there is likely a dialectal relationship between students' situational and graphing meanings, stating, "our LIT does not outline a single, linear, or developmental progression; rather, we call for repeated and connected occasions for students to engage in M.S, M.R, and [emergent reasoning] as they develop stable graphing meanings that entail EGST" (p. 205). However, the way in which students construct these connections has not been explored in detail.

Methods

We applied case study methodology (Yin, 2018) to respond to the following research question: *How might students make connections between situational quantitative and covariational reasoning (M.S) and graphical representations of covarying quantities (M.R) as they build toward EGST?* We drew our case from data collected during a teaching experiment (Steffe & Thompson, 2000) with one pair of students. We engaged the pair in tasks we intended to support their EGST. In this report, we present results from the students' activity during one session where we identified that students were engaged in reasoning to relate situational (M.S) and graphical (M.R) meanings in explicit ways.

Participants, Context, and Task

We conducted our teaching experiment with sixth grade students in a public charter school in the Northeastern United States. The first author facilitated the teaching experiment as the teacher-researcher (TR). In our selected case, our participants were Sebastian (age 11; self-identified as male and Black, Puerto Rican, and Latino) and Tom (age 11; self-identified as male and White). Both participant names are pseudonyms selected with student input.

In our case for this report, the pair collaborated on a task we designed in Desmos called *The Big Event*. We presented students with a dynamic scenario in which two teachers (Mr. K and Mrs. B) are walking away from a podium and post respectively to create a growing area where hypothetical students could stand for a presentation (the shape grew as shown in Figure 1). Throughout the task, we asked Sebastian and Tom to identify and coordinate Mr. K's distance from the podium (i.e., the horizontal base segment length) and the area of the shape formed (M.S). In this scenario, as Mr. K's distance increases, the total area of the shape increases and the amounts of change of area (hereafter AoC, see Carlson et al., 2002) decrease. Students had the option to toggle between views of the scenario changing smoothly (e.g., Figure 1) and changing in chunks (as in the triangular image in Figure 2).

Our report focuses on Sebastian and Tom's activity with respect to two prompts in the *Big Event Task* that we designed to transition students from reasoning situationally (M.S) to reasoning graphically (M.R). Prompt 1 presented students with three dynamic line segments (blue, orange, and purple) alongside the growing shape (Figure 2). Whereas the blue line segment grew by equal amounts for each jump of change in the shape, the orange segment grew by increasing amounts and the purple segment by decreasing amounts. Similar to the *Which One Task* (Liang & Moore, 2021), the text asked them to select a segment that could represent Mr. K's distance from the podium and the shape's area respectively (M.S & M.R.1). After selecting segments, Prompt 2 presented students with a blank coordinate plane and axes labeled with each focal quantity. The text requested that students consider plotting points along each axis (M.R.2) and then asked students to plot points in the plane (M.R.3).

Prior to engaging with this task, Sebastian and Tom had constructed and interpreted graphs during the teaching experiment. Specifically, they had completed the *Faucet Task*, wherein they coordinated the directional changes between the amount of water and temperature of the water depicted in a dynamic faucet applet (see also Paoletti, 2019; Paoletti & Vishnubhotla, 2022). They also graphed another scenario in the *Big Event Task* that showed a triangle growing by more each time.

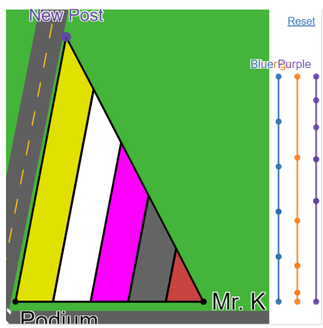


Figure 2: Screenshot from the *Big Event Task* showing the chunkily growing scenario alongside the segments (blue, orange, and purple) in Prompt 1.

Data Collection and Analysis

We audio- and video-recorded Tom and Sebastian's activity with a camera and screen capture tool. We conducted a conceptual analysis (Thompson, 2008) as we analyzed the data to build models of the students' meanings that viably explained their words and actions. With this goal in mind, after each session we constructed an event map (Green & Bridges, 2018) to organize the key events of each session. This process supported us to identify and bound our case. We transcribed the activity (i.e., dialogue, written activity, and gestures) from the segment we identified as informing our case analysis. Next, we analyzed the transcript at the utterance level, highlighting moments in which 1) the TR or text of the task prompted or 2) the pair engaged in building connections between M.S and M.R. We leveraged the patterns in our analysis to organize an account of activity that could address our research question.

Results

To explore how Tom and Sebastian might bridge their situational quantitative and covariational reasoning (M.S) with their developing graphical representations of covarying quantities (M.R), we presented prompts that we intended to provide consistent and progressively specific connections between these meanings. In this case, the students first proceeded through the task as designed (which we intended to support a shift from M.S. to M.R), which only supported limited connections between the situation and graph (Pass 1). The TR subsequently prompted Sebastian and Tom to consider more explicit connections between the situational quantities and graphical representations at each stage of their activity. Such prompts supported the students to construct a graph with increasing precision and justification (Pass 2). We illustrate how students' recursive and specific connections between situations (M.S) and graphs (M.R) across Pass 1 and Pass 2 prepared them for the transition to EGST.

Pass 1: Addressing the Task with a Linear Progression from M.S to M.R

As Tom and Sebastian engaged in their first pass through the task, they read Prompt 1 to select a segment corresponding to the situational quantities of Mr. K's total distance from the podium and the total area of the shape (i.e., to bridge M.S. with M.R.1). Sebastian identified the blue segment as representing Mr. K's total distance, gesturing between the "same jumps" he observed in the situational context (M.S) and in the blue segment. Subsequently, Tom added that the area of the triangle would match the purple segment "because the biggest jump [*gestures to Sit-A in Figure 3a*] is the first one [*gestures to Seg-A in Figure 3a*] and then it gets smaller every time." We note how Tom engaged in gestures indicative of connection between the changes in the quantity of area (M.S) and the changes in the partitions of his segment choice (M.R.1).

Next, the students advanced to Prompt 2, requesting a graph. The TR added an explicit request for students to consider how the segments on the previous slide (M.R.1) could support the construction of the graph itself (M.R.2 & M.R.3) and re-read the labels of the quantities on the axes (M.S). However, reflecting the non-trivial nature of bridging meanings across situations and graphs necessary to engage in EGST, neither Tom nor Sebastian explicitly referenced both situational quantities as they presented their strategies to graph points. For example, Sebastian described that points on the graph would begin in the top right area of the plane and proceed toward the origin to "go down smaller and smaller." Sebastian explained that this was because he expected the graph to move in the opposite direction as the increasing and concave-up graph they had constructed from the growing triangle portion of the *Big Event Task*. We note that Sebastian did not reference either of the situational quantities (Mr. K's total distance or the area) to support his conjectures.

Tom disagreed with Sebastian's strategy. Although Tom verbally described the situational quantities of Mr. K's total distance and area of the triangle, his gestures did not correspond to the quantity of area. His graphing activity focused on marking points in the situation in his constructed graph as opposed to representing the intended quantities (M.R.2 & M.R.3). First, engaging with the animation of the triangle, Tom explained:

> I would put…[Mr. K's] dot right there [*points as in (1) in Figure 3b*] and then for the area roped off, the dot right there [*points as in (2) in Figure 3b*] so I would, like, combine those [*gestures as in (3), reaching location (4) in Figure 3b*] and put it there.

To support his description, Tom plotted a first point on the graph, followed by a second point at the TR's request (Figure 3c). We note that the location of these points strongly mirrors the locations Tom identified in the situation (Figure 3b). As such, we conjecture Tom was reproducing an image (i.e., iconic graphical reasoning; see also Clement, 1989; Johnson et al., 2020) rather than engaging with explicit quantities (M.S). When the TR prompted more specifically about the match between the situational quantities of both distance and total area and the graph, Tom further acknowledged a disconnect: “Yeah, it’s not the same because it [*gestures to triangle from left to right*] increases over time.” Although we could interpret “it” in this statement to area, such an attribution is still unclear. However, the students explicitly acknowledging a disconnect provided an opportunity for them to further develop explicit connections between the situational quantities (M.S) and the graphical representation (M.R).

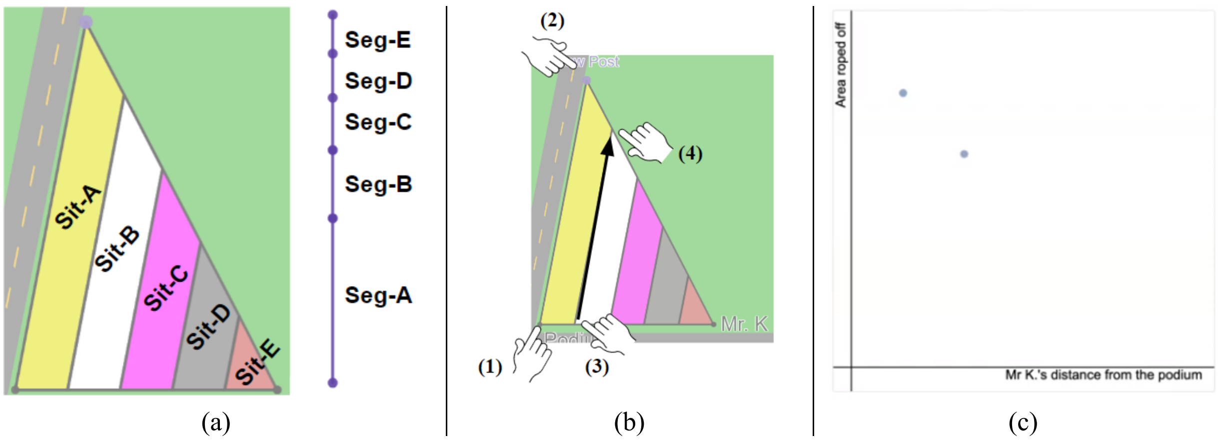


Figure 3: (a) A reference image for students’ gestures to the situation and purple segment and (b) a recreation of Tom’s gestures prior to (c) plotting initial points in the graph.

Pass 2: Addressing the TR’s Iterative Prompting to Engage M.R and M.S

The TR conjectured the students were not explicitly attending to the situational quantities (M.S). This led the TR to provide additional prompts to support increased specificity with respect to situational quantities before returning to the graph to support Sebastian and Tom developing meanings for EGST. Hence, the TR returned to Prompt 1 with the segments. On this pass, the TR’s prompts extended beyond asking students to identify a segment to match the total area of the triangle to a request to analyze how and why the segment matched in specific ways.

Viewing the screen in Figure 2, the TR focused first on a scenario in which Mr. K had only made three equal-sized jumps in his distance from the podium. The TR asked, “What would that purple segment look like?” Sebastian first described that “on the purple segment, it’s like, big [*puts hand in a “C”*] to little [*moves “C” upward while framing a smaller space*], to little [*repeats*].” The TR subsequently prompted for Sebastian to explain the parts of both the segment and the diagram he was “paying attention to.” Sebastian explained:

> Like the yellow right here [*gestures Sit-A in Figure 3a*] which is right here [*gestures from bottom to top to bottom of Seg-A in Figure 3a*] and the white [*gestures to Sit-B in Figure 3a*], right here [*points to Seg-B in Figure 3a*] and then the pink [*gestures to Sit-C in Figure 3a*], would be right here [*points to Seg-C in Figure 3a*].

We note the increasing specificity of Sebastian’s gestures between corresponding quantities in the situation (M.S) and portions of the segment (M.R.1), particularly related to distinguishing AoC of area. The TR followed Sebastian’s explanation by asking, “[If] I want to look at the total area…roped off at that point, what would we be looking at?” Tom repeated Sebastian’s connections between AoC of area and the segment components, and then described, “We’re looking at a line where it stopped…so it’d stop like [*points to the highest point of Seg-C in Figure 3a*] right there.” We highlight the explicitness of Tom’s connections between the area and the segment representation (M.S and M.R.1) in terms of both AoC and total amounts.

The TR then asked the pair to consider the AoC and total area in the situation and segment as Mr. K transitioned to his fourth jump in the diagram: "Is my [purple] line segment going to be bigger or smaller?" Tom began by gesturing across the triangle diagram from left to right: "So the triangle's going to get less but the total area is not… they are all combining." Sebastian further described that the activity felt like "reverse psychology," explaining:

> We start off with big jumps [*gestures over Sit-A in Figure 3a*] and started to decrease with smaller jumps [*moves across triangle to Sit-E in Figure 3a*] but the area starts to get bigger as those jumps [*waves three times from left to right with hand*] go on.

We take Sebastian's detailed explanation as evidence of the productive shifts in the relationship he conceived between situational quantities (M.S) and corresponding segments (M.R.1). Given this specific evidence, the TR advanced again to the graphing slide. To set up the task, the TR reminded the students of prior activity graphing with dynamic, orthogonal segments in the *Faucet Task* (M.R.2) and repeated their conclusions from the activity on the previous slide (M.R.1). Tom constructed the initial graph for the pair:

> So, I think, each jump from Mr. K [*moves over 3 times as in (1) in Figure 4a*] is going to be the exact same, but first I am gonna start out with [*moves mouse to (2) in Figure 4a*] a big jump [*moves mouse up as in (3), plotting point*] like that, and then we're going to go here [*moves mouse to (4) in Figure 4a*] and then it's a smaller jump [*moves mouse up as in (5), plotting point*].

Tom continued to plot an additional three points in this way (see Figure 4b), explaining the jumps in area were "getting smaller every time we do it." Sebastian further interpreted Tom's graph: "As Mr. K's distance is increasing and the jumps are getting smaller, [the total area]'s still getting bigger [*places hands orthogonally*], no matter how far Mr. K walks [*motions horizontal hand from left to right, repeats twice more*]."

We consider Tom and Sebastian's activity collectively to emphasize how their iterative engagement with situational quantities (M.S) and segments (M.R.1) supported shifts in their graphing activity. First, the students displayed explicit indications of conceiving orthogonal segments along each axis (e.g., Sebastian's orthogonal hand gestures, M.R.2). Furthermore, the students also indicated the simultaneity of those segments in the points they constructed (e.g., Tom's over and up gesturing "getting smaller every time we do it," M.R.3). Importantly, these graphical objects were not described as contextless; rather, Sebastian, in particular, referenced situational quantities explicitly in his interpretation of the points (M.S). In response to the TR's prompts, the pair specifically engaged with AoC and total area as they drew connections directly to segments. This activity supported their graphing with greater justification and support.

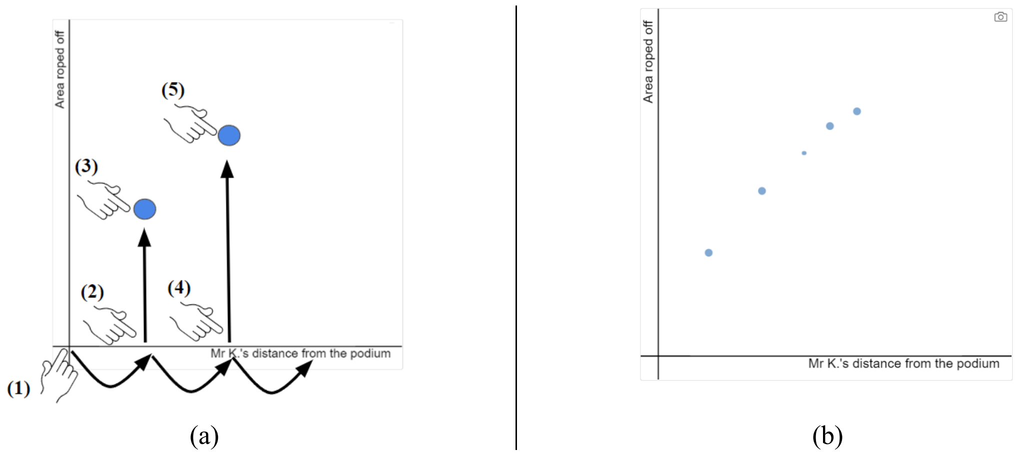


Figure 4: (a) A recreation of a subset of Tom's gestures in constructing his second graph and (b) Tom's final graph.

Discussion and Conclusion

In this paper, we presented a case study to address our RQ and describe how two students built connections between situational quantitative and covariational reasoning (M.S) and graphical representations of covarying quantities (M.R) as they progressed toward EGST. We note that the linear progression we initially designed from a situation to a graphical representation in Pass 1 did not provide sufficient opportunities for our students to reason in a coordinated way about how to represent the same two quantities across both contexts. In Pass 2, the students iteratively engaged with situations and graphs (that is, returning to M.S and repeatedly making connections between each component of M.R with M.S) in a way that resulted in more specific thinking. Thus, we conjecture that by engaging in both Pass 1 and Pass 2, the students developed robust connections between situational quantities (M.S) and graphs (M.R); we present Figure 5 to synthesize these results. Although we do not detail Tom and Sebastian's full construction of a smooth graph and EGST later in their activity here, we show evidence of robust connections between quantities in situations and graphs that prepared them for this next step in their reasoning.

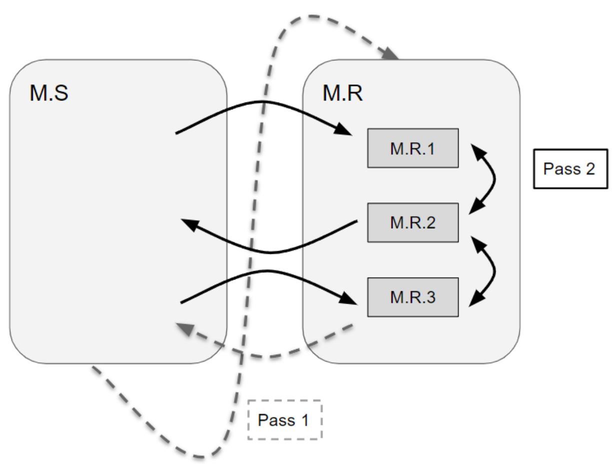


Figure 5: Visual presentation of the relationship between Pass 1 and Pass 2 in supporting students' connections between meanings in situations (M.S) and graphs (M.R).

In this report, we describe the process by which two students bridged their thinking in situational and graphical contexts. Reflecting the non-trivial nature of EGST (e.g., Thompson et al., 2017), our case provides evidence of the importance for students to have "repeated and connected occasions for students to engage in M.S [and] M.R," (Paoletti et al., 2023, p. 205) as they build towards EGST. Such findings have implications for practitioners and researchers intending to support learners' graphing meanings. Importantly, providing students recursive opportunities to make connections between meanings for situations (M.S) and graphical representations (M.R) is critical in the design of tasks and instruction. Future researchers may be interested in exploring other ways to students may connect situational quantities and graphical representations to support EGST. Such investigations could further efforts to improve the teaching and learning of graphing in ways that are attentive to the needs of STEM fields.

Acknowledgments

This material is based upon work supported by NSF under Grant No. DRL-2142000.

References

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352. https://doi.org/10.2307/4149958

Castillo-Garsow, C., Johnson, H. L., & Moore, K. C. (2013). Chunky and smooth images of change. *For the Learning of Mathematics, 33*(3), 31–37.

Clement, J. (1989). The concept of variation and misconceptions in Cartesian graphing. *Focus on Learning Problems in Mathematics, 11*(1/2), 77–87.

Glazer, N. (2011). Challenges with graph interpretation: A review of the literature. *Studies in Science Education, 47*(2), 183–210. https://doi.org/10.1080/03057267.2011.605307

Green, J. L., & Bridges, S. M. (2018). Interactional ethnography. In F. Fischer, C. E. Hmelo-Silver, S. R. Goldman, & P. Reimann (Eds.), *International handbook of the learning sciences* (1st ed., pp. 475–488). Routledge. https://doi.org/10.4324/9781315617572-46

Johnson, H. L. (2015). Secondary students' quantification of ratio and rate: A framework for reasoning about change in covarying quantities. *Mathematical Thinking and Learning, 17*(1), 64–90. https://doi.org/10.1080/10986065.2015.981946

Johnson, H. L., McClintock, E. D., & Gardner, A. (2020). Opportunities for reasoning: Digital task design to promote students' conceptions of graphs as representing relationships between quantities. *Digital Experiences in Mathematics Education, 6*(3), 340–366. https://doi.org/10.1007/s40751-020-00061-9

Liang, B., & Moore, K. C. (2021). Figurative and operative partitioning activity: Students' meanings for amounts of change in covarying quantities. *Mathematical Thinking and Learning, 23*(4), 291–317. https://doi.org/10.1080/10986065.2020.1789930

Moore, K. C. (2021). Graphical shape thinking and transfer. In C. Hohensee & J. Lobato (Eds.), *Transfer of learning: Progressive perspectives for mathematics education and related fields* (pp. 145—172). Springer. https://doi.org/10.1007/978-3-030-65632-4_7

Moore, K. C., & Thompson, P. W. (2015). Shape thinking and students' graphing activity. In *Proceedings of the 18th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education* (pp. 782-789). RUME.

Paoletti, T. (2019). Support students' understanding graphs as emergent traces: The faucet task. In Graven, M., Venkat, H., Essien, A. & Vale, P. (Eds). (2019). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol 3, pp. 185-192). Pretoria, South Africa: PME.

Paoletti, T., Gantt, A. L. & Corven, J. (2023). A local instructional theory for middle school students' emergent reasoning. *Journal for Research in Mathematics Education*, 54(*3*). https://doi.org/10.5951/jresematheduc-2021-0066

Paoletti, T., Lee, H.Y., Rahman, Z., Vishnubhotla, M., & Basu, D. (2020). Comparing graphical representations in mathematics, science, and engineering textbooks and practitioner journals.

International Journal of Mathematics Education in Science and Technology, 1-20. https://doi.org/10.1080/0020739X.2020.1847336
Paoletti, T. & Moore, K. C. (2018). A covariational understanding of function: Putting a horse before the cart. *For the Learning of Mathematics*, 38(3), 37-43.
Paoletti, T. & Vishnubhotla, M. (2022). Leveraging covariational reasoning and emergent shape thinking to distinguish nonlinear and linear relationships. In Karagöz Akar, G., Özgür Zembat, I., Selahattin, A., & Thompson, P. W. (Eds). *Quantitative reasoning in mathematics and science education* (pp. 133-168). Springer. https://doi.org/10.1007/978-3-031-14553-7
Potgieter, M., Harding, A., & Engelbrecht, J. (2008). Transfer of algebraic and graphical thinking between mathematics and chemistry. *Journal of Research in Science Teaching, 45*(2), 197–218. https://doi.org/10.1002/tea.20208
Saldanha, L. A., & Thompson, P. W. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berensen, K. Dawkins, M. Blanton, W. Coulombe, J. Kolb, K. Norwood, & L. Stiff (Eds.), *Proceedings of the 20th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 298–303).
Smith, J., & Thompson, P. (2008). Quantitative reasoning and the development of algebraic reasoning. In J. J. Kaput, W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 95–132). Erlbaum.
Steffe, L., & Thompson, P. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education* (pp. 267–307). Erlbaum.
Steffe, L. P. (1991). Operations that generate quantity. *Journal of Learning and Individual Differences, 3*(1), 61–82. https://doi.org/10.1016/1041-6080(91)90004-K
Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421–456). National Council of Teachers of Mathematics.
Thompson, P. W., Hatfield, N. J., Yoon, H., Joshua, S., & Byerley, C. (2017). Covariational reasoning among U.S. and South Korean secondary mathematics teachers. *The Journal of Mathematical Behavior, 48*, 95–111. https://doi.org/10.1016/j.jmathb.2017.08.001
von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Falmer Press.
Yin, R. K. (2018). *Case study research and applications: Design and methods* (Sixth edition). SAGE.

CONSIDERING THE ALIGNMENT BETWEEN TEACHERS' CONCEPTIONS OF JUSTIFICATION AND THEIR VISIONS FOR EQUITABLE INSTRUCTION

Kristen Bieda Michigan State University kbieda@msu.edu	Megan Staples, University of Connecticut megan.staples@uconn.edu	Kristin Doherty Michigan State kdoherty@msu.edu
Serena Anthonypillai University of Connecticut Serena.anthonypillai@uconn.edu	Alicia Matthews-Johnson Michigan State University Math325@msu.edu	Jihye Hwang Michigan State University hwangji7@msu.edu

While proving, and more broadly conceived "reasoning and sense-making," have received a great deal of attention in mathematics education research over the past three decades, recently scholars have argued for the importance of justification as a learning and teaching practice. As teachers work toward realizing goals for more equitable classroom environments, little is known about whether teachers' conceptions about mathematical practices, such as justification, reflect an understanding of how students' engagement in those practices can support more than just mathematical achievement. In this paper, we present findings from our analysis of interviews with 10 secondary mathematics teachers engaged in participatory action research to explore connections, and potential disconnections, between teachers' conceptions of justification and their visions for equitable instruction.

Keywords: Teacher Beliefs; Reasoning and Proof; Professional Development

Developing a deep understanding of mathematics is a core principle of equitable mathematics teaching (Horn, 2012). Scholars have examined, both theoretically and empirically, the role that opportunities to justify play in advancing equitable learning outcomes in K-12 classrooms (Bartell et al., 2017; Boaler & Staples, 2007). Some existing research has offered insights about whether opportunities that teachers provide students to engage in sensemaking and justification are robust, meaningful opportunities (Bieda, 2010; Henningsen & Stein, 1997), and other studies have provided insight into how classroom norms can influence students' access to and participation in argumentation practices (Klosterman, 2016; Staples, 2007; Yackel & Cobb, 1996). However, little is known about whether teachers' conceptualizations of justification and its role in school mathematics aligns with teachers' views about equitable learning outcomes. We argue that how teachers conceive of what justification is, and its role in teaching and learning mathematics, influences how they utilize justification opportunities as a tool to advance equitable learning outcomes. This paper explores the question, "In what ways do teachers' conceptions of justification align with, or deter from, their visions for equitable classrooms?"

Background

Teachers' Conceptions of Justification

Although the role of proof in the discipline of mathematics has been well-documented, there is less understanding of the role of justification and its relationship to proof and proving and other mathematical reasoning processes essential to learning mathematics (Ellis et al., 2021).

However, teachers may see a bigger role of justification in school classrooms, as they often question the role that formal proof should play in students' learning of mathematics and what students are expected to learn and be able to do (Knuth, 2002). In interviews with seventeen high school mathematics teachers, Knuth found that fourteen teachers did not feel proof should play a central role in learning mathematics, whereas all of the teachers indicated that informal proofs – such as explanations and being able to justify one's reasoning – were an important part of learning mathematics. Given the importance that teachers place on justification to support students' mathematical learning, understanding how they define justification may offer windows into the ways that they incorporate this process into the teaching of mathematics.

The Role of Justification in Teaching and Learning Mathematics

Teachers' conceptions of justification, and the role it plays in learning and teaching mathematics, influences the opportunities teachers provide for students to engage in justification (Gonzalez Thompson, 1984). The work of Staples, Bartlo, and Thanheiser (2012) specifically explored ways that teachers came to understand the role of justification through a professional development with 12 middle school teachers (grades 6 - 8) designed to co-inquire with teachers to understand justification at the middle school level and its importance for learning mathematics. Staples and colleagues discovered that teachers aligned the purposes of justification much more with its role as a teaching practice and a learning practice than its usefulness in establishing the validity of mathematical results. Specifically, teachers discussed its value in students' mathematics learning, particularly promoting conceptual understanding, fostering valued mathematical skills and dispositions. Additionally, teachers discussed justification's value to support teaching, such as gathering information about what students know, supporting students' engagement with other students and enabling more student-student interactions, and supporting students' sense of agency in and outside of the classroom.

The study reported in this paper builds upon the work of Staples, Bartlo and Thanheiser (2012) to explore how teachers see justification as playing a role in teachers' efforts to create more equitable learning environments. Specifically, we explore conceptually the connections between ways justification is defined and the dominant and critical dimensions of equity as conceptualized by Gutiérrez (2012). Further, we illustrate, based on data from 11 high school mathematics teachers participating in a study group, alignments and disconnections between teachers' conceptions of justification and their descriptions of equitable learning environments.

Theoretical Framework

Gutiérrez (2012) conceptualized equity in mathematics education as comprising dimensions that reflect a dominant perspective (*access* and *achievement*) as well as a critical one *(identity* and *power)*. The components of the dominant perspective have been the traditional focus of gaps-oriented equity work, namely the opportunities each and every student has to learn rigorous and meaningful mathematics (access) as well as how well they can demonstrate what they have learned as a gateway for academic success (achievement). In the past two decades, elements of the critical dimension have gained more prominence in mathematics education research, due in part to their role in rewriting the script of how mathematics should be taught and learned.

Gutiérrez frames *identity* using a window/mirror metaphor; "students need to have opportunities to see themselves in the curriculum (mirror), as well as have a view onto a broader world (window)" (2012, p. 19-20). As students are able to use mathematics to make sense of the world, they need agency to engage in social transformation as a result of their mathematics learning.

This component, *power*, prompts us to consider how we are promoting students with the mathematical power they need to change systems they deem unjust.

In our work, we draw upon this frame to think about how possible conceptions and purposes of justification may work to support these aspects of equity. For example, if we consider justification as "the process of supporting your mathematical claims and choices when solving problems or explaining why your claim or answer makes sense" (Bieda & Staples, 2020, p. 103), the role of justification in teaching and learning mathematics goes beyond explaining the veracity of a claim. By also seeing moments where students explain their problem-solving *choices* as the mathematical practice of justification, teachers implicitly communicate that students have agency to not only argue why they have a correct answer, but also why their solution strategy is valid (even if it is different from other strategies). This conception not only supports students' achievement (dominant axis), but also students' identity (critical axis), in that *their* ways of doing mathematics have a space to be legitimized through justification.

Building on our data, we argue first that teachers' conceptions of justification matter for how justification can be employed in their classrooms to advance equity goals. We further argue that various conceptions and purposes of justification align with components of Gutiérrez's (2012) frame on equity more principally than other components. We might imagine these conceptions imposed on the dominant and critical axes of Gutiérrez's framing of equity, with their placement highlighting how particular conceptions frame justification relative to dominant and critical mathematics within school mathematics. For instance, Figure 1 shows the placement of Bieda and Staples' (2020) conception along the axes.

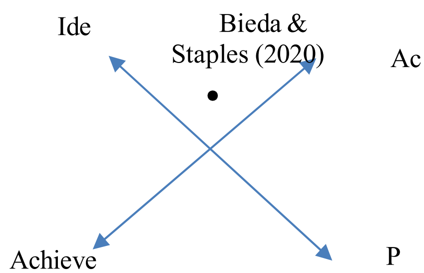


Figure 1: Plotting conceptions of justification along dimensions of equity Methods

Research Context

To better understand how teachers conceptualize justification and its relationship to equity, we have been working with ten teachers and one teacher candidate from two high schools in two different states (one in the Midwest and one in the Northeast) to conduct participatory action research over the course of the 2022-2023 academic year. These teachers volunteered to participate in this collaboration given their interests and commitments to advance equity in their classrooms. The majority of the participants are white, and the majority of their students come from minority backgrounds and homes with low income. The teaching experience of the in-service teachers range 3-33 years and averages 12.4 years.

Prior to their participation in the study group, participants engaged in two activities. First, we conducted a pre-interview with each participant to gather background information and experiences, including their initial thoughts about justification and equity. Second, during summer 2022, we held a workshop at each school (12-15 hours) with the goal of working toward shared understanding of justification and equity within each group. Since the start of the school year, we have held 1.5-hour study group meetings with the participants twice per month at each site. During the meetings, teachers report about the successes and challenges they are experiencing in supporting equitable learning opportunities and in providing students justification opportunities. Teachers' thoughts about the relationship between these two are also discussed.

Data Collection

This paper presents findings from our pre-interviews with each teacher. The pre-interviews gathered each teacher's initial conceptions of justification and equity. Teachers were asked, "How would you define justification in mathematics?", "What is the relationship you see between equity and justification?", and "When you think about creating equitable opportunities to learn, what does that mean to you?" Interviews were conducted by two members of the project team on Zoom, recorded, and transcribed.

Data Analysis

To understand the alignment between teachers' conceptions of justification and equity, we analyzed responses to each question separately and then looked across analyses to synthesize findings. To analyze responses to the questions, "How would you define justification in mathematics?" and "What is the relationship you see between equity and justification?", we first conducted open coding using a thematic analysis (Braun & Clarke, 2006) approach to teachers' definitions of justification. To analyze responses to the question, "When you think about creating equitable opportunities to learn, what does that mean to you?," we identified utterances corresponding to specific dimensions of Gutiérrez's (2012) framework for equity.

Findings

In this section, we present findings from our analysis of three cases that represent the range of responses provided by participants about their definitions of justification. In sharing these cases, we will juxtapose their definitions of justification with their thoughts about equitable opportunities to learn, to explore the alignments, and potential contradictions, between these conceptions.

Case 1: Justification as Revealing Students' Thinking

The first case, William, is an 9th-grade mathematics teacher in his fifth year of teaching. When asked to define justification, he responded:

> "I mean, as simple terms, it's their thought process. How they got their answer. Getting it on paper. Because your goal as a teacher is to kind of provide those skill sets and for them to, you know, understand the process and the math skills that are needed to answer that, to breakdown a question. And so that justification piece is really, you know… show me your work. But it's more than that. But like, you're critical, like what was your thought process? How did you know that? What did you do? Why did you do it? And just getting that on paper because that's, end of the day, is like when they take those tests you know they have to be able to get it on paper. So that's that justification piece for me" (William, Pre-Interview)

William's conception of justification is focused on assessing student's understanding, or "thought process." He discusses that justification involves how students arrived at their answer and why they solved the problem in the way they did. We also noted that his definition focuses more on individual understanding, rather than the collective understanding of the class.

We found that all of William's statements about equitable learning opportunities were coded on the dominant axis. For him, equity was about providing each individual student (*access*) what they need to be successful (*achievement*). He mentioned, "Success looks different for every student and I think giving students you know opportunities to improve their practice is crucial, but what you provide each student to be successful is going to be different... and so the equity piece is just giving students opportunities to improve in different ways at the level they're at." Thus, if justification is a means for assessing students' thought processes, it is a tool for supporting both the access and achievement elements of Guttiérez's (2012) dimensions of equity.

Case 2: Justification as Explaining and Convincing Other People

Lynda is a high school mathematics teacher (grades 10-12) in her fourth year of teaching.

Lynda described justification as:

> "basically explaining your thinking and being able to prove whatever it is that you want to claim in a way that makes sense to other people. So I think a lot of times you can justify something in your own mind, but then actually coming up with the words or the logic or the picture or whatever the case may be to prove it to somebody else." (Lynda, Pre-Interview)

In contrast to William's definition, Lynda describes justification as going beyond explaining your thinking to be able to "prove whatever it is that you want to claim in a way that makes sense to other people." Lynda's definition emphasizes how justification is a collective activity; coming up with a justification involves considering what will be convincing to somebody else.

When asked to explain about creating equitable learning opportunities, Lynda emphasized *access* and *identity*:

> "I think [the basic idea is] that everybody has access. But to expand on that, right, that everybody would be able to, like, assess the problem from the beginning, but also, like remain in the problem the entire time. So I think like some lessons are built so that you know, there's like the opener that everybody can access, but then things step up, and maybe you lose people on the way. So to create like a whole equitable lesson some way that everybody can get in. And if you kind of fall out that there's a way back in. And I don't know, just being mindful of people's different like backgrounds in terms of like background knowledge for the math needed background in terms of just like cultural experiences, that may or may not play a part in the lesson. And then also just different, like accessibility needs if people have like, different impairments that might need to be adjusted for." (Lynda, Pre-Interview)

Lynda's attention to how lessons need to provide access for each student, and her awareness of differences in students' mathematics and cultural backgrounds, reflects a commitment to ensuring that all students can participate in her classroom mathematical community. By defining justification in a way that is not just convincing to oneself but also to others, it positions justification as building a classroom where students' access to the mathematical tasks is important and students' varied backgrounds are embraced as part of their mathematical identities.

Case 3: Justification as Knowing Why

Emma is a 9th- and 10th-grade mathematics teacher in her sixth year of teaching. When asked to define justification, Emma stated:

> "I think it's being able to like to explain to somebody why you did something rather than just what you did. I think a big issue is that we tend to teach processes and how to memorize

> processes, and we don't spend as much time focusing on like why are we actually doing this, what does it connect to. And so I think that that mathematical justification is just being able to explain the why rather than just the what." (Emma, Pre-Interview)

We argue that Emma's response is somewhat of a blend of the ideas from William's and Lynda's responses. Emma's focus on justification promotes understanding the "why" behind the processes echoes William's attention to justification going beyond a student explaining how they got an answer. Yet, like Lynda, Emma also mentions that justification involves explaining to others. When asked about what it means to create equitable learning opportunities in her classroom, Emma's responses reflected attention to access in a couple of ways. She indicated:

> "And so I try to think about like not just creating learning experiences based off of what like I would be able to successfully engage in, but thinking about the different things that might hold somebody back or make them experience it differently." (Emma, Pre-Interview)

She continued: "*... it starts with just like getting to know your students and figuring out what they need."* Her attention to knowing students and their lives outside of class was predominant in her responses to questions related to creating equitable opportunities to learn, reflecting her concern for supporting students' identities, and using knowledge about their out-of-class identities to inform her instruction.

When considering if Emma's conception of justification aligns with her vision for creating equitable opportunities to learn, it is less clear whether her emphasis on being able to "explain to somebody why you did something" is supportive of her efforts to create a learning environment that connects with who they are and how they learn within and outside of the mathematics classroom.

Holistic Results

Overall, the majority of teachers (8 of 10) describe justification as an individual activity that involved some kind of written record and provided a detailed accounting of what they students knew or how they had solved a problem. Only 2 of 10 teachers suggested justification was a *practice* where students engaged one another and potentially built knowledge. Additionally, when discussing what it means to create equitable learning opportunities, the majority of teachers' responses reflected aspects of the dominant axis of equity (Gutiérrez, 2012) and only 3 of the 10 teachers discussed aspects related to supporting the critical axis. Although a noticeable minority, the responses from those three teachers reflected a vision of supporting the *identity* dimension of the critical axis (Gutiérrez).

Discussion

Although scholars have argued how justification plays a role in promoting equity (Bieda & Staples, 2020; Boaler & Staples, 2008), little empirical work has been done to show that teachers' conceptualizations of justification, and therefore the nature of the justifications and justification activity they expect from students, reflect, and align with their goals for creating equitable learning opportunities. In our findings, we discovered that most of the participating teachers conceive of justification as an activity that can promote students' access to deeper understanding of mathematics but tend to focus less on how justification can support students' mathematical identities and become a tool for exercising power in changing their worlds.

Moreover, the tendency to focus on elements of access and achievement were evident in their vision for creating equitable learning opportunities in their classrooms. What these findings suggest is that more work is needed to help teachers go beyond recognizing justification as a means for showing deeper understanding and explaining why, but to also conceptualize justification as an activity that builds mathematical identities needed for students to both step into and exercising agency with advocating for change in society.

Acknowledgments

This research was supported through funding from Blinded for Review (Blinded). Any opinions, conclusions, or recommendations contained herein are those of the authors and do not necessarily reflect the views of Blinded.

References

Bartell, T., Wager, A., Edwards, A., Battey, D., Foote, M., & Spencer, J. (2017). Toward a framework for research linking equitable teaching with the standards for mathematical practice. Journal for Research in Mathematics Education, 48(1), 7-21.

Bieda, K. N. (2010). Enacting proof-related tasks in middle school mathematics: Challenges and opportunities. Journal for Research in Mathematics Education, 41(4), 351-382.

Bieda, K. & Staples, M. (2020). Justification as an equity practice. Mathematics Teacher: Learning and Teaching PreK-12, 113(2), 102–108. https://doi.org/10.5951/MTLT.2019.0148

Boaler, J., & Staples, M. (2008). Creating mathematical futures through an equitable teaching approach: The case of Railside School. Teachers College Record, 110(3), 608-645.

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. Qualitative Research in Psychology, 3(2), 77-101.

Ellis, A. B., Staples, M., & Bieda, K. N. (2022). Justification Across the Grade Bands. In K. Bieda, A. Conner, K.Kosko, and M. Staples (Eds.), Conceptions and Consequences of Mathematical Argumentation, Justification, and Proof (pp. 287-297). Cham: Springer International Publishing.

Gonzalez Thompson, A. (1984). The relationship of teachers' conceptions of mathematics and mathematics teaching to instructional practice. Educational Studies in Mathematics, 15(2), 105-127.

Gutiérrez, R. (2012). Context matters: How should we conceptualize equity in mathematics education?. In B. Herbel-Eisenmann, D. Wagner, J. Choppin & D. Pimm(Eds.), Equity in discourse for mathematics education: Theories, practices, and policies (p.17-33). Springer Science and Business Media.

Henningsen, M., & Stein, M. K. (1997). Mathematical tasks and student cognition: Classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. Journal for Research in Mathematics Education, 28(5), 524-549.

Horn, I. S. (2012). Strength in numbers: Collaborative learning in secondary mathematics classrooms. Reston, VA: National Council of Teachers of Mathematics.

Klosterman, P. J. (2016). Identification and establishment of social and sociomathematical norms associated with mathematically productive discourse. Unpublished doctoral dissertation, Washington State University.

Knuth, E. J. (2002). Teachers' conceptions of proof in the context of secondary school mathematics. Journal of Mathematics Teacher Education, 5, 61-88.

Staples, M. (2007). Supporting whole-class collaborative inquiry in a secondary mathematics classroom. Cognition and Instruction, 25(2-3), 161-217.

Staples, M. E., Bartlo, J., & Thanheiser, E. (2012). Justification as a teaching and learning practice: Its (potential) multifacted role in middle grades mathematics classrooms. The Journal of Mathematical Behavior, 31(4), 447-462.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. Journal for Research in Mathematics Education, 27(4), 458-477.

DESIGNING MATHEMATICAL REASONING TO BE A FORM OF INOCULATION AGAINST UNWARRANTED BELIEFS

Samuel Otten
University of Missouri
ottensa@missouri.edu

Mitchelle M. Wambua
University of Missouri
mwambua@mail.missouri.edu

Emmanuel Chukwuebuka Umeh
University of Missouri
ecung6@missouri.edu

Keywords: Reasoning-and-Proving, Conspiracy Theories, Boundary Crossing

The problem of conspiracy theories and epistemically unwarranted beliefs is being addressed by media literacy specialists (Jones-Jang et al., 2021), social psychologists (Douglas & Sutton, 2018), science educators (Fasce & Picó, 2019), among others. What role can mathematics education play in reducing our societal susceptibility to unwarranted beliefs? Unwarranted beliefs are defined as beliefs not based on valid reasoning or credible data and which are maintained even in the face of countervailing evidence. Mathematics education should be able to provide skills related to validity and credibility, but there are two interrelated drawbacks which must be overcome. First, mathematical reasoning must avoid the danger of being taught narrowly so that it is constrained to a procedural exercise exclusive to school mathematics (Reid & Knipping, 2010). In other words, we must instill transferrable reasoning skills. Second, we must help students form the disposition to actually apply those analytical reasoning skills. Falling for fake news headlines, for example, seems to be more a matter of failing to apply analytical skills than it is a matter of the oft-assumed confirmation bias (Pennycook & Rand, 2019).

We employ boundary crossing (Akkerman & Bakker, 2011) as a theoretical perspective for the design of a curriculum unit intended to build students' reasoning skills in a manner relevant not only for school mathematics but also for social and political discourse. This approach explicitly attends to the building of habits within and beyond mathematical contexts so that students are more likely to actively use their analytic skills when faced with unwarranted claims. We identify the boundary as being between school mathematics situations and social situations in which political or current event arguments are made. We have designed scenarios from political debate that have mathematical substance and can be brought into the school mathematics space, while also designing a "pedagogical device" in the mathematics instruction that can be a boundary object for political debates or when consuming news media.

The pedagogical device is consistent with media literacy guidelines (Aufderheide, 1993; Jones-Jang et al., 2021) and interventions that have successfully inoculated students against unwarranted beliefs (Dyer & Hall, 2019) and is simultaneously well-suited, we believe, for spurring mathematical reasoning. It consists of two questions that we can constantly ask when a claim arises: "How do you know that is true?" and "Could that possibly be wrong?" These questions are framed not as personal attacks (which are unproductive in classrooms and in confronting unwarranted beliefs) but in the spirit of collective understanding and sense-making. The poster presentation will include multiple examples of these questions being used to spur reasoning and debate within mathematics, such as when faced with the (false) claim that an exponential function cannot "catch up" to a polynomial function with a very large exponent (Otten et al., 2023). The questions will also be applied to political claims such as the story that enough dead people voted in the 2020 U.S. presidential election to sway the outcome.

By teaching mathematical reasoning in a manner that promotes active wondering about the basis of claims ("how do you know that is true?") and invites curiosity, not confrontation, about the possible existence of counterevidence ("could that possibly be wrong?"), we hope to exemplify one possible approach to the design of learning opportunities that will resonate across mathematical and political boundaries and increase the likelihood that students activate their analytic skills in ways that inoculate them to unwarranted beliefs.

Acknowledgments

We are thankful for colleagues in social studies education, Tony Castro and Mary Adu-Gyamfi, for sharing resources and having conversations with us about these ideas.

References

Akkerman, S. F., & Bakker, A. (2011). Boundary crossing and boundary objects. *Review of Educational Research*, *81*, 132–169.

Aufderheide, P. (1993). *Media literacy. A Report of the National Leadership Conference on Media Literacy*. https://eric.ed.gov/?id=ED365294

Douglas, K. M., & Sutton, R. M. (2018). Why conspiracy theories matter: A social psychological analysis. *European Review of Social Psychology*, *29*(1), 256–298.

Fasce, A., & Picó, A. (2019). Science as a vaccine: The relation between scientific literacy and unwarranted beliefs. *Science & Education*, *28*, 109–125.

Jones-Jang, S. M., Mortensen, T., & Liu, J. (2021). Does media literacy help identification of fake news? Information literacy helps, but other literacies don't. *American Behavioral Scientist*, *65*(2), 371–388.

Otten, S., Bemke, J., & Webb, J. (2023). Viral spread of disease and disinformation. In S. Harper & D. Cox (Eds.), *Math in Action*. National Council of Teachers of Mathematics.

Pennycook, G., & Rand, D. G. (2019). Lazy, not biased: Susceptibility to partisan fake news is better explained by lack of reasoning than by motivated reasoning. *Cognition*, *188*, 39–50.

Reid, D. A., & Knipping, C. (2010). *Proof in mathematics education: Research, learning and teaching*. Sense Publishers.

FIRST YEAR UNIVERSITY STUDENTS WAYS OF REASONING AND ARGUMENTATION IN PROBLEM-SOLVING ACTIVITIES

FORMAS DE RAZONAR Y ARGUMENTAR EN ACTIVIDADES DE RESOLUCIÓN DE PROBLEMAS DE ESTUDIANTES DE PRIMER AÑO DE UNIVERSIDAD

Tania Azucena Chicalote Jiménez
CINVESTAV, IPN
tania.chicalote@cinvestav.mx

Daniel José Ortiz May
CINVESTAV, IPN
daniel.ortiz@cinvestav.mx

The aim of this study is to characterize ways of reasoning and arguing that first year university mathematics students exhibit in problem-solving activities from a course that emphasizes the importance of formulating conjectures and the search for different ways to support or validate them. The use of a Dynamic Geometry System in the representation of problems and in the formulation of conjectures or relationships that are important in the solution processes is highlighted. In this context, students have the opportunity to look for different ways to argue and support the relevance and validity of the conjectures. Results indicate that students extend their ways of reasoning so that allows them to move from empirical to formal arguments within problem solutions

Keywords: Undergraduate Education; Problem Solving; Reasoning and Proof; Technology

According to admission profiles for higher education, in Physical-Mathematical Sciences and Engineering, students are expected to have a solid knowledge of basic concepts from High School Calculus, Analytical Geometry and Algebra, as well as to be interested in problem solving. However, data from diagnostic exams for admission to the bachelor's degree in Mathematics of various institutions show that most students enter this area of study without an adequate understanding of basic mathematical concepts and with skills and strategies for problem solving focused on a basic level of reproduction. This shows that students will have significant difficulties during their integration into higher education.

This highlights the need to provide a mathematical education that encourages students to focus their interest and attention on the development of skills and strategies for the management of concepts, the resolution of mathematical problems and the elaboration of arguments that allow them to go beyond the reproduction of knowledge to build a robust and abstract mathematical thinking that enables them to formulate and solve different types of problems in any area of their academic, social and labor training.

With this in mind, it becomes important to continue researching on aspects and factors that have an influence on a successful or deficient integration at university level, mainly in the Mathematics area. Thus, in this research we sought to characterize, by means of a task focused on Problem Solving (PS) (Polya, 1965; Schoenfeld, 1985), which are the tools, skills and difficulties within the process of argumentation and mathematical reasoning that first-year undergraduate students in Mathematics show, therefore the following research question was posed How does the use of Dynamic Geometry System (DGS) within the PS promote the development of mathematical processes such as obtaining conjectures, arguing and validation, so this allows students to get in higher level mathematical activities?

Contextual framework

Research on the secondary-tertiary transition in Mathematics has considered different perspectives in order to explain and address the problems and stages that students go through during this transition period. In this regard, Leviatan (2008) states that students' difficulties are due to the fact that high school mathematics tends to focus on developing algorithmic skills for resolution of concrete and routine exercises, while at university, skills for abstraction and aspects of inquisitive questioning are required, while non-routine problem solving, and mathematical rigor are emphasized. On the other hand, Clark and Lovric (2010) define that transition from one level to another involves the process of a rite of passage divided into three phases: 1) Separation from the level, from the previous ways and routines of learning; 2) Liminality, a phase in which routines, beliefs and habits from high-school level still form part of students' attitudes within the new educational system not yet assimilated; and 3) Incorporation into the new environment. This transition implies a crisis that leads the interruption, modification, and distortion of previous routines, so this crisis is inevitable but necessary for students to develop advanced mathematical thinking and autonomy before their training. Considering this, it becomes feasible to investigate aspects that could smooth this process. Thus, Rach and Heinze (2016) identified 5 variables involved in academic success or failure during second-tertiary transition: 1) interest in mathematics, 2) self-concept as a mathematics learner, 3) previous achievements as a mathematics learner, 4) previous knowledge of mathematics, and 5) quality of learning strategies. For their part, Di Martino and Gregorio (2019) established five categories of causal attributes that lead to the difficulties presented during integration to university education: 1) Context factors; 2) Transition aspects; 3) Inadequate knowledge; 4) Inadequate way of thinking in/for mathematics; and 5) Comparison with peers. Considering these aspects allows us to generate alternatives to support the student in facing these problems.

As can be seen, problems that students will face during this transition have a multicausal nature. In this regard, Adelman (2006) suggests that students previously need examples of the activities performed during the first year of university education and the kind of future exams in order to have a better idea of what students are expected to do. Thus, it is important for students to have approaches to processes linked to argumentation and mathematical reasoning. For his part, Schoenfeld (2022) mentions that the educational challenge lies in creating robust learning environments that support students in developing not only the authentic knowledge and processes that underlie mathematics, but that promote the development of a sense of agency and authority to make sense of mathematical objects and practices within robust mathematical thinking.

Given the above, it becomes necessary to contribute to research on aspects, factors and practices that contribute to a more accessible and with greater opportunities to succeed during Mathematics education. The analysis presented in this paper is on the basis of, as mentioned before, the results of a task based on PR within a Geometry course whose methodology includes aspects related to the five dimensions to create powerful mathematics classrooms (Schoenfeld, 2014): 1) Mathematical content; 2) Cognitive demand; 3) Access to mathematical content; 4) Agency, authority and identity and 5) Use of assessments. Thus, first is given a general description of the teaching practices carried out in the course, followed by a descriptive analysis of the processes set in motion by the group of participating students. For the analysis of data obtained, we identified resources and heuristics (Schoenfeld, 1985), as well as conceptual and procedural tools (Melhuish, Vroom, et al., 2022) that students use and that bring them closer to the realization of authentic mathematical activities. For this purpose, we consider Authentic

Mathematical Proof Activity (AMPA) theoretical framework proposed by Melhuish, Vroom, et al. (2022), from the ten procedural tools expressed by the authors, we sought to identify: 1) refinement or analysis of a proof, a statement, or definition by focusing on the attainment of assumptions; 2) elaboration of formalizations, i.e., the process of translating informal ideas into formal or symbolic rhetorical forms; 3) elaboration of analogies, i.e., the process of importing proofs, statements or concepts across different domains adapting them to new schemas; 4) use of examples, a specific and concrete representation of a statement, concept or proof that represents a class of objects; and 5) elaboration of diagrams and visual representations of mathematical objects (statements, concepts or proofs) that capture structural properties.

Description of didactic methodology

Group G1 consisted of 8 undergraduate mathematics students who participated on a voluntary basis and who were taking a Modern Geometry I course from a public university.

Contents of the course were developed on the basis of problems or initial questions posed by the teacher, which students had to explore prior to the class. During the class, the teacher used the Dynamic Geometry System Geogebra (DGS) to explorate the proposed problems, as well as to simulate geometric straightedge-and-compass constructions and to verify initial conjectures, so that students were familiar with this technology and could use it to explore problems presented on their own.

The dynamics of the course sought to provide opportunities for students to have equal access to the contents developed, by means of course notes, suggested bibliography, interactive applets in Geogebra and the possibility of research via the internet. Students were encouraged to develop the ability to argue and not only the use of established formulas or algorithms; on the contrary, through the problems and questions posed, students were encouraged to generate conjectures and different ways to corroborate the validity of them, as well as the exchange of ideas in groups to meet the dimension of cognitive demand (Schoenfeld, 2014) and to generate both individual and collaborative commitment in the performance of the activities by students.

Thus, by the time the task was assigned, students had received training aimed at obtaining conjectures, arguing and exploration using Geogebra. In addition, by this time they had reviewed content related to triangles properties, triangles congruence and similarity criteria, inscribed angles and cyclic quadrilaterals properties.

Context of the Problem Solving Task Assignment

The group of students was given the following task: There is a square ABCD. If on the DA side you construct the midpoint E, then draw the segment BE and construct the perpendicular segment CF with F the perpendicular foot on BE. What kind of triangle do the points C, D and F form? Prove your conjecture in two different ways.

Students had the option of tackling the task individually or in pairs, as the latter modality of work prevailed in the dynamics of the course. Students took the problem home and had about three days for a first approach to it. Then, in a classroom class, space was provided for the group to present the conjectures obtained, the initial ideas for the demonstration of the conjecture and possible doubts or concerns. This class was part of the control elements of the RP (Schoenfeld, 1985), so the necessary feedback was given to the students.

To collect data, a logbook was requested to record the resolution processes, as well as the questions, ideas or actions that arose during the resolution of the task. For this purpose, the following elements were requested:

1. Description of the exploration, understanding of the problem and making a conjecture. For the analysis of data in this section, we sought to determine what means, instruments and processes students used or followed to explore and understand the problem, as well as to make a conjecture.
2. Description of the process of developing a plan or strategy for solving the problem. In this section we sought to identify whether students recognized the resources (concepts, mathematical content, evidence or previous results) they had or did not have to tackle the problem, so that this would lead them to determine a possible path to follow for the solution or to consider various sources of research.
3. Process of solving the problem. This section analyses arguments and processes that led to proving the conjecture obtained.
4. Problem extensions. This section analyses whether students pose new questions or problems to be solved based on what has already been solved.

From the analysis of these elements, the aim is to describe the type of reasoning and ways of acting that students put into practice in order to solve problems, as well as the difficulties they faced and the ways in which they overcame them.

Analysis of results

For the analysis of the results, it was considered the evaluation of three logbooks developed in pairs (E1, E2, E3) and one individually developed logbook (E4) of group G1. In this report, the tools and processes within the logbook developed by students are exemplified with short episodes. It should be noted that in the logs it is observed that the predominant way of working to understand and explore the problem was individual, as well as for the general writing of the log, while students worked collaboratively mainly to exchange and verify ideas during the planning of a strategy, in the process of solving it, and to obtain feedback from their peers. The results obtained in the team logbook sections are described below.

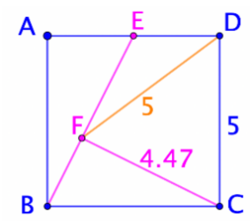
Analysis: Exploring and Understanding the Problem

Students in each team were very descriptive in terms of the acts they performed to understand the problem, and they were also open in expressing their way of acting and thinking about the processes during the RP, which allows us to identify, at least in a global way, their belief system in relation to this activity (Schoenfeld, 1985).

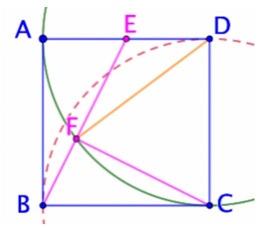
As can be seen in Table 1, all students used Geogebra to carry out the construction, and also used the software tools either to measure distances between points or the construction of circles to compare radii and thus compare lengths. The use of Geogebra helped students to represent and understand the problem, and thus to make a conjecture about the type of triangle generated in the construction.

Table 1. Responses from exploration and understanding phase

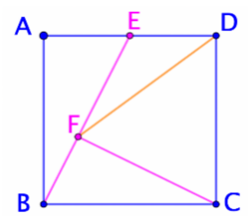
Problem representation	Processes followed by each team



E1. We first developed the construction that posed the problem with the help of the Geogebra plotter. When we made the model, we realised that the triangle CDF was isosceles, since its bisector, height and mediatrix of D coincided, which is characteristic of a triangle of this type, and we subsequently checked this by measuring the distances DF and DC with the help of the program.



E2. The first thing I did was to construct the figure in Geogebra to get a clearer idea of what was being drawn. Then when I looked at the triangle, at first glance it looked like an equilateral triangle, but after comparing the lengths of the sides, using circles, I realised that it was actually an isosceles triangle.



E3. First I traced [the figure] freehand, as I didn't get very far, I plotted the hypotheses in Geogebra and there I discovered that the triangle was isosceles.

E4. I decided to open Geogebra to build the construction step by step (I suspected it would be complicated). After building it I realized that the triangle FCD is isosceles. Now I wonder how to prove it, because I can't think how I know it is isosceles.

All the students conjectured that the triangle in question is isosceles. Thus we have that students implement the strategy of elaborating diagrams accompanied by the use of technology, and it is also observed that they have the necessary resources to elaborate the construction based on the structural characteristics of the mathematical objects. On the other hand, it is observed that students have acquired a certain degree of confidence both in making decisions regarding their actions as problem solvers and in the use of auxiliary tools or devices.

Analysis: Drawing up a Plan

In general, it can be observed that for the first demonstration of the conjecture, the students considered two possible ways, the first related to demonstrating that two sides of the triangle have the same length, the second, demonstrating that in the triangle there are two internal angles that measure the same, for which they mentioned that they could use congruence or similarity of triangles. One team highlighted as an important aspect the fact that both the angles formed by the perpendicular and those of the square are right angles, which allowed them to consider the CDEF quadrilateral and hence to consider the use of results relating to cyclic quadrilaterals. Below are fragments of what the students expressed in search of a first demonstration.

E1. At the beginning we came to the conclusion that we could find equal angles from the figures formed by the construction [...] and consider that there were congruent triangles.
E2. I felt that the best way to test this was to use triangle congruence.
E3. The fact that the sides [of the quadrilateral] measure the same and the angles are right angles is relevant, because in this way we see that it is a cyclic quadrilateral and so we

can relate this to other results such as congruence, similarity of triangles and angles inscribed in a circle.

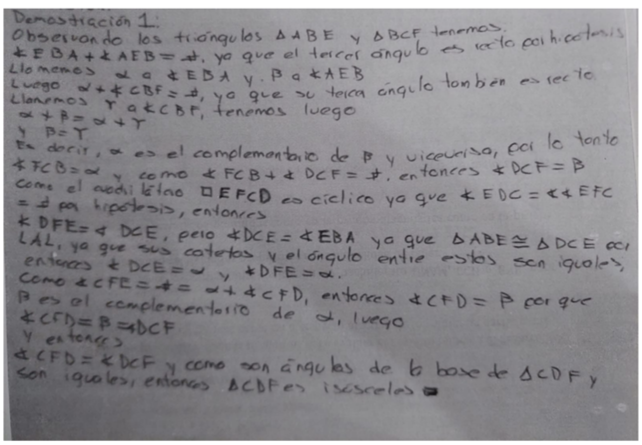
E4. First I will try to arrive at the equality of two angles of the triangle FCD.

It is worth mentioning that both in this stage and in the comprehension stage, the identification of the resources available to the students becomes evident, from which it is possible to draw up a suitable diagram and define a first approach or resolution plan.

Analysis: Problem-Solving Process

The students mentioned that after having obtained the conjecture, they faced several moments of frustration and despair, as they did not quickly find a way forward to prove their assertion. Some of them expressed "letting the ideas and frustrations rest" for a considerable time and then resuming with a calmer attitude, during this time a face-to-face class was held in which the initial ideas were expressed as a group lesson, which allowed the students to reconsider the resources they had and other possible ways of approaching the conjecture. Thus, for the demonstrations, teams E1, E2 and E4 also considered results relating to cyclic quadrilaterals and angles inscribed in a circle.

The resolution processes followed by two teams are described and analyzed below.


Demostración 1:
Observando los triángulos $\triangle ABE$ y $\triangle BCF$ tenemos
$\measuredangle EBA + \measuredangle AEB = \perp$, ya que el tercer ángulo es recto por hipótesis
Llamemos α a $\measuredangle EBA$ y β a $\measuredangle AEB$
Luego $\alpha + \measuredangle CBF = \perp$, ya que su tercer ángulo también es recto
Llamemos γ a $\measuredangle CBF$, tenemos luego
$\alpha + \beta = \alpha + \gamma$
y $\beta = \gamma$
Es decir, α es el complementario de β y viceversa, por lo tanto
$\measuredangle FCB = \alpha$ y como $\measuredangle FCB + \measuredangle DCF = \perp$, entonces $\measuredangle DCF = \beta$
Como el cuadrilátero $\square EFCD$ es cíclico ya que $\measuredangle EDC = \measuredangle EFC = \perp$ por hipótesis, entonces
$\measuredangle DFE = \measuredangle DCE$, pero $\measuredangle DCE = \measuredangle EBA$ ya que $\triangle ABE \cong \triangle DCE$ por LAL, ya que sus catetos y el ángulo entre estos son iguales,
entonces $\measuredangle DCE = \alpha$ y $\measuredangle DFE = \alpha$.
Como $\measuredangle CFE = \perp = \alpha + \measuredangle CFD$, entonces $\measuredangle CFD = \beta$ porque β es el complementario de α, luego
$\measuredangle CFD = \beta = \measuredangle DCF$
y entonces
$\measuredangle CFD = \measuredangle DCF$ y como son ángulos de la base de $\triangle CDF$ y son iguales, entonces $\triangle CDF$ es isósceles ∎

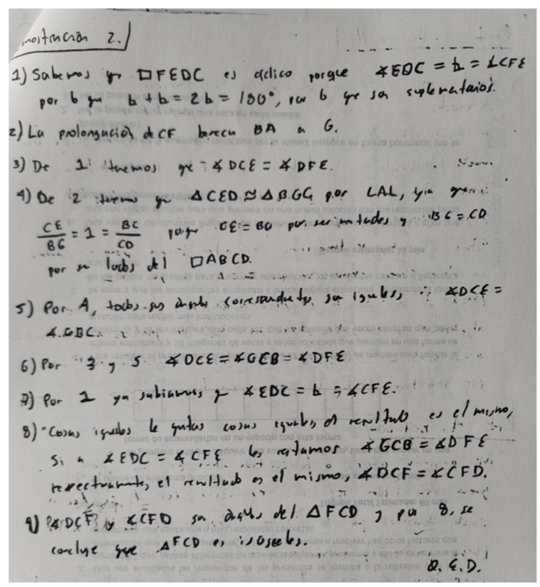

Demostración 2.
1) Sabemos que $\square FEDC$ es cíclico porque $\measuredangle EDC = b = \measuredangle CFE$ por lo que $b + b = 2b = 180°$, con lo que son suplementarios.
2) La prolongación de CF hace BA a G.
3) De 1 tenemos que $\measuredangle DCE = \measuredangle DFE$.
4) De 2 tenemos que $\triangle CED \cong \triangle BGC$ por LAL, ya que
$\frac{CE}{BG} = 1 = \frac{BC}{CD}$ porque $CE = BG$ por ser mitades y $BC = CD$ por ser lados del $\square ABCD$.
5) Por 4, todos sus ángulos correspondientes son iguales, ∴ $\measuredangle DCE = \measuredangle GBC$.
6) Por 3 y 5 $\measuredangle DCE = \measuredangle GCB = \measuredangle DFE$
7) Por 1 ya sabíamos que $\measuredangle EDC = b = \measuredangle CFE$.
8) "Cosas iguales le quitas cosas iguales el resultado es el mismo,
Si a $\measuredangle EDC = \measuredangle CFE$ les restamos $\measuredangle GCB = \measuredangle DFE$ respectivamente, el resultado es el mismo, $\measuredangle DCF = \measuredangle CFD$.
9) $\measuredangle DCF$ y $\measuredangle CFD$ son ángulos del $\triangle FCD$ y por 8, se concluye que $\triangle FCD$ es isósceles.
Q.E.D.

Figure 1. Answers corresponding to the resolution of the problem (on the left side demonstration 1 of E1, on the right side demonstration 2 of E4).

In general, it is observed that students used results related to characteristics of cyclic quadrilaterals, criteria of congruence and similarity of triangles, and inscribed angles in a circle. In addition, they expressed using Geogebra to verify the ideas that emerged in this process.

In the argumentation generated by E1, it can be observed that they use (although it is not explicitly mentioned) the property that the internal angles of a triangle add up to 180° together with the fact that the angles of the initial square are right angles in order to obtain the value of other angles. In addition, it is observed that one of the (almost immediate) ways of acting of the students is to obtain results by means of mathematical calculations even when the relevance of performing certain calculations has not been established, i.e. more calculations are performed than necessary (not for that reason incorrect), so that the process of monitoring and refining the elaborated demonstration could be improved. Subsequently, they use the fact that if in a convex quadrilateral its opposite angles are supplementary (add up to 180°) then the quadrilateral is cyclic, this result is also not explicitly expressed. Finally, they also implicitly use the property that in a cyclic convex quadrilateral the measure of an angle formed by one side and a diagonal is equal to that of the angle formed by the side opposite to the first and the other diagonal.

In E4's answer we can see that their argumentation takes as a starting point that the quadrilateral FEDC is cyclic, this is argued because the [opposite] angles of the quadrilateral are supplementary. On the other hand, in point 2, it is not argued why point G is the midpoint of the segment AB. Then, like E1, in (3) they implicitly use the fact that in a convex cyclic quadrilateral the measure of an angle formed by one side and one diagonal is equal to that of the angle formed by the side opposite to the first and the other diagonal. In (4) it is observed that although students handle the concept of congruence and the criteria that allow them to establish the congruence of triangles, they do not correctly use the notation to express this fact, which subsequently leads them to compare sides and corresponding angles correctly or incorrectly (point 5). In (8) it is established that by considering equal angles and subtracting other angles whose measure is equal, equal angles will be obtained, which will make it possible to establish the equality of two internal angles of the triangle in question in order to demonstrate that it is an isosceles triangle. Although the students' reasoning is correct and allows them to reach the desired conclusion, it is observed that the equality considered in (7), the expressions corresponding to the angles and the corresponding subtractions are incorrect. This shows that there is still a need to work on the process of transferring the ideas [oral or thought] to a written and formal argumentation.

Analysis: Problem extensions

In group G1 it was observed that the extension phase was not developed by most of the students as they omitted this section, those who elaborated an answer (E3) considered asking about some properties that are generated by other objects in the construction or if the circumcircle of the triangle DEB is considered, how many other circumcircles present in the construction are going to be cut by the first one? None of these questions are answered. On the other hand, one student in particular, expressed that by means of Geogebra she built (and replicated) the initial construction, with which she observed that "a repetitive figure" was formed, with which she asks herself if this construction forms a fractal, "how does it look like to repeat this process", "how is it demonstrated that point D is the midpoint of LK and that D is the vertex of ED? These questions are left open for further exploration.

Conclusions

The dynamics implemented in group G1 course encouraged students to use Geogebra as a means for exploration, understanding, obtaining conjectures, and verifying some initial ideas. This leads us to conclude that it is important to provide physical and temporal spaces in high-school and university mathematics courses for students to use GDS not only as a means of representation but also to manipulate the elements that make up the construction to obtain conjectures and verify them, and even more so, to obtain their own or additional results of the problem from this manipulation.

With regard to argumentation, on the one hand, it is important for students to first recognize the mathematical resources they have and then to be able to apply them in problem solving; on the other hand, it is also important that during mathematics education, spaces are created to develop oral and written arguments that allow them to develop an informal mathematical thinking, but with a logical sequence that brings them closer to the process of abstraction and demonstration. Thus, it was observed that the students had the opportunity to put into action various mathematical skills and seek different ways of arguing and supporting the relevance and validity of the conjectures. The results indicate that the students extended their ways of reasoning, allowing them to move from empirical arguments to formal arguments in the presentation of solutions to the problems.

On the other hand, it was observed that within the group dynamics there is a lack of space for students to pose their own problems, either based on those proposed by the teacher or not. Thus, the role of the teacher in guiding and providing feedback during the monitoring and control processes is considered to be of utmost importance.

Finally, it is crucial that dynamics in mathematics courses encourage students to make decisions regarding the way in which the mathematical problems presented are approached, the manipulation of auxiliary tools such as technologies for the exploration and construction of mathematical objects in order to obtain conjectures and create spaces for the generation of formal arguments, the use of mathematical notation and symbology and the refinement of proofs to strengthen mathematical reasoning during the transition to university.

Acknowledgments

The authors wish to thank Department of Mathematics, Faculty of Sciences, National Autonomous University of Mexico (UNAM) and their colleagues at the participating universities for making this study possible by facilitating the data collection.

References

Clark, M., Lovric, M. Suggestion for a theoretical model for secondary-tertiary transition in mathematics. *Mathematics Education Research Journal, 20*, 25–37 (2008). https://doi.org/10.1007/BF03217475

Di Martino, P., Gregorio, F. (2019). The Mathematical Crisis in Secondary–Tertiary Transition. *International Journal of Science and Mathematics Education, 17*, 825–843. https://doi.org/10.1007/s10763-018-9894-y

Galindo, C. (2016). Desarrollo de habilidades básicas para la comprensión de la geometría*. EMA*, 2(1), 49-58.

Hull, S. H., Seeley, C. L., & Hirsch, C. (2010). High School to Postsecondary Education: Challenges of Transition. *The Mathematics Teacher*, *103*(6), 442–445. http://www.jstor.org/stable/20876659

INEE. (2019). *Panorama Educativo de México 2018. Indicadores del Sistema Educativo Nacional. Educación básica y media superior.* INEE, México. https://www.inee.edu.mx/wp-content/uploads/2019/08/P1B117.pdf

Leviatan, T. (2008). Bridging a cultural gap. *Mathematics Education Research Journal, 20*, 105–116. https://doi.org/10.1007/BF03217480

Liljedahl, P. (2016). Building thinking classrooms: Conditions for problem solving. En P. Felmer, J. Kilpatrick y E. Pekhonen (Eds.). *Posing and solving mathematical problems: Advances and new perspectives,* (361-386). Springer.

Liljedahl, P. (2019). Conditions for Supporting Problem Solving: Vertical Non-permanent Surfaces*.* En Liljedahl, P., Santos-Trigo, M. (Eds.). *Mathematical Problem Solving, Current Themes, Trends, and Research.* Springer.

Melhuish, K., Vroom, K., Lew, K., & Ellis, B. (2022). Operationalizing authentic mathematical proof activity using disciplinary tools. *Journal of Mathematical Behavior*, *68*. https://doi.org/10.1016/j.jmathb.2022.101009

OECD. (2019). *PISA 2018 Results (Volume I): What Students Know and Can Do*. PISA, OECD. https://doi.org/10.1787/5f07c754-en

Polya, G. (1965). *Cómo plantear y resolver problemas*. (Trad. J. Zugazagoitia). México: Trillas (Trabajo original publicado en 1945).

Rach, S. & Heinze, A. (2016). The Transition from School to University in Mathematics: Which Influence Do School-Related Variables Have?. *International Journal of Science and Mathematics Education*, *15*. https://doi.org/10.1007/s10763-016-9744-8

Santos Trigo, M. (2019). *Mathematical problem Solving and the Use of Digital Technologies*. En P. Liljedahl, M. Santos-Trigo (Eds.). *Mathematical Problem Solving. Current Themes, Trends, and Research. ICME-13 Monographs*. Springer.

Santos Trigo, M. (2020). La resolución de Problemas Matemáticos: Conectando el trabajo de Polya con el desarrollo del razonamiento digital. En Yuri Morales-López y Ángel Ruíz (Eds.), *Educación Matemática en las Américas 2019* (pp.29-40). CIAEM.

Santos-Trigo, M., Aguilar-Magallón, D., Reyes-Rodríguez, I. (2019). A mathematical problem-solving approach based on digital technology affordances to represent, explore, and solve problems via geometric reasoning. En P. Felmer, P. Liljedahl y B. Koichu (Eds.). *Problem Solving in Mathematics Instruction and Teacher Professional Development, Research in Mathematics Education*, (pp. 145-166). Springer.

Santos Trigo, M. y Reyes Martínez, I. (2019). High school prospective teachers' problem-solving reasoning that involves the coordinated use of digital technologies. *International Journal of Mathematical Education in Science and Technology,* 50(2), 182-201. https://doi.org/10.1080/0020739X.2018.1489075

Schoenfeld, A. H. (1985). *Mathematical Problem Solving*. Academic Press.

Schoenfeld, A. H. (2014). What makes for powerful classrooms, and how can we support teachers in creating them? A story of research and practice, productively intertwined. *Educational Researcher*, 43(8), 404–412. https://doi.org/10.3102/0013189X1455

Schoenfeld, A. H., Thomas, M., y Barton, Bill. (2016). On understanding and improving the teaching of university Mathematics. *International Journal of STEM Education* 3(4). https://doi.org/10.1186/s40594-016-0038-z

Schoenfeld, A. H. (2022). Why Are Learning and Teaching Mathematics So Difficult? In *Handbook of Cognitive Mathematics* (pp. 1–35). Springer International Publishing. https://doi.org/10.1007/978-3-030-44982-7_10-1

IMPACT OF EMBEDDED MENTAL MODELS ON MODELING PROCESS

Sarah Gilchrist
The Ohio State University
gilchrist.42@osu.edu

Azita Manouchehri
The Ohio State University
manouchehri.1@osu.edu

Keywords: modeling, mathematical representations, cognition, preservice teacher education

The Realistic Mathematics Education (RME) approach was developed and popularized in the Netherlands in the 1970s, built on Freudenthal's observation that mathematics is a human activity (Treffers, 1993). RME involves students working on context problems, problems whose context is experientially real to them. When looking at student work on context problems, Gravemeijer and Doorman (1999) highlighted different models generated by students and ways that students might transition between them. Realistic modeling, an important perspective on mathematical modeling, focuses on students' abilities and skills to model and understand real-world and authentic problems (Pollak & Garfunkel, 2013; Abassian, 2020). Research in both RME and realistic modeling emphasizes the importance of the models students produce and use when solving realistically grounded contexts. In this work we considered how preservice teachers' knowledge of the problem context and their interest in the setting influence their choices when engaged in mathematical modeling.

This report is part of a broader study whose primary goal is to unpack the quality and content of interactions around mathematical modeling tasks. We conducted clinical interviews with participating teachers. The data was analyzed to determine what factors influenced the choices that the participants made. We found distinct ways the participants' use of embedded mental models impacted the choices they made while working on modeling tasks. Participants working on one of the tasks used in the study, "the classroom task" is discussed below as an illustration.

The task involved determining the number of desks that could fit into a classroom based on the CDC COVID guidelines. No model was included as part of the task; instead personal experiences shaped the assumptions made in exploring the problem. These personal experiences enforced particular conditions on the models produced and used by the participants. These particularities are what we call embedded mental models. Drawing on previous research around implementation of RME amongst young learners, we further note that working from an embedded model had many affordances for our participants. They engaged quickly and easily with the task and stayed engaged until they felt they had solved it. They quickly sketched a picture and discussed variables and parameters worthy of consideration. These practices were consistent with what they had experienced as students in regular classrooms.

Working with an embedded model imposed distinct constraints on participants' modeling process. It prevented them from considering alternative design approaches. Although the task could be seen as a packing problem, participants did not consider layouts other than rows and columns or table shapes other than rectangular model. When asked about a different layout for classroom set up, they expressed that they were unable to visualize a different arrangement. As such, embedded mental models limited the variables the participants considered or viable approaches that could move them towards constructing a generalized model.

When working with participants on mathematical modeling tasks (whether teachers or students), it is important to identify when they might be using an embedded mental model and what specific affordances and constraints they will experience as a result.

References

Aline Abassian, Farshid Safi, Sarah Bush & Jonathan Bostic (2020) Five different perspectives on mathematical modeling in mathematics education, Investigations in Mathematics Learning, 12:1, 53-65, DOI: 10.1080/19477503.2019.1595360

Gravemeijer, K., Doorman, M. Context Problems in Realistic Mathematics Education: A Calculus Course as an Example. *Educational Studies in Mathematics* **39**, 111–129 (1999). https://doi.org/10.1023/A:1003749919816

Pollak, H., & Garfunkel, S. (2013). A View of Mathematical Modeling in Mathematics Education. *Journal of Mathematics Education at Teachers College*. https://doi.org/10.7916/jmetc.v0i0.658

Treffers, A. Wiskobas and Freudenthal realistic mathematics education. *Educ Stud Math* **25**, 89–108 (1993). https://doi.org/10.1007/BF01274104

MUTABILITY OF STEM MAJORS' ABSTRACTED QUANTITATIVE STRUCTURES

Jennifer A. Czocher
Texas State University
czocher.1@txstate.edu

Andrew Baas
Texas State University
andrew.baas@txstate.edu

Elizabeth Roan
Texas State University
eaw109@txstate.edu

Abigail Quansah
Texas State University
alq8@txstate.edu

Recently, abstracted quantitative structures (AQS), a construct from quantitative reasoning, has been offered as a means to conceptualize and study mathematization during mathematical modeling. Extending this theoretical work, we provide empirical evidence that an intervention targeting participants' AQS can assist in aligning modelers' models with normatively correct models. We report on a pre/post intervention study designed to elicit alignment between symbolic forms and AQS and alignment between AQS and modeling scenarios. We used the Sørenson-Dice coefficient and cluster analysis to identify shifts in student associations of symbolic forms with modeling scenarios.

Keywords: modeling, mathematical representations, undergraduate education

Developing students' capacity to apply their mathematical knowledge to real-world scenarios is a central goal of mathematics education, especially for undergraduate STEM majors. Transfer of mathematics knowledge to a non-mathematical domain is difficult for students to do (Lesh & Zawojewski, 2007; Wake, 2014) and is challenging for researchers to study (Carraher & Schliemann, 2002; Evans, 1999; Lobato, 2006). Within research on the teaching and learning of mathematical modeling, the idea of transfer is captured by the idea of *mathematizing* or recognizing mathematical structure within a real-world scenario (Maaß, 2006; Zbiek & Conner, 2006). Mathematization has been difficult to study because it largely occurs as a modeler's mental actions, and despite this fact, has been under-theorized (Cevikbas et al., 2021). Recently, the field has made considerable efforts towards finding theoretically-grounded ways to operationalize *mathematization* such that it can be studied; specifically, scholars have made progress in explaining modeler's mental actions from a cognitive constructivist perspective by examining modeling from a quantitative reasoning lens (Czocher & Hardison, 2019; Czocher et al., 2022; Kularajan, 2023; Niss, 2010). Such approaches are necessarily based in the students' own interpretations of real-world scenarios and representations of them. Some hypotheses have emerged. One is that transfer of mathematical knowledge between scenarios can be traced by attending to individuals' *abstracted quantitative structures* (AQS), which are networks of quantitative operations an individual has interiorized to such an extent that it is independent of figurative material (Moore et al., 2022). Another is that operationalizing mathematization through the theoretical machinery of quantitative reasoning provides leverage for designing interventions and supports that may improve students' overall modeling skills. This paper adds to the conversation by addressing both hypotheses.

If the approach of operationalizing modeling in terms of quantitative reasoning is viable, then the field needs empirical evidence that abstracted quantitative structures are mutable during modeling and that real-world scenario can be assimilated into schema associated with modelers' AQSs. In this study, we operationalize AQS in terms of Sherin's (2001) symbolic forms and report on a pre/post intervention conducted with undergraduate STEM majors. We answer the research questions: *Are participants' ways of reasoning with abstracted quantitative structures stable, when subjected to a learning environment focused on modeling with those structures?* and *Is there an impact on participants' associations between symbolic forms and scenarios*?

Theoretical Perspectives on Modeling & Empirical Background on Mathematical Concepts

We adopt the cognitive perspective on mathematical modeling which is suitable for studying modelers' processes of rendering a real-world problem as a mathematical problem to solve (Kaiser, 2017). The cognitive perspective articulates the phases a modeler may pass through to specify a mathematical problem. These phases include anticipating mathematical structures that may be useful in representing and solving the problem, carrying out a solution process, and interpreting and validating the solution in terms of real-world constraints. *Mathematizing*, one phase of mathematical modeling, refers specifically to introducing conventional representational systems (e.g., equations, graphs, and tables) to represent mathematical "properties and parameters that correspond to the situational conditions and assumptions that have been specified" (Zbiek & Conner, 2006, p. 99). Others have pointed out that mathematizing entails "anticipating mathematical representations and mathematical questions that, from previous experience, have been successful when put to similar use" (Stillman & Brown, 2014, p. 766). That is, modelers need an idea of what to try out as a model that might be adequate. While the idea of *anticipation* and *implemented anticipation* (Niss, 2010) has gained traction in modeling research for their descriptive power, they lack both explanatory and predictive power with regards to how to aid students during mathematization. We view the idea of implemented anticipation through the lens of quantitative reasoning to explain *what* is anticipated and *how* anticipated structures are formed.

Sherin (2001) elaborated on the construct of *symbolic forms* that contemplates modelers' associations between algebraic *templates* and *conceptual schema*. The template is a format (e.g., $\_ \times \_ = \_$) that can express a mathematical idea or relationship (e.g., rate of change is proportional to amount present). Symbolic forms help explain how and why a modeler might choose to use $\times$ instead of $+$ when constructing an equation to represent a scenario. Theoretical work on quantities, quantification, and quantitative reasoning helps explain how individuals imbue the templates, variables, and conceptual schema with situationally relevant meanings. Quantitative reasoning means conceiving quantities and relationships among quantities; those relationships may be arithmetic (numerically evaluated) or quantitative (mental operations) (Thompson, 1990). Thompson gave an example using two individuals' heights to demonstrate that an *additive comparison* (mental operation) can be evaluated using *subtraction* (arithmetic operation). However, a comparison of the difference in two individuals' heights to the difference in another two individuals' heights (the difference between A and B is N times more than the difference between C and D) does not require evaluation of the differences in order to conceive the quantitative meaning. An abstracted quantitative structure (AQS) is a network of quantitative operations that an individual has interiorized and can operate *as if* it is independent of a figurative material (Moore et al., 2019). The core idea is that when a modeler has constructed an AQS, it is available to the individual when the scenario that engendered its construction is no longer present. Moore et al. (2019) clearly demonstrated evidence of the construction of AQSs and evidence of assimilation of new-to-the-student scenarios to previously constructed AQSs. Moore et al. (2019) also hypothesized that AQSs play an important role in transfer due to cognitive reorganization of previous experiences. Here, by transfer, we mean recognizing the applicability of symbolic forms for the purpose of mathematizing a given scenario, that is, *implemented anticipation* (Niss, 2010).

Mathematization of a real-world scenario is notoriously difficult for students across grade bands and content areas. Even students in advanced mathematics find it challenging because they face more complex scenarios where *rate of change* takes on dual roles as both dependent and independent variables. In such scenarios, students struggle with the idea that time is an implicit variable (Keene, 2007) and often interchange a differential equation with its solution (Donovan, 2002). Scaffolding for these students includes placing emphasis on the system being modeled and making explicit connections between the equation and the system it represents (Baker, 2009; Czocher, 2017; Myers et al., 2008; Pennell et al., 2009). For these reasons, we designed a study to generate empirical evidence as to whether participants' ways of reasoning with AQSs are stable throughout a learning environment focused on modeling with those structures. With reference to previous work on differential equations, our study focused on the templates of symbolic forms in Figure 3 and the learning environment emphasized these forms as representing conceptual schema like *rate of change is constant (A), rate of change is proportional to quantity present (B), and net rate of change (C)*. Throughout, we will refer to the answer choices in the Matching Tasks as "templates," intending to correspond to a family of *symbolic forms* and we will use "AQS" to refer to the *abstracted quantitative structures* we observed participants create, use, and re-use during the open modeling tasks comprising the learning environment. The templates operationalize the AQS as generic formats.

Methods

We conducted a pre/post intervention study within a design research project, which we evaluated using a paired samples t-test and a follow-up study using cluster analysis techniques. The intervention was a learning environment – task sequence and scaffolding – designed to build participants' mathematizing competencies. The learning environment comprised 10 hour-long task-based interviews with each of 23 undergraduate STEM majors recruited from courses listing differential equations as a prerequisite. The 10 sessions were organized so that the first and last sessions – our focus in this report – featured prompts intended to (a) document the criteria participants used to classify scenarios and (b) document participants' association of templates with the given scenarios. The middle 8 sessions consistently emphasized recognizing symbolic forms and associating them with quantitative structures participants recognized in the real-world scenarios presented in the tasks.

We describe the intervention's tasks using Yeo (2007) classification framework. The tasks were mathematical modeling problems with well-defined goals (develop a model for the scenario) but ill-defined answers (multiple valid models). Earlier tasks in the sequence provided built-in guidance and were relatively simple in that they called for fewer quantities and quantitative operations. Later tasks were not guided and were also more complex, being open to constructing many quantities and manipulating them with quantitative operations and relationships. In this paper, we focus on the first and final sessions, which featured pre/post items intended to document the participants' associations between scenarios and quantitative structures. In Sessions 1 and 10 we gave the Matching Task to document stability of reasoning with the target templates.

The Matching Task prompt (Figure 3) asked participants to match scenarios to the templates (a) – (e) based on what *they* saw as relevant within the problem scenario. For example, Item 9 is an abbreviated prompt based on a canonical salty tank problem from differential equations. It was an open problem during the modeling sessions and as a pre/post item was reduced to indicating the template perceived as appropriate to the participant. Participants were asked for explanation of their answers, information which we use to enrich our analysis.

> Consider the following mathematical expressions:
>
> a) $\frac{dQ}{dt} = k$ b) $\frac{dQ}{dt} = k \cdot Q(t)$ c) $I = a \pm b$ d) $I = k \cdot a \cdot b$ e) $\frac{dQ}{dt} = kP(t)Q(t)$
>
> where $Q(t)$ and $P(t)$ are quantities, $Q'(t)$ is the rate $Q(t)$ changes with respect to time, k is a constant, and a, b, and I are quantities that possibly depend on time (or not) or may be composed of other quantities (or not). Which of the mathematical expressions above (**either individually or as a composition**) can be used to model the real-world scenarios below?
>
> Item 4: Consider a natural habitat where bobcats are natural predators of rabbits. Bobcats are very good hunters, but they aren't perfect. Therefore, not all of the bobcat/rabbit encounters result in a rabbit's death. Model *the number* of rabbit deaths due to only predation by bobcats.
>
> Item 9: Consider a tank where water and a salty solution enters the tank while the well-mixed liquid inside the tank exits the tank. Model *the rate* at which the amount of salt in the tank changes with respect to time.

Figure 3: Abbreviated statement of Matching Task

A total of $N = 23$ undergraduate STEM majors participated in the design research project whose result was the modeling intervention reported in this paper. The participants were generally high preforming, reporting high grades in mathematics ($M = 3.4, SD = 0.5$) and a high level of confidence with relevant mathematical concepts ($M = 364, SD = 69$ out of 500 points on a confidence scale). We describe the sample composition by gender and major, but do not analyze the data according to these subgroups because of the small resulting sample sizes. Approximately 22% of participants identified as female, 74% as male, and 4% as non-binary. Approximately 30% of participants were pursuing a degree related to computer science, 35% in electrical engineering, 17% in civil engineering, 13% in physics, and 4.3% in mechanical engineering. Because the overall project used design methodology to develop the intervention, the Matching Task was revised after Implementation 1 ($N = 6$) to provide response E. We exclude $N = 9$ participants' responses from the t-test and pre/post response comparisons because the pre/post response options were not the same for participants in the earliest implementations ($N = 7$), and some participants did not complete the post-test ($N = 2$). However, we did include all participants' responses in the pre-test cluster analysis ($N = 23$) and all except the latter in the post-test analysis ($N = 21$) by using the response similarity metric (described below).

To quantify similarity of participants' ways of reasoning with the templates, we established a metric for response pattern proximity, as follows. Participants were encouraged to consider the templates as composites. For example, the normatively correct response for Item 9 was keyed as B&C, where the template B reflects the fact that the rate at which salt leaves the tank is proportional to the current amount of salt in the tank, and C reflects additive comparison of the inflow and outflow (net rate). Selecting C was regarded as closer to the keyed answer than selecting A and scoring reflected this. We chose the Sørenson-Dice coefficient (SDC) as a similarity metric to calculate scores because it both emphasizes similarities in response patterns and ignores template options which were neither included in the student answer nor the keyed answer. The SDC provides a 0 to 1 score for each item which can be interpreted as a percent overlap of the participant and keyed answers and is computed as $SDC = 2|P \cap K|/(|P| + |K|)$ for student answers P and keyed answers K (Sørenson, 1948). We calculated SDC per item and then summed to generate a participant pre- and post-test score between 0 and 9. We conducted a paired samples t-test on these scores, which met all assumptions.

We then explored shifts in associations per-participant and per-item using hierarchical cluster analysis techniques available in SPSS. Cluster analysis is a "data-driven approach" that allows clumping together participants who respond similarly, instead of using *a priori* groups. The goal is to organize heterogeneous samples into smaller groups that are maximally similar within-group and maximally dissimilar across groups (Woods et al., 2020). We estimated similarity by applying the SDC metric on participant responses to the Matching Task items to determine similarity between sets of participant responses. Finally, we teased apart the influence of item- and individual-level response patterns to articulate an account of shifts in participants' associations between symbolic forms (represented by the response choices) and scenarios.

Results

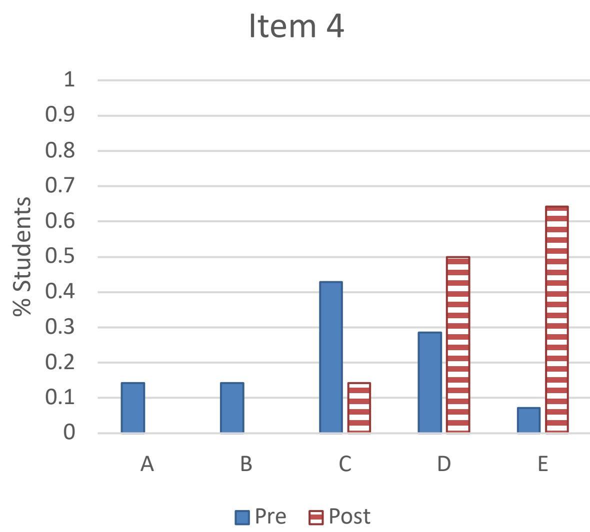
A paired t-test of the scores of the pre-tests ($M = 4.37,\ SD = 1.54$) and post-test ($M = 6.42,\ SD = 1.29$) suggest that the intervention resulted in additional normatively-correct associations between the templates and problems ($t(13) = 4.8, p < 0.001$), with a paired samples effect size of $d = 1.25$. In the aggregate, participants' associations between scenarios and templates shifted. Calculating the SDC between students' pre and post answers to each item (n = 126 pairs of pre/post responses from 9 items and 14 participants), we found that 41% of the post-test responses had no similarity with the corresponding pre-test answer, with an average pre/post-test similarity of 0.50. The cluster analysis revealed that similarity of participants' responses was lower on the pre-test than on the post-test (Table 1). Clustering performed on pre-test response sets yielded low average similarity. The two primary pre-test clusters included 11 of the 23 participants with average internal similarities of 0.63 ± 0.08 and 0.66 ± 0.08. The post-test revealed two clear primary clusters containing 8 and 7 participants respectively, and the 6 remaining participants were clear outliers from either cluster. Table 1 provides the similarity details of the post-test clusters, as well as average performance of students within these clusters.

Table 1: Post-Test Cluster Statistics

		Internal Similarity		Matching Task Performance		
Group	n	Mean	Range	Pre[2]	Post	Improvement[2]
Cluster 1	8	0.76 ± 0.07	(0.59, 0.93)	5.27 ± 1.31	7.43 ± 0.68	2.16 ± 1.65
Cluster 2	7	0.76 ± 0.08	(0.55,0.91)	3.87 ± 1.33	6.10 ± 0.64	2.25 ± 1.58
Outliers	6	0.40 ± 0.10	(0.19,0.56)	3.15 ± 0.52	4.31 ± 0.35	1.17 ± 0.17

- The cluster analyses revealed overall shifts towards the normatively correct response patterns and also indicated non-conforming item- and participant-level response patterns. For example, pre/post response patterns on Item 4 and Item 9 are shown in

[2] These columns only include scores on the final form of the pre/post test, resulting in $n_1 = 6, n_2 = 6, n_{out} = 2$.



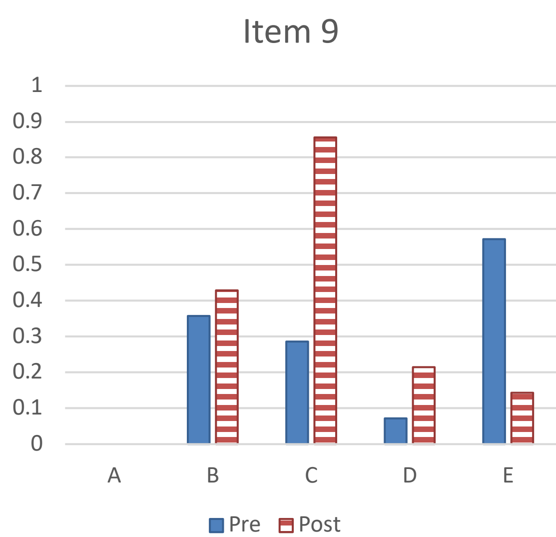


Figure 4. For the $n = 14$ participants who took the same version of the pre/post tests, the graphs illustrate a general shift towards the normatively correct responses for each item. On Item 9, just shy of 60% of participants included E in their responses, while only 29% included C, and 14% selected both. Participants who provided a justification for selecting only E mentioned it looked similar to what they recalled from their differential equations class and that the multiplicative factors represented inflow and outflow. On the post-test, 86% associated template C with the scenario, and only 14% kept E, indicating that many participants learned to associate the conceptual schema for the superposition of flows with template C as necessary to model the salt tank problem.

On some items, participant response patterns did not shift towards the keyed answers. In item 4, the response frequency for E increased though D was the keyed answer. Item 4 describes a predator-prey scenario. The item requests a model for the *number* of rabbit deaths due to predation, explicitly prompting the participant for an amount rather than a rate. The keyed answer D reflects a proportion of the possible interactions between the bobcat population and the rabbit population. On the pre-test, the most selected template was C (43%). Participant justifications indicated they were conceiving the predator-prey interactions as being one-to-one, implying that the number of rabbits killed by bobcats was equivalent to the number of bobcats. On the post-test, the percentage of participants selecting C dropped to 14% while the proportion who chose E rose from 7% to 64% and the percentage who chose D rose from 29% to 50%. One of the open modelling tasks during the sessions featured a predator-prey scenario in which participants leveraged a symbolic form which fits into template E (as in the Lotka-Volterra equations). In comparison, Item 5 presented another predator-prey scenario and requested a model for the species interactive dynamics. The keyed answer was B&C&E to reflect two equations for the two species. On this item, participant responses did shift towards the keyed answer, with B rising from 21% to 50%, C from 29% to 57%, and E from 36% to 86%. No participants selected A in pre or post, and D decreased from 36% to 29%. Thus, the response patterns suggest that many participants learned to associate the template with the predator-prey scenario, but that the finer conceptions involved in transferring an AQS that call for distinction between amounts and rates-of-change of amounts were obscured.

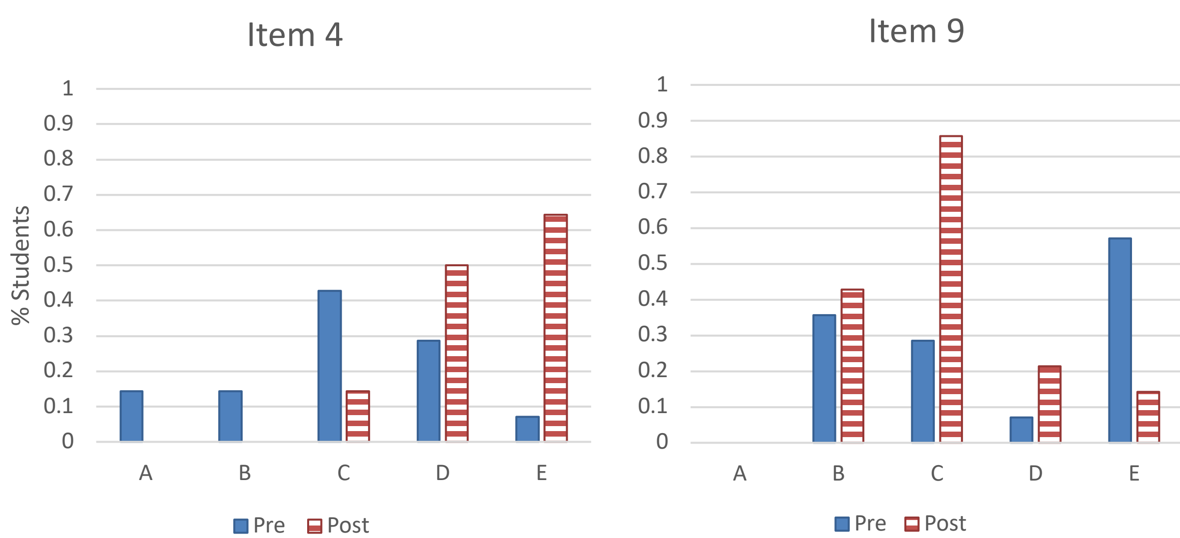


Figure 4 Relative Frequency of Multi-Select Responses to Items 4 and 9

To better understand shifts in patterns of reasoning in relation to the item contexts, we closely examined Yixli, Tien, Niali, and Khriss. These participants were chosen because they exhibited four archetypal cases arising from a 2 × 2 configuration: associations were (not) shifted × associations were (not) normatively correct. The existence of the four archetypal cases demonstrates that shifting associations does not imply a shift to a normative association. For each participant, we describe how their ways of reasoning with AQS may have changed (or not) from pre to post. Yixli revised his responses to only Item 4 and 5 from pre to post; his post score was in the lower third of the sample (5.5/9). Explaining his change from C to E on Item 4, he stated, “because at this point it’s, like, beaten into my skull from all the problems that we did” [in the sessions]. Yixli successfully developed a system of differential equations for the predator-prey scenario during the open modeling sessions. Thus, his statement supports the inference that he learned to associate predator-prey scenarios with template E but still inconsistently associated an AQS for rate of change in amount with a prompt requesting an amount, a common conflation when learning to model with differential equations (Rasmussen & King, 2000; Rowland & Jovanoski, 2004). More generally, Yixli demonstrated high levels of recognition of key features of the scenarios but did not associate normatively correct AQS with those features. For example, in items 2 and 4, he selected non-normative responses because he furnished a rate of change equation instead of an amount equation. In item 6, he selects one keyed response accounting for only part of the information presented in the prompt. Finally, on item 8, Yixli associated the scenario with exponential growth (B) instead of linear growth (A) based on the real-world setting rather than its properties. Considering Yixli's response patterns across items explains why the majority of his responses – which were non-normative – did not shift from pre- to post. There were few non-normative associations for the intervention to address and Yixli's tendency to focus on the real-world setting rather than its features was unperturbed by the intervention.

Tien changed responses to 4 items, selecting additional templates on the post while maintaining the templates he selected on the pre. He scored in the top third of the sample on post (7.3/9). On his pre-test, Tien selected at least one keyed response on all except two items; on items 4 and 9 he selected options C and E, respectively. He changed his response on item 4 to C&D&E, explaining that, for him, D reflected the number of rabbit deaths due to bobcats, while C and E reflected other quantities in the scenario. He similarly revised his responses to items 5 and 6 to select C&D&E, which overlapped with the keyed response B&C&E. Items 5 and 6 treated population dynamics, indicating that Tien viewed the C&D&E composition of templates as important for modeling population dynamics. Finally, Tien revised his response to item 9 from C to C&E, overlapping with the keyed response B&C. However, Tien explained that C reflected the sum of salt content and solution content, yielding the total amount of substance in the tank, not the additive comparison of inflow to outflow. We interpret Tien's response patterns as indicating that he began the intervention with mostly normative associations between scenarios and productive symbolic forms and also gained additional association with quantitative structures that would aid him in modeling population dynamics. Additionally, his associations of forms with the salty tank scenario were correct for his conceptions, but non-normative.

Unlike the previous two, Khriss revised his answers on all but 3 items but only slightly improved his score by a single point from 3.7 to 4.7, ending up scoring in the bottom tenth of the sample. On the pre-test, Khriss exclusively utilized the templates B, C, and E. On problem 2 he wrote out a normative equation but selected a non-normative template. Similarly, on problems 3 and 8 he selected the template E to model an exponential growth problem, demonstrating that he had a normatively correct answer on 3 and a common non-normative answer on 8, but simply selected a non-normative template to describe those models. This was continued in the post-test on which Khriss was able to identify normative equations which modelled every problem, but then associated the scenarios with non-normative templates. The primary distinction between Khriss's pre and post tests was that the equations Khriss wrote for the post-test responses were more detailed and normatively-correct than the equations Khriss wrote while taking the pre-test. This indicates that, while Khriss's associations of templates with scenarios did not generally become more normative, the symbolic forms used by Khriss did.

Finally, Niali revised his responses to nearly all items, improving from 2 to 7.2, and scored in the top third of the sample on the post-test (7.2/9). During the sessions, Niali indicated a robust association between template A and scenarios featuring linear growth and between template B and scenarios featuring exponential growth. Thus, we infer that his low pre-test scores were not due to a lack of transferrable AQS's adequate to distinguishing those scenarios mathematically. Instead, he over-selected E for exponential growth scenarios rather than B. By the post-test, Niali associated E with scenarios featuring a rate of change dependent upon interaction between two quantities. Like Yixli, he responded to the real-world setting of item 8 rather than the scenario-specific features in the item.

Conclusions

The paired samples t-test indicates that undergraduate STEM majors' ways of reasoning with abstracted quantitative structures can be modified through modeling with those structures. The cluster analyses also revealed overall shifts towards the normatively correct response patterns and also indicated non-conforming item- and participant-level response patterns. We observed that for some participants, the context was a stronger indicator of response selection than scenario-specific features. In previous literature, this phenomenon has been referred to as focusing on "surface features" vs. "deep structure" (Schoenfeld & Herrmann, 1982) However, we argue that the conclusion is not so simple. Our participants proffered "deep structural" explanations based on their structural conceptions of quantities and relationships among quantities even when seemingly focused on contextual features. That is, many selecting non-normative responses demonstrated evidence of structural ways of reasoning rather than superficial reasoning.

One limitation of our approach is that template-matching is not perfectly predictive of associations between scenarios and symbolic forms, as Khriss's response patterns revealed. Teasing this apart might require developing a more nuanced similarity metric that accounts for the idea that template B is actually a sub-template of E, both of which can be expressed as instances of C (though many participants did not evidence awareness of these insights). However, overall, the Sorensen-Dice coefficient enabled a complex metric that modeled the response pattern data well. It supported a robust comparison of *similarities* in participants' ways of reasoning with sets of responses, providing an additional tool for evaluating complex reasoning patterns. Thus, we are optimistic about the approach to measuring the assimilation of scenarios to AQS' in this way because it opens possibilities for future work on AQS.

Our objective in the present study was to examine the stability of participants' ways of reasoning with symbolic forms when engaging with modeling tasks designed to help them assimilate scenarios to those forms. Recently, Kularajan (2023) argued that a promising approach to studying (and subsequently improving) students' capacity for mathematizing is to examine and respond to their emergent quantitative reasoning about the scenario. Our contribution is providing empirical support to this conjecture. We conclude that participants showed evidence of *changing* which scenarios are assimilated to a given AQS by matching it with a template as a consequence of engaging with modeling scenarios that reinforced the use of symbolic forms matching those AQSs. The next steps in this line of research are articulating the contours of learning environments that may be fruitful for students with differing ways of reasoning.

Acknowledgments

This research was supported by National Science Foundation Grant No. 1750813.

References

Baker, G. (2009). An Introduction to Differential Equations for Scientists and Engineers.
Carraher, D., & Schliemann, A. c. D. (2002). The Transfer Dilemma. The Journal of the Learning Sciences, 11(1), 1-24.

Cevikbas, M., Kaiser, G., & Schukajlow, S. (2021). A systematic literature review of the current discussion on mathematical modelling competencies: state-of-the-art developments in conceptualizing, measuring, and fostering. Educational Studies in Mathematics, 109(2), 205-236. https://doi.org/10.1007/s10649-021-10104-6
Czocher, J. A. (2017). How Can Emphasizing Mathematical Modeling Principles Benefit Students in a Traditionally Taught Differential Equations Course? Journal of Mathematical Behavior, 45, 78-94. http://dx.doi.org/10.1016/j.jmathb.2016.10.006
Czocher, J. A., & Hardison, H. (2019). Characterizing Evolution of Mathematical Models Proceedings of the 41st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, St Louis, MO.
Czocher, J. A., Hardison, H. L., & Kularajan, S. S. (2022). A bridging study analyzing mathematical model construction through a quantities-oriented lens [OriginalPaper]. Educational Studies in Mathematics, 1-23. https://doi.org/doi:10.1007/s10649-022-10163-3
Donovan, J. E., II. (2002). Students' Understanding of the First-Order Differential Equations (Publication Number 3063116) University of New York].
Evans, J. (1999). Building Bridges: Reflections on the problem of transfer of learning in mathematics. Educational Studies in Mathematics, 39(1), 23-44. https://doi.org/10.1023/A:1003755611058
Kaiser, G. (2017). The Teaching and Learning of Mathematical Modeling. In J. Cai (Ed.), Compendium for Research in Mathematics Education (pp. 267-291). National Council of Teachers of Mathematics.
Keene, K. A. (2007). A Characterization of Dynamic Reasoning: Reasoning with Time as Parameter. Journal of Mathematical Behavior, 26, 230-246. https://doi.org/10.1016/j.jmathb.2007.09.003
Kularajan, S. S. (2023). STEM Undergraduates' Structural Conception of Situational Attributes [PhD, Texas State University]. San Marcos, TX.
Lesh, R., & Zawojewski, J. (2007). Problem solving and modeling. In J. Frank K. Lester (Ed.), Second Handbook of Research on Mathematics Teaching and Learning (pp. 763-802). Information Age Publishing.
Lobato, J. (2006). Transfer Strand: Alternative Perspectives on the Transfer of Learning: History, Issues, and Challenges for Future Research. The Journal of the Learning Sciences, 15(4), 431-449.
Maaß, K. (2006). What are Modelling Competencies? ZDM, 38(2), 113-142. https://doi.org/https://doi.org/10.1007/BF02655885
Moore, K. C., Liang, B., Stevens, I. E., Tasova, H. I., & Paoletti, T. (2022). Abstracted Quantitative Structures: Using Quantitative Reasoning to Define Concept Construction. In G. Karagöz Akar, İ. Ö. Zembat, S. Arslan, & P. W. Thompson (Eds.), Quantitative Reasoning in Mathematics and Science Education (pp. 35-69). Springer International Publishing. https://doi.org/10.1007/978-3-031-14553-7_3
Moore, K. C., Liang, B., Stevens, I. E., Tasova, H. I., Paoletti, T., & Ying, Y. (2019). A Quantitative Reasoning Framing of Concept Construction. Twenty-Second Annual Conference on Research in Undergraduate Mathematics Education, Boston, MA.
Myers, J., Trubatch, D., & Winkel, B. (2008). Teaching Modeling with Partial Differential Equations: Several Successful Approaches. PRIMUS, 18(2), 161-182. https://doi.org/10.1080/10511970701643889
Niss, M. (2010). Modeling a Crucial Aspect of Students' Mathematical Modeling. In R. Lesh, P. L. Galbraith, C. Haines, & A. Hurford (Eds.), Modeling Students' Mathematical Modeling Competencies (pp. 43-59). Springer. https://doi.org/10.1007/978-1-4419-0561-1
Pennell, S., Avitabile, P., & White, J. (2009). An Engineering-Oriented Approach to the Introductory Differential Equations Course. PRIMUS, 19(1), 88-99. https://doi.org/10.1080/10511970701474111
Rasmussen, C., & King, K. (2000). Locating Starting in Differential Equations: A Realistic Mathematics Education Approach. International Journal of Mathematical Education in Science and Technology, 31(2), 161-172.
Rowland, D. R., & Jovanoski, Z. (2004). Student Interpretations of the Terms in First-Order Ordinary Differential Equations in Modelling Contexts. International Journal of Mathematical Education in Science and Technology, 35(4), 503-516.
Schoenfeld, A. H., & Herrmann, D. (1982). Problem perception and knowledge structure in expert and novice mathematical problem solvers. Journal of Experimental Psychology: Learning, Memory, and Cognition, 8, 484-494.
Sherin, B. L. (2001). How Students Understand Physics Equations. Cognition and Instruction, 19(4), 479-541. www.jstor.org/stable/3233857
Sørenson, T. (1948). A method of establishing groups of equal amplitude in plant sociology based on similarity of species and its application to analyses of the vegitation on Danish commons. Biologiske Skrifter, 5(41-34).

Stillman, G., & Brown, J. P. (2014). Evidence of Implemented Anticipation in Mathematising by Beginning Modellers. Mathematics Education Research Journal, 26, 763-789. https://doi.org/10.1007/s13394-014-0119-6

Thompson, P. (1990). A theoretical model of quantity-based reasoning in arithmetic and algebra.

Wake, G. (2014). Making Sense of and With Mathematics: The Interface Between Academic Mathematics and Mathematics in Practice. Educational Studies in Mathematic, 86, 271-290. https://doi.org/10.1007/s10649-014-9540-8

Woods, A. D., Ahmed, S. F., Katz, B. D., & Morrison, F. J. (2020). How Stable Is Early Academic Performance? Using Cluster Analysis to Classify Low Achievement and EF. J Learn Disabil, 53(1), 19-35. https://doi.org/10.1177/0022219419881632

Yeo, J. B. W. (2007). Mathematical tasks: Clarification, classification and choice of suitable tasks for different types of learning and assessment. https://repository.nie.edu.sg/bitstream/10497/949/3/MathematicalTasks.pdf

Zbiek, R. M., & Conner, A. (2006). Beyond Motivation: Exploring Mathematical Modeling as a Context for Deepening Students' Understandings of Curricular Mathematics. Educational Studies in Mathematics, 63, 89-112. https://doi.org/10.1007/s10649-005-9002-4

PRACTICING MATHEMATICS COLLABORATION IN STEM CAREERS

Daniel Heck
Horizon Research, Inc.
dheck@horizon-research.com

Jill V Hamm
UNC Chapel Hill
Jill.Hamm@unc.edu

Jessica Dula
Horizon Research, Inc.
j.dula@horizon-research.com

Keywords: Informal education, Communication

Deliberate Practice of Mathematics Collaboration

CASCADE is designing simulations of collaboration on tasks involving algebra and statistics that arise in STEM careers. Players engage with aspects of tasks, discourse, and social dynamics that shape mathematics collaboration (Hamm, Farmer, Lambert, & Gravelle, 2014; Heck & Hamm, 2016; Lotan, 2003; Smith & Stein, 1998). The simulations are designed in partnership with BIPOC professionals in STEM fields and leaders of programs serving grades 6-12 youth historically underrepresented in STEM careers. PROJECT's design research first draws on frameworks of mathematics collaboration and deliberate practice to design simulations, then adapts the design based on student playtesting.

Set in STEM career fields, the simulations engage players with processes of communication and problem solving to practice collaboration skills (Heck & Hamm, 2016) with virtual work partners. Following tenets of deliberate practice (Ericsson, Krampe & Tesch-Römer, 1993), players have repeated chances to try specific skills of collaboration, experience consequences, and receive feedback. Players' practice is low stakes in terms of achievement or consequences.

Designing the Simulation as a Learning Environment with Practice and Feedback

PROJECT simulations place players in virtual environments of STEM career fields with a narrative built around a mathematics task common in these careers. In first-person perspective, players encounter virtual partners with whom they collaborate. We utilize the three central features of mathematics collaboration—tasks, discourse, and social dynamics—to engage players with targeted challenges. As each challenge arises, players choose among options reflecting varying levels of collaborative functioning. Players receive two types of feedback: implicit feedback—a direction the narrative takes in response, and explicit feedback—explanations and guidance delivered by a mentor character (Folsom-Kovarik, Newton, Haley, & Wray, 2014).

Adapting the Simulation Design through Lessons Learned

Seven pairs of adolescents from a college pathway program playtested a simulation prototype. Their play was recorded using screencapture and a camera with audio, and researchers conducted debriefing interviews. Rather than having their choices judged "right" or "wrong," players appreciated receiving feedback and the experience of seeing both positive and negative consequences play out. Analysis of playtesting data generated two additional design elements. First, players are explicitly encouraged to explore choices they feel are non-productive, so they can see what consequences arise. Second, when players choose the most productive options, resulting feedback includes consequences that would have resulted from non-productive choices.

Acknowledgments

This research is funded by the National Science Foundation (NSF) to Hamm [2048993] and Heck [2048985]. The findings reported here do not necessarily reflect the opinions of the NSF. Hamm and Heck are married and Heck is employed by and co-owns Horizon Research, Inc. UNC reviewed these arrangements, concluding that the possible benefit to Hamm or Heck is not

likely to affect participant safety or the scientific quality of the study.

References

Ericsson, K. A., Krampe, R. T., & Tesch-Römer, C. (1993). The role of deliberate practice in the acquisition of expert performance. *Psychological Review, 100*(3), 363-406. http://dx.doi.org/10.1037/0033-295X.100.3.363

Hamm, J. V., Farmer, T. W., Lambert, K., & Gravelle, M. (2014). Enhancing peer cultures of academic effort and achievement in early adolescence: Promotive effects of the SEALS intervention. *Developmental Psychology*, *50*(1), 216-228. https://doi.org/10.1037/a0032979

Heck, D. J. & Hamm, J. V. (2016). Small group learning environments in middle grades classrooms. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1325–1328). Tucson, AZ: University of Arizona. https://doi.org/10.51272/pmena.38.2016

Lotan, R. A. (2003). Group-worthy tasks. *Educational Leadership, 60*(6), 72-75.

Folsom-Kovarik, J.T., Newton, C., Haley, J., & Wray, R.E. (2014). Modeling proficiency in a tailored, situated training environment. In *Proceedings of the Conference on Behavior Representation in Modeling and Simulation*.

Stein, M. K., & Smith, M. S. (1998). Mathematical tasks as a framework for reflection: From research to practice. *Mathematics Teaching in the Middle School, 3*, 268-275. https://doi.org/10.5951/mtms.3.4.0268

PROBLEM POSING IN INVESTIGATIONS: THE CASE OF BRANDI AND BEN

Andrew Kercher
Simon Fraser University
andrew_kercher@sfu.ca

Canan Güneş
Simon Fraser University
canan_gunes@sfu.ca

Rina Zazkis
Simon Fraser University
zazkis@sfu.ca

Extant research has demonstrated that problem-posing and problem-solving mutually affect one another. However, the exact nature and extent of this relationship requires a detailed elaboration. This is especially true when adidactical problem-posing arises within a problem-solving context. In this study, we analyze the scripting journey used by two students to record their investigation of sums of consecutive integers. We analyze the adidactical problem-posing found within the scripting journey using three facets of a problem posing framework: mathematical knowledge base, problem-posing heuristics, and individual considerations of aptness. Our analysis reveals how these aspects of problem-posing emerge within a mathematical investigation, how they are related to surrounding problem-solving, and what types of activity act as catalysts to promote further problem-posing activity.

Keywords: Problem Solving; Undergraduate Education; Preservice Teacher Education

Introduction

"How are problem-posing skills related to problem-solving skills?" (Cai et al., 2015, p. 14). According to Cai et al. (2015), this question is yet unanswered, but exploring it could lead to advances in the collective understanding of students' mathematical activity. We note that this is not a question of the existence of a relationship; at the time, extant research already supported the hypothesis that successful problem-posers were also successful problem-solvers, and vice versa (cf. Silver & Cai, 1996; Cai & Hwang, 2002; 2003). Instead, the question suggests that it is the fundamental character of this relationship that remains unclear—and it calls for mathematics education researchers to explore the nature and extent of the connection between problem-posing and problem-solving.

This call was not ignored. Liljedahl and Cai (2021) reported on advances in both problem-solving and problem-posing from the intervening years—and in particular, those studies that sought to understand better how the two fields intersect. For example, Elgrably & Leikin (2021) discovered that initial problem-solving efforts can spark later problem-posing creativity; conversely, Hartmann et al. (2021) noted that initial problem-posing can lead to unexpected student success in later problem-solving.

Although the two studies cited above combined problem-solving and problem-posing, in each case, all problem-posing activity was incited directly by the task itself. That is, participants were explicitly instructed to pose mathematical problems. However, Koichu (2020) exemplified that problem-posing can be instigated *adidactically*, "as an activity necessitated for the posers by the need to find or create problems that would serve another goal" (p. 3). In Koichu's work, this other goal was pedagogically oriented; problem-posing arose adidactically as participants prepared to teach a difficult topic. We wondered to what extent adidactical problem-posing might naturally arise when the ultimate goal is not to teach a particular mathematical idea, but instead, to investigate a particular mathematical phenomenon through problem-solving.

To this end, we developed an investigative task without an explicit problem-posing component. The data in this study is a self-reported dialogue inspired by participants' firsthand experiences as they use problem-solving to explore the task; we call this type of dialogue a scripting journey. We address the following research question: *How does adidactical problem-posing emerge when engaging with a problem-solving investigation?* A framework for the analysis of problem-posing, an overview of scripting journeys, and the task itself are provided in the following sections.

A Theoretical Framework for Problem-Posing Analysis

To address our research questions, we draw on a framework proposed by Kontorovich et al. (2012). The framework consists of attributes that attend to the cognitive, affective, and social dimensions of problem posing. In this paper, we analyze episodes of adidactical problem-posing activity using three of these attributes: mathematical knowledge base, problem-posing heuristics and strategies, and individual considerations of aptness.

A problem poser's *mathematical knowledge base* includes mathematical definitions, facts, procedures, prototypical problems, and competencies related to mathematical discourse and writing. *Problem-posing heuristics and strategies* refer to the systematic approaches that a problem poser adopts to analyze and transform a mathematical situation and, later, to pose problems. Drawing on previous research, Kontorovich et al. (2012) composed a provisional list of such strategies, three of which are of interest to this report:

1. *Numerical manipulation*: Posing a new problem by assigning different numerical values to given constraints.
2. *What-if-notting*: Posing a new problem by removing or changing either a given constraint or an underlying assumption about the mathematical setting.
3. *Generalization*: Posing a new problem "for which the given problem is a special case" (Kontorovich et al., 2012, p. 152).

Finally, throughout a problem-posing task, problem posers consider the suitability of posed problems for a particular audience. This audience might include themselves, a real or hypothetical evaluator, or an intended audience who will be tasked with solving the problem. *Individual considerations of aptness* are the problem posers' conceptualizations of the explicit and implicit criteria by which this audience will judge the posed problem (or by proxy, the problem-poser) and how necessary it may be to meet these criteria. For example, when considering whether a problem is appropriate for a later problem solver, the problem poser could try to anticipate whether the problem will be mathematically challenging, engaging, or capable of successfully teaching a desired concept to the solver.

The Task

Scripting Tasks and Scripting Journeys

A scripting task is an activity centered around the construction a mathematical dialogue, typically involving some combination of teacher- and student-characters. Sometimes, a scripting task provides a *prompt*—a few introductory lines of dialogue that introduce the topic of the script. When a prompt is included, it typically introduces a mathematical question of a student (e.g., Bergman et al., 2022; Kontorovich & Zazkis, 2016; Marmur & Zazkis, 2018), a misunderstanding (e.g., Zazkis et al., 2013), or a disagreement (e.g., Marmur et al., 2020; Zazkis & Zazkis, 2014) that the script should attend to and eventually resolve.

Recently, researchers have explored the application of scripting tasks that produce a particular type of dialogue referred to as a *scripting journey* (Kercher et al., in press). Unlike dialogues resulting from other types of scripting tasks, a scripting journey is not a continuation of a prompt; rather, the scriptwriters use their own mathematical activity as a model for constructing the scripting journey. Kercher et al. (in press) observed that student-written scripting journeys contain robust mathematical activity and are thus appropriate for analysis in mathematics education research. With this result in mind, we leverage scripting journeys to capture and analyze the emergent adidactical problem-posing activity within a problem-solving investigation.

The Consecutive Integers Task

The activity used to stimulate adidactical problem-posing was the *Consecutive Integers task* (CI task). Its introductory instructions are presented in Figure 1.

> The number 294 has the following interesting property:
>
> $$294 = 39 + 40 + 41 + 42 + 43 + 44 + 45$$
>
> That is, it can be written as the sum of 7 consecutive integers.
>
> **Your task is to investigate this property and its variations.** That is, your task is to investigate the sums of consecutive integers. The following questions can guide the beginning of your investigation. **You do not need to address all the suggested questions.** You can choose a few or you can proceed with your own problems/ questions.

Figure 1: The introduction of the Consecutive Integers task.

Participants were then provided with a selection of example questions to direct their explorations, which included: Consider any number of your choice. What values are possible for K? Can you find all of them? Can you generalize further? That is, given any natural number, can it be written as the sum of consecutive integers? If so, in what ways? If not, why?

Thus, the CI task can be thought of as a modular, semi-structured investigation in which participants were free to select from a number of smaller problem-solving tasks of varying degrees of open-endedness and mathematical sophistication. The task also invites participants to explore sums of consecutive integers independently of the suggested problems, and in doing so, guides participants towards posing and solving their own problems. Completion of the CI task required participants to record their mathematical activity as a scripting journey.

Participants, Data Collection, and Analysis

The data in this study is part of a larger study on the adidactical problem-posing of prospective elementary school teachers enrolled in a mathematics methods course. We focus on the work of Brandi and Ben (pseudonyms), who completed the CI task outside of usual classroom hours. Both scripting tasks and mathematical investigations were a typical component of the course, but the CI task was the first assignment in which the participants had been required to record their work as a scripting journey. The course did not include explicit problem-posing activities or instruction.

The research team first read and reread Brandi and Ben's scripting journey, in which they featured themselves as characters, to become familiar with their investigation. Then, the first author coded the script for instantiations of adidactical problem-posing. Considering that the participants were not asked to set aside a comprehensive list of what they considered to be the problems that they posed over the course of their investigation, two different methods were used to distinguish posed problems within the dialogue. First, we identified some explicit questions as posed problems—such as when Brandi asks, “But if 294 is X, what happens when X is unknown? For example, $7N + 21 = X$?” On the other hand, some posed problems were inferred from statements of intent or from a character’s musings. These problems did not always appear in the form of a question, but the remark was coded as a posed problem if it was suggestive of particular constraints and goals that the speaker had in mind. For example: “I wonder if this can be written as a formula to work with other patterns.”

The second author then independently coded the scripting journey for adidactical problem-posing, and these codes were compared with the first author’s codes. Discrepancies were resolved by discussion until complete agreement was reached. Brandi and Ben’s scripting journey was then subjected to analysis using the framework of Kontorovich et al. (2012). Throughout the application of the analysis framework, the entire authorial team met regularly to discuss their interpretations of the observed problem-posing activity.

Findings: Brandi and Ben

In this section, we first present a summary of Brandi and Ben’s scripting journey in the form of a short vignette. Following the vignette, we analyze the recorded problem-posing behavior using the framework of Kontorovich et al. (2012).

Vignette

Immediately upon beginning their investigation, Brandi and Ben work together to establish a way of representing 294 as the sum of 7 consecutive integers algebraically. First, Brandi wonders if the property that 294 can be written as the sum of 7 consecutive integers “can be written as a formula to work with other patterns.” In response, Ben proposes a formulation using the variable N:

> I think it should only go to $N + 6$ because N is the first integer and $N + 1$ is the second. Therefore, seven consecutive integers would end at $N + 6$. It should be: $N + (N + 1) + (N + 2) + (N + 3) + (N + 4) + (N + 5) + (N + 6)$.

Note that Ben uses the variable N to represent the smallest integer in the sequence even though the exact sequence of integers is given by the task (see Figure 1). This behavior, taken in context with Brandi’s stated desire to work with “other patterns,” suggests that the student-characters in this excerpt are working to solve an implicitly posed problem: namely, they are attempting to discover an algorithm that will allow them to locate a sequence of integers that add to a given sum without guessing and checking.

In support of this overarching problem, Brandi and Ben go on to pose a number of follow-up problems that they anticipate will help them better understand the functionality of their developing algorithm:

Table 1: Posed problems in response to the developing algorithm

	Speaker	Dialogue
[1]	Brandi	But if 294 is X, what happens when X is unknown? For example, $7N + 21 = X$.
[2]	Ben	[After noticing the sum is divisible by the number of divisors] Do you think that can work if the number of consecutive integers is an even number?
[3]	Ben	What happens when we solve for N and it's not a round whole number?

In the process of answering these problems, Ben eventually realizes procedure they have been using can be improved by simply dividing the desired sum by the number of consecutive integers. Then, "any quotient that is a whole number will have an odd number of consecutive integers, […] and any quotient that is a half number will have an even number of consecutive integers." The student-characters decide to leverage technology and construct a spreadsheet that will handle these computations. Freed from the responsibility of manual arithmetic, Brandi and Ben pose a sequence of problems that attend to more abstract concerns. These include:

Table 2: Posed problems in response to the creation of the spreadsheet

	Speaker	Dialogue
[4]	Brandi	What are you going to put in the table?
[5]	Brandi	How will you know which numbers [of consecutive integers] to check?
[6]	Brandi	Can it go to an unlimited number of even or odd consecutive integers?
[7]	Brandi	What happens when we have a huge number of consecutive integers?
[8]	Brandi	Well, I guess we can go to negative numbers?

Brandi wonders first about the information that should be provided to the spreadsheet; that is, she considers which information in the task should be considered a constraint and which should be a goal. The student-characters then explore the boundaries of what reasonable values for K, the number of consecutive integers, might be. In particular, Brandi wonders about a hypothetical upper bound on K; in response, the student-characters consider the inclusion of negative numbers and zero as one avenue for generating a "huge number of consecutive integers."

After attending to these considerations and building the spreadsheet (Figure 2), Brandi suggests that they test it with a newly posed problem: to find a list of all possible values of K for a completely different number, 165. Using their spreadsheet, they divide 165 by consecutive integers starting at 2, consider the quotient, and, where possible, produce the sequence of consecutive integers that sum to 165 by using the quotient as the midpoint of the sequence.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	AA	AB	AC	AD	AE	AF	AG	AH
1	Dividend	Divisor/ Number of Consecutive Integers	Quotient	Should this work?	Sum																													
2						1		2		3		4		5		6		7		8		9		10		11		12		13		14		15
3	165	2	82.5	Y	165	82	+	83																										
4	165	3	55	Y	165	54	+	55	+	56																								
5	165	4	41.25	N	0																													
6	165	5	33	Y	165	31	+	32	+	33	+	34	+	35																				
7	165	6	27.5	Y	165	25	+	26	+	27	+	28	+	29	+	30																		
8	165	7	23.5714	N	0																													
9	165	8	20.625	N	0																													
10	165	9	18.3333	N	0																													
11	165	10	16.5	Y	165	12	+	13	+	14	+	15	+	16	+	17	+	18	+	19	+	20	+	21										
12	165	11	15	Y	165	10	+	11	+	12	+	13	+	14	+	15	+	16	+	17	+	18	+	19	+	20								
13	165	12	13.75	N	0																													
14	165	13	12.6923	N	0																													
15	165	14	11.7857	N	0																													
16	165	15	11	Y	165	4	+	5	+	6	+	7	+	8	+	9	+	10	+	11	+	12	+	13	+	14	+	15	+	16	+	17	+	18
17	165	16	10.3125	N	0																													
18	165	17	9.70588	N	0																													
19	165	18	9.16667	N	0																													
20	165	19	8.68421	N	0																													
21	165	20	8.25	N	0																													

Figure 2: The first 20 of the 450 rows in Brandi and Ben's spreadsheet.

Compiling this list of ways in which 165 is expressed as the sum of integers inspires Brandi to pose a new collection of problems:

Table 3: Posed problems in response to the application of the spreadsheet to an example

	Speaker	Dialogue
[9]	Brandi	Is there something to do with the factors of the number?
[10]	Brandi	Why won't that [choosing $K = 18$] work?

In what follows, the student-characters attempt to uncover an underlying mathematical justification for the acceptable values of K identified by the exhaustive search they undertook with the aid of their spreadsheet. For example, the student-characters readily accept that 18 consecutive integers will not sum to 165 after consulting their spreadsheet; to supplement this conclusion, they examine the prime factorization of 165 and attempt to rationalize why 18 did not work but, on the other hand, 15 did.

Analysis

Mathematical knowledge base. Brandi and Ben used mathematical knowledge when they addressed the validity of using negative numbers in their sequence of consecutive integers [Problem 8]. Furthermore, much of Brandi and Ben's work before the creation of their spreadsheet leveraged their knowledge of divisibility rules and properties. After the spreadsheet, they used their knowledge of divisibility and prime factorizations to attempt to find a relationship between valid choices of K for a number and its prime factors [Problems 9, 10].

In addition to their knowledge of mathematical content, Brandi and Ben also demonstrated an awareness of mathematical norms for justification, generality, and efficiency. For example, problems in the dialogue attended to multiple cases [Problems 2, 6], and Ben's initial suggestion to work with the variable N can be seen as a way of ensuring that these separate cases can be united under a sufficiently general notation. We also draw attention to the characters' attempts to streamline their spreadsheet application [Problems 4, 5, 7], which we take as evidence of their desire to avoid an impractical guess-and-check approach for locating possible values of K. Brandi and Ben seem to feel that such an approach would not sufficiently address the mathematical goal of the task.

Problem-posing heuristics and strategies. Brandi and Ben engaged with the *generalization* heuristic when they attempted to capture the mathematical situation using equations and variables; their attempts to unite multiple cases under a single algorithm are another form of generalization. Considering different cases typically required the *numerical manipulation* heuristic, as did testing conjectures on different examples [Problem 10]. The symbolic representation proposed by Ben also allowed the characters to easily manipulate not only the number of consecutive integers but also the value of the target sum [Problem 1]. Later, the spreadsheet would serve the same goal of allowing Brandi and Ben to apply the numerical manipulation heuristic as quickly as possible.

Brandi and Ben also employed the *what-if-not* strategy to challenge their implicit assumptions about the CI task. First, when they realized that they had taken for granted that K must be less than the desired sum and posed problems questioning whether that really was its upper bound [Problems 5, 7]; and second, when they wondered whether they should be allowed to use a sequence of consecutive numbers that included negative values [Problem 8]. Both of these applications of the what-if-not strategy were instigated by the use of the spreadsheet, which removed the computational barrier and enabled them to examine the reasonable bounds of K.

Individual considerations of aptness. The primary consideration of aptness made by Brandi and Ben concerned whether or not a problem would be immediately appropriate for furthering their understanding of the mathematical situation at hand. That is, a good problem in the context of the scripting journey was one that would yield progress in the ongoing investigation. In this sense, Ben's suggestion to consider even numbers [Problem 2] was apt in that it addressed a case that the group had not yet attended to; similarly, Brandi's problem attempting to integrate prime factorization [Problem 9] was considered apt because it allowed the characters to refine their procedure for locating appropriate values of K. This latter example highlights that Brandi and Ben valued efficiency as an appropriate criterion for their posed problems.

Brandi and Ben also considered a problem apt when it invoked a symbolic representation of the CI task, as demonstrated early in the vignette. Although neither character said outright why a symbolic representation was valuable, we might interpret this preference in multiple ways. First, a symbolic representation might be considered apt in that it is a more efficient way to test possible values of K; this interpretation is in line with Brandi and Ben's other evident priorities and explains why their symbolic system was abandoned once they had constructed the more efficient spreadsheet. Second, it could be that the characters expected that the required "general solution" to the CI task would only be sufficiently general if its steps could be demonstrated symbolically. In this sense, the characters would consider representing the task symbolically to be apt not only because it furthered their understanding but because it was an implicit requirement of a correct solution.

Discussion

The research question guiding our study was: *How does adidactical problem-posing emerge when engaging with a problem-solving investigation?* In addressing this research question, we focus on the process of problem-posing, the purpose of problem-posing, and the conditions under which problem-posing appears.

When we interpret this question in terms of the mechanical process of forming a novel mathematics problem, we see that there are many similarities between didactical and adidactical problem posing—in both cases, the problem poser relies on their mathematical knowledge base and a selection of heuristics to construct a problem that is relevant to the mathematical setting. This report illustrates the nuanced relationship between these three components of the analysis framework. For example, Brandi and Ben applied the what-if-not heuristic when they questioned their implicit assumption that the sequence of consecutive integers should consist of only positive numbers. But in order to apply this heuristic, they first had to consult their mathematical knowledge base and clarify the elements of the set of integers. Finally, they considered whether a problem that makes use of negative integers would even be appropriate for making progress in the investigation at hand.

Another aspect of addressing our research question lies in examining not only *the means by which* problems are posed but also *for what purpose* they are posed in an investigation. Certainly, one purpose of posed problems was to complete the CI task. We note that this goal necessarily reduced the amount of variation in problem type that Brandi and Ben felt compelled to explore. That is, participants' problem-posing activity was limited by the aptness of a problem for contributing to their ongoing investigation into sums of consecutive integers. We note that Problem 6 received very little follow-up in the scripting journey; this could be because Brandi and Ben recognized that sequences of odd or even numbers was beyond the scope of their investigation of sums of consecutive integers.

Although some of the constraints of the consecutive integer task were immutable, participants decided independently when their investigations had reached a natural conclusion. The point at which a group found their work on the CI task to be satisfactory illuminated what kinds of mathematical artifacts they perceived to be normatively valued within their mathematics course. Brandi and Ben were not satisfied with their work until they had answered some subset of the example problems provided by the task; however, they also posed and endorsed their own problems that dealt with formal mathematical representations, the efficiency or reliability of algorithms and formulae, or justifications that demonstrate generalizability. In this way, the CI task prompted problem-posing to emerge not only *in service of* solving a given problem but also *as a consequence* of solving that problem.

Finally, we address our research question by considering *when* in the course of problem-solving it is likely that students will engage in a didactical problem-posing. We observed that the creation of the groups' spreadsheet engendered a flurry of problem-posing unrestrained by computational limitations. These posed problems explored how the new spreadsheet might become even more efficient with a better understanding of the mathematical setting [Problems 4-8]. Because it triggered a burst of novel problem-posing activity, we identify the creation of the spreadsheet as an example of what we call a *problem-posing catalyst*. A problem-posing catalyst is a shift in perspective brought about by the removal of constraints or a mathematical realization. The problem-posing activity that follows a catalytic event is more concentrated because the catalyst creates a "fresh" problem space; problems occur more easily to the posers because they appear from previously unexplored directions. Consequently, the fact that post-catalyst problem-posing takes place within a newly conceptualized mathematical setting means more potential for further insights—that is, more catalysts. In this way the cycle of didactical problem-posing fuels itself.

Concluding Remarks

This report contributes to problem-posing literature by describing different ways in which a problem-poser's mathematical knowledge base, problem-posing heuristics and strategies, and individual considerations of aptness might play a role in the didactical problem-posing they exhibit during an investigation involving problem-solving activity.

Additionally, we introduce the construct of a problem-posing catalyst to provide a touchstone for future explorations of a didactical problem posing. This study illustrates problem-posing catalysts that arise through investigative problem-solving activity; however, we note that because catalysts are characterized by a shift in perspective, they might emerge in other settings. Future studies could describe catalysts in other types of tasks with potential fora didactical problem-posing, such as when students must generate a variety of examples or search for visual patterns.

Because scripting journeys are participants' self-reported retellings of problem-solving activity, conclusions that we are able to draw about their problem-posing come with caveats. By attempting to capture their engagement with the CI task as a narrative dialogue, the participants could be expected to selectively include only those problems that they perceived as contributing to their mathematical progress. This can be seen as a limitation. That is, the scripting journeys may have included only those problems, both solved and unsolved, which already met the scriptwriters' individual considerations of aptness. Future research might use other methods of monitoring problem-posing activity, such as video recordings of group work, to capture problems that were posed by the group but not included in their scripting journey.

References

Bergman, A. M., Gallagher, K., & Zazkis, R. (2022). Prospective teachers' responses to students' dialogue on fractions: attribute substitution and heuristic approaches. *Research in Mathematics Education, 25*(1), 85-104. https://doi.org/10.1080/14794802.2021.2020155

Cai, J., & Hwang, S. (2002). Generalized and generative thinking in US and Chinese students' mathematical problem solving and problem posing. *The Journal of Mathematical Behavior*, *21*(4), 401-421. https://doi.org/10.1016/S0732-3123(02)00142-6

Cai, J., & Hwang, S. (2003). A perspective for examining the link between problem posing and problem solving. *International Group for the Psychology of Mathematics Education*, *3*, 103-110.

Cai, J., Hwang, S., Jiang, C., & Silber, S. (2015). Problem-posing research in mathematics education: Some answered and unanswered questions. In *Mathematical Problem Posing* (pp. 3-34). Springer. https://doi.org/10.1007/978-1-4614-6258-3_1

Elgrably, H., & Leikin, R. (2021). Creativity as a function of problem-solving expertise: Posing new problems through investigations. *ZDM - Mathematics Education*, *53*(4), 891-904. https://doi.org/10.1007/s11858-021-01228-3

Hartmann, L. M., Krawitz, J., & Schukajlow, S. (2021). Create your own problem! When given descriptions of real-world situations, do students pose and solve modelling problems?. *ZDM - Mathematics Education*, *53*(4), 919-935. https://doi.org/10.1007/s11858-021-01224-7

Kercher, A., Bergman, A. M., Zazkis, R. (in press) Counting the stars: Advancing mathematical activity in a scripting journey. *Mathematical Thinking and Learning*.

Koichu, B. (2020). Problem posing in the context of teaching for advanced problem solving. *International Journal of Educational Research*, *102*. https://doi.org/10.1016/j.ijer.2019.05.001

Kontorovich, I., Koichu, B., Leikin, R., & Berman, A. (2012). An exploratory framework for handling the complexity of mathematical problem posing in small groups. *The Journal of Mathematical Behavior*, *31*(1), 149-161. https://doi.org/10.1016/j.jmathb.2011.11.002

Kontorovich, I., & Zazkis, R. (2016). Turn vs. shape: Teachers cope with incompatible perspectives on angle. *Educational Studies in Mathematics*, *93*(2), 223-243. https://doi.org/10.1007/s10649-016-9699-2

Liljedahl, P., & Cai, J. (2021). Empirical research on problem solving and problem posing: A look at the state of the art. *ZDM - Mathematics Education*, *53*(4), 723-735. https://doi.org/10.1007/s11858-021-01291-w
Marmur, O., & Zazkis, R. (2018). Space of fuzziness: Avoidance of deterministic decisions in the case of the inverse function. *Educational Studies in Mathematics*, *99*(3), 261-275. https://doi.org/10.1007/s10649-018-9843-2
Marmur, O., Moutinho, I., & Zazkis, R. (2020). On the density of Q in R: Imaginary dialogues scripted by undergraduate students. *International Journal of Mathematical Education in Science and Technology*, 1-29. https://doi.org/10.1080/0020739X.2020.1815880
Silver, E. A., & Cai, J. (1996). An analysis of arithmetic problem posing by middle school students. *Journal for Research in Mathematics Education*, *27*(5), 521-539. https://doi.org/10.5951/jresematheduc.27.5.0521
Zazkis, R., Sinclair, N., & Liljedahl, P. (2013). *Lesson play in mathematics education: A tool for research and professional development*. Springer. https://doi.org/10.1007/978-1-4614-3549-5
Zazkis, R., & Zazkis, D. (2014). Script writing in the mathematics classroom: Imaginary conversations on the structure of numbers. *Research in Mathematics Education*, *16*(1), 54-70. https://doi.org/10.1080/14794802.2013.876157

PROSPECTIVE ELEMENTARY SCHOOL TEACHERS' PERCEPTIONS OF STEM THINKING AND THEIR ORIENTATIONS FOR STEM THINKING

Marta T. Magiera	Mohammad S. Al-younes
Marquette University	Marquette University
marta.magiera@marquette.edu	mohammad.al-younes@marquette.edu

Drawing on a concept-map methodology, we investigated how 18 prospective elementary teachers (PSTs) conceptualize STEM thinking as habits of mind shared across STEM domains in the context of problem-solving prior to explicit classroom discussions about STEM thinking. A 28-question, 5-point Likert-scale survey was used to explore PSTs' orientations for STEM thinking in elementary school mathematics classrooms. Our results show that PSTs come to teacher education with many general ideas about STEM thinking in problem-solving contexts but do not necessarily see STEM ways of thinking as common habits of mind supporting problem-solving across STEM domains. Our data also reveals that PSTs come with positive overall STEM thinking orientations, but they tend to be hesitant to think about themselves as future teachers who foster STEM thinking in their elementary school classrooms. We discuss implications for teacher preparation.

Keywords: Integrated STEM / STEAM, Preservice Teacher Education, Problem Solving, Teacher Beliefs

Background

The importance of STEM education, which refers to teaching and learning science, technology, engineering, and mathematics, has been established for many years. STEM education contributes to developing a STEM-literate society (Bybee, 2013). Early on, STEM education was interpreted through the lens of improving student learning in isolated STEM disciplines, primarily science, and mathematics (Breiner et al., 2012; Sanders, 2009; Wang et al., 2011). Over the years, the meaning of the acronym STEM has shifted from thinking about STEM as a collection of isolated disciplines—S, T, E, and M to thinking about STEM as an interdisciplinary domain centered around authentic problems that allow for integrating two or all of the STEM content (Tytler et al., 2019). Recently, some STEM education researchers advocated for expanding thinking about STEM education from content integration to focusing on common habits of mind that link STEM disciplines (Bennett & Ruchti, 2014; Kelley & Knowles, 2016; Maiorca & Roberts, 2022; Roberts et al., 2022; Williams & Roth, 2019). Kelley and Knowles (2016), for example, proposed interpreting STEM integration through the lens of problem-solving practices shared across the STEM disciplines. Bennett and Ruchti (2014) argued for interpreting STEM integration through the lens of reasoning skills foundational to all STEM disciplines. Roberts et al. (2022) viewed integrated STEM as problem-solving practices and ways of thinking that support problem-solving across the STEM domains.

Advocates of more integrated approaches to STEM education argue that teaching STEM in a more connected manner in the context of solving real-world problems helps students understand the characteristics and features of STEM disciplines as forms of human knowledge and inquiry and generates student awareness of how STEM fields shape human environments (Roberts et al., 2022; National Research Council, [NRC], 2014).

There is an increased trend to prepare STEM-focused teachers. However, despite the emphasis on STEM integration in K-12 education and calls for providing elementary students with early experiences with STEM, limited research addresses prospective elementary school teachers (PSTs') preparation for STEM integration. Our research explores perceptions about habits of mind across STEM problem-solving that elementary PSTs hold prior to engaging them in discussions about STEM thinking in a teacher education program. This research was guided by the following research questions: (1) What is elementary PSTs' initial understanding of habits of mind shared across STEM domains in the context of problem-solving before explicit instruction about STEM thinking? and (2) What is PSTs' initial orientation for STEM thinking in elementary school mathematics classrooms?

Conceptual Foundations

We drew on phenomenography as our research methodology to investigate PSTs' perceptions of STEM thinking (Marton, 1986; 1988). Our goal was to construct an initial framework that describes our PSTs' perceptions of STEM thinking in the context of problem-solving and track changes in our PSTs' understandings of STEM thinking. A central focus of phenomenography studies is the examination of *learners' conceptions* of a phenomenon of interest. However, we find it useful to briefly discuss how STEM thinking is conceptualized in the existing research literature.

Like other researchers (e.g., Bennett & Ruchti, 2014; Denick et al., 2013, Kelley & Knowles, 2016; Williams & Roth, 2019), we view STEM thinking as a way of STEM integration through the lens of problem-solving practices shared across the STEM disciplines. Consistent with NRC (2014) descriptions, we conceptualize STEM thinking as purposeful thinking in problem-solving situations that incorporates concepts, methods, attitudes, and practices from science, technology, engineering, and mathematics. We operationalize STEM thinking as the use of disciplinary practices related to problem-solving that transcend across STEM disciplines. Examples of these practices include asking questions, evaluating information, defining and interpreting problems, planning, carrying out investigations, developing and using solution models, analyzing and interpreting information, testing ideas, constructing and justifying arguments, constructing explanations, or communicating information.

Given the intricacies of STEM thinking and practices, we drew on concept mapping to allow PSTs to externalize their interpretations of STEM thinking. Concept maps are visual displays that show how individuals represent their knowledge by organizing their thoughts and experiences about a phenomenon of interest. The existing literature on concept mapping in teacher education provides evidence that concept maps can serve as useful and valid assessment tools for gaining insights into student knowledge (Brinkmann, 2003; Kinchin et al., 2000; Schmittau, 2009). Past researchers also documented that concept maps provide valid and reliable research tools (Miller et al., 2009).

Our conceptualization of prospective teachers' orientation for fostering STEM thinking in their future practice was grounded in the Theory of Planned Behavior (Ajzen, 1991). The Theory of Planned Behavior describes behavioral intentions as the predictive variables of one's behavior. We also drew on the existing research that provides evidence that beliefs, understandings, and intentions form a predictive platform on which prospective teachers build their orientations for fostering STEM subjects in their future classrooms (e.g., Kurup et al., 2019; Wu et al., 2022). For our study, we selected four behavioral domains as a platform for describing PSTs' orientations for STEM thinking: prospective teachers' (a) emotional readiness for STEM thinking practices in problem-solving contexts, (b) visions of the importance of STEM thinking, (c) perceptions of classroom implementation of STEM thinking, and (d) self-efficacy beliefs about their ability to engage in STEM thinking and foster STEM thinking in elementary classrooms.

Method

Participants and study context. The study was conducted in the context of a larger project that produced curricular materials designed to support teacher candidates' learning about STEM thinking in elementary mathematics classrooms. Participants were 18 PSTs enrolled in *Problem Solving and Reasoning for Teachers* course. The vast majority were in their first year at the university (12 PSTs), and the remaining participants were either in their second year (3 PSTs) or in their third year (3 PSTs). For all PSTs, the course that provided the context for the study was their first mathematics course in a 3-course mathematics sequence for elementary education majors. The 75 minutes long class sessions met twice a week for 14 weeks. This paper reports data collected during the first week of the course, during which PSTs engaged in a problem-solving activity from the first module. The initial class activity asked PSTs to build the tallest freestanding tower, given 20 pieces of uncooked spaghetti, clear tape, one yard of string, scissors, and a measuring tape. By working in groups of three, PSTs had 18 minutes to accomplish their task. Following the activity, the instructor introduced Polya's (2004) problem-solving framework as a general guide for organizing problem-solving activities and invited PSTs to reflect on their problem-solving approaches, strategies, and thinking in the *Spaghetti Tower* problem context. Individual students and groups shared how they made sense of the problem, planned solution approaches, evaluated and tested their ideas and strategies, what knowledge they drew upon, how they decided which materials to use, etc.

Data and Data Analysis. As part of their homework, following the *Spaghetti Tower* activity, PSTs were asked to construct and explain a concept map that illustrates their interpretation of ways of thinking that support solving problems across STEM disciplines (STEM thinking). We analyzed concept maps and reflections on STEM thinking (n = 18) to answer RQ #1.

At the beginning of the semester, PSTs were also asked to respond to 28 (5-point) Likert-scale survey questions designed to gain insights into their initial orientations for STEM thinking (n = 18). Consistent with our conceptual framework, the survey questions addressed each of the four predictive behavior categories described earlier. The survey questions were adapted from the existing literature (e.g., Kurup et al., 2019; Wu et al., 2022) and included seven questions per category. Included in Figure 1 are sample survey questions from each question category.

Emotional readiness questions (ER): Q3*: I enjoy thinking about becoming a teacher who takes on a STEM thinking approach in my lessons.* Q6: *I feel it is important for a teacher to be able to foster STEM thinking in the elementary school mathematics classroom.*	Visions of importance questions (VI) Q9*: I think the focus on STEM thinking in elementary mathematics classrooms can improve students' employment competitiveness.* Q10. *I think the focus on STEM thinking in elementary mathematics classrooms helps to cultivate students' ability to solve real-life problems.*
Perceptions of implementation questions (PI)	Self-efficacy questions (SE)

Q16: *I do not think implementing STEM thinking in my elementary school classroom will make me feel stressed and anxious.* Q 19: *If I have an option, I would like to teach in a place where STEM thinking and STEM integration are valued.*	Q23: *I feel confident that I will be able to motivate students who have a low interest in activities that facilitate STEM thinking*. Q 28: *I feel confident in my ability to understand and apply concepts and methods from STEM subjects in different problem situations.*

Figure 1: Examples of survey questions from each behavioral category

Analysis of PSTs' reflections and concept maps. We first utilized qualitative content analysis methods and open coding (Saldaña, 2016) to identify PSTs' initial views about thinking strategies that transcend STEM problem-solving. This data analysis stage comprised multiple passes through the data, during which each response was carefully annotated and scored. The identified concepts were then organized into broader categories that emerged from patterns in the data set. While revising our coding, we kept track of concepts that did not fit the initial list of the broader categories and further expanded our initial list of categories using patterns from the analysis. Ultimately, concepts that PSTs connected to STEM problem-solving were organized into five broader categories: (1) Analyzing, (2) Planning, (3) Executing, (4) Evaluating, and (5) Other. The *Other* category included more general concepts identified across the data that did not fit the first four categories. We then tabulated code frequencies to provide a collective summary of concepts that our group of PSTs connected to STEM thinking in problem-solving.

We also applied a concept map scoring rubric adapted from (Watson et al., 2016) to provide the overall picture of each PST's views of STEM thinking. Each map was scored on (a) the overall breadth of ideas related to STEM thinking in problem-solving included in the map, (b) the level of interconnectedness of ideas, and (c) the overall map design. Each aspect was scored on a 4-point scale (max. 12). The overall map score for each PST was computed as the average across concept map score categories. The overall map score provided a measure of the strength of each PSTs' vision of STEM thinking in problem-solving. We then computed the average concept map score for our PSTs cohort.

Analysis of survey responses. We first constructed survey scale sub-scores by combining PSTs' responses to questions from each group. For each PST, the survey sub-scores were defined as a sum of the PST's responses to all questions from a respective category (Sullivan & Artino, 2013; Norman, 2010). The sub-scores provided a measure of variables associated with each category: ER, emotional readiness for STEM thinking; VI, visions of the importance of STEM thinking; PI, perceptions on classroom implementation of STEM thinking; and SE, self-efficacy with STEM thinking (scale 7-35). We then conducted a non-parametric Friedman test to examine the distribution of median sub-scores and explore the extent to which each sub-score contributed to our PSTs' STEM thinking orientation. We conducted a Wilcoxon post hoc test with a Bonferroni adjustment for multiple comparisons to identify possible differences between sub-score pairs. For each PST, we also examined the percentage of survey questions for which the PST responded with a "strongly agree" score of 5 or "agree" (score of 4) to provide the overall summary of STEM thinking orientation for PSTs in our cohort.

Results

RQ1. *How do elementary PSTs understand habits of mind shared across STEM domains in the context of problem-solving before explicit instruction about STEM thinking?*

Included in Table 1 is a summary of ideas identified across the analyzed concept maps and accompanying explanations that our PSTs associated with STEM thinking.

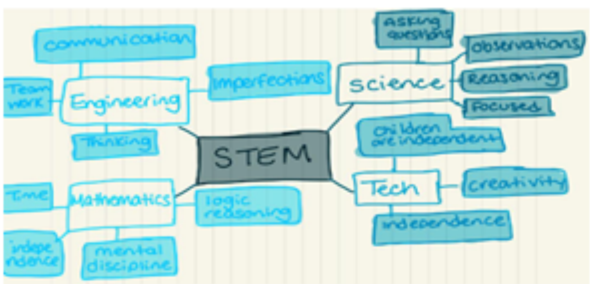
Table 1 shows that, as a group, collectively, PSTs associated with STEM thinking a broad range of habits of mind. About 60% of our PSTs considered some aspect of thinking related to analyzing problem information as a common thinking habit across STEM problem-solving. Over 70% of our PSTs considered some aspect of thinking related to solution planning as common habits across STEM problem-solving. As summarized in Table 1, half of our PSTs also thought about STEM problem-solving in terms of creativity. Less often, PSTs in our cohort associated STEM thinking in problem-solving with ways of thinking focused on executing or evaluating problem-solving activity. While, as a group, our PSTs associated a broad range of concepts with STEM thinking, the visions of STEM thinking shared by individual PSTs were more limited. PSTs' maps included between 1-12 concepts. The average map score for the group was 1.85, and the distribution of individual overall map scores was as follows: 5 PSTs (scores between 3-4); 4 PSTs (scores between 2-3); 9 PSTs (scores between 1-2). Most of the analyzed concept maps had limited breadths (number of concepts included) and minimal or no connections illustrating how the included concepts support thinking across STEM problem-solving.

Table 1: PSTs' collective views of STEM habits of mind in problem-solving

Category	Included Concepts	#PSTs (%)
Analyzing	•Thinking about problem information, what is known, interpreting problem situation, thinking about the knowledge needed to generate problem solution, making a mental picture of the problem and what is needed to address/solve, organizing information	11 (61%)
	•Generating and asking questions, wondering about	6 (33%)
	•Thinking about similarities and differences in problems and solutions, considering patterns,	5 (27%)
	•Making observations	3 (16%)
Planning	•Anticipating how a solution could look like, brainstorming, thinking about/considering options, having an open mind to possibilities, experimenting with ideas	14 (77%)
	•Thinking out of the box, thinking creatively/flexibly, being open to risk-taking, thinking in innovative ways	9 (50%)
	•Thinking about different tools	7 (39%)
Executing	•Testing options and strategies, creating new strategies/prototypes/models	7 (39%)
	•Engaging with others in a team, communicating/explaining solutions	7 (39%)
Evaluating	•Reflecting on understanding, reflecting on what works and what does not, thinking about progress, thinking about better worse ideas/solutions	5 (28%)

Other	•Using logic and reasoning, thinking in systematic ways	6 (33%)

Figure 2 shows an example of a concept map drawn by Alice (pseudonym). Alice's map and her explanation (Figure 1) illustrate that she might not perceive STEM thinking in terms of habits of mind that *connect* the STEM disciplines in problem-solving. Like many other PSTs, Alice considered a limited number of concepts overall and associated those concepts with isolated disciplines. Her star-like map design and the explanation she provided for her map show that she might think about STEM as a collection of isolated disciplines and not see the STEM habits of mind as overarching ways of thinking applicable to problem-solving situations across the STEM disciplines.



…Science allows to experiment and explore outside world. Observe things around, ask questions and wonder. Being pushed to find reasoning. Technology allows to be creative and independent when figuring out on the computer. Engineering allows for communication and team work. Mathematics brings mental discipline. …

Figure 2: An example of a concept map and an excerpt from the accompanying explanation (Alice)

RQ 2. *What is PSTs' initial orientation for STEM thinking in elementary school mathematics classrooms prior to the instruction about STEM thinking?*

The analysis of survey responses revealed that, overall, our PSTs had positive STEM thinking orientations. Eleven of the PSTs (61%) responded "strongly agree" or "agree" to more than 50% of survey questions. The analysis of survey responses, disaggregated by question group, showed that PSTs' STEM orientations differed along the four behavioral domains of interest.

Figure 3 summarizes distributions of PSTs' survey responses disaggregated by each of the four behavioral domains of interest: Emotional readiness for STEM thinking. ER; Visions of the importance of STEM thinking, VI; Perceptions of classroom implementation of STEM thinking, PI; and Self-efficacy beliefs about one's ability to engage in STEM thinking and foster STEM thinking in elementary classrooms, SE.

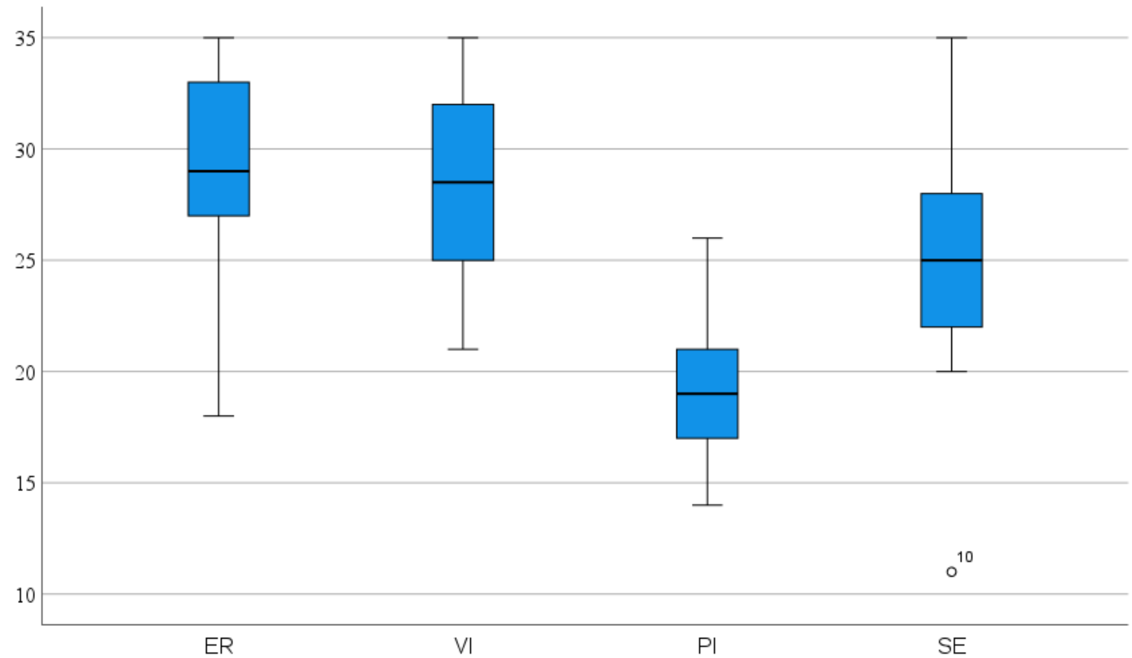


Figure 3: Distribution of median survey sub-scores

PST's orientations for STEM thinking differed across the four behavioral domains: ER (median = 29, range 18-35); VI (median = 28.5, range 21 - 35); PI (median = 19, range 14 - 26), and SE (median = 25, range 11-35). Friedman's test revealed statistically significant differences across PSTs' responses to these four groups of survey questions: $\chi2(3, N = 18) = 33.16$, $p = 0.01$. Kendal's coefficient of concordance of 0.614 indicated that the observed differences in PSTs' orientations for STEM thinking were fairly large across the four domains.

We conducted follow-up comparisons using a Wilcoxon test and controlling for Type I error across these comparisons at the 0.05 level using the LSD procedure. The median for perceptions of implementation (PI) was significantly lower than the median for emotional readiness (ER), $p < 0.01$, the median for visions of importance (VI), $p < 0.01$, and the median for self-efficacy (SE), $p < 0.01$. The median for self-efficacy (SE) was significantly lower when compared to the median for emotional readiness (ER), $p = 0.02$, and the median for visions of importance (VI), $p = 0.006$.

Discussion

There is an increased focus on providing elementary school students with STEM experiences and engaging them in STEM problem-solving practices and ways of thinking (Estapa & Tank, 2017). At the same time, little is known about ideas about STEM thinking that prospective elementary school teachers bring to their preparation programs and what initial orientations for STEM thinking they have. Below, we discuss the implications of our research for teacher educators and future studies.

The analysis of concept maps our PSTs generated following the first class activity suggests that PSTs come with some ideas about STEM ways of thinking in problem-solving contexts. When considered as a group, our PSTs included many concepts and ideas about STEM thinking that were consistent with the existing literature (e.g., Bennett & Ruchti, 2014; Denick et al., 2013, Kelley & Knowles, 2016; Roberts et al., 2022; Williams & Roth, 2019). For example, thinking about structural similarities or differences across problems and solutions, anticipating how a problem solution could look like, asking questions, and wondering. But when considered individually, our PSTs shared more limited perspectives about STEM thinking in problem-solving contexts. Their maps were not well developed and generally included a limited number of concepts. In addition, the overall designs of their maps suggested that many of our PSTs might not think about STEM thinking as a form of STEM integration. To prepare PSTs for meeting the challenges of teaching mathematics in a way that supports STEM integration, teacher educators need to engage PSTs in explicit discussions about shared ways of thinking and practices that support STEM problem-solving. A possible entry point for supporting PSTs in developing views of habits of mind that connect STEM disciplines in problem-solving contexts could be explicit discussions about ways of thinking that contribute to problem analysis, planning, and execution across STEM problem-solving situations.

Our study shows that elementary PSTs come to their preparation programs with many positive orientations toward STEM thinking overall. About two-thirds of our participants documented positive emotional readiness for STEM thinking, viewed STEM thinking as important for students to develop, and many of the PSTs had positive self-efficacy beliefs about their ability to engage in STEM thinking in problem-solving contexts.

Our analysis of survey responses also revealed that our PSTs were most concerned about classroom implementation of STEM thinking. Their responses suggested that they appear to be hesitant to think of themselves as future teachers who can foster STEM thinking in their elementary school classrooms. This finding deserves special attention from teacher educators. Our results show that even though PSTs might think highly about the importance of engaging students in STEM thinking, they might not facilitate this engagement in their future classroom practice without proper support.

In future research, we are interested in seeking more information about ways in which prospective elementary school teachers make sense of STEM thinking in problem-solving contexts as a way of integrating STEM domains. We are also interested in exploring how to best support the development of PSTs' knowledge about STEM thinking as a way of STEM integration in problem-solving contexts and their dispositions for STEM thinking.

Acknowledgments

This work was supported by the National Science Foundation Grant, NSF IUSE #2142646. Opinions, findings, and conclusions are those of the authors and do not necessarily reflect the views of the funding agency.

References

Ajzen, I. (1991). The theory of planned behavior. *Organizational Behavior and Human Decision Making*, *50*(2), 17-211. https://www.sciencedirect.com/science/article/abs/pii/074959789190020T?via%3Dihub

Bennett, C. A., & Ruchti, W. (2014) Bridging STEM with mathematical practices*, Journal of STEM Teacher Education*, *49*(1), 17-28. DOI: doi.org/10.30707/JSTE49.1

Breiner, J., Harkness, M., Johnson, C. C., & Koehler, C. (2012). What is STEM? A discussion about conceptions of STEM in education and partnerships. School Science and Mathematics, 112(1), 3–11.

Brinkmann, A. (2003). Graphical knowledge display - mind mapping and concept mapping as efficient tools in mathematics education. *Mathematics Education Review*, *16*, 35-48.
Bybee, R. W. (2013). *The Case for STEM Education: Challenges and Opportunities.* National Science Teachers Association, NSTA Press, Arlington, Virginia.
Denick, D., Dringenberg, E., Fayyaz, F., Nelson, L., Pitterson, N., Yatchmeneff, M., & Cardella, M. (2013). STEM thinking in informal environments: Integration and recommendations for formal settings. *In Proceedings of the 2013 ASEE IL-IN Section Conference*, Angola, IN.
Kelley, T. R., & Knowles, J. G. (2016). A conceptual framework for integrated STEM education, *International Journal of STEM Education*, *3*(11), 1-11. https://doi.org/10.1186/s40594-016-0046-z
Kinchin, I. M., Hay, D. B., & Adams, A. (2000). How a qualitative approach to concept map analysis can be used to aid learning by illustrating patterns of conceptual development. *Educational Research*, *42*(1), 43-57
Korup, P., Li, X., Powell, G., & Brown, M. (2019). Building future primary teachers' capacity in STEM: based on a platform of beliefs, understandings and intentions. *International Journal of STEM education,* 6:10, https://doi.org/10.1186/s40594-019-0164-5
Marton, F. (1986). Phenomenography: A research approach to investigating different understandings of reality. *Journal of Thought*, *21*(3), 28–49.
Marton, F. (1988). Phenomenography: Exploring different conceptions of reality. In D. M. Fetterman (Ed.), *Qualitative approaches to evaluation in education: the silent scientific revolution* (pp. 176–205). New York: Praeger.
Maiorca, C., & Roberts, T. (2022). Problem-solving by design: The integrated STEM practices. *The Elementary School Journal*, 20-21.
Miller, K., Koury, K., Fitzgerald, G. E., Hollingsead, C., Mitchem, K., Tsai, H., & Park, M. (2009). Concept mapping as a research tool to evaluate conceptual change related to instructional methods. *Teacher Education and Special Education*, *32*(4), 365-378.
National Research Council (2014). *STEM integration in K-12 education: Status, prospects, and an agenda for research*. Washington, DC: The National Academies Press. https://doi.org/10.17226/18612
Norman, G. (2010). Likert scales, levels of measurement and the "laws" of statistics. *Advances in Health Science Education*, *15*, 625-632.
Pólya, G. (2004). How to solve it: A new aspect of mathematical method (No. 246). Princeton University Press.
Roberts, T., Maiorca, C., Jackson, C., Mohr-Schroeder, M. (2022). Integrated STEM as problem-solving practice. *Investigations in Mathematics Learning*, *14*(1), 1-13.
Saldaña, J. (2016). *The coding manual for qualitative researchers*, Thousand Oaks, CA: Sage Publications Inc.
Sanders, M. (2009). STEM, STEM education, STEMmania. The Technology Teacher, 68(4), 20–26.
Schmittau, J. (2009). Concept mapping as a means to develop and assess conceptual understanding in secondary mathematics teacher education. In K. Afamasaga-Fuata'I (Ed.). *Concept mapping in mathematics: Research into practice,* (pp. 137-147). Springer Science+Business Media, New York, NY.
Sullivan, G., M., & Artino, A. R. (2013). Analyzing and interpreting data from Likert-type scales. *Journal of Graduate Medical Education*, *5*(3), 541-542.
Tytler, R., Williams, G., Hobbs, L., & Anderson, J. (2019). Challenges and opportunities for STEM interdisciplinary agenda. In B. Doing, J. Williams, D. Swanson, R. Borromeo Ferri, P. Drake (Eds.). Interdisciplinary mathematics education. The state of the art and beyond. (pp. 51-84). Cham, Switzerland: Springer.
Wang, H., Moore, T. J., Roehrig, G. H., & Park, M. S. (2011). STEM integration: teacher perceptions and practice. *Journal of Pre-College Engineering Education Research*, *1*(2), 1–13.
Watson, M. K., Pelkey, J., Noyes, C., Rodgers, M. (2016). Assessing conceptual knowledge using three concept maps scoring methods. *Journal of Engineering Education*, *105*(1), 118-146.
Williams, J., & Roth, W. M. (2019). Theoretical perspective on interdisciplinary mathematics. In B. Doing, J. Williams, D. Swanson, R. Borromeo Ferri, & P. Drake (Eds.). *Interdisciplinary mathematics education. The state of the art and beyond*, (pp. 13-34). Cham, Switzerland: Springer.
Wu, P., Yang, L., Hu, X., Li., B., Liu, Q., Wang., Y., & Huang, J. (2022). How K-12 teachers' readiness influences their intentions to implement STEM education: Exploratory study based on the decomposed theory of planned behavior, *Applied Science*, *12*, 11989. https://doi.org/10.3390/app122311989

QUANTITATIVE OPERATORS IN MATHEMATICAL MODELING

Sindura Subanemy Kularajan	Elizabeth Roan	Jennifer A Czocher
Utah State University	Texas State University	Texas State University
Sindura.kularajan@usu.edu	eaw109@txstate.edu	czocher.1@txstate.edu

In this report, we present cases where students constructed new quantities through operating on quantities that does not fit the definitions of existing theories on quantitative operations. As a result, we identified five quantitative operators—operators that can be used on single qualities in order to transform the quantity to a new quantity—students used while constructing mathematical models for real-world scenarios.

Keywords: Quantitative Reasoning, Mathematical Modeling, Operations on Quantities

Mathematical modeling is an important skill for students to learn. However, it is still a challenging subject for students (Stillman et al., 2010; Jankvist & Niss, 2020). In an effort to mitigate some of these challenges, recently, mathematical modeling scholars have adopted theories from quantitative reasoning (Thompson, 1994; 2011) to operationalize mathematical modeling competencies (e.g. Czoher et al., 2022; Larsen, 2013; Roan & Czocher, 2022). In this adaptation, quantities are viewed as building blocks of a mathematical model (Larsen, 2013). That is, a new quantity can be constructed through operating on one or more existing quantities. As a result, a mathematical model maybe viewed as a network of such operations on quantities (Thompson, 1990). If this perspective on *mathematical models* and *mathematical modeling* is to be taken to investigate students' mathematical modeling, then more work needs to be done on the mechanisms involved in the conception of a new quantity through operating on existing quantity or quantities. These mechanisms have been explicated by scholars as quantitative operations (Thompson, 1990) and (co)variational reasoning (Carlson et al., 2002) through investigating, predominantly, K-12 students' mathematical reasonings. However, it is still not clear exactly how theories from quantitative reasoning explain students' reasoning as students mathematically model dynamic, complex situations, especially those that require differential equations. For example, students have constructed rate of change through operating on existing quantities in ways that cannot be explained by the current theories of quantitative reasoning (Kularajan & Czocher, 2022). In this report, we share examples from a study of students' reasoning during mathematical modeling that demonstrate the need for amending and extending theories of quantitative reasoning to include *quantitative operators*.

Operation on Quantities

Quantities are conceptual entities that exist in the mind of an individual. Thompson (1994) defined *quantity* as a mental construct consisting of three interdependent entities: an object, a measurable attribute, and a *quantification*. Quantification involves conceiving a measurable attribute of an object and a unit of measure and forming a proportional relationship between the attribute's measure and the unit of measure (Thompson, 2011). While objects are constructions taken as given, the attributes that one conceives as measurable are imbued by the individual conceiving them (Thompson, 1994). Thompson (1990) explains this phenomenon through the ideas of motion and distance moved in an amount of time. For example, for a young child watching a cat running to hunt a bird in the backyard, the cat probably is an object, and the young child may have imbued the attribute motion to the running cat. However, for this young child, the running cat probably does not have the attribute distance moved in a corresponding amount of time.

A relationship among measurable attributes is established through operating on quantities. Thompson (1994) defines *quantitative operation* as the "mental operation by which one conceives a new quantity in relation to one or more already-conceived quantities" (p.10). As a result of a quantitative operation a *quantitative relationship* is created: the quantities operated upon along with the quantitative operation are in relation to the result of operating (Thompson, 1994). In other words, a quantitative relationship is the "conception of three quantities, two of which determine the third by a quantitative operation (Thompson, 1990, p. 12)." Examples of quantitative operations include *combining two quantities additively*, *comparing two quantities additively, combining two quantities multiplicatively*, *comparing two quantities multiplicatively*, *instantiate a rate*, *generalize a ratio*, and *composing two rates or ratios* (Thompson, 1994). For example, how many more cats visited my backyard on Saturday than on Sunday is a quantity that we may construct by *additively comparing* the number of cats that visited my backyard on Saturday and the number of cats that visited my backyard on Sunday. At the same time, we may construct the total number of cat-bird interactions on Saturday during the time period 9am to 5pm by *instantiating a rate* of 10 cat-bird interactions per hour for 8 hours.

Although Thompson (1994) defined quantitative operations as the mental operations on "one or more already-conceived quantities" to construct a new quantity, the definition of a quantitative relationship (Thompson, 1990) and the examples given for quantitative operations (Thompson, 1994) emphasize operations on two quantities to conceive a third new quantity. Therefore, it is not clear through the definition of quantitative operations (Thompson 1994a), quantitative relationships (Thompson, 1990), and the examples of quantitative operations, whether mental operations performed on one quantity to construct a new quantity fall within the scope of Thompson's quantitative operations. In addition, students may engage in operations on quantities without clear evidence of the operations having a situationally relevant quantitative meaning, but the resultant quantity has a quantitative meaning for the student. We refer to these borderline instances as pseudo-quantitative operations and present examples in our findings.

Methods

Data for this report were drawn from a larger study of effective scaffolding for promoting modeling competencies. We worked with 34 undergraduate STEM majors who were enrolled in or had already taken differential equations at the time of the interviews. The students participated in 10 hour-long task-based clinical interviews (Goldin, 2000) where they developed mathematical models for real-world systems. We present examples, to illustrate our case, from four students' work—Ivory, Szeth, Pattern, and Winnow—on The Cats and Birds Task, The Tropical Fish Task, The Pruning Task, and The Tuberculosis Task.

> *The Cats and Birds Task:* Cats, our most popular pet, are becoming our most embattled. A national debate has simmered since a 2013 study by the Smithsonian's Migratory Bird Center and the U.S. Fish and Wildlife Service concluded that cats kill up to 3.7 billion birds and 20.7 billion small mammals annually in the United States. The study blamed feral "unowned" cats but noted that their domestic peers "still cause substantial wildlife mortality." In this problem, we will build a model (step-by-step) that predicts the species' population dynamics, considering the interaction of the two species.

The Tropical Fish Task: To regulate the pH balance in a 300L tropical fish tank, a buffering agent is dissolved in water and the solution is pumped into the tank. The strength of the buffering solution varies according to $1 - e^{\frac{-t}{20}}$ grams per liter. The buffering solution enters the tank at a rate of 5 liters per minute. Create an expression that models how quickly the amount of buffering agent in the tank is changing at any moment in time.

The Pruning Task: Imagine you have a hedge in your garden of some size, S, and you want it to increase its size even more. You hire a gardener for some advice on growing this particular plant. She advises you that the overall rate of growth will depend both on the extent of pruning and on the regrowth rate, which is particular to the plant species and environmental conditions. Both rates can be measured as a percentage of the size of the plant. The pruning rate can be adjusted to result in a target overall growth rate. Can you derive a model for the rate of change of the size of the plant?

The Tuberculosis Task: Tuberculosis (TB) is a serious infectious disease caused by a bacterium that originated in cattle, but can affect all mammals including humans. It typically affects the lungs, causing a general state of illness, coughing, and eventual death. Many infected individuals carry a latent (inactive) infection for a long time before their lungs succumb to the damage caused by the bacteria. The disease is highly contagious; it is spread from person to person when an infected individual coughs, spits, speaks, or sneezes. Because transmission rates are so high, TB outbreaks are frequently associated with poverty conditions – locations where overcrowding is common. In these communities, spread (rate of new infections) can be very high and decimate a community rapidly. Imagine a community where sick and well members move about freely among one another. Create a mathematical model for the rate that the disease will spread through the community.

The interviews were retrospectively analyzed to construct second-order accounts (Steffe & Thompson, 2000) of students' reasonings via inferences made from students' observable activities such as verbal descriptions, language, written work, discourse, and gestures. The retrospective analysis consisted of multiple passes of the data to arrive at examples that illustrate the different ways students engaged in Pseudo-Thompsonian Quantitative Operations. First, we watched the videos in MAXQDA in chronological order and paraphrased each interview by chunks. Next, we created accounts of students' mathematics and the reasons they attributed to their mathematics. Next, we reviewed the accounts and the videos at the same time and refined our accounts by adding details using theories from quantitative reasoning. We credited a student to have instantiated a quantity if we were able to infer from his reasonings that he had conceived an object, attribute, and a measurement process for the attribute. As evidence of student to have conceived a measurement process, we checked if at least one of the quantifications criteria was met (see Czocher & Hardison, 2021). We used segments of transcripts, where the students engaged in quantitative reasoning along with inscriptions and gestures as evidence for our claims. Next, from these accounts, we selected instances where the students constructed a new quantity by operating on a singular quantity or the operation itself (to construct the new quantity) did not have clear evidence of a situationally relevant quantitative meaning. Finally, we refined our second-order accounts by triangulating with utterance and gestures to support our claims.

Findings

We identified five examples where students engaged in pseudo-quantitative operations: (1)

constructing rate of change through taking the derivative, (2) constructing total amount through taking the integral, (3) constructing percent through considering parts of a whole, (4) constructing amount through considering a proportion of the whole, and (5) constructing rate of change through negation. We illustrated the first example in Kularajan & Czocher (2022), where we showed two modelers taking the derivative to construct the rate of change of the bird population due to predation by cats, in The Cats and Birds Task. In this report, we present examples of (2)-(5).

Taking the Integral to Construct Total Amount

To illustrate this operation, we present Ivory's work from The Tropical Fish Task. In the Tropical Fish Task, Ivory was working towards constructing a model for the amount of buffering agent in the tank at time t. To accomplish this goal, Ivory first constructed expression 1 to represent the rate at which the amount of buffering agent enters the tank.

$$m_E(t) = 5 \cdot (1 - e^{\frac{-t}{20}}) \tag{1}$$

In expression 1, Ivory defined $m_E(t)$ as the rate at which the amount of buffering agent enters the tank at time t. After constructing expression 1, she stated that she would take the integral of $m_E(t)$ in order to construct an expression for the amount of buffering agent in the tank at time t. Following this reasoning, Ivory constructed the expression below to represent the amount of buffering agent inside the tank at time t.

$$M(t) = \int m_E(t) \tag{2}$$

The interviewer pointed out that the expression 2, as written, only accounts for the amount of buffering agent that had entered the tank and ignores the amount of buffering agent that exits the tank. In response, Ivory modified expression 2 to the one shown in Figure 1 while confidently voicing that expression 2 "does work." In Figure 1, Ivory defined $m_L(t)$ as the rate at which the amount of buffering agent leaves the tank at time t. Through the expression in Figure 1, we infer that Ivory constructed the amount of buffering agent in the tank at time t, by *additively comparing* the amount of buffering agent that enters the tank and the amount of buffering agent that leaves the tank.

$$M(t) = \int m_E(t) - \int m_L(t)$$

Figure 1: Ivory's model for the amount of buffering agent in the tank at time t

We infer that Ivory first constructed the rates at which the amount of buffering agent enters and leaves the tank to operate on them further through taking the integral to construct the amount of buffering agent that enters the tank and the amount of buffering agent that leaves the tank, respectively. Even though Ivory constructed a quantity that had situationally relevant meaning to her (amount of buffering agent that enters (leaves) the tank at time t), the operation (taking the integral) on the singular quantity (rate at which the amount of buffering agent enters (leaves) the tank) did not have clear evidence of a situationally relevant quantitative meaning. For Ivory, taking the integral was an operation that could be performed on the measurable attribute rate of change to construct the amount.

Envisioning Parts of the Whole to Construct Percent

To illustrate this operation, we present Szeth's work from the Pruning task. Szeth first constructed Expression 3 where Szeth defined R' as the rate at which the plant would be growing, P as the "pruning," G' as the "regrowth rate," and E as "environmental conditions."

$$R' = P + G' + E \tag{3}$$

After Szeth constructed expression 3, he mathematized R' and G' as $R' = \frac{S}{100}$ and $G' = \frac{S}{100}$ because "both rates can be measured as a percentage of the size of the plant." We interpret that when Szeth read the task "the overall rate of growth [of the plant] will depend both on the extent of pruning and on the regrowth rate…Both rates can be measured as a percentage of the size of the plant," he interpreted "both rates" to be the rate at which the plant is growing (R') and the regrowth rate (G'), as oppose to regrowth rate and the extent of pruning. Therefore, Szeth constructed R' and G' as a percentage of the size of the plant, S, through considering $\frac{1}{100}^{th}$ of the size of the whole plant S. In this instance, for Szeth, considering $\frac{1}{100}^{th}$ of the size of the plant S was an operation on the quantity S in order to construct the new quantity "regrowth rate." Szeth said that he would substitute $R' = \frac{S}{100}$ and $G' = \frac{S}{100}$ in expression 3. Following this, the conversation below was exchanged among us.

Interviewer: You have $R' = \frac{S}{100}$ and $G' = \frac{S}{100}$. So, does that say that R' and G' are both equal to each other, or can they be different percentages?

Szeth: I guess it does say they're equal. I wouldn't take them to be equal. In real life perspective, they are supposed to be different things.

Interviewer: And how would you modify it so that they're not equal?

Szeth: Think I would just get rid of this [*scratches off* $R' = \frac{S}{100}$], because G' is already in an equation that affects R'. So, if I put this, let's substitute that [*pointing at* $G' = \frac{S}{100}$] into there [*pointing at* G' *in* $R' = P + G' + E$], then it's still true that this rate [*referring to* R'] can be measured as a percentage of the size. It still be involved in this equation up here.

In the above excerpt, when the interviewer asked Szeth if R' and G' are equal, Szeth responded that "in real life…they are supposed to be different things." By that, we interpret that he meant R' and G' measure different qualities of the plant, that may (or may not) have different values. We take this as evidence that for Szeth considering a fraction of the size of the plant—in particular considering $\frac{1}{100}^{th}$ of S—was a mental operation performed on the size of the plant, to construct the quantities that can be written as a percentage as a size of the plant. In response, when we asked how he would modify R' and G' such that they wouldn't be equal, Szeth indicated that he would substitute $G' = \frac{S}{100}$ in expression 3 and modified expression 3 as below.

$$S' = P + \frac{S}{100} + E \tag{4}$$

In expression 4, Szeth replaced $R' = S'$ and indicated that S' is implicitly written as a percentage of the size of the plant because S' is written in terms of $G' = \frac{S}{100}$.

In this example, we illustrated how Szeth constructed the quantity G'—"regrowth rate"—through operating on the quantity S—"the size of the plant"—by considering a portion of S. In other words, we illustrated how Szeth constructed G' by structurally conceiving percent through imagining a part, in particular $\frac{1}{100}^{th}$, of a whole size of the plant.

Envisioning a Proportion of the Whole to Construct Amount

To illustrate this operation, we present Pattern's work from the Cats and Birds task. In The Cats and Birds Task, Pattern was working towards constructing a model for the number of cat-bird encounters that result in a bird's death. To accomplish this goal, Pattern first constructed an expression to calculate the maximum number of cat-bird encounters at time t (Figure 2(a)). Pattern was then asked to consider how he might modify his expression to account for the fact that only a proportion of the maximum encounters are realized. In response to this, Pattern elected to multiply the number of maximum number of cat-bird encounters at time t by some percentage α, shown in Figure 2(b). Pattern explained his reasoning for multiplying by α as follows.

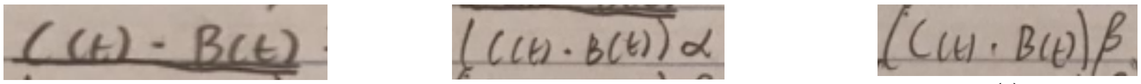
> Pattern: So this [*referring to the expression in Figure 2(a)*] is the total possible encounters that could possibly happen if perfect conditions are met for each cat to meet each bird, and then you're going to take a percentage of that total, and that would be your total.

In this instance, Pattern constructed the number of encounters that actually happened by envisioning a proportion of the maximum number of cat-bird encounters at time t. Here, for Pattern, α acted as a multiplicative scaling factor to quantify the proportion of cat-bird interaction that were realized. There was no clear evidence that α had a situationally relevant quantitative meaning for Pattern. That is, we were not able infer a situational referent (an object) and the attribute α was measuring.

Later, Pattern was asked to adapt his model (Figure 2(a)) to account for the fact that sometimes a bird might escape. In response to the request, Pattern decided to multiply the maximum number of cat-bird encounters at time t by some percentage β as shown in Figure 1(c). Pattern explained his decision to multiply by β as follows.

> Pattern: Because what I did here was I made it really easy for myself by creating this baseline [*referring to* $C(t) \cdot B(t)$*)*]. And from here, you can... Since they're not giving it to me, I can add whatever I want. So that gives me the freedom of being like, well, since you want to know how many birds die, we can just create this percentage [*referring to* $\big(C(t) \cdot B(t)\big) \cdot \beta$].

In this instance, Pattern constructed the number of encounters that result in a bird's death by taking a different proportion of the maximum number of cat-bird encounters at time t. Through Pattern's reasoning above, we interpret that, $C(t) \cdot B(t)$ acted as a "baseline" to consider different proportions of the maximum number of cat-bird interactions—number of cat-bird interactions that realized and number of cat-bird interactions that resulted in a dead bird. Pattern accomplished this by using multiplicative scaling factors (e.g. α% and β%, respectively) that reduced the size of the whole—maximum number of cat-bird interactions. In both of these instances, Pattern constructed new quantities by considering a proportion of the "baseline" via using multiplicative scaling factors . However, α% and β% were not associated with a discernable or situationally relevant attribute.



(a) **(b)** **(c)**

Figure 2: Pattern's models for (a) maximum number of cat-bird encounters, (b) number of actual cat-bird encounters, (c) number of cat-bird encounters that resulted in a birds death

Negating a Rate of Change to Construct a New Rate of Change

To illustrate this operation, we present Winnow's work from The Tuberculosis Task. Winnow constructed expression 5 as a model for the rate of change of the sick people with respect to time.

$$\frac{dS}{dt} = \frac{m}{S(t) \times H(t)} \times H(t) \quad (5)$$

In expression 5, Winnow defined $H(t)$ as the number of healthy people at time t, $S(t)$ as the number of sick people at time t, and m as the number of contacts between healthy and sick people that actually occur. When the interviewer asked for a model for the rate of change of the healthy people with respect to time. Winnow noted immediately that the rate of change of the healthy people "would be a negative number because the healthy people would be—the number of healthy people would be decreasing." Following this reasoning, Winnow constructed an initial model for the rate of change of healthy people with respect to time as shown in expression 6.

$$\frac{dH}{dt} = -\frac{m}{S(t) \times H(t)} \times S(t) \quad (6)$$

After constructing expression 6, Winnow validated his model by checking against specific conditions, $S(t) = 10$ and $H(t) = 10$. He was happy that substituting for $S(t) = 10$ and $H(t) = 10$ in expressions 5 and 6 yielded the same value, but the negation of one another, asserting that the rate of change with respect to time for sick people would be the same value as the rate of change with respect to time for healthy people.

To perturb Winnow, the Interviewer asked him to validate his model against the values $S(t) = 2$ and $H(t) = 10$. Realizing that $\frac{dS}{dt}$ and $\frac{dH}{dt}$ would not yield the equal values, Winnow modified his model for the rate of change of healthy people with respect to time to be $\frac{dH}{dt} = -\frac{dS}{dt}$. Winnow justified his modification to his model as "the number of new sick people and the decrease in healthy people should be the same." Through this we infer that winnow negated the rate of change of sick people with respect to time to construct the rate of change of healthy people with respect to time, because for Winnow, the number of *new* sick people, directly corresponds to the *decrease* in the number of healthy people. However, there was no clear evidence whether the " $-$ " in $\frac{dH}{dt} = -\frac{dS}{dt}$ represented a quantity, let alone -1, for Winnow.

Discussion

We introduced the construct pseudo-quantitative operations to account for five instances of modelers constructing quantitative relationships without clear evidence that the operations performed on quantities had situationally relevant quantitative meaning. In this report, we presented four such instances. First, we illustrated how Ivory constructed the amount of buffering agent entering and leaving the tank by taking the integral of the rate at which the buffering agent enters the tank and the rate at which the buffering agent leaves the tank, respectively. Next, we presented an example of how Szeth constructed the "regrowth rate" of the plant by considering a fraction of the whole size of the plant, by using a multiplicative scaling factor of $\frac{1}{100}$. Third, we illustrated how Pattern constructed the number of birds that died due to predation by cats by *shrinking* the total number of cat-bird interactions that realized through using a multiplicative scaling factor α. Finally, we presented how Winnow considered the negation of the rate of change of the sick people with respect to time in order to construct the rate of change of healthy people with respect to time.

What do we mean by *the operations on quantities having no situationally relevant quantitative meaning?* Recall the example where Winnow constructed the rate of change of healthy people with respect to time by negating the rate of change of sick people with respect to time. If Winnow had indicated that the $-$ in $\frac{dH}{dt} = -\frac{dS}{dt}$ represented $\frac{-1\, healthy\ person}{1\, sick\ person}$ — a measurable attribute of the system of healthy and sick people—then we would have credited Winnow to have engaged in the *multiplicative combination* of two quantities (number of people removed from the healthy population for each person getting sick and the rate of change of the sick people with respect to time), making it a quantitative operation as Thompson (1990) defined it. For Winnow, the $-$ in $\frac{dH}{dt} = -\frac{dS}{dt}$ acted as an operator to transform $\frac{dS}{dt}$, in turn quantifying $\frac{dH}{dt}$.

Similarly, when Pattern constructed the number of birds that died due to predation by cats by shrinking the total number of cat-bird interactions that realized through using a multiplicative scaling factor α, we were not able to infer whether for Pattern α carried any situationally relevant quantitative meaning. We would have credited Pattern to have engaged in a quantitative operation—*instantiating a rate*—between α and $C(t) \cdot B(t)$, if he had shown clear evidence that α measured the number of cat-bird interactions that resulted in a dead bird for every 100 cat-bird interactions, an attribute of the system of cats and birds. However, for Pattern α acted as an operator that *shrank* the size the total number of possible cat-bird interaction to represent a subset of that amount. Likewise, when Szeth constructed a percentage of the size of the plant as $\frac{S}{100}$, the $\frac{1}{100}$ acted as an operator to transform S in order quantify the "regrowth rate." At the same time, for Ivory, the integral acted as an operator to transform a *rate* to an *amount*; there was no clear evidence whether the integral involved the coordination of two quantities.

Even though taking the derivative, integral, using a multiplicative scaling factor, and negating in and of itself may not be credited as quantitative operations, they are operations that the students performed on existing quantities to construct a new quantity. In addition, the aforementioned operations on existing quantities are an image that is prevalent in undergraduate learners and the result of that action is often useful for mathematical modeling. Adhering to the definitions of quantitative operations, as Thompson defined it, instances such as illustrated in this report will be missed. Therefore, to investigate students' quantitative reasoning in mathematical modeling, we propose to extend the definition of *operations on quantities* to include operations on singular quantities to construct new quantities through the aid of *quantitative operators*. We identified five such *quantitative operators*: $\frac{d\,\blacksquare}{dt}$, $\int \blacksquare$, $\alpha \times \blacksquare$, $\frac{1}{100} \times \blacksquare$, and $-\blacksquare$. These *quantitative operators* can be viewed as functions that take in a quantity and output a new quantity, where a quantity itself can be operationalized as a function consisting of three variables (object, attribute, and quantification). The inclusion of these *quantitative operators* allows us to make better sense of the mechanisms involved in the construction of new quantities in students' mathematical modeling. Our findings reaffirm that quantitative reasoning is essential to mathematical modeling, but also caution that not every parameter, variable, or operation in the student's work needs to signify a situational referent in order for their modeling activities to be productive.

Acknowledgments

This material is based upon work supported by the National Science Foundation under Grant No. 1750813.

References

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352–378. https://doi.org/10.2307/4149958

Czocher, J. A., & Hardison, H. L. (2021). Attending to quantities through the modelling space. In F. Leung, G. A. Stillman, G. Kaiser, & K. L. Wong (Eds.), *Mathematical modelling education in East and West* (pp. 263–272). Springer.

Czocher, J.A., Hardison. H., Kularajan, S. (2022). A bridging study analyzing mathematical modeling through a quantity-oriented lens. *Education Studies in Mathematics, 111*(2), 299–321. https://doi.org/10.1007/s10649-022-10163-3

Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 517–545). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Kularajan, S., & Czocher, J.A. (2022). STEM undergraduates' structural conception of rate of change. In Lischka, A. E., Dyer, E. B., Jones, R. S., Lovett, J., Strayer, J., & Drown, S. (Eds). *Proceedings of the forty-fourth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Middle Tennessee State University.

Larson, C. (2013). Modeling and quantitative reasoning: The summer jobs problem. In R. Lesh, P. L. Galbraith, C. R. Haines, A. Hurford, *Modeling students' mathematical modeling competencies* (pp. 111–118). Springer.

Stillman, G., Brown, J., & Galbraith, P. (2010). Identifying challenges within transition phases of mathematical modeling activities at year 9. In R. Lesh, P. Galbraith, C. R. Haines, & A. Hurford (Eds.), *Modeling students' mathematical modeling competencies* (pp. 385–398). Springer.

Jankvist, U. T., & Niss, M. (2020). Upper secondary school students' difficulties with mathematical modelling. *International Journal of Mathematical Education in Science and Technology*, *51*(4), 467–496. https://doi.org/10.1080/0020739X.2019.1587530

Roan, E., & Czocher, J.A. (2022). Justifications students use when writing an equation during a modeling task. In Lischka, A. E., Dyer, E. B., Jones, R. S., Lovett, J., Strayer, J., & Drown, S. (Eds). *Proceedings of the forty-fourth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Middle Tennessee State University.

Thompson, P. W. (1990). A theoretical model of quantity-based reasoning in arithmetic and algebra. Unpublished manuscript, Center for Research in Mathematics & Science Education, San Diego State University, San Diego, CA.

Thompson, P. W. (1994). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics*, *26*(2–3), 229–21. https://doi.org/10.1007/BF01273664

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In L. L. Hatfield, S. Chamberlain & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education: Papers from a planning conference for WISDOM^e* (Vol. 1, pp. 33–57). Laramie, WY: University of Wyoming.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education* (pp. 267–307). Hillside, NJ: Erlbaum.

SECONDARY TEACHERS' CONCEPTIONALIZATIONS OF THE RELATIONSHIPS BEWEEN MATHEMATICAL MODELING AND PROBLEM SOLVING

Rui Kang
Georgia College and State University
rui.kang@gcsu.edu

The purpose of this study is to gain understanding of how secondary teachers conceptualize the relationships between mathematical modeling and problem solving. Eight secondary teachers participated in semi-structured, think-aloud individual interviews. Some conceptualizations include (a) modeling and problem solving are two distinct processes; (b) modeling is a subset of, or a tool for, problem solving; (c) the two processes share similar characteristics such as a real-life scenario but are different in terms of whether a single correct answer exists; (d) the two processes are inseparable, completely enmeshed in each other, and co-dependent; and (e) problem solving is a reduced process of modeling. Teachers' conceptualizations are related to their preferred instructional sequence and the types of problems or activities they'd rather use.

Keywords: Modeling, Problem Solving, Teacher Knowledge, Teacher Educators

Mathematical modeling, or responding to real-world problems mathematically, meets the needs and interests of 21st century learners by offering abundant opportunities for developing critical skills such as adaptability, systems thinking, nonroutine problem solving, and complex communication skills (Bybee, 2010). As an example, Model Eliciting Activities (MEAs) are open-ended and context-rich activities that challenge learners to generate models or systems as useful solutions for complex real-world situations (Aguilar, 2021; Lesh & Doerr, 2003). Lesh and colleagues (e.g., Lesh & Harel, 2003; Lesh & Lehrer, 2003) implemented MEAs with students who were enrolled in remedial mathematics and were from poor, large urban school districts. Learners in these studies were competent in developing conceptual tools as they simultaneously built "communities of mind" and "the thinking of teams" (p.187). Despite the potential of mathematical modeling for actively engaging and motivating diverse learners, there is evidence that teaching of mathematical modeling is limited in P-12 classrooms (Doerr, 2007; Zbiek, 2016; Zbiek & Conner, 2006).

Purposes of Study

It has been difficult for the mathematical modeling community to come to an agreement on (a) a unified definition of mathematical modeling, (b) the characterization of the modeling process, or (c) how modeling is differentiated from traditional application problems. This lack of agreement has become one of the major challenges for teaching, learning, and research of modeling in the context of P-12 mathematics education (Cai et al., 2014).

The boundary between problem solving and modeling has never been clear. Some researchers (e.g., Blum & Niss, 1991; Lesh & Doerr, 2003) emphasize the differences between mathematical modeling and *traditional* problem solving and avoid using the term "problem solving" without qualification. These researchers often point out that the facts and rules in traditional application problems are restricted artificially so that the problems can be solved with readily available algorithms or basic operations. Therefore, traditional application problems are also called “dressed up” word problems (Blum & Niss, 1991) or “pre-modeled problems” (Burkhardt & Pollak, 2006). While traditional problem solving often requires one cycle of straightforward interpretation of a real-world scenario, mathematical modeling requires multiple cycles of adapting, modifying, and refining ideas (Lesh & Harel, 2003).

Other researchers such as Selden et al. (1999) call all non-routine or novel problems that require higher-order reasoning simply “problems,” and routine problems that can be solved using clear procedures “exercises”. In their conception of problems, mathematical modeling *is* problem solving, whereas traditional problem solving is just an exercise.

Although we have some knowledge of how researchers perceive the relationship between mathematical modeling and problem solving, limited studies focused on teachers’ perspectives and how their perspectives might help explain their instructional practices and decisions. The purpose of this study is to gain understanding of teachers’ perspectives of the relationship between mathematical modeling and problem solving. Two research questions guide this study:

(1) How do secondary teachers conceptualize the relationships between mathematical modeling and problem solving? (2) How are teachers’ conceptualizations related to their perceptions of teaching mathematical modeling in mathematics classrooms?

Perspectives

Doerr and English (2003) define modeling as a process to develop “systems of elements, operations, relationships, and rules that can be used to describe, explain, or predict the behaviour of some other familiar system” (p. 112). Several common and essential features of the modeling process include (a) modeling is a cyclic process that usually requires multiple iterations, (b) the process typically begins with a real situation, (c) modeling typically ends with the report of a successful result or the decision to revise the initial model to achieve a better result, and (d) the entire modeling process contains several common steps such as formulation, computation, interpretation, and validation (Blum & Leiß, 2007; Blum & Niss, 1991; Galbraith & Stillman, 2006; Pollak, 2003; Zbiek & Conner, 2006). Early depictions of modeling are often rooted in applied mathematics and do not differentiate between the modeling process and the problem-solving process (Burkhardt & Pollak, 2006).

However, more recent characterizations of the modeling process tend to point out the various distinctions between traditional application problems and modeling. Not all applied mathematical problem solving qualifies as modeling. This is because traditional problem solving usually takes a single cycle from givens to goals; whereas modeling tasks involve iterative cycles during which the emerging/initial model is subject to refinement, revision, adaptation, and modification (Dossey, 2010; Lesh & Doerr, 2003; Lesh & Harel, 2003; Lesh & Yoon, 2007; Lesh & Zawojewki, 2007). Solving traditional application problems may also be considered a reduced process of the modeling cycle (Blum & Niss, 1991; Lesh & Doerr, 2003; Lesh & Yoon, 2007). In MEAs, making symbolic descriptions of meaningful real problem situations, or *mathematization*, is at the heart of the tasks. However, in traditional application problems, students make meaning of symbolically described situations (Lesh & Doerr, 2003; Lesh & Harel, 2003). In other words, traditional application problems have already mathematized the situations for students.

In addition, traditional application problems and modeling often lead to very different products or outputs. Many traditional problems typically require a short answer in the form of a sentence or a number. On the other hand, for modeling tasks, students create conceptual tools and artifacts such as new techniques, examples, systems, approximations, and algorithms.

(Dossey, 2010; Lesh & Doerr, 2003; Lesh & Harel, 2003; Pollak, 2003)The criteria for judging the quality of modeling and traditional problem solving tend to be different as well. The solution to an application problem is typically judged based on correctness.

The criteria for judging a mathematical model may include reusability, modifiability, and shareability, as well as model's generalizability beyond the specific problem situation (Doerr, 2016; Lesh & Harel, 2003; Lesh & Yoon, 2007; Zawojewski, 2013). Lesh and Zawojewki (2007) have pointed out the "end in view" nature of MEAs, where models are judged based on the expressed needs of the client given at the beginning of the modeling task.

One subtle difference between mathematical modeling and solving traditional application problems is reflected in the relation between the modeling process and the world outside of mathematics. A modeling task arises from the real world. Modeling is a process to *mathematize the real world*, i.e., bringing the real world into contact with mathematics. On the other hand, a traditional problem is a process of *realizing mathematics*, i.e., given the mathematical knowledge and problem-solving strategies and heuristics, apply them to solve real world problems (Lesh & Doerr, 2003; Lesh & Lehrer, 2003). The best characterization of this difference is given by Dossey (2010), who states:

Modeling involves standing outside mathematics and looking into mathematics to find things that conceivably might help resolve the driving question. Applications, on the other hand, come from standing inside mathematics and noting that particular pieces can be used to better understand or highlight objects outside of mathematics. (p. 88)

Finally, Models and Modeling Perspectives (MMP) proposed by Lesh and colleagues (e.g., Lesh & Doerr, 2003; Lesh & Harel, 2003; Lesh & Zawojewki, 2007) is in direct contrast with the traditional view of the relationships between modeling, applied problem solving, and traditional problem solving. In the traditional view, applied problem solving is treated as a subset or special case of traditional problem solving. Within this view, modeling is a type of applied problem solving. In contrast, MMP treats traditional problem solving as a subset or a special case of "applied problem solving as modeling activities" (Lesh & Yoon, 2007, p.783).

Methods

Participants and Contexts

Participants included eight secondary (6-12) math teachers (Alexis, Alicia, Ann, Brian, Carrie, Eric, Katherine, Sarah, pseudonyms) who taught at eight different schools, and their ages ranged from mid-20s to late-40s. Prior to this study, the eight teachers were enrolled in a two-year secondary teacher preparation program (6-12) at a public university located in the Southeastern United States. The program was a non-traditional program designed for those who have a bachelor's degree in a content area. Half of the eight teachers held a bachelor's degree in an economics or business field: Katherine (Business Management), Ann (Accounting), Sarah (Finance), and Eric (Econometrics). Brian held a bachelor's degree in Aeronautics. The rest of the participants (Alicia, Alexis, and Carrie) majored in mathematics.

Prior to this study, the eight teachers participated in another study (about 1.5 years before this study) during which they had developed conceptual understanding of (a) the meanings of mathematical modeling, (b) the modeling cycle, and (c) criteria for judging the products of mathematical modeling. The eight teachers also solved one MEA and analyzed two sets of MEAs together during the earlier study. This study engaged the teachers in reasoning about the relationships between problem solving and mathematical modeling, an aspect that was treated briefly and given only tentative interpretations in the earlier study.

Data Collection

Alexander and Dochy's (1995) graphic catalyst (displayed in Figure 1) was adapted to elicit the eight teachers' conceptualizations of the relationship between mathematical modeling and problem solving. Teachers were asked to indicate which option best represented their understanding of the relationship between modeling and problem solving. In addition, teachers were free to create their own model of this relationship as an alternative. Alexander and Dochy's categories: *separate*, *overlapping*, *inseparable*, *knowledge subsumption*, and *belief subsumption*, along with their meanings, were borrowed directly for this study except that the last two categories were changed to *modeling subsumption* and *problem-solving subsumption* according to the purpose of this study.

The meanings of these categories were presented to the teachers using the exact words from Alexander and Dochy (1995): (a) *Separat*e (Option 1) shows modeling and problem solving are "two distinct and unrelated entities;" (b) *Modeling subsumption* (Option 2) suggests that modeling is a component of problem solving; (c) *Problem-solving subsumption* (Option 3) shows modeling "as embedded within" problem solving; (d) *Inseparabl*e (Option 4) means modeling and problem solving are "completely overlapping and indistinguishable constructs;" and finally,

(e) *Overlapping* (Option 5) indicates "some dimensions of modeling are integrated with problem solving, still allowing for some aspects of each construct to remain separate and distinct" (p. 417, 419). "Problem solving" was intentionally left ambiguous or unqualified to elicit potentially diverse responses from the teachers. In addition, although each of the graphical representations in Figure 1 and their associated meanings were provided at the beginning of the study, teachers were also free to give their own interpretations.

Each of the eight teachers participated in a semi-structured think-aloud individual interview. Think-aloud interviews are likely to unveil underlying mental processes and knowledge structures, and therefore, are appropriate for a study that seeks to understand teachers' mental representations of the relationship between modeling and problem solving (Kelly & Lesh, 2000). Open-ended and broad prompts were used to elicit a wide range of productive thinking as naturally as possible. Each interview lasted for about an hour.

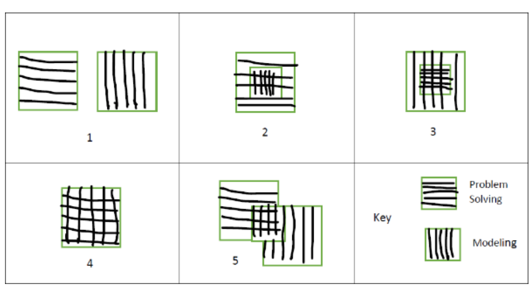


Figure 1: Graphic Representations of Various Relationships Between Modeling and Problem Solving

At the beginning of the interview, each teacher was given (a) Figure 1 and (b) transcript of their verbal response to the question: "What does mathematical modeling mean to you?" (At the end of the earlier study mentioned above). Each teacher was asked to choose one representation or draw an alternative one that best captured their understanding of the relationship based on their response to the question above, and then explain and justify their choice. Each teacher was also given the chance to change their choice based on their thinking at the time of the interview, an opportunity that none of the eight teachers took. In addition, each teacher was asked to explain how their choice might influence their perceptions of teaching mathematical modeling in 6-12 mathematics classrooms.

Data Analysis

A hybrid approach of inductive (emergent) and deductive (theoretically guided) coding (Fereday & Muir-Cochrane, 2006) was used to analyze the data. Alexander and Dochy's (1995) graphic representations and the corresponding categories existed *a priori* and were applied to the data directly as provisional codes. However, codes that emerged during the analysis were also allowed to avoid premature closure, and to uncover new ideas (Saldaña, 2016; Strauss & Corbin, 1990). Several rounds of coding were performed on the data. The first round of coding focused on identifying and labeling excerpts of interest. The second round of coding refined the initial codes as guided by the research questions. Codes that were considered as trivial or less relevant to the study were removed. As analysis continued, codes that shared similar attributes were grouped together into categories or themes during the third round of coding. Finally, a fourth round of coding was conducted to make appropriate modifications to achieve a better fit between the final codes and the data and to make sure that the coding categories had stabilized.

Results

Teachers' Conceptualizations of the Relationship between Modeling and Problem Solving

Each of the final coding categories and sample responses from the eight teachers are presented in Table 1. Alexis was the only teacher who chose Option 1. She was hesitant to claim that the two processes are completely unrelated, but she was certain that they are distinct from each other due to the very different natures of the two processes. Although problem solving has an obvious path; with modeling, the learner needs to find the "entire mathematics approach."

Brian and Alicia both chose option 2. For both teachers, problem solving is the bigger purpose. Modeling also serves the purpose of solving problems. When asked how traditional application problems are different than MEAs, Brian explained that traditional application problems are "very basic," then he shared the example of giving students a constant speed of a train and a travel distance, and ask them: "How long is it going to take to go this far?" On the other hand, Brian viewed MEAs as more complicated: "take a step outside of this [real-life scenario], and we can start applying some math concepts to it, then step back into our real-world scenario" to find out whether a solution "fits the bill." Alicia described MEAs as requiring "more depth of knowledge," and "really, really difficult, like having them [students] think outside of the box." For both teachers, the larger box contains all kinds of problem solving, including context-free problems, such as solving or manipulating an equation. Applied problem solving that involves real-world contexts belongs to the box in the middle. MEAs or other advanced modeling activities also belong to the box in the middle and constitute a more difficult subset [not shown on the figure] of all applied problem solving.

Ann was the only teacher who chose Option 5. She explained that the part that belongs to only problem solving but not modeling represents solving "cut and dry equations" using a given formula [i.e., context-free problems]. The part that belongs to modeling but not problem solving are problems or activities like MEAs that are "open-ended" and "extensive," and may not always lead to an accurate answer. The part where modeling and problem-solving intersect must "making sense of real-world situations" and require interpretation before coming up with a solution, but still must have an "accurate" answer or "one true answer."

Sarah chose Option 3. Carrie was debating between Option 3 and Option 4. Both teachers believed that problem solving in a mathematics classroom must involve mathematical symbols or expressions; and therefore, only after a real-world situation is mathematized, problem solving begins. The steps before mathematization such as interpreting and visually representing a scenario are part of modeling or preparation for problem solving but not problem solving.

Katherine and Eric chose Option 4 as their best representation of the relationship between modeling and problem solving. These teachers emphasized the cyclic and iterative nature of the modeling process and how problem solving, and modeling are integrated into the same process and are mutually dependent on each other. Carrie (who chose both Option 3 and Option 4) also pointed out even "simple little problem" like 3 + 2 can be a mathematical representation of a real-world scenario and therefore, math is all about describing the real world. Eric shared a similar view and argued that addition and subtraction in early grades, as well as using multivariate functions to represent and predict an economic phenomenon in econometrics are all mathematical modeling except for their "varying degrees of difficulty and fluidity."

Table 1: Sample Responses for Each of the Coding Categories

Coding Categories	Sample Responses
Straightforward versus open-ended	Classroom-based problems, I'm literally only looking for a solution. I am not really looking for the in between…simply do what the formula is asking me to do…whereas with modeling…still given a given, but you have to produce the goal, and you have to produce everything in between, like your approach to how to get your solution. (Alexis, Option 1)
Modeling as a tool for problem solving	Modeling can be used for problem solving. (Brian, Option 2) The whole purpose of modeling is to assist with problem solving… Problem solving would be the bigger picture. (Alicia, Option 2)
Overlapping but different	The problem solving [part], I feel like is just like handing them an equation to solve, that's problem, and it's got to be right or wrong answer… the modeling part I kind of…realized that's even more real-world type situations in context … A lot of times, one student may interpret it differently than another student. They're not necessarily always getting the same answer…Modeling is not always accurate, I guess, because you can get, you know, different opinions from groups or students…modeling could help solve the problem, so maybe that would be the middle…[but] also has to be accurate. (Ann, Option 5)
Problem solving embedded in modeling	Problem solving is embedded within modeling. (Sarah, Option 3) There's more to modeling the real world than just problem solving. (Carrie, Option 3 & 4)

Modeling and problem solving as an integrated process	Can’t have one without the other…Modeling is just like math in general…used to describe the real world…you can’t have modeling without math, and you can’t really have math without modeling. (Carrie, Option 3 & 4) Enmeshed with each other. (Katherine, Option 4) They're one in the same…varying degrees of difficulty and fluidity. (Eric, Option 4)

Teaching Mathematical Modeling in Mathematics Classrooms

In response to the question: “How do you think your conceptualization of the relationship between modeling and problem solving influences your decisions regarding teaching mathematical modeling in mathematics classrooms,” the eight teachers focused on two dimensions of instruction: instructional sequence and problem/activity types (see Figure 2). Among all the teachers, Alexis was the most hesitant to teach mathematical modeling to her students. She felt that she was not prepared to enact modeling activities in her classrooms and emphasized that formal training was needed before she would be ready. She was also concerned about her students, stressing: “I don’t feel like everybody will be able to grasp the concept [of mathematical modeling].” She admitted that she was more comfortable with the traditional instructional sequence in which formal demonstration of mathematical concepts should happen before asking students to use the concepts to solve traditional application problems.

Instructional Sequence / Problem /Activity Types	Traditional Instructional Sequence (Demonstration -> Application Problems)	Modeling Approach (Modeling with Problem Solving -> Direct Instruction)
Traditional Applied Problems	Alexis	Brian, Alicia
MEAs or Interdisciplinary Projects		Ann, Katherine, Eric, Sarah, Carrie

Figure 2: Teachers’ Preferred Instructional Sequences and Types of Problems

The rest of the teachers all shared the belief that teaching mathematical modeling was both important and lacking in secondary mathematics education. In addition, they all preferred a modeling approach to teaching mathematics, which is, having students engage and struggle with a mathematical modeling activity first (to develop mathematical concepts naturally) before formally introducing mathematical concepts. Especially Alicia, when reflecting on the changes in her own instructional sequence, stated “introducing a model first could be really beneficial for the students.” With the traditional approach, Alicia observed, “they [her students] were lost,” and “they’re going through the motions, and they don’t really understand the concept of the function.” However, both Brian and Alicia were also satisfied with using traditional application problems as modeling activities.

The rest of the teachers were more open to activities like MEAs. Katherine acknowledged that although she was using a pilot program that required more problem solving from her students than before, the problems were still just requiring students to consider one variable at a time; MEAs require the learner to consider multiple factors at the same time and ideally should be used instead. Ann and Eric both felt positively about how the math teams at their schools were regrouping their standards so that one project can address multiple standards at the same time.

All three teachers pointed out (a) the benefits of working with colleagues in other disciplines, and (b) the similarities between MEAs and the interdisciplinary projects at their own schools.

Discussion and Conclusion

The eight teachers' conceptualizations of the relationship between modeling and problem solving are diverse and mostly consistent with research literature. For example, Alexis's focus is on whether the path from givens to goals is obvious or hidden. This is also pointed out by Lesh and colleagues as one of the key differences between traditional application problems and MEAs (Lesh et al., 2000). Brian's and Alicia's view of modeling as a subset of problem solving is consistent with what Lesh and colleagues (Lesh et al., 2000; Lesh & Doerr, 2003) call "the traditional view." This view considers all application problems, including MEAs, as a subset of all kinds of problem solving. Ann's review is more consistent with the MMP advocated by Lesh

and colleagues (Lesh et al., 2000; Lesh & Doerr, 2003). If we do not consider the part in Option 5 of Figure 1 that belongs to only problem solving but not modeling (i.e., according to Ann, context-free problems that require using formulas only), the rest of the graphic representation is consistent with MMP in terms of viewing traditional application problems as a subset of modeling activities in general. Sarah's and Carrie's view of solving traditional application problems as a reduced process of the modeling cycle is also consistent with Blum and Niss (1991). The view that modeling and problem solving are completely enmeshed, inseparable and mutually dependent on each other is a new conceptualization not found in research literature.

There is some preliminary evidence that a teacher's conceptualization of the relationship between modeling and problem solving is likely to be related to the instructional sequence and type of problems they prefer. For example, one teacher's (Alexis) view seems to create a barrier between the two processes, which makes the teacher feel the transition from the traditional approach to the modeling approach more challenging. In addition, when modeling is viewed as a more advanced form of problem solving, MEAs are often seen as another subset of all applied problems. This view does not seem to deter a teacher (e.g., Alicia) from successfully adopting the modeling approach or teaching mathematical modeling. However, under this view only traditional application problems (e.g., use an initial fee and fixed monthly payment to model linear functions) are likely to be used to teach mathematical modeling. Teachers who view modeling and problem solving as two integrated and mutually dependent processes or whose view is consistent with the MMP are more open to not only the modeling approach but also teaching mathematical modeling using open-ended, complex activities.

In conclusion, this study is an exploratory study that involves only eight teachers. Although the eight teachers' conceptualizations of the relationships between modeling and problem-solving are diverse, it is possible that other conceptualizations are missed. The relationship between these conceptualizations and instructional preferences is tentative due to the small number of teachers studied and needs further research. Despite these limitations, this study offers some implications for how to increase the use of modeling to teach mathematics in 6-12 classrooms. First, the view that mathematical modeling and problem solving are two distinct processes seems to be the least ideal for encouraging modeling in the mathematics classroom. In this study, the teacher who held this view was the most hesitant to teach modeling because she perceived considerable risks for using modeling in her teaching. Second, all the other conceptualizations including (a) modeling as a subset of problem solving, (b) the two are overlapping, (c) problem solving is embedded in modeling, and (d) modeling and problem solving is an integrated process, seem to be compatible with the instructional approach and sequence that integrate modeling throughout the curriculum rather than saving modeling for the end of a unit or a course. This instructional approach and sequence often allow students to explore mathematics problems collaboratively before direct instruction and has shown to be more effective than the traditional approach with teacher demonstration followed by practicing traditional problems (Boaler, 2015). Third, some teachers may hold "the traditional view" that all application problems, including MEAs, constitute a subset of all kinds of problem solving, and therefore, do not make a clear distinction between traditional application problems and MEAs.

These teachers may feel content with using traditional application problems even if these problems often focus on developing procedural skills rather than reasoning ability. Teacher educators may steer teachers away from conceptualizations that are counterproductive to the use of modeling activities that require higher-order thinking and help teachers develop MMP by building upon productive conceptualizations.

References

Aguilar, J. J. (2021). Modeling through model-eliciting activities: An analysis of models, elements, and strategies in high school. The cases of students with different level of achievement. *Mathematics Teaching Research Journal, 13*(1), 52-70.

Alexander, P. A., & Dochy, F. J. R. (1995). Conceptions of knowledge and beliefs: A comparison across varying cultural and educational communities. *American Educational Research Journal, 32*(2), 413-442.

Blum, W., & Leiß, D. (2007). How do students and teachers deal with modeling problems? In C. P. Haines, P. Galbraith, W. Blum, & S. Khan (Eds.), *Mathematical modeling (ICTMA 12): Education, engineering and economics* (pp. 222-231). Chichester, UK: Horwood.

Blum, W., & Niss, M. (1991). Applied mathematical problem solving, modeling, applications, and links to other subjects: State trends and issues in mathematics instruction. *Educational Studies in Mathematics, 22*, 37-68.

Boaler, J. (2015). *Mathematical mindsets: Unleashing students' potential through creative math, inspiring messages and innovative teaching* (1st ed.). Jossey-Bass.

Burkhardt, H., & Pollak, H. (2006). Modeling in mathematics classrooms: Reflections on past developments and the future. *ZDM: The International Journal on Mathematics Education, 38*(2), 178-195. https://doi.org/10.1007/BF02655888

Bybee, R. W. (2010). Advancing STEM education: A 2020 vision. *Technology and Engineering Teacher, 30*, 30-35. Cai, J., Cirillo, M., Pelesko, J. A., Borromeo Ferri, R., Borba, M., Geiger, V., Stillman, G., English L. D., Wake, G.,

Kaiser, G., & Kwon, O. N. (2014). Mathematical modeling in school education: Mathematical, curricular, cognitive, instructional, and teacher education perspectives. In P. Liljedahl, C., Nicol, S. Oesterle, & D. Allan (Eds.), *Proceedings of the 38th Conference of the International Group for the Psychology of Mathematics Education [PME]*, vol. 1 (pp. 145-172). Vancouver, CA: PME.

Doerr, H. M. (2007). What knowledge do teachers need for teaching mathematics through applications and modeling? In W. Blum, P. Galbraith, H. Henn, & M. Niss (Eds.), *Modeling and applications in mathematics education: The 14th ICMI Study, New ICMI Studies Series no. 10* (pp. 69-78). Springer.

Doerr, H. M. (2016). Designing sequences of model development tasks. In C. R. Hirsch, & A. R. McDuffe (Eds.), *Mathematical modeling and modeling mathematics* (pp. 197-206). The National Council of Teachers of Mathematics [NCTM].

Doerr, H., M. & English, L. (2003). A modeling perspective on students' mathematical reasoning about data. *Journal for Research in Mathematics Education, 34*, 110-136. https://doi.org/10.2307/30034902

Dossey, J. (2010). Mathematical modeling on the catwalk: A review of modeling and applications in mathematics education: The 14th ICMI Study. *Journal for Research in Mathematics Education, 41*(1), 88-95.

Fereday, J., & Muir-Cochrane, E. (2006). Demonstrating rigor using thematic analysis: A hybrid approach of inductive and deductive coding and theme development. *International Journal of Qualitative Methods, 5*(1), 80-92. https://doi.org/10.1177/160940690600500107

Galbraith, P., & Stillman, G. (2006). A framework for identifying student blockages during transitions in the modeling process. *ZDM: The International Journal on Mathematics Education, 38*(2), 143-162. https://doi.org/10.1007/BF02655886

Kelly, A. E., & Lesh, R. A. (Eds.). (2000). *Handbook of research design in mathematics and science education*. Routledge.

Lesh, R. A., & Doerr, H. M. (Eds.) (2003). *Beyond constructivism: A models and modeling perspectives on mathematical problem solving, learning, and teaching*. Lawrence Erlbaum Associates.
Lesh, R. A., & Harel, G. (2003). Problem solving, modeling, and local conceptual development. *Mathematical Thinking and Learning, 5*(2-3), 157-189. https://doi.org/10.1080/10986065.2003.9679998
Lesh, R. A., Hoover, M., Hole, B., Kelly, A., & Post, T. (2000). Principles for developing thought-revealing activities for students and teachers. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 591–646). Lawrence Erlbaum Associates, Inc.
Lesh, R. A., & Lehrer, R. (2003). Models and modeling perspectives on the development of students and teachers. *Mathematical Thinking and Learning, 5*(2-3), 109-12. https://doi.org/10.1080/10986065.2003.9679996 Lesh, R. A., & Yoon, C. (2007). What is distinctive in (our views about) models and modeling perspectives on mathematical problem solving, learning, and teaching? In W. Blum, P. Galbraith, H. Henn, & M. Niss (Eds.), *Modeling and applications in mathematics education: The 14th ICMI Study, New ICMI Studies Series no. 10* (pp. 161-170). Springer.
Lesh, R. A., & Zawojewski, J. S. (2007). Problem solving versus modeling. In R. A. Lesh, P. L., Galbraith, C. R. Haines, & A. Hurford (Eds.), Modeling students' mathematical modeling competencies (pp. 237-243). Springer.
Pollak, H. O. A. (2003). A history of the teaching of modeling. In G. M. A. Stanic, & J. Kilpatrick, (Eds.), A history of school mathematics (pp. 647-669). National Council of Teachers of Mathematics [NCTM].
Saldaña, J. (2016). The coding manual for qualitative researchers (3rd ed.). Sage.
Strauss, A., & Corbin, J. (1990). Basics of qualitative research: Grounded theory procedures and techniques (2nd ed.). Sage.
Selden, A., Selden, J., Hauk, S., & Mason, A. (1999). Do calculus students eventually learn to solve non-routine problems? Technical report. No. 1999-5. Cookeville, TN: Tennessee Technological University.
Zawojewski, J. S. (2013). Problem solving versus modeling. In R. A. Lesh, P. L., Galbraith, C. R. Haines, & A. Hruford (Eds.), Modeling students' mathematical modeling competencies (pp. 237-243). Springer.
Zbiek, R. M. (2016). Supporting teachers' development as modelers and teachers of modelers. In C. R. Hirsch, & A.R. McDuffe (Eds.), Mathematical modeling and modeling mathematics (pp. 263-272). The National Council of Teachers of Mathematics [NCTM].
Zbiek, R. M., & Conner, A. (2006). Beyond motivation: Exploring mathematical modeling as a context for deepening students' understanding of curricular mathematics. Educational Studies in Mathematics, 63(1), 89-112. https://doi.org/10.1007/s10649-005-9002-4

SOCIOMATHEMATICAL SCAFFOLDING AS STUDENTS ENGAGE IN DISCIPLINARY PRACTICES

Kristen Vroom
Michigan State University
vroom@msu.edu

Instructors manage several tensions as they support students to engage in mathematical disciplinary practices such as defining, conjecturing, and proving. These tensions include honoring students' contributions while simultaneously apprenticing students to following mathematical norms. I present a case study of a teacher-researcher in a laboratory setting who was particularly skilled at this endeavor. I found that the teacher-researcher engaged in a pattern in which the teacher-researcher cycled between inquiring into the students' thinking about their draft of a definition, conjecture, or proof and then engaged in scaffolding, including scaffolding of mathematical norms. I exemplify this pattern with an episode of students writing a conjecture equivalent to the Archimedean Property that served as a warrant for one of their proofs. I close the paper discussing complexities of apprenticeship into the norms of the discipline.

Keywords: Instructional Activities and Practices, Reasoning and Proof, Undergraduate Education

One common goal of many undergraduate mathematics classes is to engage students in disciplinary practices that support them in creating definitions, conjectures, and proofs. This allows students to experience mathematics as a creative activity, rather than a static product handed down to them by their instructor. For example, a student from a class that used curriculum materials that were specifically designed for this goal commented on their experience:

> "You're not being handed a finished package and being told to just take it as is. You're building it. It's the difference between being given an assembled Lego set and building it yourself…"

Many scholars have discussed the tensions that instructors manage as they support students to 'build it themselves' (e.g., Ball, 1993; Melhuish et al., 2022; Rasmussen & Marrongelle, 2006; Speer & Wagner, 2009). At the heart of this complex work is honoring the students' contributions, or using the Lego analogy, valuing what the students do with the Legos. At the same time, instructors aim to support students to make progress on the mathematical goals that are set by the course. Using the Lego analogy, this means the "it" in "build it themselves" is a specific structure that the instructor has in mind such a Lego-car. Further, instructors are often obligated to support students to adopt norms from the mathematics discipline. Put in another way, instructors are compelled to support students to build their Lego-car with four wheels touching the ground, for instance, rather than fulfilling the wheels' purpose in some other inventive way.

There have been notable discussions around the teaching and learning of mathematics discipline's norms (Dawkins & Weber, 2017; Weber & Melhuish, 2022). Mathematical norms have been discussed as a matter of access that can operate to equip or exclude students in mathematics and mathematics classrooms (Dawkins & Vroom, in press). And thus, students who desire to interact with the mathematics discipline should be given meaningful experiences with them, such as those that support students to see norms as reasoned choices (Larsen et al., 2022; Vroom, 2022). Mathematical norms have also been discussed in relation to identity since the adoption of disciplinary norms may overshadow students' personal, cultural, or linguistic capacities (Goffney et al., 2018). And so, an instructor must "strike a balance between opportunities to reflect on oneself and others as part of the mathematics learning experience" (Gutiérrez, 2009, p. 5). The complexities of apprenticeship into the norms of the discipline warrant more research on the different ways this might be enacted. This research could better inform when this apprenticeship is (and is not) appropriate, how instructors can craft meaningful experiences with disciplinary norms, and how instructors can strike a balance between providing opportunities to reflect on oneself and others.

This paper makes some progress on understanding apprenticeship into mathematical norms. I present a case study of a teacher-researcher in a laboratory setting who was particularly skilled at building on students' initial drafts of definitions, conjectures, and proofs. He supported the students to refine the drafts so that they both captured the students' ideas and followed mathematical norms. I investigated what the teacher-researcher was doing to support the students in this way. In what follows, I discuss the constructs that I used to describe the teacher-researcher's moves, including sociomathematical scaffolding. Then, I present a cyclic pattern that I noticed in the teacher-researcher's moves and exemplify it with an episode from the laboratory experiment.

Theoretical Perspective

I take scaffolding to mean the process that enables students to carry out a task that they may have not otherwise been able to accomplish without assistance (Wood et al., 1976). In this report, I investigate the teacher-researcher's scaffolding of students' creation of definitions, conjectures, and proofs that are consistent with mathematical norms. Mathematical norms are expectations on practice accepted by the mathematics community to uphold mathematics community's shared orientations and goals that underlie shared activity (Dawkins & Weber, 2017). For instance, conventions in formal mathematical language are examples of norms that uphold the goal of effectively communicating and using definitions, theorems, and proofs. Norms can also include expectations such as excluding irrelevant statements from a proof in order to uphold that proofs should increase understanding of the mathematical theory involved (Dawkins & Weber, 2017).

Williams and Baxter (1996) distinguished between analytic and social scaffolding. *Analytic scaffolding* is "the scaffolding of mathematical ideas for students" (p. 24) and is a tool to support students to create mathematical knowledge. Whereas *social scaffolding* is "the scaffolding of norms for social behavior and expectations regarding discourse" (p. 24) and influences the "ritual of classroom life" (p. 36). Nathan and Knuth (2003) note that social scaffolding is not particular to mathematics and can apply to any instruction regardless of the subject. An example of social scaffolding is requesting students to take turns sharing their rough draft ideas with their groupmates.

Inspired by Cobb and Yackel's (1996) emergent perspective, I distinguish a third category, *sociomathematical scaffolding*, as the scaffolding of mathematical norms. For instance, an instructor might let students know that it is normative in mathematics for "there exists" and "there exists at least one" to be interchangeable as students interpret a mathematical statement with the phrase (Vroom, 2022). Such scaffolding was included in Nathan and Knuth's (2003) analytic scaffolding; however, this differentiation is crucial as students engage in creating definitions, conjectures, and proofs as such products are heavily influenced by mathematical norms. It is a widely shared goal that students should be supported to create products that are conceptually reasonable (potentially supported by analytic scaffolding). But there is critical debate about the extent to which and how students should be supported to adopt the norms of the mathematics discipline (potentially supported by sociomathematical scaffolding).

There are a variety of ways in which instructors can scaffold. I highlight two categories of ways because of their relevance to the data that I share in the Results section: (a) creating problematic situations and (b) pedagogical content tools. *Creating problematic situations* is connected to Harel's (2008, 2013) theory that students should have an intellectual need to learn what we intend to teach them, where an intellectual need refers to a problematic situation that motivates the construction of the piece of knowledge. An instructor who aims to create intellectual needs to refine students' definitions, conjectures, and proofs would focus efforts on shining light on an issue with the draft (i.e., creating a problematic situation).

Pedagogical content tools (PCTs) can be powerful tools in instructor scaffolding. A PCT is "a device, such as a graph, diagram, equation, or verbal statement, that a teacher intentionally uses to connect to student thinking while moving the mathematical agenda forward" (Rasmussen & Marrongelle, 2006, p. 389). Rasmussen and Marrongelle discuss two types of PCT's: transformational records and generative alternatives. *Transformational records* are "notations, diagrams, or other graphical representation that are initially used to record student thinking and that are later used by students to solve new problems" (p. 389). *Generative alternatives* are "alternate symbolic expressions or graphical representations that a teacher uses to foster particular social norms for explanation and that generate student justifications for the validity of these alternatives" (p. 389). For instance, the teacher might introduce a teacher-initiated alternative to the class with the intention of soliciting justifications, or a teacher might highlight a student-initiated alternative and then engage students in a discussion about merits of the alternative.

Methods

The data that I present in this report comes from a laboratory design experiment (Cobb & Gravemeijer, 2008). The primary design goal was to refine tasks that supported students in reinventing foundational concepts in real analysis using the instructional design theory of Realistic Mathematics Education (Gravemeijer, 1999). A secondary goal was to test the conjecture that the task sequence could be leveraged to support students in learning about proof-related activity such as defining, conjecturing, and proving (see Larsen et al., 2022 and Vroom, 2020 for details about the task design). The experiment was ten 1.5-hour long teaching sessions.

The design experiment involved two students who I refer to as Chloe and Gabe, a teacher-researcher who I refer to as Bryan, an observer (myself), and a camera operator. Chloe and Gabe typically earned high marks in their mathematics coursework and were recruited from the last two courses of the calculus sequence at their university. Bryan was particularly experienced at teaching in such a way that built on students' thinking. He had previously conducted many teaching experiments in this role (in both laboratory and classroom settings) to develop inquiry-oriented curricula and instructor support materials. The reason I selected the design experiment for further investigation was because Bryan seemed to play a key role in supporting students to refine their contributions (draft of definitions, conjectures, proofs) in such a way that both captured the students' ideas and followed disciplinary norms.

The data analysis for this study began with creating content logs of each teaching session. The logs were a detailed chronological account of the students' activity and discourse. The content logs were supplemented with pictures of the students' work and selected transcribed excerpts. After the end of the experiment, I reread the logs and identified episodes of the students engaging in writing definitions, conjectures, and proofs. I considered an episode to be the activity that occurred between the students' first and last draft of a particular definition, conjecture, or proof. I then selected three focal episodes that varied in activity type (defining, conjecturing, proving) to further analyze based on Bryan's seemingly crucial role. To do so, I identified instances in which Bryan inquired into the students' thinking, engaged in scaffolding, created problematic situations, or used a pedagogical content tool. I used these codes because they were either instructional moves that Bryan and I specifically talked about while collecting data (inquiring into the students' thinking, PCTs) or I perceived there to be relevant afterwards (scaffolding, creating problematic situations). As I analyzed the data, I noticed an emerging pattern in Bryan's moves. After I analyzed the focal episodes, I then returned to the remaining episodes to both observe whether the pattern appeared in the other episodes and add more nuanced understanding of this pattern. In what follows, I first describe this pattern and then I illustrate it with one of the focal episodes.

Results

Throughout the teaching sessions, Bryan's moves followed a cyclic pattern in which he repeatedly inquired about the students' intended meaning of their draft (of a definition, conjecture, or proof) and then followed with scaffolding to support the students in refining their draft. I illustrate the cyclic patten in the episode that follows in which Bryan supported the students to write and refine a conjecture equivalent to the Archimedean Property by repeatedly engaging in sociomathematical scaffolding.

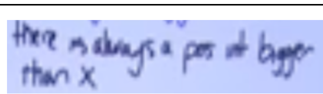
First Cycle

Prior to this episode, Chloe and Gabe engaged in writing a proof that the sequence $\{2^n\}$ tended to infinity. Their emerging proof let k be a real number, let $x = \log_2 k$, and then "let $m > x$ where m is a positive integer". The episode began with Bryan following up with the line "let $m > x$ where m is a positive integer" by asking why they could "pick a positive integer that is bigger than x". Chloe responded with the first draft of their conjecture (see Figure 1.)

"there is always a bigger number"

Figure 1. Draft 1.

Bryan began sociomathematical scaffolding, saying "Add a little more precision to this, what kind of number is a bigger number and what is it bigger than?" This question implicitly shared the mathematical norm that statements should be unambiguous. The question exposed a problematic situation in that Chloe's intended meaning was unclear because it did not state what type of number she was referring to, nor did it state what sort of number she was comparing it to. This led Chloe to refine the statement to Draft 2 (Figure 2).



"there is always a pos int bigger than x"

Figure 2. Draft 2.

Second Cycle

Next, Bryan inquired into Chloe's intended meaning of Draft 2 by asking clarifying questions such as: "And what is x?", "And what do we know about x?", "So, are you saying that there is always a positive integer that is bigger than that particular x, the one that is $\log_2 k$? Or is some more general principle of numbers that you're trying to say? So, is this x specific to our problem?". With this inquiry, Chloe revealed that she intended for the statement to be true for the particular x-value that they introduced in their proof ($x = \log_2 k$). She also explained that the statement was also true for an arbitrary x-value, explaining "but like it is applicable for any x".

The conversation continued with Bryan sociomathematical scaffolding by exposing a mathematical norm that the statement should justify the line in the proof:

Teacher-researcher: So, what kind of number should we say that it is in our statement of this property? If we said that 'there is always a positive integer that is bigger than x where x is an integer' would that be saying the same thing as saying 'where x is a real number'?

Chloe: No, but they are both true.

Teacher-researcher: And which are you using here?

Chloe: It doesn't matter.

Teacher-researcher: Well, let's test you on that. Write 'where x is an integer'.

(Chloe altered the statement as requested: "there is always a pos int bigger than x where x is an integer")

Teacher-researcher: Now if we were somehow able to prove that true, what you just wrote, would you be able to use that to explain why you can choose an m over here? (referring to "let $m > x$ where m is a positive integer" in the students' emerging proof)

Gabe: No.

Observer: Why not?

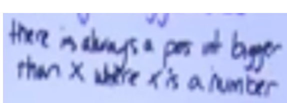
Gabe: Because this isn't necessarily going to be an integer (points to $x = log_2 k$).

Chloe: Yeah, true.

Gabe: And so it won't cover all the potential values.

Teacher-researcher: So, if you say that 'there is always a positive integer bigger than x where x is an integer' then that's not strong enough to use in this case.

(Chloe refined the statement to Draft 3, see Figure 3.)



"there is always a pos int bigger than x where x is a number"

Figure 3. Daft 3.

During the above exchange, Bryan leveraged a generative alternative by focusing the students' attention on two specification options for x: a real number or an integer. Chloe explained that both options were acceptable since both versions of the statement were true. Bryan then refocused the students to not only attend to the truth of the statement but also the need to explain the step in their proof. Using the alternative in relation to the line of the proof created a problematic situation: the version of the statement that specified x to be an integer was true but did not function as a warrant because “it won't cover all the potential values” of $x = \log_2 k$. This motivated Chloe to refine the statement by adding “where x is a number” to the drafted conjecture (Figure 3).

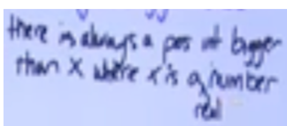
Third Cycle

Bryan continued the conversation by inquiring into the students’ thinking about Draft 3, asking “Do you mean real number? Or a…” Chole responded, “I don’t see why it matters what number it is because there is going to be a positive integer that is larger than it.” I interpret Chloe’s comment to mean that she didn’t see why there was a need to specify x to be real number, and that she thought her statement (that said ‘where x is a number’) captured her idea.

Bryan continued the discussion by sociomathematical scaffolding by sharing a mathematical norm. He explained:

> Well that’s why we say real numbers when we mean that. When we mean that it doesn’t matter what kind of number it is, we say real number. To do two things. To one to say that we mean it more than just naturals, or integers, or rationales, or irrationals, we mean it for all of them. That’s the set that we say when it doesn’t matter, we say it’s true for all real numbers. It also lets them know that we are not thinking of imaginary numbers.

With this comment, teacher-researcher shared that the mathematics community would use ‘real numbers’ instead of ‘numbers’ to communicate the students’ meaning. Chole responded with “ok” and then altered the draft by adding the word “real” before number. (See Figure 4.)



“there is always a pos int bigger than x where x is a real number”

Figure 4. Draft 4.

Fourth Cycle

Bryan then considered Draft 4 before he continued the conversation by sociomathematical scaffolding:

Teacher-researcher: Alright, so… [reads statement under breath]. Alright, so one thing that I want to do for practice if you go back to your definition we wrote that one (pointing to a definition that they previously wrote for a sequence converging to infinity) in terms of ‘for any real number there exists’, so it’s sort of like ‘for any… there exists’. But this one over here that Chloe just wrote doesn't quite... it seems like it is trying to say the same sort of thing but not using the same kind of language. So, can we write this with saying like ‘there exists’ and ‘for any’ or ‘for all’? I think that ‘for all’ is more common but ‘for any’ is also kosher.

(Chloe starts writing.)

Teacher-researcher: It's not that what you are saying is wrong, but if we kind of write them all in the same way then it is easier to see how we map them in proofs and stuff. And maybe if you use variables that would be cool too.

(Chloe and Gabe give teacher-researcher questioning look.)

Teacher-researcher: Like letters like you did before you had the $a'_x s$ and the $k's$ (pointing to their previously constructed definition of sequence converging to infinity), things like that. So, you have an x for one of them but you didn't use a variable for your integer.

Chloe: Yeah.

Teacher-researcher: So maybe use a letter for both of them.

Chloe: Yeah.

(Crosstalk while Chloe writes)

(Chloe writes: for any real number x there exists a positive integer m that is larger than x)

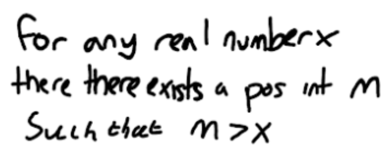
Chloe: Happy now?

Teacher-researcher: Well I am always happy, I am also always happy to ask for more.

(Bryan and students laugh)

Teacher-researcher: So, is there a mathy way to say 'is larger than'?

Gabe: Yeah. (Gabe finished refining the statement to Draft 5, see Figure 5.)



"for any real number x
there there exists a pos int m
such that $m > x$"

Figure 5. Draft 5.

In the previous segment, Bryan leveraged transformational records. First, Bryan continued to connect the students' statements with a conventional version of the statement by suggesting that they use the phrases 'there exists' and 'for any' or 'for all'. Then, Bryan suggested that they use symbols to encode 'is larger than'. With these requests, Bryan suggested that such phrases are normative in mathematics discourse. These requests motivated the students to refine their draft to what's in Figure 5. Bryan appeared to be satisfied with Draft 5 ending the refining activity.

Conclusion

In this report, I presented the cyclic pattern in which Bryan supported the students in refining their definitions, conjectures, and proofs. In the episode that I presented, Bryan cycled between inquiring into the students' thinking about a draft of a conjecture and engaged in sociomathematical scaffolding. The episode that I offered exemplified that this is complex work. He drew on a variety of tools to support his scaffolding including creating problematic situations and using PCTs. I do not present this pattern as necessarily an idealistic way to support students to write and edit definitions, conjectures, and proofs. The appropriateness of this pattern is highly contextual and minimally depends on the students' learning goals. Rather, I present the pattern to further contextualize the notion of sociomathematical scaffolding to promote critical discussions on the teaching of mathematical norms. I highlight serval points about Bryan's sociomathematical scaffolding in what follows.

There were instances in which Bryan was able to create problematic situations to support his sociomathematical scaffolding (during the first and second cycle). Sociomathematical scaffolding seemed particularly productive when paired with creating problematic situations in the sense that such moves emphasize reasoning for such refinements. During the first cycle, this pairing revealed that Draft 1 was unclear what number was being reference and compared. During the second cycle, this pairing exposed that the version of the statement that specified x to be an integer was true but did not function as a warrant in the students' proof. Both of these instances focused on the need for a refinement that was connected to a mathematical norm. I suspect that sociomathematical scaffolding paired with problematic situations would promote student access in mathematics classrooms (the extent to which this instruction allows students to "play the game", Gutiérrez, 2009, p. 6). Future work could further explore this conjecture.

I also predict that the way in which sociomathematical scaffolding is enacted can promote or constrain the re-making of mathematical norms through negotiation with students and their contexts (the extent to which this instruction allows students to "change the game", Gutiérrez, 2009, p. 6). I conjecture that sociomathematical scaffolding paired with creating problematic situations would affirm students' identity in the mathematics classroom because creating problematic situations focuses on the need for a refinement, not the actual refinement itself. Thus, students' have the ability to negotiate how they want to (re-)make mathematical norms for their classroom. For instance, the second cycle highlighted the need for the statement to warrant a line in their proof. The students refined the draft so that it functioned as a warrant by stating that x was a number, which later it was clarified that the students intended "number" to mean how others might mean "real number". During the third cycle, Bryan choose to share the norm that mathematicians use "real number" when they reference the sort of numbers that the students were referring to. I argue that the appropriateness of this move depends on the students. If the students desired to abide by mathematical norms then this scaffolding seems appropriate. If the students did not have this desire, then it may be more advantageous to allow students to use "number" instead of "real number" to affirm students' own linguistic capacities in the mathematical setting.

I did not see evidence of the instructor engaging in social scaffolding as he supported students to refine their definitions, conjectures, proofs. This is likely because of the laboratory setting and the episodes that I analyzed were later in the sessions likely after social norms had been established. I did, however, find evidence of the teacher-research engaging in analytic scaffolding in other episodes that I analyzed. I suspect that the interplay between social, sociomathematical, and analytic scaffolding would be important in the whole class setting and future research could investigate the ways in which these interact.

This paper takes some first steps to document how the apprenticeship into mathematical norms can be enacted as students engage in defining, conjecturing, and proving. I hope that future work can build on it to further explore contexts for which this apprenticeship is (and is not) advantageous, the ways to create meaningful experiences with disciplinary norms, and how instructors can balance supporting students to play and change the game of mathematics through classroom mathematical norms.

Acknowledgments

This work is part of the Advancing Students' Proof Practices in Mathematics through Inquiry, Reinvention, and Engagement project (NSF DUE #1916490). The opinions expressed do not necessarily reflect the views of the NSF.

References

Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *The elementary school journal*, *93*(4), 373-397.

Cobb, P., & Gravemeijer, K. (2008). Experimenting to support and understand learning processes. *Handbook of design research methods in education: Innovations in science, technology, engineering, and mathematics learning and teaching*, 68-95.

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational psychologist*, *31*(3-4), 175-190.

Dawkins, P. C., & Weber, K. (2017). Values and norms of proof for mathematicians and students. *Educational Studies in Mathematics*, *95*, 123-142.

Dawkins, P. C., & Vroom, K. (in press). Researching proof viewed as a genre of text. In M. Savić & K. Weber (Eds.), *New directions for research on proving: Honoring the legacy of John and Annie Selden.*

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical thinking and learning*, *1*(2), 155-177.

Goffney, I., Gutiérrez, R., & Boston, M. (2018). Mathematics for Black, Indigenous, and Latinx Students.

Gutiérrez, R. (2009). Framing equity: Helping students "play the game" and "change the game.". *Teaching for excellence and equity in mathematics*, *1*(1), 4-8.

Harel, G. (2008). DNR perspective on mathematics curriculum and instruction, Part I: focus on proving. *ZDM*, *40*, 487-500.

Harel, G. (2013). Intellectual need. *Vital directions for mathematics education research*, 119-151.

Larsen, S., Alzaga Elizondo, T., & Brown, D. (2023). Following in Cauchy's Footsteps: Student Inquiry in Real Analysis. In *Practice-Oriented Research in Tertiary Mathematics Education* (pp. 431-447). Cham: Springer International Publishing.

Larsen, S., Strand, S., & Vroom, K. (2022). How undergraduate students think about summation (sigma) notation. *International Journal of Research in Undergraduate Mathematics Education*, 1-23.

Melhuish, K., Vroom, K., Lew, K., & Ellis, B. (2022). Operationalizing authentic mathematical proof activity using disciplinary tools. *The Journal of Mathematical Behavior*, *68*, 101009.

Nathan, M. J., & Knuth, E. J. (2003). A study of whole classroom mathematical discourse and teacher change. *Cognition and instruction*, *21*(2), 175-207.

Rasmussen, C., & Marrongelle, K. (2006). Pedagogical content tools: Integrating student reasoning and mathematics in instruction. *Journal for research in mathematics education*, *37*(5), 388-420.

Schleppegrell, M. J. (2004). *The language of schooling: A functional linguistics perspective*. Routledge.

Speer, N. M., & Wagner, J. F. (2009). Knowledge needed by a teacher to provide analytic scaffolding during undergraduate mathematics classroom discussions. *Journal for Research in Mathematics Education*, *40*(5), 530-562.

Vroom, K. (2020). *Guided reinvention as a context for investigating students' thinking about mathematical language and for supporting students in gaining fluency* (Doctoral dissertation, Portland State University).

Vroom, K. (2022). A functional perspective on student thinking about the grammar of multiply quantified statements. *The Journal of Mathematical Behavior*, *67*, 100992.

Weber, K., & Melhuish, K. (2022). Can we engage students in authentic mathematical activity while embracing critical pedagogy? A commentary on the tensions between disciplinary activity and critical education. *Canadian Journal of Science, Mathematics and Technology Education*, *22*(2), 305-314.

Williams, S. R., & Baxter, J. A. (1996). Dilemmas of discourse-oriented teaching in one middle school mathematics classroom. *The Elementary School Journal*, *97*(1), 21-38.

Wood, D., Bruner, J. S., & Ross, G. (1976). The role of tutoring in problem solving. *Child Psychology & Psychiatry & Allied Disciplines*.

URBAN GIRLS' VISUOSPATIAL REASONING: MAPS AS ECO-CULTURAL TOOLS TO LEVERAGE LIVED EXPERIENCES AND SPATIAL REASONING

Elizabeth Suazo-Flores	Lisa Nuguid	Signe E. Kastberg
University of North Dakota	Purdue University	Purdue University
lizsuazo@gmail.com	mnuguid@purdue.edu	skastber@purdue.edu

Keywords: mathematical processes and practices, informal education, ethnomathematics.

Significance and Context

Visuospatial reasoning is malleable (Heckman, 2006; Sinclair et al., 2016; Uttal et al., 2013), could be developed before attending school (Sinclair et al., 2013), and is conducive to science, technology, engineering, and mathematics (STEM) careers (Chen & Mix, 2013; Gathercole & Pickering, 2000; Grissmer et al., 2013; Newcombe, 2010; Wai et al., 2009). Owens (2020) described visuospatial reasoning as involving space and place, recognizing shapes, combinations, and their properties, symmetry thinking, comparing quantities and using ratios, spatial capabilities, locating, intention, attention, and noticing. As part of an afterschool club led by researchers from a Midwestern research-intensive university in an urban elementary school, we invited 14 typically marginalized girls (TMGs) to draw maps from their homes to the school. Maps have been used to gain insight into spatial thinking (Cohrssen & Pearn, 2021). Using Owen's (2020) work as a lens, we learned that girls' maps went beyond spatial representation, showing links across time, perspectives, and relationships. We argue that inviting learners to draw their lived experiences invites the use of their cultural experiences, e.g., school-family experiences, that further develop their visuospatial reasoning.

Analysis and Findings

We sought to answer the question: What elements of Owens' framework are present in 14 TMGs' maps? Building from the task of Cohrssen and Pearn (2021), we invited 14 TMGs from one urban elementary school to draw maps representing the route from their homes to school. We analyzed the maps using Owen's (2020) framework. Preliminary findings illustrate how the girls experienced their journeys. Maps represented the time TMGs spent on their car rides to school, 2D and 3D perspectives from different physical locations or aerial views, characteristics of their communities including representations of various shops they saw on their way to the school, and features of roadways including bumps, curves, angles, and reduced speed.

Conclusion

The TMGs' maps provided insight into the interplay of lived experience and the development of spatial reasoning. Using visuospatial reasoning as an interpretive lens for analyzing geometric tasks elevates the significance of lived experience in cognitive development (Owens, 2020). Girls showed advanced visuospatial reasoning aligned with existing research (Sinclair et al., 2013). Findings from our study support the argument that visuospatial reasoning develops from lived experience, representations of experience, and communication about experiences that encourages abstractions. Further, using visuospatial development, rather than spatial reasoning alone supports learners in seeing themselves as doers and thinkers of STEM disciplines.

References

Cheng, Y.-L., & Mix, K. (2013). Spatial training improves children's mathematics ability. Journal of Cognition and Development, 15(1), 2-11. https://doi.org/10.1080/15248372.2012.725186

Cohrssen, C. & Pearn, C. (2021). Assessing preschool children's maps against the first four levels of the primary curriculum: Lessons to learn. Mathematics Education Research Journal, 33(1), 43-60. https://doi.org/10.1007/s13394-019-00298-7

Gathercole, S. E. & Pickering, S. J. (2000). Assessment of working memory in six- and seven-year old children. Journal of Educational Psychology, 92, 377-390.

Grissmer, D. W., Mashburn, A. J., Cottone, E., Chen, W. B., Brock, L. L., & Murrah, W. M. (2013). Play-based after-school curriculum improves measures of executive function, visuospatial and math skills and classroom behavior for high risk K-1 children. [Paper presentation] Society for Research in Child Development, Seattle, WA.

Heckman, J. J. (2006). Skill formation and the economics of investing in disadvantaged children. Science, 312(5782), 1900–1902. https://doi.org/10.1126/science.1128898

Newcombe, N. S. (2010). Picture this: Increasing math and science learning by improving spatial thinking. American Educator, 34(2), 29–35. https://www.aft.org/sites/default/files/periodicals/Newcombe_1.pdf

Owens, K. (2020). Transforming the established perceptions of visuospatial reasoning: Integrating an ecocultural perspective. Mathematics Education Research Journal, 32(2), 257-283. https://doi.org/10.1007/s13394-020-00332-z

Sinclair, N., de Freitas, E. & Ferrara, F. (2013). Virtual encounters: The murky and furtive world of mathematical inventiveness. ZDM – The International Journal on Mathematics Education, 45(2), 239-252.

Sinclair, N., Bartolini Bussi, M. G., de Villiers, M., Jones, K., Kortenkamp, U., Leung, A., & Owens, K. (2016). Recent research on geometry education: An ICME-13 survey team report. ZDM, 48(5), 691–719. https://doi.org/10.1007/s11858-016-0796-6

Uttal, D. H., Meadow, N. G., Tipton, E., Hand, L. L., Alden, A. R., Warren, C., & Newcombe, N. S. (2013). The malleability of spatial skills: A meta-analysis of training studies. Psychological Bulletin, 139(2), 352–402. https://doi.org/10.1037/a0028446

Wai, J., Lubinski, D., & Benbow, C. P. (2009). Spatial ability for STEM domains: Aligning over 50 years of cumulative psychological knowledge solidifies its importance. Journal of Educational Psychology, 101(4), 817–835. https://doi.org/10.1037/a0016127

USING MANIPULATIVES TO FOSTER MATHEMATICAL REASONING: DIDACTIC VARIABLES AND AFFORDANCES

Doris Jeannotte
Université du Québec à Montréal
jeannotte.doris@uqam.ca

Claudia Corriveau
Université Laval
claudia.corriveau@fse.ulaval.ca

Keywords: Rational numbers, Elementary school education,

Like other curricula around the world (OECD, 2019), Québec curriculum (Gouvernement du Québec, 2002) prescribes the use of manipulatives as a way to foster mathematical reasoning in elementary classes. Yet, most research on manipulatives focuses on whether students are more successful in math when using them (e.g., Carbonneau et al, 2013). And, we still have very little information about the use itself and how different teachers' practices shape mathematical reasoning. This poster uses data from the project MathéRéaliser, which aims to address this issue by studying how students use manipulatives to solve different tasks in relation to teachers' practices. More specifically, we will present the analysis of students' mathematical reasoning when solving a word problem involving fractions in relation to different pedagogical choices. To qualify the different pedagogical choices, we use Brousseau's (1981) definition of didactic variables, i.e., parameters, linked to the design and setting of a teaching situation, which can be varied by the teacher and lead to different mathematical activities. The concept of affordance, which is the relational properties of an object within a certain environment that are bound to certain actions (Gibson, 1979), helps link the students' mathematical reasoning to those choices.

Four elementary classes (two 3rd, one 5th, and one 6th) solved the following problem using different manipulatives: *Your neighbor wants to make a garden and plant carrots, tomatoes, and lettuce. He wants to give half the area to carrots and give a larger area to tomatoes than lettuce. He wonders what fraction of the garden to reserve for each of the vegetables. Offer him two solutions.* Figure 1 shows three examples of manipulatives. In each class, field notes and pictures were collected in addition to the videos of 7 dyads and of the whole class.

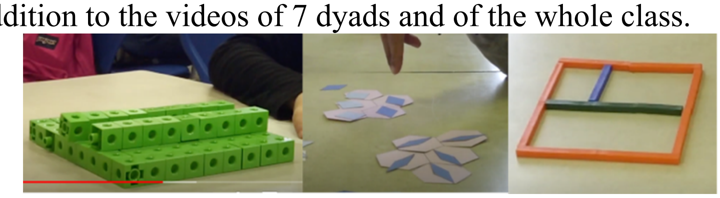


FIGURE 1: Three different manipulatives used to solve the problem.

The analysis allows noticing how students exploit different properties of manipulatives to solve, communicate and justify their answers in relation to the different pedagogical choices made. For example, we can see in figure 1 (left) that the dyad could not rely on color to distinguish the different parts of their gardens as only one color was available. They instead relied on the creation of two fences to partition their garden. Most dyads used manipulatives in very creative ways, different than what was intended by the designers of the manipulatives, to produce and communicate a valid solution, helping understand how different choices lead to different opportunities to reason mathematically.

Acknowledgments

This project is financed by [anonymized]. Thank you to all the teachers and students without

whom this project would not have been possible.

References

Brousseau, G. (1981). Problèmes de didactique des décimaux. *Recherches en didactique des mathématiques*, *2*(3), 37-127.

Carbonneau, K. J., Marley, S. C., & Selig, J. P. (2013). A meta-analysis of the efficacy of teaching mathematics with concrete manipulatives. *Journal of Educational Psychology*, *105*, 380–400.

Gibson, J. J. (1979). *The ecological approach to visual perception*. Psychology Press.

Gouvernement du Québec. (2001). *Programme de formation de l'école québécoise éducation préscolaire, enseignement primaire*. Ministère de l'Éducation.

OECD (2019), PISA 2018 Mathematics Framework, *in PISA 2018 Assessment and Analytical Framework*, OECD Publishing, Paris. https://doi.org/10.1787/b25efab8-en

WHAT CAN SEMIOTIC THEORY CONTRIBUTE TO AN ENACTIVIST ANALYSIS OF SENSE MAKING WITH MULTIPLE ARTIFACTS?

Steven Greenstein
Montclair State University
greensteins@montclair.edu

Denish Akuom
Montclair State University
akuomd1@montclair.edu

Erin Pomponio
Montclair State University
pomponioe1@mail.montclair.edu

Allison L. Gantt
University of Delaware
agantt@udel.edu

This project seeks to understand the emergence of mathematical meanings mediated by learners' interactions with multiple artifacts. Extending our prior work which took an enactivist approach and revealed the dynamics of embodied interactions fundamental to understanding fraction division, we now employ a semiotic lens to illuminate how learners make personal meanings from their engagement with multiple artifacts and translate them into more generalized mathematical meanings. We are doing so by taking a semiotic approach to tracking the emergent phenomenon of two learners' meaning making as it arises from the complex interplay of signs. We rely on our findings to argue that semiotic theory can be used as a resource to complement and enhance an enactive analysis of the unfolding of sense making with multiple artifacts. Implications for the design of learning experiences with multiple artifacts are proposed.

Keywords: Mathematical Representations, Cognition, Learning Theory, Rational Numbers

Hiebert and Grouws (2007) synthesized evidence from a number of studies to argue that the conceptual learning of mathematics is associated with teachers' and students' "explicit attention to the development of mathematical connections among ideas, facts, and procedures" (p. 391). Indeed, the Principles and Standards laid out by the National Council of Teachers of Mathematics (NCTM, 2000) emphasize the value of representing mathematical ideas in a variety of ways, and that these representations are fundamental to how we understand and apply mathematics. Its more recent Principles to Actions (NCTM, 2014) identifies, "Use and connect mathematical representations" as one of only eight "high-leverage practices and essential teaching skills necessary to promote deep learning of mathematics" (p. 9). Much research has been done regarding the ways in which teachers can support students' engagement with multiple representations. What is less well understood is the process by which multiple representations of a concept can be leveraged in order to contribute to learners' meanings for concepts. The better we understand this process, the better prepared we are to *engage and support all learners.*

Findings from an enactivist analysis of strategy development in mental mathematics contexts suggest that the nature of the processes at play cannot be understood in terms of solvers' flexibility and choice of strategies (Proulx, 2013). Rather, these processes are dynamic, emergent, and contingent on "an ongoing loop" (p. 319) of interactions between the problem and the solver. Since meaning making results from problem solving, and since problem solving has been shown to be dynamic, emergent, and contingent (Proulx, 2013), it would seem that meaning making might be, as well. Furthermore, research has also shown that mathematical meaning making is inextricably linked to the material and symbolic tools that mediate its learning (Verillon & Rabardel, 1995). Following these lines of thought, in our prior work (Greenstein et al., 2021) we sought to determine what an enactivist analysis could reveal about the nature of processes at play involved in mathematical meaning making in the context of multiple artifacts. We did so through the analysis of the mathematical activity of a father and daughter as they aimed to make sense of an algorithm for fraction division using a manipulative designed for learners' engagement with fraction concepts. What that work revealed was the potential power of manipulatives to provoke particular forms of embodied experience (Abrahamson & Sánchez-García, 2016; Nathan 2021) fundamental to the conceptual learning of mathematics. It also revealed that although students' use of multiple representations is increasingly prevalent in their learning of mathematics, the process by which learners connect these representations is actually quite complex (Greenstein, 2013). That process can be supported by teachers' and students' commitments to the integrity of a model of knowing as making sense, and to the availability of artifacts such as manipulatives that provide learners with immediate feedback they can use to self-assess the viability of their reasoning.

This work seeks to complement the findings of the benefits of these embodied experiences by further illuminating how learners make personal meanings from their interaction with multiple representations of a concept and translate those meanings into more generalized mathematical meanings. We do so by employing a semiotic lens to track the evolution of these emergent meanings as we address the question, "*What can a semiotic lens contribute to an enactivist perspective on fraction division meaning making with multiple artifacts?*"

Theoretical Framework

Two perspectives ground this study, the enactivist theory of cognition (Maturana & Varela, 1987; Varela, et al, 1992) and the theory of semiotic mediation (Bartolini Bussi & Mariotti, 2008). We first use enactivism to understand the evolutionary dynamics of two learners' embodied interactions as they aim to make sense[3] of fraction division using a manipulative designed for the exploration of fraction concepts. Then, we take a semiotic approach to analyzing these interactions in order to elucidate how the personal meanings that emerge from their engagement with each artifact are coordinated and ultimately converge upon a mathematical meaning of fraction division.

Two principles underlie the enactive approach: "1) Perception consists in perceptually guided action, and 2) cognitive structures emerge from the recurrent sensorimotor patterns that enable action to be perceptually guided" (Varela et al., 1992, pp. 172-173). Thus, from this perspective, knowing is not "outward manifestations of some inner workings" (Davis, 1995, p. 4) but a dynamically co-emergent phenomenon that arises and is made visible within a world of significance brought forth (Maturana & Varela, 1987) through one's goal-directed, "embodied (enacted) understandings" (Davis, 1995, p. 4).

By viewing knowing in the interactivity of learners (Davis, 1995), the enactivist perspective offers an alternative to the conventional view of knowledge as a static accumulation of facts, strategies, and ideas. Cognition, or active knowing, is "not something that happens 'inside' brains, bodies, or things; rather, it emerges" in the interactions between them (Malafouris, 2013, p. 66). In Davis's (1995) adaptation of Maturana and Varela's (1987) words, "*Knowing* is *doing* is *being*" (p. 7). We can consider one's way of knowing/doing/being as driven by the evolutionary imperative for an organism to maintain a functional, or harmonious, relationship

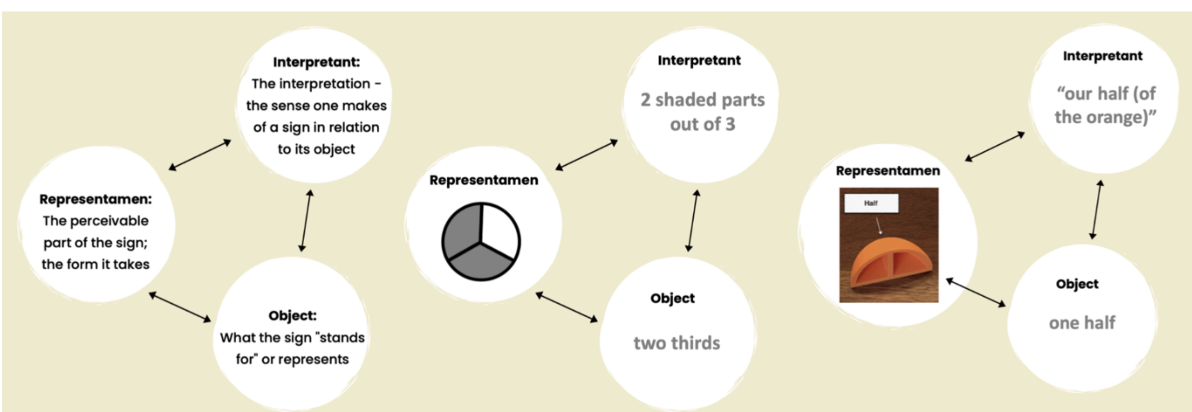
[3] We tend to use "sense making" in the context of our enactivist analysis, since cognition is defined in relation to our senses. We tend to use "meaning making" elsewhere in its conventional sense. However, we will also intentionally use them interchangeably to endorse the enactivist view that the terms are synonymous.

with its environment. "In other words, to know is to respond adequately; it is a doing that *fits* into the context where it emerges as the organism (e.g., a student, a researcher) interacts effectively with their environment" (Maheux & Proulx, 2015, p. 212). This drive toward a "good enough" solution rather than an optimal one is theoretically linked to the concepts of structural coupling and structural determinism.

Structural coupling is associated with the Darwinian concept of co-evolution, whereby an organism and its environment experience mutual structural changes through recursive and repeated interactions (Maturana & Varela, 1987) that allow their co-evolution to continue. Thus, their fit is dynamic and contingent upon their unique histories of recurrent interactions and structural changes (Maturana, 1988, as cited in Reid & Mgombelo, 2015, p. 175). So, as the environment provides a source of "triggers" that occasion the evolution of the organism, that evolution is determined by the organism's structure, a phenomenon that Maturana and Varela (1987) conceive of as *structural determinism*.

As mentioned earlier, in our prior work we took this enactivist perspective on mathematical activity as knowing-in-action to investigate the in-the-moment problem solving of a father and daughter as they seek to understand fraction division. From this perspective, their understanding is made evident as they seek to find harmony in mathematical meanings across initially conflicting interpretations of the various elements of two artifacts: 1) the paper-based, "flip-and-multiply" algorithm for fraction division, and 2) a manipulative that the daughter designed for engagement with fraction concepts. As Maffia and Maracci (2019) have found, this use of multiple artifacts within problem solving contexts can elicit the production, transformation, and coordination of personal meanings (that resolve dissonances into harmonies) toward more culturally established mathematical signs.

In order to understand how meanings emerge from contexts in which multiple artifacts mediate one's meaning making, we leverage Maffia and Maracci's (2019) concept of *semiotic interference,* which relies on Presmeg's (2006) use of Peirce's (1998) triad of sign relations. As depicted in Figure 1 (left), Peirce's *sign* is a triadic relationship among a *representamen* (the perceivable part of a sign; the perceiver's personal representation[4]), an *object* (what the sign stands for), and an *interpretant*, which "is the result of trying to *make sense* of the relationship… [between] the object and the representamen" (Presmeg, 2006, p. 170, emphasis added).



[4] This conception of "representation" is distinct from its typical use in mathematics education (e.g., tables, graphs, expressions, manipulatives; NTCM, 2000, 2014). Thus, in the context of our semiotic analysis, we will use "representation" in the Peircean sense (1998) and "artifact" in the sense that it is typically used in our field.

Figure 1: The triadic relationship of a sign (adapted from Peirce, 1998, and Presmeg, 2006); the example of a sign for "two-thirds"; and an example using a piece of the Fraction Orange, presented in Figure 3 below.

Now, semiotic interference describes the phenomenon that arises whenever "the *interpretant* of a *sign* whose *object* belongs to the context of [one] artifact is translated by a student in a new sign whose object belongs to the context of another artifact" (Maffia & Maracci, 2019, p. 3-58; see Figure 2, right). Thus, semiotic interference provides a window into learners' chaining of signs (Presmeg, 2006; Bartolini Bussi & Mariotti, 2008) as they negotiate their interpretations in order to converge upon a meaning for fraction division. Presmeg (2006) envisions a semiotic chain as a nested relationship of signs (see Figure 2, left) such that once a sign is established (e.g., S_1), it can then be regarded as the object of a new sign. This new sign (e.g., S_2) "comprises everything in the entire chain to that point" (p. 169). The ongoing process of enchaining signs (see Figure 2, center) is meant to depict the emergent phenomenon of meaning making that arises from this "complex interplay of signs" (Maffia & Maracci, 2019, p. 3-57).

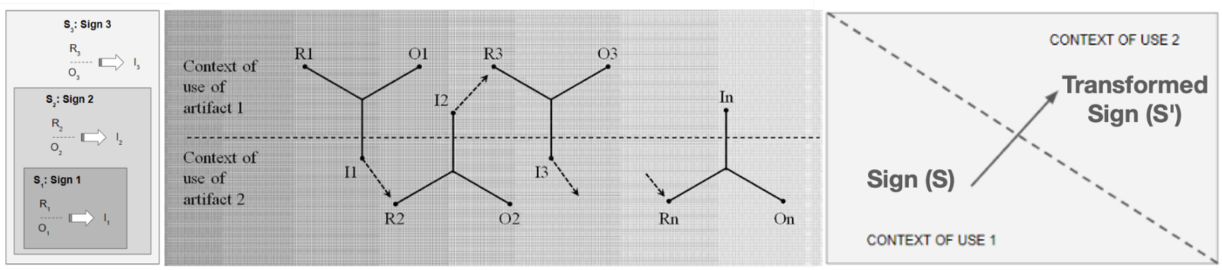


Figure 2: Left: Depiction of Presmeg's nested model of semiotic chaining (2006, p. 171); Center: Semiotic interference in a semiotic chain (Maffia & Maracci, 2019, p. 3–59); Right: Depiction of semiotic chaining across contexts.

Methods

This project is part of a larger study (Greenstein & Seventko, 2017; Greenstein & Olmanson, 2018; Greenstein et al., 2019; Greenstein et al., 2020; Greenstein et al., 2021; Akuom & Greenstein, 2022) that tested the hypothesis that a pedagogically genuine Making experience (Halverson & Sheridan, 2014) of a physical manipulative for mathematics learning would be formative for the development of practicing and prospective mathematics teachers' (PMTs') inquiry-oriented pedagogy. The study took place in a specialized mathematics course for PMTs at a university in the northeastern United States. For the project reported here, we took a revelatory case study approach (Yin, 2014) in order to determine what an enactivist-semiotic perspective might reveal about the problem-solving activity of "Dolly" and her father, "Lyle" (both pseudonyms).

Dolly was a participant in the larger study. She calls the manipulative she designed a "Fraction Orange" (Figure 3) and in designing it, she aimed to create a tool embedded with mathematical meanings that afford the mediated learning of fraction concepts (e.g., part-whole meaning of fractions, measurement meaning of division). The orange is a sphere partitioned into two hemispheres. One hemisphere is further partitioned into fourths, eighths, and sixteenths of the whole, and the other is further partitioned into thirds, sixths, and twelfths. Dolly designed the fraction orange to support a student's learning of fraction division. However, she and the instructor (first author on this paper) agreed that an adult would be fine, too. Many adults have not yet had the privilege or experienced the joy of making sense of fraction division.

Dolly's manipulative and the thirteen-minute problem-solving interview she conducted with Lyle constitute the data for this case study. Three researchers, including Dolly, analyzed the data both individually and in collaborative dialogue. We undertook the analysis by analyzing "verbal utterances through line-by-line analysis of the transcripts; stud[ying] body language and intonation by viewing video tapes...; and inferr[ing] mathematical forms and objects from the participants' actions, utterances and notations" (Simmt, 2000, p. 154), which constitute Dolly and Lyle's knowings-in-action as they aimed to coordinate meanings for fraction division in the manipulative and in an algorithm that ostensibly substantiates those meanings.

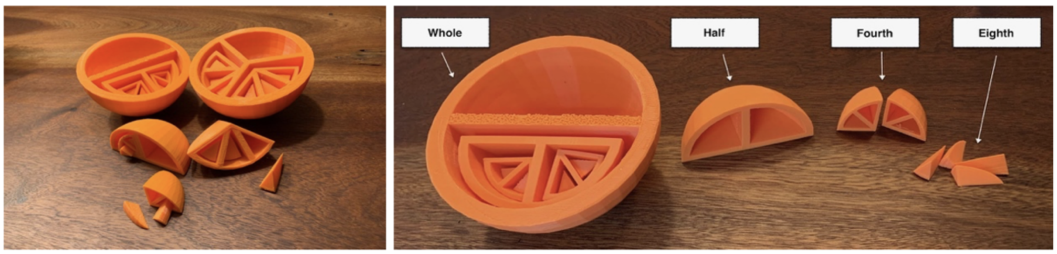


Figure 3: Dolly's Fraction Orange

Findings

Here we present excerpts from our semiotic analysis of the data, which we undertook in order to assess its viability for complementing an enactive analysis that revealed the interactions that were fundamental to two learners' conceptual learning of fraction division. The intent of this complementary analysis is to understand and depict *how* their mathematical meanings were achieved through those interactions.

The Initial Emergence of Personal Mathematical Meanings

At the outset of the interview, Dolly asks Lyle to solve the problem, ½ ÷ ¼, which she presents to him on paper alongside her fraction orange (Figure 4a). Although both the orange and the pen and paper are made available to him, Lyle chooses the pen, performs the standard, flip-and-multiply algorithm, and declares his answer to be 2, having simplified the 4/2 to 2/1. With enactivist principles framing our interpretations, we suggest that these are structurally-determined actions informed by Lyle's history of structural coupling with traditional school mathematics, where the answer derived from solving a fraction-division task by flipping and multiplying was deemed good enough to "survive." It constituted what Lyle needed to do to achieve harmony within the milieu of mathematics classroom environments.

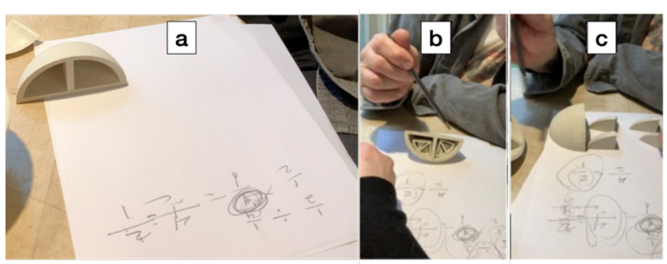


Figure 4: a) The Algorithm, b) the Half-Piece, and c) the Fourth-Pieces

Next, because Dolly is conducting the interview in order to assess the efficacy of her manipulative, she points to her fraction orange and asks Lyle, *"Can you show me with this?"* It is at this moment where we observe (a) the launching point on an emergent path of non-linear and unfolding (Proulx, 2019) problem-solving interactions bounded by alternating moments of what we refer to as *harmony* (a pleasing fit) and *dissonance* (a displeasing conflict or lack of fit), and (b) the emergence and evolution of personal meanings toward established mathematical meanings as the learners strive to harmonize their interpretations across the two artifacts: pieces of the orange and symbolic forms on paper. Mindful of the view of cognition as effective action from an enactivist perspective, we deliberately chose "harmony" and "dissonance" to reflect the cognitive and affective constitution of moments of fit or lack thereof, respectively. In the excerpt below we observe Lyle's enacted responses to the task Dolly posed to him.

Lyle: A half divided by a quarter… <*removes what he considers to be a half-piece*> a half divided by a quarter <*he counts the 4 fourth-pieces inside the 2 sections of the half-piece* (see Figures 4b and 4c)> is four.

Dolly: <*pointing to Lyle's written work*> But that's not what you got.

Lyle: Uh oh. <*Lyle pulls out the 4 fourth-pieces and looks back-and-forth between the paper and the orange as a timbre of concern grows in his voice:*>. Uh oh. A half, <*Lyle points to the now-empty half-piece.*> divided by a quarter. Why doesn't that work?

We see this as a moment of dissonance in meanings originating from Lyle's contrasting interpretations of his representations of the orange (i.e., manipulatable pieces inside pieces) and the algorithm (i.e., manipulatable symbols in expressions; Landy & Goldstone, 2007). Although he declares that the answer "is four," the somatic marker of his facial expression conveys the dissonance he was experiencing. When pressed for an explanation – *"But that's not what you got"* – a verbal marker of that dissonance is expressed: *"Uh oh…Why doesn't that work?"* Apparently, Lyle's knowing of fraction division as he actively perceives it (i.e., his engagement is both active and perceptually guided; Harvey, 2017) in the algorithm is discordant with his knowing of fraction division as he actively perceives it in the fraction orange.

The Messiness of Multiple Representations

In this next exchange, Dolly is surprised to find that she's not sure of her meanings, either. Thus, we observe both father and daughter striving for harmony in meanings as they attempt to make sense of the fraction division expression, $\frac{1}{2} \div \frac{1}{4}$.

Dolly: Here's our half. <*She takes the half-piece, places it next to the algorithm carried out on paper, and asks Lyle:*> And how many quarters go into a half?

Lyle: <*Looking at the orange*> Two. <*Shifting his attention to the written work and restating that outcome:*> Four. <*Shifting his attention back to the orange, and then back again to the paper:*> Is that half of a quarter, though? It's half <*pointing to the* $\frac{1}{2}$ *in the expression,* $\frac{1}{2} \div \frac{1}{4}$> of a quarter. <*now pointing to the* $\frac{1}{4}$ *on the paper*> It's not half of a whole thing.

Dolly: <*Sensing that Lyle is still deliberating:*> It's a quarter of a half, right?

Lyle: <*Lyle refers to the orange, then to the paper, and then back to the orange. With uncertainty:*> Yeah?

Dolly: How many quarters of a half are there? <*laughs and smiles*> Why is this so hard?

Here, Dolly begins by enacting her interpretation of the object "one half" as a half-piece of the orange (her representamen) and physically placing the piece on the paper, as if to propose a common meaning between them. Thus, her interpretation of the posed problem, $\frac{1}{2} \div \frac{1}{4}$, as *"How many quarters go into a half?"* is both meaningful and en-actionable to her. Lyle, referencing the orange and evoking his own interpretations of the objects "one fourth" and "one half," determines that two quarter-pieces fit into one half-piece (his representamen) and declares the answer to be "2." Immediately thereafter, however, he shifts his attention to the algorithm on the page and changes his answer to "4," presumably trying to match the "4" in the "4/2" that is the outcome of the algorithm and that at one point he calls "4 halves." In doing so, he seems to privilege the algorithm over the embodied actions that are reflective of interpretations he attributed to representamens in the orange. Next, he checks his answer of "4" against his interpretation of the division expression: *"Is that half of a quarter, though?"* Then, Dolly steps in to suggest a different interpretation of the expression: *"It's a quarter of a half, right?"* Lyle's unsure: *"Yeah?"* To which Dolly responds with yet another interpretation presumably informed by Lyle's: *"How many quarters of a half are there?"*

Lyle's actions at this moment are directed at finding harmony in meanings across multiple interpretations of multiple elements in the fraction division expression and the two different values he derives from it. The dissonance Dolly's feeling is expressed through her utterance, *"Why is this so hard?"* Her dad is feeling it, too. Their collective words and actions express the messiness of their engagement with incompatible interpretations of multiple representations and what it feels like as they strive to make a generalizable mathematical meaning for fraction division through what amounts to a complex interplay of signs.

By this point we've seen the value of using a semiotic lens to track the complex interplay of Dolly and Lyle's multiple interpretations and the evolution of emergent meanings. We share this next excerpt in order to demonstrate the utility of a semiotic depiction of the enchaining of signs. Here, Dolly and Lyle find harmony in their meanings for fraction division by resolving some of the many dissonances that were swirling around this last excerpt.

Traversing the Semiotic Landscape: The Enchaining of Signs

This excerpt took place in the final moments of Dolly and Lyle's problem solving. Just prior to this moment, they enchained their signs for – and thus made sense of – $\frac{1}{2}$ and $\frac{1}{4}$ by translating their interpretations of representamens in the orange to elements of the algorithm. Now, they are engaged in similar meaning-making activity as they aim to find interpretations for the $\frac{1}{2}$ and $\frac{1}{4}$ in the posed problem, $\frac{1}{2} \div \frac{1}{4}$.

> *Dolly:* We wanna take a half of one and divide it by a quarter of one, right?
> *Lyle:* Yes.
> *Dolly:* Take a half of one and divide – oh, that's what it is! … We wanna take this <*half-piece of the orange*> and see how many of those <*quarter-pieces*> fit in there <*the half-piece. Then, confidently:*>And that's why our answer is 2.
> *Lyle:* Yes.
> *Dolly:* There's still two halves in a whole, 'cuz this <*the expression, $\frac{1}{2} \div \frac{1}{4}$*> is in regards to a whole. <*rephrasing*> This is in regards to 1. So a half of 1 divided by a quarter of 1 is 2, because 2 quarters fit into 1 half.
> *Lyle:* Yeah. It makes sense that way.

In this excerpt, we observe the meaning Dolly makes of the expression, $\frac{1}{2} \div \frac{1}{4}$, by enchaining interpretations of $\frac{1}{2}$ and $\frac{1}{4}$ according to the measurement meaning of division she and Lyle enacted earlier. Next, Lyle proceeds to re-enact the interpretation for himself.

Lyle: <pointing with his pencil to ½ on the page:> So this is half of a whole *<then pointing to the ¼ on the page.>* and this is a quarter of a whole. *<Next, he turns his attention to the orange (as in Figure 4b) and points to the half-piece:>* Half of a whole. *<Next, he points to each quarter-piece in a sweeping motion across the pieces:>* Quarter of a whole *<Then, pointing to the two quarter-pieces, he continues:>* is 2. *<Thus, he appears to be establishing that the number of quarter-pieces he's identified, 2, is the answer to the posed problem, ½ ÷ ¼>.*

Dolly: <pointing to the 2 quarter-pieces and agreeing with Lyle> Yeah, 'cause there's two quarters of a whole.

Lyle: *<with a sigh of relief>* Yeah, that makes sense.

As if to establish his own meanings for fraction division and its coherence in representations across artifacts as Dolly has just done, Lyle uses the pencil to re-enact a physical bridge between the elements of the problem and the pieces of the orange. We interpret his subsequent actions as an effort to match his interpretations of half of a whole and quarter of a whole in the symbolic expression to representamens he's identified in the orange (the half-piece and the quarter-piece, respectively; see Figure 5, left). These actions reify the harmony that has finally emerged from recursive interactions that culminate in an enchaining of signs signifying the sense he and Dolly have made to form a newly coupled structure of fraction division (see Figure 5, right).

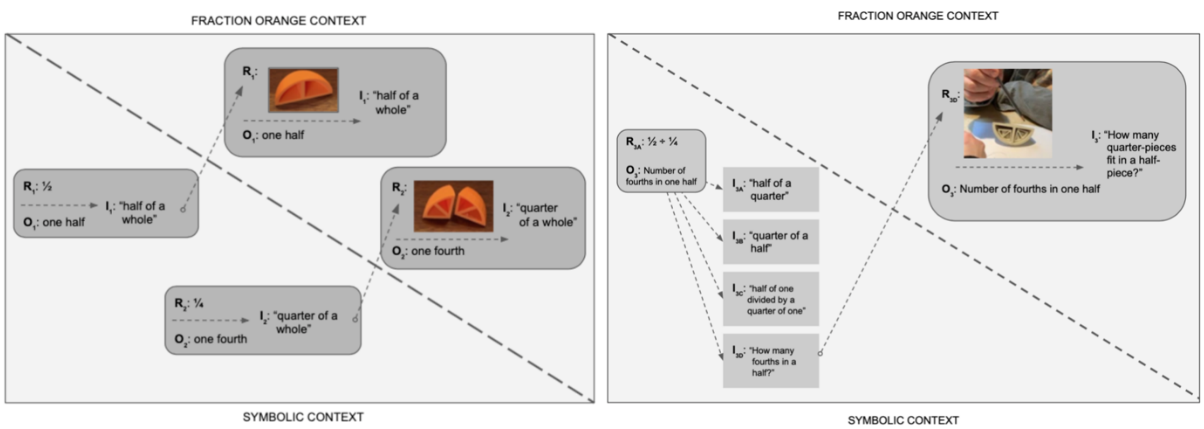


Figure 5: Enchained and unchained signs for one half and one fourth (Left), and for the number of fourths in one half (Right)

Concluding Discussion

A taken-as-shared principle of mathematics education is that the learning of mathematics entails using and connecting multiple representations of mathematical ideas (e.g., Hiebert & Grouws, 2007). This work presented here reminds us that learners' paths of problem solving are contingent on their particular mathematical structures and interactions (Proulx, 2013), and thus the process by which learners connect these representations is actually quite complex (Greenstein, 2013; Lesh et al., 1987; Maracci & Mariotti, 2019). In order to make sense of this complexity and thereby offer implications for the designs of tools and tasks that have the potential to support the evolution of all learners' mathematical meanings, this work set out to discern the contributions of a semiotic lens to an enactivist analysis of fraction division meaning making with multiple artifacts. That enactivist analysis revealed the mediated interactions with multiple artifacts that were productive for developing mathematical meanings for fraction division. The findings presented here in the depictions of Dolly and Lyle's chaining of signs through semiotic interference revealed just how their mathematical meanings were achieved through the interactions revealed by the enactivist analysis. In other words – and this is the contribution we propose to have made – the semiotic analysis was successful for explicating the interpretive constitution of two learners' enacted meanings. Thus, we take these findings to argue that semiotic theory can be used as a resource to complement and enhance an enactive analysis of the unfolding of meaning making with multiple artifacts.

Acknowledgments

This material is based upon work supported by the National Science Foundation under Award No. 1812887

References

Akuom, D. O., & Greenstein, S. (2022). The Nature of Prospective Mathematics Teachers' Designed Manipulatives and their Potential as Anchors for Conceptual and Pedagogical Knowledge. Journal of Research in Science, Mathematics and Technology Education, 5, 109-125.

Abrahamson, D., & Sánchez-García, R. (2016). Learning Is Moving in New Ways: The Ecological Dynamics of Mathematics Education. Journal of the Learning Sciences, 25(2), 203-239.

Bartolini Bussi, M. G., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: Artefacts and signs after a Vygotskian perspective. In L. English, M. Bartolini Bussi, G. Jones, R. Lesh, & D. Tirosh (Eds.), Handbook of International Research in Mathematics Education (2nd ed., pp. 746–805). Lawrence Erlbaum.

Davis, B. (1995). Why Teach Mathematics? Mathematics Education and Enactivist Theory. For the Learning of Mathematics, 15(2), 2–9.

Greenstein, S. (2013) Connecting Rate Using Dynamic Representational Technologies. The New Jersey Mathematics Teacher, 71(2), 13-20.

Greenstein, S., & Seventko, J. (2017). Mathematical Making in Teacher Preparation: What Knowledge is Brought to Bear? In E. Galindo & J. Newton (Eds.), Proceedings of the 39th Annual Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education (pp. 821–828). Hoosier Association of Mathematics Teacher Educators.

Greenstein, S., & Olmanson, J. (2018). Reconceptualizing pedagogical and curricular knowledge development through making. Emerging Learning Design Journal, 4(1), 1–6

Greenstein, S., Fernández, E., & Davidson, J. (2019). Revealing Teacher Knowledge Through Making: A Case Study of Two Prospective Mathematics Teachers. Proceedings of the 41st Annual Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education.

Greenstein, S., Jeannotte, D., Fernández, E., Davidson, J., Pomponio, E., & Akuom, D. (2020). Exploring the Interwoven Discourses Associated with Learning to Teach Mathematics in a Making Context. In A. I. Sacristán, J. C. Cortés-Zavala, & P. M. Ruiz-Arias (Eds.), Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (pp. 810–816). Cinvestav/AMIUTEM/PME-NA

Greenstein, S., Pomponio, E., & Akuom, D. (2021). Harmony and Dissonance: An Enactivist Analysis of the Struggle for Sense Making in Problem Solving. Proceedings of the 43rd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.

Halverson, E. R., & Sheridan, K. (2014). The maker movement in education. Harvard Educational Review, 84(4), 495-504.

Harvey M. I. (2017) "posing | solving" can be explained without representations, because it is a form of perception-action. Constructivist Foundations 13(1), 169–171. http://constructivist.info/13/1/169

Hiebert, J., & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. In F. K. Lester, Jr. (Ed.), Second Handbook of Research on Mathematics Teaching and Learning (pp. 371-404). Information Age Publishing.

Maheux, J.-F., & Proulx, J. (2015). Doing|mathematics: analysing data with/in an enactivist-inspired approach. ZDM, 47(2), 211-221.

Landy, D., & Goldstone, R. L. (2007). How abstract is symbolic thought? Journal of Experimental Psychology: Learning, Memory, and Cognition, 33, 720-733. https://doi.org/10.1037/0278-7393.33.4.720

Lesh, R., Post, T. R., & Behr, M. (1987). Representations and translations among representations in mathematics learning and problem solving. In C. Janvier (Ed.), Problems of representations in the teaching and learning of mathematics (pp. 33-40). Lawrence Erlbaum.

Malafouris, L. (2013). How things shape the mind. MIT Press.

Maturana, H. R., & Varela, F. J. (1987). The tree of knowledge: The biological roots of human understanding: New Science Library/Shambhala Publications.

Nathan, M. J. (2021). Foundations of embodied learning: A paradigm for education. Routledge. https://doi.org/https://doi.org/10.4324/9780429329098

National Council of Teachers of Mathematics. (2000). Principles and standards for school mathematics. Reston, VA: Author.

National Council of Teachers of Mathematics. (2014). Principles to actions: Ensuring mathematical success for all. Reston, VA: Author.

Peirce, C. S. (1998). The Essential Peirce, Volume 2: Selected Philosophical Writings, 1893-1913 (Peirce Edition Project, Ed.). Indiana University Press.

Presmeg, N. (2006). Semiotics and the "Connections" standard: Significance of semiotics for teachers of mathematics. Educational Studies in Mathematics, 61(1), 163-182.

Proulx, J. (2013). Mental mathematics, emergence of strategies, and the enactivist theory of cognition. Educational Studies in Mathematics, 84(3), 309-328.

Proulx, J. (2019). Mental mathematics under the lens: Strategies, oral mathematics, enactments of meanings. The Journal of Mathematical Behavior, 56, 1-16.

Reid, D. A., & Mgombelo, J. (2015). Survey of key concepts in enactivist theory and methodology. ZDM, 47(2), 171-183.

Simmt, E. (2000). Mathematics Knowing in Action: A Fully Embodied Interpretation. Proceedings of the Annual Meeting of the Canadian Mathematics Education Study Group (pp. 153-159), University of Quebec at Montreal, Canada.

Varela, F. J. (1987) Laying down a path in walking. In W. I. Thompson (Ed.) Gaia: A Way of Knowing (pp. 48-64). Lindisfarne Press.

Varela, F. J., Rosch, E., & Thompson, E. (1992). The Embodied Mind: Cognitive Science and Human Experience: MIT Press.

Verillon, P., & Rabardel, P. (1995). Cognition and Artifacts: A Contribution to the Study of Though in Relation to Instrumented Activity. European Journal of Psychology of Education, 10(1), 77-101.

Yin, R. K. (2014). Case study research: Design and methods (5 ed.). Sage Publications.

CURRICULUM, ASSESSMENT, AND RELATED TOPICS

A MODIFIED DEPTH OF KNOWLEDGE FRAMEWORK FOR WORD PROBLEMS

Jonathan D. Bostic
Bowling Green State Univ.
Bosticj@bgsu.edu

Timothy Folger
Bowling Green State Univ.
tdfolge@bgsu.edu

Kristin Koskey
Drexel Univ.
kk3436@drexel.edu

Gabriel Matney
Bowling Green State Univ.
gmatney@bgsu.edu

Toni May
Drexel Univ.
Tas365@drexel.edu

Gregory Stone
MetriKs Amérique
gregory@metriks.com

Depth-of-knowledge (DOK) is a means to communicate the cognitive demand of tasks and is often used to categorize assessment items. Webb's (2002) framework has been applied across content areas. The aim of this two-phase iterative study was to modify Webb's DOK framework for word problems. Through work with school partners, this iterative design-research based study provides supportive evidence for a modified DOK framework reflecting levels of complexity in word problems. The resulting modified DOK framework presents an opportunity for mathematics educators to reflect on various aspects of cognitive complexity.

Keywords: assessment, problem solving

Alignment between student learning standards and assessments can be evaluated based on content and cognitive complexity (AERA et al., 2014; Webb, 2007). Webb (1997; 2002) has argued for models that categorize assessment items by the cognitive demand, or depth-of-knowledge (DOK), required to successfully complete the item. Webb (2002) classified DOK into the four levels: (1) Recall, (2) Skills and concepts, (3) Strategic thinking, and (4) Extended thinking (see table 1). Kilpatrick and colleagues (2001) define a mathematical exercise as a task that promotes efficiency with a known procedure, which aligns with level one DOK. This study examined the DOK required to solve mathematical problems (i.e., not exercises). For instance, word problems may require translating from text to a mental model, and later to a mathematical model prior to applying a mathematical strategy (Verschaffel et al., 2000). A multi-step process such as this is likely to extend beyond rote procedure; suggesting such word problems are unlikely to be classified as a level one DOK.

Table 1: Webb's DOK (2002) and Associated Descriptions

Level	Description
Level 1: Recall	"The recall of information such as a fact, definition, term, or simple procedure, as well as performing a simple algorithm or applying a formula" (Webb, 2002, p. 3).
Level 2: Skills and Concepts	"The engagement of some mental processing beyond a habitual response. A Level 2 assessment item requires students to make some decisions as to how to approach the problem or activity… These actions imply more than one step" (Webb, 2002, p. 4).
Level 3: Strategic Thinking	"Requires reasoning, planning, using evidence, and a higher level of thinking than the previous two levels. In most instances, requiring students to explain their thinking is a Level 3… [Level 3 items may require] developing a logical argument for concepts; explaining phenomena in terms of concepts, and using concepts to solve problems" (Webb, 2002, p. 4).
Level 4: Extended Thinking	"Requires complex reasoning, planning, developing, and thinking most likely over an extended period of time… At Level 4, the cognitive demands of the task should be high and the work should be very complex. Students should be required to make several connections - relate

ideas *within* the content area or *among* content areas - and have to select one approach among many alternatives on how the situation should be solved" (Webb, 2002, pp. 4-5).

Problem statement

A team of researchers developed Problem-Solving Measures for grades 3-8 (PSM 3-8) that align with the Common Core State Standards for Mathematics (CCSSM; e.g., Bostic & Sondergeld, 2015, Bostic et al., 2020). Each word problem was developed using two synergistic frameworks for problems: Schoenfeld (2011) and Verschaffel et al. (1999). These frameworks suggest that tasks are problems if (a) a task is complex enough such that a strategy is not readily apparent, (b) it is unknown whether the task has a solution, and (c) it can be solved using multiple strategies. These elements were drawn upon to design mathematics word problems aligned to the CCSSM, which ultimately became the sample space for the study. Two sample items are shared in Figure 1. It was hypothesized that items categorized as problems might be classified at a DOK higher than level one DOK. The word problems transcended level one DOK and led to a question: How might DOK better capture unique aspects of lower complexity compared to higher-complexity grades 3-8 DOK word problems? An intended outcome from this work is to develop a modified DOK framework that has potential to inform researchers and practitioners about DOK within the context of mathematics word problems and extend prior scholarship on DOK. A second outcome was to obtain sufficient rater agreement using that modified DOK framework. Two phases framed this study. Phase I entailed developing a modified DOK framework such that DOK level descriptors reflected variation in the cognitive complexity of mathematics word problems. Phase II examined rater reliability applying the modified DOK framework to the grades 3-8 PSMs.

PSM7 (7.NS.2): Rosalinda plans to make marshmallow treats to share with her class. The recipe requires $2\frac{1}{8}$ cups of marshmallows, $3\frac{1}{4}$ cups of rice cereal, and $\frac{1}{4}$ cup of butter. It serves eight people. How many cups of rice cereal will she need if she must make treats for 28 people?

PSM4 (4.OA.3): Josephine sold tickets to the fair. She collected a total of $1,302 from the tickets she sold. $630 came from the adult ticket sales. Each adult ticket costs $18. Each child ticket costs $14. How many child tickets did she sell?

Figure 1: Sample items from PSM4 and PSM7

Phase I

Development of a modified DOK framework was an iterative process with numerous feedback loops, drawing upon a design-based research approach (Middleton et al., 2008). To develop the framework, a team of five panelists were selected based on content expertise and prior experience developing PSM items. The panel consisted of two mathematics education faculty, one doctoral student, and two masters-level students. One of the two faculty member served as an external other throughout the process.

All participants read and reflected on material's related to Webb's DOK (2002). The group conducted a literature search, moved possible reading materials into a shared folder, scanned materials for content overlap and uniqueness, then reflected on materials that were essential and those that were recommended for becoming familiar with DOK. The result was three materials recommended for further review (i.e., Hess, 2006; Petit & Hess, 2006; Webb, 2002). The team read, discussed, and reflected on each reading to better understand Webb's (2002) framework (see table 1). Once the team reached consensus in operationalizing the modified DOK language and classifying a subset of grade seven PSM items, then they applied it to a larger set of 15 grade 6 PSM items in pairs. One faculty member and one masters student formed a group while the doctoral student and a second masters student formed a second group. Memos were made during this application about limitations in using Webb's DOK framework, word problems where it was easier to classify than others, and potential modifications to Webb's DOK framework. Discussions between two team members were documented. A goal from those discussions was for each team to have agreement on a DOK score for each item. Those scores were then compared between the two pairs of team members. Disagreements were resolved through discussions and each item received a single score. Results from DOK coding were shared with the external expert as a way to check the credibility of the process as rigorous, well documented, and logically applying modifications to Webb's DOK framework.

Next in the process, the team sought areas to revise the emergent DOK framework with the intention of improving the DOK classification process for word problems. Consensus was reached that some word problems had less complexity (suggesting a lower DOK score), while other word problems required students to draw upon multiple developmentally appropriate, content-focused strategies to solve the task (suggesting a higher DOK level). The team began the process of creating a modified DOK framework by revising Webb's DOK (2002) framework. Discussions of memos from earlier in the process generated big ideas, which informed DOK modifications. The team developed a draft revised DOK framework for word problems. The four team members sought feedback from the external other, which was integrated into the revision. The team tested the draft revised DOK framework with four randomly selected items from a pool of grades 6-8 items. Each team member worked individually and memoed about their rationale for each rating. Then, they met with their team partner for agreement and discussion of memos, followed by a whole-group meeting. Revisions were made to the draft modified DOK framework based upon emergent patterns from memos. The team shared DOK scores, memos, and patterns with the external other as part of the process. After further rounds of revisions in this fashion, the team developed a revised DOK framework specifically for mathematics word problems, which was again shared with the external other. The team applied this revised DOK framework to a set of 10 randomly sampled grades 3-8 word problems that were new. The degree of inter-rater reliability agreement (κ) was interpreted using Cohen's guidelines (1960): x< 0.20 (none to slight), 0.21-0.40 (fair), 0.41-0.60 (moderate), 0.61-0.80 (substantial), and x> 0.81 (almost perfect). There was agreement across nine items and disagreement with one item generating Cohen's $\kappa = .9$. On that one item, pairs of coders had agreement, but the codes differed by one adjacent unit. Discussion ensued and the team ultimately reached agreement. Thus, there was almost perfect agreement during the final iteration across four team members. At this time, the team developed training materials that for future coders and/or re-calibration. Developed materials included a manual, sample items with explanations of coding, as well as a set of items to be used for calibration coding. The modified DOK framework is shared in Table 2.

Table 2: Modified DOK Framework for Word Problems

Level	Description
Level 1: Recall & Reproduction	Respondents are expected to recall or reproduce knowledge and/or skills while problem solving. This typically involves working with facts, terms, details, calculations, and/or simple procedures or formulas. A solution to a level 1 item does not need to be "figured out" or "solved".

Level 2: Skills & Foundational Concepts	A student is expected to decontextualize from a situational context when solving items. A level 2 DOK item requires students to process information regarding the concept being measured.
Levels 3: Strategic Thinking & Reasoning	A level 3 DOK item requires mathematical reasoning and higher order thinking processes. Students are expected to combine multiple skills or heuristics and conceptual understanding to reach a solution. There may be multiple viable solutions.

Phase II

The modified DOK framework for mathematics word problems was then used by three raters (mathematics education graduate students) not familiar with the development process. The training manual and selected readings (e.g., Hess, 2006; Webb, 2002) were shared with them in advance of their training session. A 90-minute training with the faculty member from phase one included discussing readings and sample PSM items. Following coder training and calibration, the raters were asked to work collaboratively on three items and come to consensus using the revised DOK framework. One week later, the faculty member and three raters debriefed about how they reached decisions with the framework, nuances related to each word problem, and concerns they had while applying codes. Following this debriefing, the raters engaged in independent coding. The raters were asked not to disclose ideas with each other related to their independent codes. They received six-word problems and a copy of the revised DOK framework. All three raters agreed on four items, yet there was a discrepancy on two items by one adjacent level (i.e., DOK 2 vs DOK 3). This indicated substantial agreement using Cohen's (1960) guidelines (Cohen's $\kappa = .71$); however, the team wanted to have a stronger understanding of the revised DOK framework. The raters convened with the faculty member to discuss coding results and their memos from applying the modified framework. Emerging from their use of the framework were challenges understanding developmental differences of content knowledge at a specific grade level (e.g., grade 7 vs grade 8) and what counted as two distinct strategies. After coming to agreement and finalizing discussions with the lead faculty member and external other, this team completed independent coding as a confirmation of their ability to reliably code items. They were provided with seven randomly selected grades 3-8 PSM items and asked to (a) code each item using the modified DOK framework and (b) provide a rationale for their code. Results from their independent coding showed that the three raters applied the same codes for five of seven items. One rater differed from the other two raters on two of seven items by an adjacent code. Rater agreement (Cohen's $\kappa = .89$) yielded almost perfect agreement. Coding memos across the three raters communicated further support for differentiating between levels two and three was warranted to help improve agreement in classifying DOK level.

Discussion

The present work serves as a means to reconsider applications of Webb's (2002) DOK framework for word problems. Through a focused, design-research process, we developed a modified DOK framework that helps to magnify potential differences between word problems. Thus, this modified framework extends prior DOK scholarship. Much like a magnifying glass can highlight unique features not seen with the naked eye, we believe this framework may do the same for word problems: highlighting the unique features of word problems. A second intention in this design was to test the potential of others using this framework. An initial use with a small sample demonstrates potential for effectiveness (i.e., scaling up) trials this revised DOK framework with a larger sample of coders and other items. One limitation with this framework is that the word problems reviewed did not lend themselves towards Webb's (2002) level four

DOK. Thus, our revised DOK framework did not reveal changes in this area. However, we believe reviewing tasks that promote modeling with mathematics (CCSSI, 2010) may lend themselves to this level and result in revisions to the framework. Another limitation is that the team has only reviewed word problems associated with a project that designed word problems. A future study might explore use of the revised DOK framework with samples from textbooks used in classroom instruction.

References

American Educational Research Association, American Psychological Association, & National Council on Measurement in Education. (2014). Standards for educational and psychological testing. Washington, DC: American Educational Research Association.

Bostic, J., & Sondergeld, T. (2015). Measuring sixth-grade students' problem solving: Validating an instrument addressing the mathematics Common Core. *School Science and Mathematics Journal, 115, 281-291*.

Bostic, J., Matney, G., Sondergeld, T., & Stone, G. (2020, March). *Measuring what we intend: A validation argument for the grade 5 problem-solving measure (PSM5). Validation: A Burgeoning Methodology for Mathematics Education Scholarship*. In J. Cribbs & H. Marchionda (Eds.), Proceedings of the 47th Annual Meeting of the Research Council on Mathematics Learning (pp. 59-66). Las Vegas, NV.

Common Core State Standards Initiative. (2010). Common core standards for mathematics.

Cohen, J. (1960). A coefficient of agreement for nominal scales. Educational and Psychological Measurement, 20, 37-46.

Hess, K. (2006). Exploring cognitive demand in instruction and assessment. National Center for the Improvement of Educational Assessment, Dover NH.

Kilpatrick, J., Swafford, J., & Findell, B. (2001). Adding it up: Helping children learn mathematics. Washington, DC: National Academy Press.

National Council of Teachers of Mathematics. (2000). Principles and standards for school mathematics. Reston, VA: Author.

Petit, M., & Hess, K. (2006). Applying Webb's depth of knowledge and NAEP levels of complexity in mathematics.

Schoenfeld, A. H. (2011). How we think: A theory of goal-oriented decision making and its education applications. New York, NY: Routledge. https://doi.org/10.4324/9780203843000

Verschaffel, L., De Corte, E., Lasure, S., Van Vaerenbergh, G., Bogaerts, H., & Ratinckx, E. (1999). Learning to solve mathematical application problems: A design experiment with fifth graders. Mathematical Thinking and Learning, 1(3), 195-229. https://doi.org/10.1207/s15327833mtl0103_2

Verschaffel, L., Greer, B., & De Corte, E. (2000). Making sense of word problems. Lisse, Netherlands.

Webb, N. L. (1997). Criteria for Alignment of Expectations and Assessments in Mathematics and Science Education. (Council of Chief State School Officers and National Institute for Science Education Research Monograph No. 6). Madison: University of Wisconsin, Wisconsin Center for Education Research.

Webb, N. L. (2002, March). Depth of knowledge levels for four content areas. Wisconsin Center for Education Research, University of Wisconsin-Madison, Madison, WI.

Webb, N. L. (2007). Issues related to judging the alignment of curriculum standards and assessments. Applied measurement in education, 20(1), 7-25. https://doi.org/10.1080/08957340709336728

A COMPARATIVE ANALYSIS OF FRACTION PROBLEMS WITHIN THE IRANIAN CURRICULUM AND GO-MATH TEXTBOOKS

Seyedehkhadijeh Azimi Asmaroud
Virginia State University
Sazimi@vsu.edu

Textbooks play an important role in teachers' instructional decisions (Jones & Tarr, 2007), which consequently affects students' learning. This paper reports on a comparison of the elementary mathematics textbooks used in Iran and the United States, the Go-Math textbook. I analyzed topic sequences, frequency of the tasks, and cognitive demands of the fraction task in second and third-grade textbooks, employing the framework developed by Smith and Stein (1998) regarding the Levels of Cognitive Demands (LCD). Findings showed that Iran's textbooks devoted more percentage of pages to fractions in second grade than Go-Math textbooks. LCD of the tasks in second grade in both courtiers were in lower levels. Also, the presentation of the fraction concepts varied in different countries and Go-Math covered more fraction concepts in third grade. Recommendations for future research were offered.

Keywords: Textbook analysis; Fraction; Levels of cognitive demand; Curriculum; Content analysis.

Fractions are one of the topics in elementary school textbooks used in many countries (Edwards et al., 2023) and methods and the presentation of this concept vary in different countries. In Iran, there is only one mathematics textbook published by the Ministry of Education (ME) for each grade, and teachers and students rely on the textbooks published by ME. Thus, the underlying rationale for focusing on the concept of fractions in textbooks was that the analyzed Iran's textbooks were used nationwide and reflected a good picture of the nature of tasks implemented by teachers. This study was part of a larger study that compared the type of questions in mathematics textbooks in different countries and discuss the result of the study about comparing fraction tasks in the textbook series in Iran and a commonly used textbook in the U.S., Go-Math textbook.

Literature Review

Learning and teaching fractions is one of the difficult areas in mathematics (Edwards et al., 2023; Getenet & Callingham, 2019). Research showed that students struggle with the concept of fractions and these difficulties with fractions have been observed across all levels (Charalambous & Pitta-Pantazi, 2006). Furthermore, some research shows a relation between differences in students' performance and the approaches of the textbooks to mathematics topics (e.g. Li (2000)). In their study of 340 fifth graders and 306 sixth graders Charalambous and Pitta-Pantazi (2006) argued that "the differences in students' performance on the five interpretations of fractions mirror the imbalance in the emphasis placed on them during instruction" (p. 309). For example, students' performance in part-whole subconstruct of fractions was better than other five interpretations of fractions, and it was the subconstruct that students met more frequently in their mathematics textbooks. Furthermore, the research emphasizes that textbooks in many ways can contribute to improved learning outcomes (Milligan et al., 2018) and the nature of the mathematics tasks in textbooks can influence students' thinking and also can limit or broaden their views (Henningsen & Stein 1997).

In mathematics education, textbooks have a great influence on the education scene (Fan et al., 2013; Milligan et al., 2018). Textbooks play an important role in teachers' instructional decisions (Jones & Tarr, 2007), the materials they use in the classroom (Chang & Silalahi, 2017), and the concepts that are imparted in the classroom (Jones & Tarr, 2007). Chang and Silalahi (2017) stated that analysis of textbooks can aid in understanding the effectiveness of specific approaches and identify what is required in terms of teaching and identify areas for improvement in curriculum development. Thus, this study aimed to analyze Iran's mathematics textbooks in different grades.

Iran's elementary textbooks, published by the ME, are used nationwide, and national assessments use these textbooks as the main sources of learning material. There is only one mathematics textbook published by the ME for each grade, thus they reflect a good picture of the nature of the tasks that are implemented by teachers in the classrooms. This paper sets out to analyze the introduction to the fraction concept used in elementary textbooks to establish the frequency and problem types of the tasks. Furthermore, students need opportunities "to engage with tasks that lead to deeper, more generative understandings about the nature of mathematical concepts" (Stein et al., 2001, p. 15). Thus, this study also reports on the result of the analysis of the level of the cognitive demands required by tasks in Iran's elementary textbooks and compares it with a commonly used textbook in the U.S., Go-Math.

Research Questions

In this study, I addressed the following research questions:

1- What are the cognitive demands of the fraction tasks in Iran's textbooks in comparison to a commonly use textbook in the U.S., Go-Math?

2- What are the nature and frequency of the fraction tasks in second and third grades mathematics textbooks in Iran and Go-Math in the U.S.?

Theoretical Framework

This study draws from a framework developed by Smith and Stein (1998) in considering what kind of thinking tasks the textbooks demand of the students. Smith and Stein (1998) categorized mathematical tasks in terms of four cognitive features. They provided a scoring rubric that can be applied to all kinds of mathematical tasks and helps to rate the tasks based on the required cognitive demand. Characteristics of the tasks in the Lower-level Memorization (LM) were "reproducing previously learned facts, rules, formulas, or definitions" (Smith & Stein, 1998, p. 348). This type of task cannot be solved by using procedures (Smith & Stein, 1998). Lower-level Procedure without connections (LP) task is defined as a task that requires "use of the procedure either is specifically called for or is evident from prior instruction" (p. 348) and requires limited cognitive demand. Higher-level Procedures with connections (HP) refers to a task that focuses students' attention on the use of procedures, suggests explicit or implicit pathways, and requires some degree of cognitive effort. Higher-level Doing Mathematics (HDM) task requires "students to explore and understand", and requires "complex and nonalgorithmic thinking" and considerable cognitive effort (p. 348). In this study, I used the list of the characteristic of the tasks at each level and examples provided by Smith and Stein (1998) to analyze questions related to fraction tasks in second and third-grade elementary textbooks, which will be elaborated more in the next part.

Methods

This study was a part of research in analyzing mathematics textbook questions in different countries and intended to examine the extent that second and third-grade mathematics textbooks in Iran and the U.S. attend to fraction concepts. The subjective data of this study was all questions related to the fraction concept in two textbook series in the U.S. and Iran. Two grade-level textbooks published under the approval of the ME in Iran for grades two and three, the tenth edition published in 2022, were selected and compared to the Go-Math textbook series in the U.S., the student e-book 2015 Common Core. The selection of the second and third garde was done purposefully, because subconstructs of the fractions were not covered in earlier grades.

Tasks that were explored in this study were all of the questions from both textbook series, Iran's textbooks and Go-Math textbooks, related to the fraction concept, and included questions listed as activities, introduction, problem-solving, application, practice and review questions. In counting the number of the tasks for both textbook series, a set of exercises that build on one another were considered as one task, as illustrated by an example in Figure 1 part A. Also, a set of questions that attend to the same topic but had different question numbers, and can be answered in isolation, was considered as a single task (see Figure 1 part B).

A B

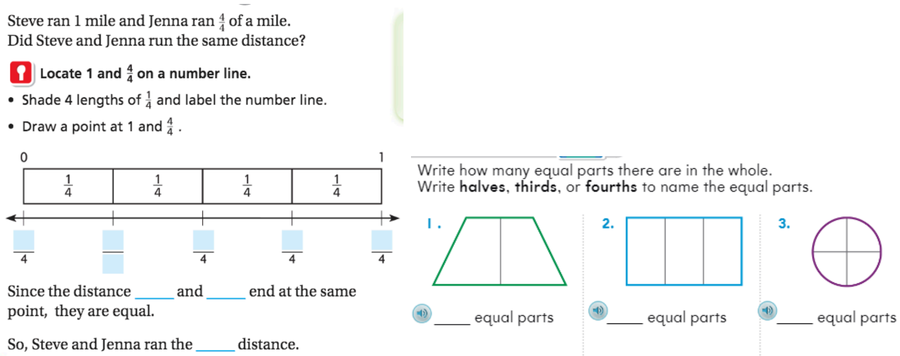


Figure 1: Example of Multiple Questions Considered as One Task in Coding Process

In the first stage, the topic sequences, frequency of fraction questions, and page position of the questions were compared in both textbook series. In the second stage, the level of cognitive demands of all the questions was analyzed employing the framework developed by Smith and Stein (1998). Finally, the contents of selected textbooks were compared in their pedagogical approach to their focus on learning fractions.

In the second stage, I used the list of characteristics in Smith and Stein (1998) to assign a level for each task. Figure 2 shows examples of the tasks coded in each LCD. In this process, if a task appeared for the first time in the textbook that required using a procedure and represented the concept in multiple ways or made connections among multiple representations, it was coded at HP. Similar tasks that were observed later in the textbook were coded as LP because they reproduced previously learned facts (Smith & Stein, 1998). For example, finding a fraction on the number line that was introduced in the third-grade Go-Math textbook was coded as HP in the first appearance in the textbook and as LP later because it required limited cognitive demand (Figure 2). To ensure the validity of the data, I asked an expert college (Creswell & Miller, 2000) to do coding independently for random questions, then we compared assigned codes.

Lower-Level Demands

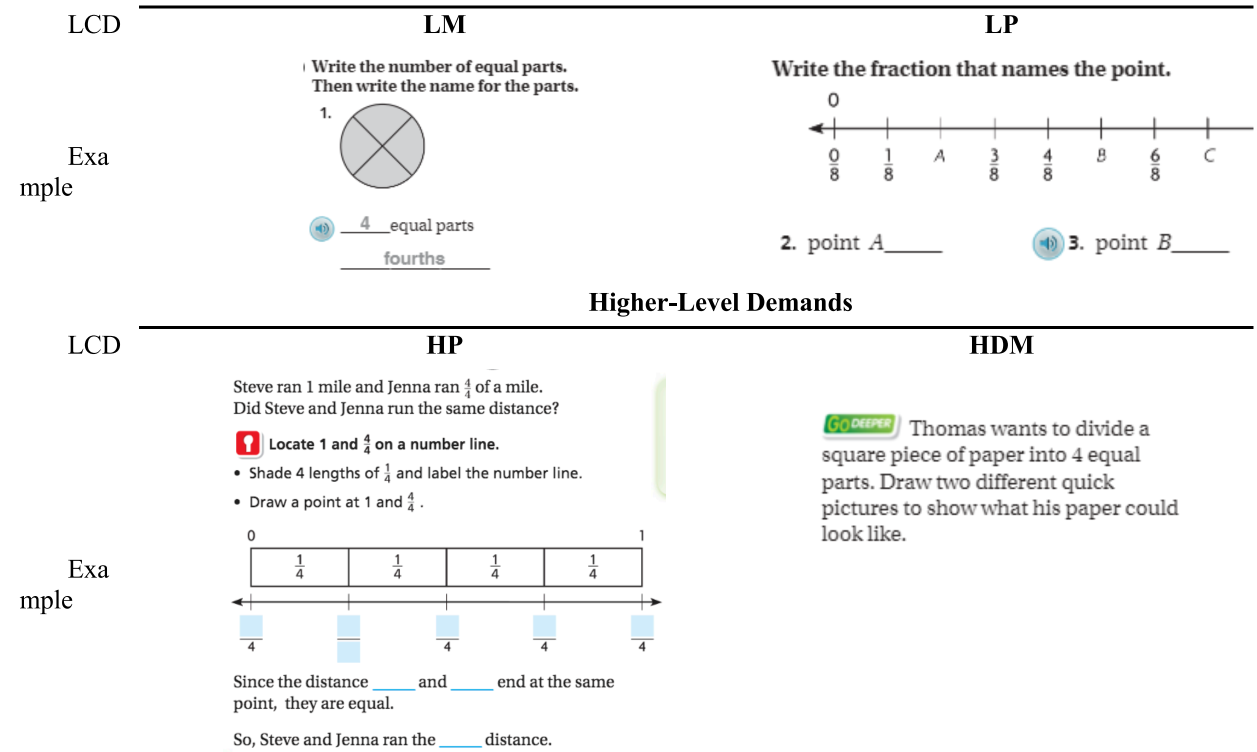


Figure 2: Example of the Tasks Coded in Each LCD

Findings

Comparison of the Number of Pages

In general, the total number of pages in Iran's textbooks was less than the total number of pages in Go-Math textbooks (145 pages in comparison to 774 pages in second grade and 151 pages in comparison to 758 pages in third grade). The number of pages that included fraction tasks in the third-grade Go-Math textbook was higher than the number of pages in second grade (117 and 28 respectively), while in Iran's textbook, a relatively equal number of pages were allocated to fraction tasks in both grades (15 pages in third grade and 17 pages in second grade which were 11% of the total pages in the both grades' textbooks). Furthermore, in comparing the percentage of the pages that were allocated to the fraction tasks in second grade, in the Go-Math textbook fraction tasks were 3% of the total number of gages in the textbook, while this percentage was 0.11 in Iran's textbook. In third grade, the percentage of the pages that were allocated to the fraction tasks in Go-Math textbooks was higher than in Iran's textbook (15 % in comparison to 11%). Table 1 shows the number of pages and page position in fraction tasks in both textbook series.

Table 1: Number of Pages and Page Position in Fraction Tasks

Textbook	Grade Level	Number of Pages	Number of Fraction Pages	% of pages with Fractions	First Fraction Page Position (Unit/ch/pg)	Final Fraction Page Position (Unit/ch/pg)
Go Math! 2015	2nd	774	28	3%	p. 747 ch 11out of 11 Lesson 8, 9, 10, 11	p. 774 ch 11 out of 11 Lesson 8, 9, 10, 11

	3rd	758	117	15%	p. 440 ch. 8 Out of 12	p. 556 ch 9. Out of 12
Iran's Math Textbook 2022	2nd	145	15	11%	p. 110 ch 7 out of 8	p. 125 ch 7 out of 8
	3rd	151	17	11%	p. 41 ch 3 out of 8	p. 57 ch 3 out of 8

Comparison of the LCD

The findings of this study indicated that the levels of the tasks related to fraction concepts in both textbook series in second grade required low cognitive demand. In Go- Math textbook only 6 tasks that required students to compare different parts of a whole or to find different ways of showing equal parts were coded as HDM and 86% of the tasks required low-level cognitive demand. All of the tasks in Iran's second-grade textbook required low-level cognitive demand; there was not any task that required high-level demand (0%, see Table 2). Also, the Go-Math textbook included more memorization tasks than Iran's textbook (81% in comparison to 60%).

Both third-grade textbook series included more high-level tasks in comparison to the second grade; 40% of the tasks in third-grade Iran's textbook in comparing to 0% in second grade, and 27% of the tasks in third grade in Go-Math in comparison to 14% in second grade (see Table 2) were in HP or HMD levels. Table 2 shows the number of tasks at each level.

Table 2: LCD of the Tasks in Go-Math Textbooks and Iran's Textbooks

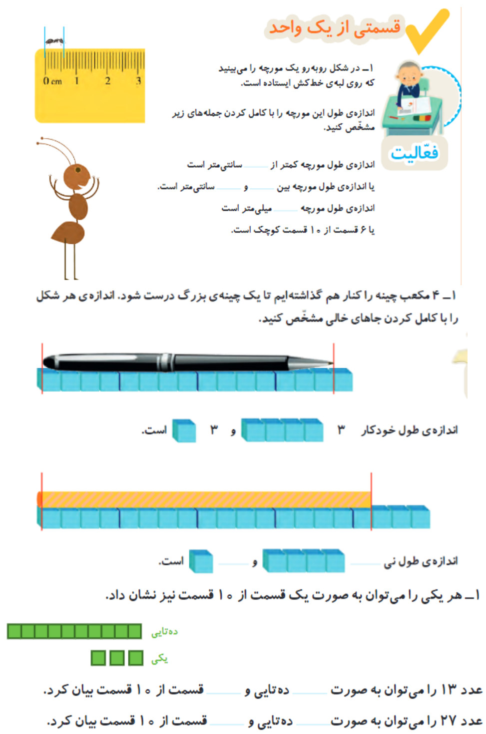
Textbook	Grade Level	Number of Fraction Pages	Number of Tasks	LM	LP	HP	HDM
Go Math 2015	2nd	28	43	35	2	0	6
				86%		14%	
	3rd	117	358	88	173	84	13
				73%		27%	
Iran's textbook 2022	2nd	15	25	15	10	0	0
				100%		0%	
	3rd	17	35	3	18	7	7
				60%		40%	

Pedagogical Analysis

Grade 2. Both textbook series introduced the concept of fractions by dividing the whole unit to equal parts in second grade. Go-Math introduced the concept of equal parts by partitioning geometric shapes in lesson eight in chapter 11, which is the last chapter of the textbook (see Table 1), and related this concept to the geometric figures that were discussed earlier in this chapter. Iran's textbook introduced this concept in chapter seven (out of eight chapters) by referring to the concept of measurement, discussed in chapter five, and also referring to the ten's and one's place values in two-digit numbers, discussed in chapter two of the textbook. Alajmi (2012) found that Japan's textbook similar to Iran's books introduce fractions with referring to the concept of measurement.

Furthermore, while in the Go-Math textbook, equal parts of a whole is the last chapter of the textbook, Iran's' textbook followed this concept with probability questions; connecting the likelihood of the occurrence to the number of equal parts that were colored in a shape. Figure 4 shows examples of how these textbook series approached the concept of equal parts in connection to previously introduced concepts.

Iran's Textbook



Go-Math Textbook

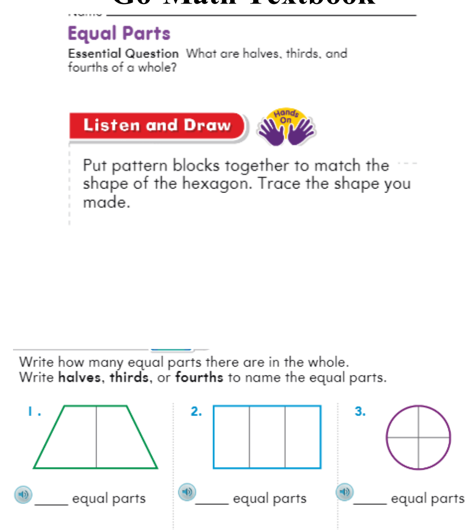


Figure 4. Introduction to The Concept of Fraction in Iran's Textbook and Go-Math Textbook, Grade 2

Grade 3. The presentation of the fraction concepts varied in different countries. The number of concepts that were covered in the Go-Math textbook was higher than in Iran's textbook (see Table 3). However, Iran's Textbook included Part of a Part of a Whole, writing a fraction for 3D figures, and Estimated Fraction that were not discussed in Go-Math (see Figure 4). Figure 3 shows the example of the Part of a Part of a Whole in which one part of a whole/unit where considered as part of a bigger whole.

The Go-Math textbook, introduced the fraction concepts with LM and LP tasks while the first pages of the fraction chapter in Iran's textbook had more HP and HDM tasks. For example, Figure 3 shows the questions on the first page of the chapter about fractions in which students were required to work on Part of a Part of a Whole and Comparing Parts of a Whole, in HDM tasks. The concept of Comparing Parts of a Whole, as presented in Iran's Textbook in the form of dividing a whole to different pieces and comparing created parts (see Figure 3) was discussed on page 519 (the first page of the chapter is 440) in the concept of Compare Fractions with the Same Numerator in Go-Math in a form of HP task and later on in the form of LP tasks.

Table 3: Concepts Related to The Fraction in Iran's Textbook and Go-Math Textbook

	Go-Math	Iran's Textbook
2nd	Equal Parts (the whole, halves, thirds, and fourths).	Equal parts of a whole unit (cm as a unit and mm as parts, equal parts in a geometric shape, equal parts of a segment)
3rd	Equal Parts of a Whole Equal Shares Unit Fractions of a Whole (First fraction p. 455) Fractions of a Whole Fractions on a Number Line Relate Fractions and Whole Numbers whole number and a fraction greater than 1(such as 2=8/4) Fractions of a Group Find Part of a Group Using Unit Fractions Problem Solving Find the Whole Group Using Unit Fractions Compare Fractions with the Same Denominator Compare Fractions with the Same Numerator Compare Fractions with Different Numerators and Numerators (Missing pies strategy) Compare and Order Fractions (order fractions with same numerator) Model Equivalent Fractions	Equal Parts of a Whole , Model Equivalent Fractions Compare Fractions with the Same Numerator Compare Fractions with the Same Denominator Compare Fractions with Different Numerators and Denominator (Same whole and different parts) Two or Three Whole units and fraction (greater than 1, Ex: 2 wholes and 1/3 of a whole)

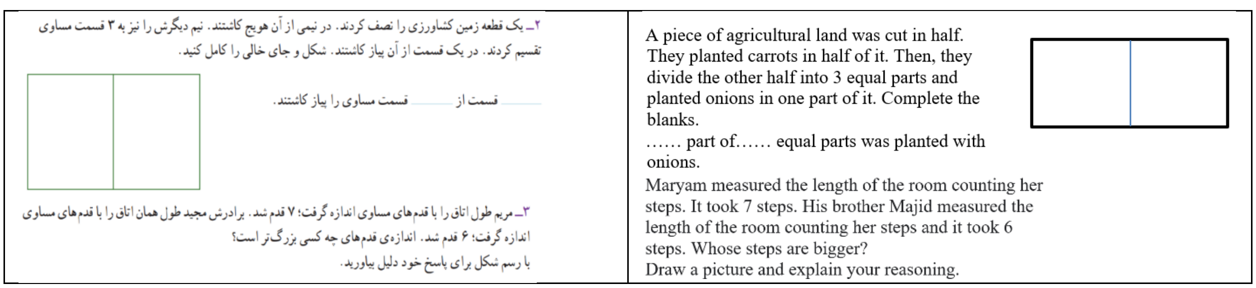


Figure 3: Example of the Questions in the First Page of chapter 3 in Iran's Textbook, Grade3

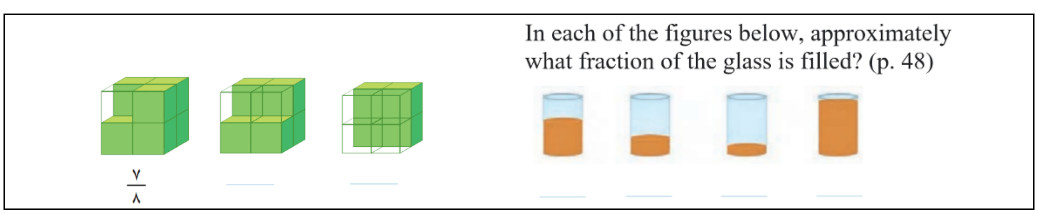


Figure 4: Fraction for 3D Figures and Estimated Fraction in Iran's Textbook

Discussion and Implications

This study contributes to the field of research on elementary school mathematics curricula with respect to fraction concept analysis.

As mentioned in the data analysis section, the total number of textbook pages allocated to fractions in Iran's textbook was considerably less than the U.S. textbooks. A study by Alajmi (2012) showed a similar result in comparing Kuwait, Japan, and U.S. mathematics textbooks. Further studies are required to study textbooks from different countries in an attempt to gauge how textbooks approach the concept of fractions in elementary grades.

Students should have the opportunity to engage with higher-level tasks that lead them to a deeper understanding of the concept (Stein et al., 2000). On the other hand, research showed that the LCD of tasks might declines throughout teachers' implementation in the classroom (Stein & Smith, 1998; Stein et al., 2001). Considering the findings of the current study that showed a higher number of lower-level tasks in both countries, and the result of similar studies such as Bütüner (2021) in Turkish and Singaporean textbooks, including more high-level tasks is required in the school's curriculum. Furthermore, according to Smith and Stein (1998) setting up a high-level task does not guarantee students' engagement at a high level. Thus, further studies are required to analyze and compare how teachers implement these tasks in different countries. Moreover, professional developments are avenues that could prepare teachers to implement lessons with high cognitive demands.

References

Alajmi, A. H. (2012). How do elementary textbooks address fractions? A review of mathematics textbooks in the USA, Japan, and Kuwait. Educational studies in mathematics, 79, 239– 261.

Bütüner, S. Ö. (2021). Content and problem analysis in Turkish and Singaporean mathematics textbooks: The case of multiplying fractions. Journal of Research in Mathematics Education, 10(2), 117– 151.

Chang, C. C., & Silalahi, S. M. (2017). A review and content analysis of mathematics textbooks in educational research. Problems of Education in the 21st Century, 75(3), 235.

Edwards, D., Vale, P., & Westaway, L. (2023). A Comparison of Traction Word Problem Types Within the South African Curriculum and Three Mathematics Texts. In 31st Annual Conference of the Southern African Association for Research in Mathematics, Science and Technology Education (p. 196).

Fan, L., Zhu, Y., & Miao, Z. (2013). Textbook research in mathematics education: development status and directions. ZDM, 45(5), 633-646. doi:10.1007/s11858-013-0539-x

Getenet, S. & Callingham, R. (2019). Teaching interrelated concepts of fraction for understanding and teachers' pedagogical content knowledge. Mathematics Education Research Journal. https://doi.org/10..1007/s13394-019-00282-1.v

Jones, D. L., & Tarr, J. E. (2007). An Examination of the Levels of Cognitive Demand Required by Probability Tasks in Middle Grades Mathematics Textbook. Statistics Education Research Journal, 6(2).

Li, Y. (2000). How textbooks support students' understanding of mathematics: A case study of fraction multiplication. Journal for Research in Mathematics Education, 31(2), 157–177.

Milligan, L. O., Koornhof, H., Sapire, I. & Tikly, L. (2018). Understanding the role of learning and teaching support materials in enabling learning for all. Compare: A Journal of Comparative and International Education, 49 (4), 529-547

Smith, M. S., & Stein, M. K. (1998). Reflections on practice: Selecting and creating mathematical tasks: From research to practice. Mathematics teaching in the middle school, 3(5), 344–350.

Stein, M. K., & Lane, S. (1996). Instructional tasks and the development of student capacity to think and reason: An analysis of the relationship between teaching and learning in a reform mathematics project. Educational Research and Evaluation, 2(1), 50– 80.

Stein, M. K., & Smith, M. S. (1998). Mathematical tasks as a framework for reflection: From research to practice. Mathematics teaching in the middle school, 3(4), 268– 275.

Stein, M. K., Smith, M. S., Henningsen, M. A., & Silver, E. S. (2001). Implementing standards-based mathematics instruction. Book Reviews, 33(3), 99– 101.

ANALYSIS OF MIDDLE SCHOOL MATHEMATICS TEXTBOOKS TO UNCOVER THE PRESENCE OF SOCIAL ISSUE CONTEXTS

Debasmita Basu
Eugene Lang College of Liberal Arts, The New School
basud1@newschool.edu

Madhavi Vishnubhotla
Virginia Tech University
madhaviv@vt.edu

Annie Hua
Eugene Lang College of Liberal Arts, The New School
huaa162@newschool.edu

When mathematics is taught in the context of social issues, it provides students with mathematical skills to analyze and think critically about the world. Using the theoretical perspective of Opportunity to Learn, in this study we analyzed five middle school mathematics textbooks to investigate the extent to which school mathematics textbooks incorporate contextualized situations related to social issues. Findings of the study suggest that there are an overwhelming number of decontextualized problems in every textbook. The contextualized problems covered a range of categories, but a very few of them were on social issues that could offer students the chance to engage in critical thinking.

Keywords: Curriculum, Social Justice, Middle school education

Although mathematics is often considered a subject which is less context-dependent (Janelle, 1990; Gutstein, 2003), research suggests that the context of a mathematical task supports student learning (Boaler, 1993). Presenting mathematical concepts in real-life contexts connected to students' culture and local communities helps students engage in mathematics and understand their reality (Boaler, 1993). Furthermore, mathematics when grounded in social issues such as, climate change, race, gender biases, socioeconomic status, resource disparity, environmental injustice, and other meaningful real-life contexts provides students with mathematical skills that help them analyze and think critically about the world (Frankenstein, 1989, 2009; Gutstein, 2003). This study aims to investigate what role school mathematics textbooks play in familiarizing students with social contexts and developing students' social consciousness. We analyzed five middle school mathematics textbooks in this study and investigated, *to what extent do middle school mathematics textbooks present contextualized situations that align with social issues?*

A Brief on Textbook Analysis

Textbooks are essential vehicles for learning. It is a primary source of resources for teachers and strongly influences their instructional practices (O'Keeffe, 2014; Weiland, 2019). Although teachers often modify curricular materials according to their idiosyncratic requirements, textbooks still offer insight into what is happening in classrooms (Brown & Edelson, 2003). In the past, researchers have analyzed mathematics textbooks to investigate how textbook features impact students' conceptual understanding of mathematics (O'Keeffe & O'Donoghue, 2011), how differentiated mathematical activities challenge students' cognitive abilities (Brändström, 2005), and how procedural complexity, type of solving processes, and degree of repetition of mathematics problems impact eighth grade students' learning of mathematics (Vincent & Stacey, 2008). However, few researchers have explored the contextualized situations in mathematics textbooks. Rivers (1990) compared five textbooks adopted in first-year algebra classes in South

Carolina and analyzed the contexts used in the algebra problems to examine if the contexts are responsive and inclusive towards women and people of ethnic minorities. Weiland (2019) analyzed statistics problems in two high school mathematics textbook series and examined their contextual situations. Weiland (2019) found that the contexts were predominantly neutral and fictional, and provided students limited opportunities to engage in discourse around relevant and crucial topics in today's society. Building on Weiland's (2019) work, in this study we spanned across all content in five textbooks to examine the presence of social issue contexts.

Theoretical Perspectives

Carroll (1963) originally proposed the concept of Opportunities to Learn (OTL) which refers to the inputs and resources in the education system that are provided to students that support their learning. Carroll (1963) identified time spent in class as a key opportunity that shapes students' learning. Subsequently, researchers (Stevens, 1993; Hadar, 2017; Hwang & Ham, 2021) recognized other factors, such as textbook content, cognitive demand of mathematics tasks, teachers' instructional practices, and teachers' selection of content as opportunities that influence students' academic performances. Furthermore, Wijaya et al. (2015) pointed that mathematics tasks embedded in relevant contexts provide students with an opportunity to learn mathematics in meaningful ways and build problem solving skills. Drawing on the construct of OTL, we conjecture that mathematics problems designed in the context of social issues can provide students with opportunities to think critically about the world they live in. Accordingly, we began our investigation of the contexts presented in middle school mathematics textbooks.

Methods

To address our research question, we analyzed five selected mathematics textbooks (Table 1) based on the results of a questionnaire we sent to a group of middle school teachers via a social media platform. We asked them to list the textbooks they used and chose to investigate five mathematics textbooks that were most frequently used. Two coders went through every example and practice set problem of each textbook and identified statements describing contextualized situations. We based our coding on the categories established by Weiland (2019). When a particular context did not fit into a category proposed by Weiland (2019), we created new categories. Considering the focus of our study, we searched for problems that were based on social contexts, and created a new category called "Social Issues."

Table 1: Textbook title, publisher, % of decontextualized and contextualized problems

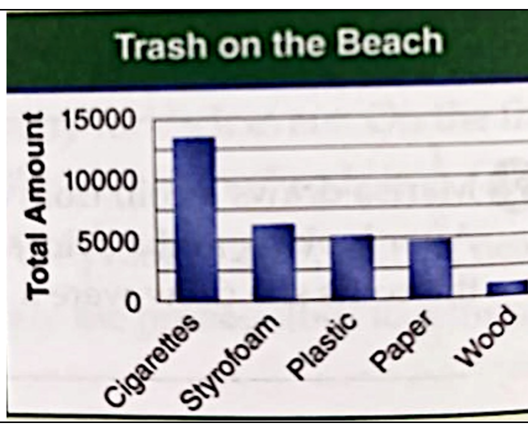
Name of the book	Publisher	% of Decontextualized	% of Contextualized
McGraw-Hill Math 7, Second Edition	McGraw Hill Education	73.81	26.19
McGraw-Hill Math 8, Second Edition	McGraw Hill Education	75.50	24.50
enVisionmath2.0 (Grade 7)	Pearson	36.98	63.02
enVisionmath2.0 (Grade 8)	Pearson	42.91	57.09
Big Ideas Math	Houghton Mifflin Harcourt	58.51	41.49

Also, we merged categories when we identified contextualized situations that fit a broader context. For example, we merged the Business/Sales & the Work/Salary/Savings categories to create the "Finance and Monetary" category. The categories are summarized in the results section. We do not present all categories and examples from every contextualized category for space constraints.

Results

In analyzing the five textbooks, we noted that all textbooks contained a significant number of problems that were devoid of any context. We coded them as decontextualized problems. The percentages of decontextualized problems are shown in Table 1. The rest of the problems were identified under different categories. We note that the categories we identified in the contexts are not mutually exclusive and several contextualized problems could be assigned to more than one category. For example, the problem "how much more does the ham and cheese omelet cost than the cheese omelet" (Big Ideas, p. 82) fits into both food and the finance and monetary categories.

With regards to the "Social Issues" category, all the five textbooks contained a minimal number of questions on social issues. As shown in Figure 2, in enVision 8, 3% of the contextualized problems were on social issues, while in the rest of the books, the percentage of contextualized problems on social issues was lower than 1.5%. These problems were presented in the context of uneven consumption of oil across different countries, depletion of natural resources, trends in the temperature of a town, amount of garbage produced in a city, and community activities, and sustainability. We observed that many of these problems had accompanying graphs and images, and students were asked to solve the problems based on the visuals. For instance, Figure 1 displays a problem from McGraw Hill 7 on the accumulation of trash on a beach that asks students to use a given bar graph to calculate the number of cigarettes, styrofoam, plastic, paper, and wood found on the beach.



1. Is there more trash on the beach due to plastic or paper?
2. Which material accounts for more than twice as much trash on the beach as Styrofoam?
3. What material accounts for the least amount of trash on the beach?

Figure 1: Example of a social issue context from McGraw Hill 7, page 136

In terms of the other categories, we found an overwhelming number of problems in the context of Sports and Entertainment, Finance and Monetary, Science/Weather/Astronomy, School/Testing, and Food. As shown in Figure 2, in three textbooks, the majority of the contextualized problems were centered around Sports and Entertainment (McGraw Hill 7:

16.9%, McGraw Hill 8: 16.44%, and Big Ideas: 17.64%). Across the other two textbooks, the contextualized problems were mostly in the Finance and Monetary (enVision 7, 28%, enVision 8: 19.08%) and Science/Weather/Astronomy (enVision 7: 18.57%, enVision 8, 24.4%) categories. In all textbooks except enVision 8, approximately 10% of the contextualized problems were related to Food.

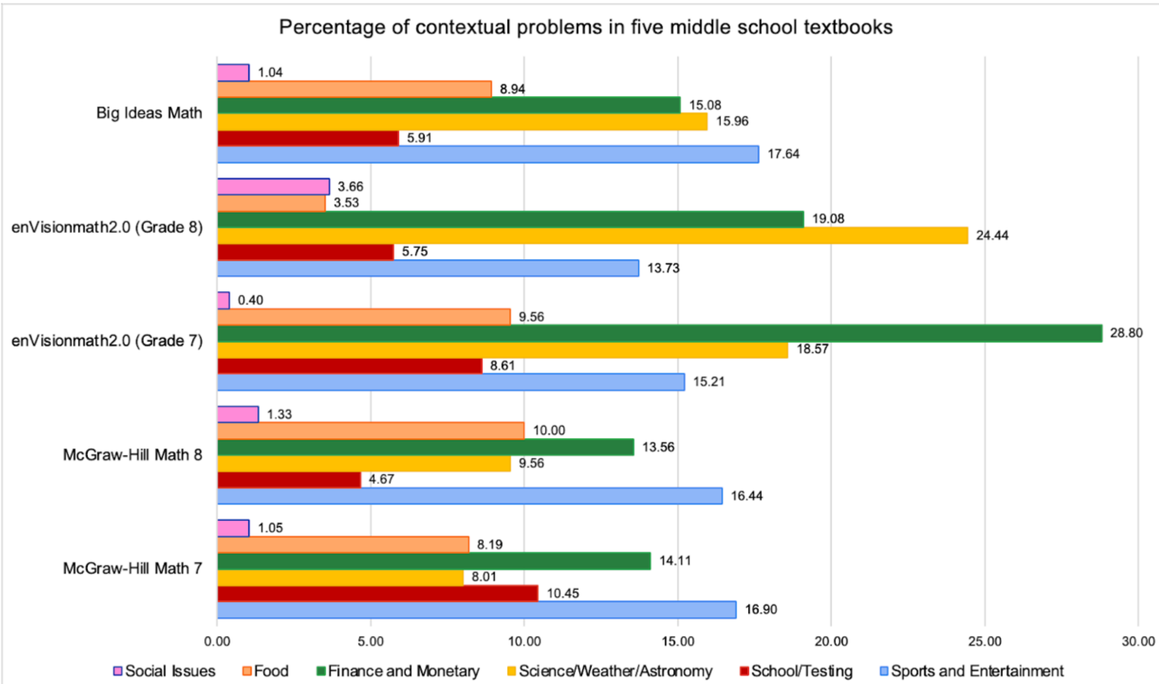


Figure 2: Bar graph with the percentage of contextualized problems in five textbooks

Preliminary Discussion and Implications

The aim of this study was to investigate how often social issues are used as contexts in mathematics textbooks. While we note that the contextualized situations presented in the textbooks referred to sports, theme parks, birthday parties, clothing, school supplies, finance, and pets, social issues prevalent in society such as race, ethnicity, gender, socio-economic conditions are completely absent from the textbooks. This omission is significant and noteworthy. Although a very few questions (refer to example in Figure 1) provide a potential to think about issues such as garbage collection and energy conservation, we agree with Weiland (2019) that most contexts presented are neutral and do not offer students many opportunities to get curious and think critically about social issues. For example, page 11 in enVision 7 asks students to determine the difference when the temperature changes from 8F to 0F. The context presented in this problem is fictitious. It does not prompt students to think critically about climate change caused due to human activities. We propose that instead of providing arbitrary temperatures, if the problem asked students to determine how the average air temperature of the United States has changed with time due to human activities, that would provide students with an opportunity to think about a pressing social issue. As educators, if we want to prepare students to be critical citizens and use mathematics to understand and analyze social issues, we recommend reforms of textbooks that foster such social literacy at the school level.

References

Brändström, A. (2005). *Differentiated tasks in mathematics textbooks: An analysis of the levels of difficulty* (Doctoral dissertation, Luleå tekniska universitet).

Brown, M., & Edelson, D. (2003). Teaching as design: Can we better understand the ways in which teachers use materials so we can better design materials to support their changes in practice. Evanston, IL: *The Center for Learning Technologies in Urban Schools*.

Boaler, J. (1993). The Role of Contexts in the Mathematics Classroom: Do they Make Mathematics More" Real"? *For the learning of mathematics*, 13(2), 12-17.

Carroll, J. B. (1963). A model of school learning. *Teachers college record*, 64(8), 1-9.

Frankenstein, M. (1989). *Relearning mathematics: A different third R--radical math (s)*. Free Association Books.

Frankenstein, M. (2009). Developing a criticalmathematical numeracy through real real-life word problems. In L. Verschaffel, B. Greer, W. Van Dooran, & S. Mukhopadhyay (Eds.), Words and worlds: Modeling verbal descriptions of situations (pp. 111–130). Rotterdam, The Netherlands: Sense Publishers.

Gutstein, E. (2003). Teaching and learning mathematics for social justice in an urban, Latino school. *Journal for Research in Mathematics education*, 34(1), 37-73.

Hadar, L. L. (2017). Opportunities to learn: Mathematics textbooks and students' achievements. *Studies in Educational Evaluation*, 55, 153-166.

Hwang, J., & Ham, Y. (2021). Relationship between Mathematical Literacy and Opportunity to Learn with Different Types of Mathematical Tasks. *Journal on Mathematics Education*, 12(2), 199-222.

O'Keeffe, L. (2014). Mathematics textbook analysis: Supporting the implementation of a new mathematics curriculum. In K. Jones, C. Bokhove, G. Howson & F. Lianghuo (Eds.), Proceedings of the International Conference on Mathematics Textbook Research and Development (ICMT) 2014, (pp. 365-370), Southampton, University of Southampton.

O'Keeffe, L., & O'Donoghue, J. (2011). The use of evidence based research on mathematics textbooks to increase student conceptual understanding. *International Journal for Cross-Disciplinary Subjects in Education*, 2(1), 304-311.

Rivers, J., (1990), Contextual Analysis of Problems in Algebra 1 Textbooks. In *Proceedings of the Annual Meeting of the American Educational Research Association*. Boston, MA.

Stevens, F. I. (1993). Opportunity to learn: Issues of equity for poor and minority students. Washington, DC: *National Center for Education Statistics*.

Vincent, J., & Stacey, K. (2008). Do mathematics textbooks cultivate shallow teaching? Applying the TIMSS video study criteria to Australian eighth-grade mathematics textbooks. *Mathematics Education Research Journal*, 20(1), 82-107.

Weiland, T. (2019). The contextualized situations constructed for the use of statistics by school mathematics textbooks. *Statistics Education Research Journal*, 18(2), 18-38.

Wijaya, A., van den Heuvel-Panhuizen, M., & Doorman, M. (2015). Opportunity-to-learn context-based tasks provided by mathematics textbooks. *Educational Studies in Mathematics*, 89, 41-65.

ANALYZING THE IMPACT OF COREQUISITE REMEDIATION ON DEVELOPMENTAL STUDENTS IN STATISTICS

Shawnda Smith
Texas Woman's University
Ssmith115@twu.edu

Brandi Falley
Texas Woman's University
bfalley@twu.edu

Lorianne Gillander
Texas Woman's University
lgillander@twu.edu

Jaquelyn Hoermann-Elliott
Texas Woman's University
jhoermannelliott@twu.edu

Keywords: Undergraduate Education, Data Analysis and Statistics, Assessment, Curriculum

In our study, we analyze the performance of students enrolled in a corequisite Elementary Statistics course compared to students in the traditional (on-level) Elementary Statistics course. Students deemed not college-ready, based on the standardized state placement exam, were placed in an additional three-hour developmental mathematics course (corequisite course) paired with an Elementary Statistics course. They received instruction covering developmental mathematics topics necessary for their understanding of the topics immediately being addressed in the on-level course. The preliminary study consisted of 2 corequisite sections (38 students) and 2 traditional credit-bearing on-level statistics sections (85 students). The Elementary Statistics courses administered three common exams and a common cumulative final exam.

Table 1: Average by Exam for Each Group

	Corequisite Sections	On-Level Sections
Exam 1 Averages	71.9	72.8
Exam 2 Averages	74.4	76.3
Exam 3 Averages	73.5	72.2
Final Exam Averages	60.9	64.2
Overall Course Average	70.2	71.4

Table 2: Proportion Comparisons by Letter Grade

Grades	Corequisite	On-Level
A	8.16%	14.35%
B	22.45%	22.78%
C	22.45%	25.11%
D	14.29%	10.55%
F	12.24%	14.56%
D/W	20.41%	12.66%

We analyzed the performance of the corequisite students and the on-level students' exam scores and found no significant difference between the grades. The authors intend to expand this analysis using data from Fall 2022 and Spring 2023 for all sections of Elementary

Statistics. This includes both corequisite sections, in-person on-level sections, hybrid, and online sections.

Introduction

In this study, we are interested in the performance of students enrolled in the corequisite course for Elementary Statistics compared to the on-level Elementary Statistics courses to assess the effectiveness of the current corequisite structure.

This study aims to address the following research questions:

1. How do the exam grades and overall course average of students enrolled in the corequisite course compare to those not enrolled in the corequisite?
2. How do the corequisite students compare in their overall performance to the on-level students?

Methodology

Setting and Participants

This study was conducted in the Spring of 2022 at a 4-year public university in the southern United States. The preliminary study consisted of 2 corequisite sections (38 students) and 2 traditional credit-bearing on-level statistics sections (85 students) taught by two different instructors.

Data Collection and Analysis

Instructors of the Elementary Statistics courses used in this study conducted a secondary data analysis by collecting three exam grades and one final exam grade for each enrolled student. We used independent two-sample t-tests to compare the performance of corequisite students on all exams and overall semester grades to the students enrolled in the traditional, on-level statistics course. Additionally, we use a two-sample proportions test to compare the proportion of each letter grade for the corequisite students (2 sections, 49 students) and the on-level statistics students (11 sections, 474 students).

Findings

We report the preliminary findings for our research questions regarding Spring 2022 Elementary Statistics courses.

Research Question 1

How do the exam grades and overall course average of students enrolled in the corequisite course compared to those students not enrolled in the corequisite?

Table 1 shows the averages for each exam for both the corequisite classes (38 total students) and the on-level classes (85 total students).

Independent two-sample t-tests were used to compare the two groups for each exam and the overall course grade. All t-tests concluded that the differences were statistically insignificant. In other words, there is not a significant difference in the averages for each exam and the overall course average between the two groups. For this preliminary analysis, we only compared classes taught by two instructors. The first instructor taught one corequisite section and the second instructor taught one corequisite section and two on-level sections.

Research Question 2

How do the corequisite students compare in their overall performance to the on-level students?

Table 2 shows the percentages for each grade for the corequisite classes (2 sections, 49 total students) and the on-level classes (11 sections, 474 total students).

Table 2: Proportion Comparisons by Letter Grade

Grades	Corequisite	On-Level
A	2.92%	4.99%
B	8.03%	7.93%
C	8.03%	8.74%
D	5.11%	3.67%
F	4.38%	5.07%
D/W	7.3%	4.41%

In this comparison, all sections of Elementary Statistics were used. This includes 11 sections, of which, five were full in-person, four were hybrid, and two were 100% online. A two-sample proportions test indicates that all letter grades are insignificant. There is no statistically significant difference between the proportions of each letter grade when comparing the corequisite classes to the on-level elementary statistics classes.

Conclusion and Future Plans

Our analysis shows that the corequisite Elementary Statistics courses performed just as well as the on-level sections in both the exam grades and overall letter grade. The corequisite sections have a student population that is deemed not college-ready, so they are having to catch up to the on-level material in the class. One would assume that these students would struggle more than the on-level students, but we are successfully addressing the gaps in learning through our just-in-time statistics remediation. Our model is proving to serve the students well and helping them complete the on-level Statistics course required for their degree plan.

This preliminary analysis was based on four sections of Elementary Statistics - two corequisite sections and two on-level sections. The authors intend to expand this analysis using data from Fall 2022 and Spring 2023 for all sections of Elementary Statistics. This includes both corequisite sections, in-person on-level sections, hybrid, and online sections.

References

Attewell, P.A., Lavive, D.E., Domina, T., & Levey, T. (2006). New Evidence on College Remediation. *Journal of HIgher Education, 77(5), 886-924*

Baldwin, A. (2018). *Scaling co-requisite supports at the University of Central Arkansas ...* Retrieved February 21, 2023, from https://dcmathpathways.org/file-download/download/public/699

Bettinger, E.P., and Long, B.T. (2005). Addressing the Needs of Underprepared Students in Higher Education: Does College Remediation Work? NBER Working Paper No. 11325. Cambridge, MA: National Bureau of Economic Research (NBER).

Boatman, A. (2021) Accelerating College Remediation: Examining the Effects of Math Course Redesign on Student Academic Success, *The Journal of Higher Education, 92:6, 927-960*

Chen, X. (2016). Remedial Coursetaking at U.S. Public 2- and 4-Year Institutions: Scope, Experiences, and Outcomes (NCES 2016-405). U.S. Department of Education. Washington, DC: National Center for Education Statistics.

Logue, A. W. (2018, July 17). *The extensive evidence of co-requisite remediation's effectiveness - IR*. Inside Higher Ed. Retrieved February 22, 2023, from http://ir.westcliff.edu/wp-content/uploads/2018/08/The-Extensive-Evidence-of-Co-Requisite-Remediations-Effectiveness.pdf

THE TREATMENT OF LINEAR FUNCTIONS IN JAPANESE CURRICULA THROUGH QUANTITATIVE AND COVARIATIONAL REASONING

Nurdan Turan
New York University
nt2124@nyu.edu

Gülseren Karagöz Akar
Boğaziçi University
gulseren.akar@boun.edu.tr

This study investigated how Japanese curricula represent functional relationships through the lenses of quantitative and covariational reasoning. Utilizing both macro and micro textbook analyses, we examined the tasks, questions, and representations in the Japanese elementary and lower secondary course of study, teachers' guide, and textbooks. Findings showed that, starting from the 4th grade by the end of 8th grade, Japanese curricula focus on iteratively improving learners' quantitative and covariational reasoning gradually raising up to continuous covariation level. This pointed out that Japanese curricula have a spiral nature in the learning of functional relationships involved in proportional and linear situations. We discuss the implications of findings for teaching, learning, and teacher education.

Keywords: curriculum, quantitative reasoning, covariational reasoning, linear functions

Background

School curriculum has a role in improving students' learning and acvement (Cai, 2017). The term curriculum is a broad notion including intended, implemented, attained, and potentially implemented curriculum. *Intended curriculum* is a planned set of actions for students to achieve a specific goal. *Implemented curriculum* refers to how the goals of the intended curriculum are implemented within classrooms and the *attained curriculum* is considered as what is learned by students in classrooms. Within the span of curriculum research, textbooks are considered a potentially *implemented curriculum* as they build a pathway "… between the intentions of the designers of curriculum policy and the teachers that provide instruction in classrooms" (Valverde et al., 2002, p.2). Being mediators between intended and implemented curriculum, textbooks are considered as having a strong impact on what occurs in classrooms (Valverde et al., 2002). In addition, textbooks provide information about pedagogical predispositions, nature of subject matter, arrangements of topics, and level of complexity. Therefore, textbooks have been accepted as tools to be investigated (Cai, 2017) in order to gain some possible insights about opportunities for students to learn mathematics (e.g., Son & Senk, 2010), teacher's teaching and learning (e.g., Son & Kim, 2015), and educational preferences of a particular country (e.g., Usiskin, 2013). Son and Diletti (2017) further argued that teachers' guidebooks and other supplementary materials (e.g., assessments) might be classified as potentially implemented curriculum since they also have potential impact on teaching. It is in this regard that, in this study, we examined Japanese curricula including the course of study, teachers' guide and textbooks because of their potential impact on learning of mathematics topics and classroom teaching.

Textbook analysis is mainly done either on the overall structure of textbooks, which is named as macroanalysis (i.e., horizontal analysis) or on the treatment of specific mathematics topics, which is called as microanalysis (i.e., vertical analysis). Researchers recommended to use both the macro and micro analysis to investigate the treatment of a mathematics topic so that the topic could be examined in terms of its place in specific educational system, related topics to it, and learning opportunities provided in it (Charalambous et al., 2010; Watanabe et al., 2017; Li, 2000). Regarding microanalysis, the treatment of functions is one important notion to study because functions are the backbone of the domain of algebra and lays the foundation for advanced mathematics (Kaput, 1994). Functional relationships between quantities can be modeled by linear, quadratic, polynomial, exponential functions etc. Although scarce in number (Son & Diletti, 2017), there are some studies focusing on secondary school mathematics such as quadratic functions (e.g., Sağlam & Alacacı, 2012) and linear functions (e.g., Wang et al., 2017). Particularly, Wang et al. (2017) examined mathematics textbooks in Shanghai and in England by looking at the hierarchical levels of students' understanding of linear functions through some aspects, such as dependent relationship and connecting representations. Functional relationships can mainly be expressed in two ways: (i) by a correspondence relationship between elements of two sets and (ii) by representing covariation in quantities (Blanton et al., 2011). Though, the topics of functions were suggested to be developed through quantitative and covariational reasoning and to do it starting with earlier grades rather than waiting until high school and college (e.g., Oehrtman et al., 2008). Thompson and Carlson (2017) defined:

A function, covariationally, is a conception of two quantities varying simultaneously such that there is an invariant relationship between their values that has the property that, in the person's conception, every value of one quantity determines exactly one value of the other. (p. 436)

Lloyd et al. (2010) pointed to the essentiality of understanding functions as means to outline how related quantities covary. In addition, studies showed that students who cannot reason variationally and covariationally seemed experiencing difficulties about functional and proportional relationships (e.g., Carlson et al., 2002). It is also found that undergraduate students start their collegiate study with weak understanding of functions, thus experiencing deficiency about making sense of dynamic events (e.g., Carlson et al., 2002). Therefore, research suggested designing instruction and curriculum for improving the learners' understanding of functional relationships (Thompson & Carlson, 2017; Carlson et al., 2002). Thus, in compliance with earlier research (e.g., Karagöz et al. 2022; Taşova et al., 2018), we acknowledge that quantitative and covariational reasoning might be a possible avenue for analyzing curricular materials. Similarly, Thompson and Carlson (2017) argue that Japanese curricula exemplifies how quantitative and covariational reasoning can be integrated in standards and textbooks.

It is in this regard that, in this study, we investigated the topic of functional relationships in Japanese curricula between the grades of 4 to 8 through the lenses of quantitative and covariational reasoning. We also particularly attended to the call for integrating macro and micro analyses to investigate the research questions of "How might Japanese curricula materials depicted in the topic of functional relationships potentially trigger quantitative reasoning and covariational reasoning? In what ways tasks, problem situations, questions, and the use of representations (including algebraic, graphical, tabular, and verbal) in functional relationships might potentially trigger quantitative and covariational reasoning in Grades 4 to 8?"

Theoretical Framework

Thompson (1990) defined quantitative reasoning as "the analysis of a situation into a quantitative structure—a network of quantities and quantitative relationships" (p. 12). A quantity is a measurable quality of an object coming into being with a person's conception of a situation by considering the measurable quality of an object (Thompson, 1994a). Both quantitative and covariational reasoning is about conceiving a situation where the former requires a person to conceive the situation in terms of a quantitative structure and the latter is necessary for a person to conceive the dynamic situation which involves quantities' varying values (Thompson &

Carlson, 2017). Carlson et al. (2002) defined covariational reasoning as “the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other” (p. 354). They argued that these activities (i.e., images of covariation) are developmental, meaning that “the images of covariation can be defined by level and that the levels emerge in an ordered succession” (Carlson et al., 2002, p. 354). In their covariational reasoning framework, Carlson et al. presented five mental actions originating in five covariational reasoning levels (see Figure 1).

Covariational Reasoning Level	Description of the Covariational Reasoning Level	Necessary Mental Actions for the reasoning level	Description of the highest mental action
Smooth continuous covariation	“The person envisions increases and decreases (thereafter, changes) in one quantity’s or variable’s value, and the person envisions both variables varying smoothly and continuously.” (Thompson and Carlson, 2017, p. 435)	MA5 MA4 MA3 MA2 MA1	**MA5:** “Coordinating the instantaneous rate of change of the functions with continuous changes in the independent variable for the entire domain of the function.” (Carlson *et al*., 2002, p. 357)
Chunky continuous covariation	“The person envisions changes in one variable’s value as happening simultaneously with changes in another variable’s value, and they envision both variables varying with chunky continuous variation.” (Thompson and Carlson, 2017, p. 435)	MA4 MA3 MA2 MA1	**MA4:** “Coordinating the average rate-of-change of the function with uniform increments of change in the input variable.” (Carlson *et al*., 2002, p. 357)
Coordination of values	“The person coordinates the value of one variable (x) with values of another variable (y) with the anticipation of creating a discrete collection of pairs (x,y).” (Thompson and Carlson, 2017, p. 435)	MA3 MA2 MA1	**MA3:** “Coordinating the amount of change of one variable with changes in the other variable.” (Carlson *et al*., 2002, p. 357)
Gross coordination of values	“The person forms a gross image of quantities’ values varying together, such as “this quantity increases while that quantity decreases.” The person does not envision that individual values of quantities go together. Instead, the person envisions a loose, nonmultiplicative link between the overall changes in two quantities’ values.” (Thompson and Carlson, 2017, p. 435)	MA2 MA1	**MA2:** “Coordinating the direction of change of one variable with changes in the other variable.” (Carlson *et al*., 2002, p. 357)
Precoordination of values	“The person envisions two variables’ values varying, but asynchronously- one variable changes, then the second variable changes, then the first, and so on. The person does not anticipate creating pairs of values as multiplicative objects.” (Thompson and Carlson, 2017, p. 435)	MA1	**MA1:** “Coordinating the value of one variable with changes in the other.” (Carlson *et al*., 2002, p. 357)
No coordination	“The person has no image of variables varying together. The person focuses on one or another variable’s variation with no coordination of values.” (Thompson and Carlson, 2017, p. 435)	-	-

Figure 1: Covariational reasoning levels and corresponding mental actions

Method

In this study we examined the Mathematics International (MI) textbook series published in 2011 globally by Tokyo Shoseki in collaboration with Global Educational Resources. Tokyo Shoseki is a leading textbook publisher in Japan and its textbook series are one of the six most widely used series in elementary mathematics (Watanabe et al., 2017). The MI series cover grades 1 to 9 which includes both elementary (grades 1 to 6) (Fujii & Iitaka, 2012) and lower secondary (grades 7 to 9) school mathematics (Fujii & Matano, 2012). In addition, we investigated the course of study (COS) published in 2008, which is the national curriculum standards in Japan, and teachers’ guides, which is defined as “a guidebook on Japanese curriculum standards” (Isoda, 2010a, p. i) for teachers. Both were provided by the Ministry of National Education of Japan.

To analyze the curricula materials, we utilized a content analysis method (Weber, 1990). First, we determined the grade levels in the Japanese COS where the functional relationships were covered. We found out that under the topic of functional relationships, rate, ratio, proportion, and linear functions were covered between the grades 4 and 8. In this paper we specifically focus on linear functions and proportional relationships. We conducted our analysis in the order of the course of study (COS), teachers’ guide, and the units of textbooks. Our unit(s) of analysis in all the curricular materials were a statement or a set of statements, graphs, diagrams, tables and symbols. Using both quantitative reasoning and covariational reasoning frameworks as our theoretical constructs, we examined how quantities were introduced and how the relationships between them were triggered in the statements, tasks, questions, problem situations and representations throughout curricula. As covariational reasoning enables examining quantitative reasoning in dynamic situations, using the categories in Figure 1, we examined which mental actions were targeted on the part of students to engage in covariational reasoning. This way, we determined the tasks, questions, problem situations, and representations that seemed potentially to trigger the mental actions, and eventually covariational reasoning.

Findings

Findings has shown that the level of covariational reasoning and complexity of task variables increases with the increase in grade level (see Figure 2).

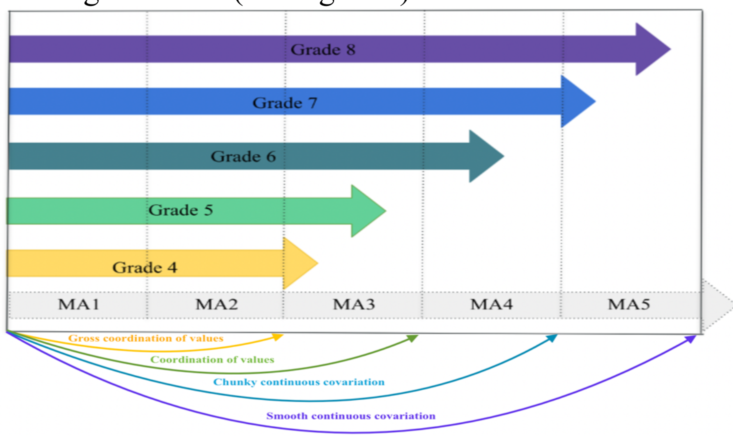


Figure 2: Spiral and iterative nature of Japanese curricula with respect to the metal actions in covariational reasoning

Particularly, in the context of functional relationships, the gross coordination level of covariational reasoning has been supported starting from the 4th grade to the 8th grade. Although the tasks and questions in each grade mostly focus on and delve into specific mental actions with the corresponding covariational reasoning level, there is a spiral nature such that the covariational reasoning level in the upcoming grade builds on and deepens on the previous ones. This suggests that mental actions in the previous covariational reasoning level are iteratively triggered to build the next level of reasoning. Particularly, *gross coordination of values* level of covariational reasoning seems to be aimed in the 4th grade by particularly triggering MA1 and MA2; *coordination of values* level seems to be targeted in the 5th grade by particularly triggering MA1, MA2, and MA3; and *continuous covariation* seems to be targeted in the 6th and 7th grades by particularly triggering MA1, MA2, MA3, and MA4; and (chunky and smooth) *continuous covariation* seems to be targeted in the 8th grade by particularly triggering MA1, MA2, MA3, MA4 and MA5. In lieu of space, we do not share detailed analysis, rather we aim to present different mental actions, reasoning levels, or representations that's not included in the previous grade.

In *4th grade*, in the COS, students are expected to "represent and investigate the relationship between two quantities as they vary simultaneously" (Takahashi et al., 2008, p. 11). Teachers are suggested to use "activities to find two quantities in everyday life that vary in proportion to each other, and to represent and investigate the relationships of numbers/ quantities in tables and graphs." (Isoda, 2010a, p. 116). Also, the Japanese curriculum explicitly differentiates quantities from numbers as stating "a quantity expresses size of an object…A number of objects can be expressed in integers, for example by counting them. On the other hand, in measuring length of strings or weight of water, the quantity can be divided infinitely and cannot always be expressed integers" (Isoda, 2010, p.34). Particularly, teachers are suggested to present activities having quantities of actual objects compared to make it easier for students "to grasp what quantity is being compared, and the meaning of the quantity will become gradually clear to students …For length, for example, "long/short"; … for speed, "fast/slow" (Isoda, 2010, p. 35). All these suggested that there is emphasis on the fact that quantities are measurable qualities of objects such that speed measures whether an object is fast or slow while length measures whether an object is long or short. In the textbook, students are given a broken-line graph to represent monthly temperature changes of Tokyo to build the gross coordination of values level of covariational reasoning. Particularly, students are expected to verbally indicate the axes and coordinate the values, such as, the highest temperature and the corresponding specific month (MA1). They are also expected to focus on the direction of change (i.e., increase, decrease, constancy) of temperature in months (MA2). Thus, the goal in this grade seems to trigger *gross coordination of values* level of covariational reasoning.

In *5th grade*, the objectives and teachers' guide explanations aim for students to (i) deepen their understanding about covarying quantities, and (ii) consider and express quantitative relationships through using different representations. The main goal seems to deepen the *gross coordination of values* level; but also *coordination of amount of changes* of variables (MA3) has seemed to be introduced. In the textbook, through a task asking to use sticks of the same length side by side to make 30 squares, students are expected to recognize the pattern of an arithmetic sequence by examining the change in quantities and their relationships. As an example of two different students' ideas, the following figure is given in the textbook.

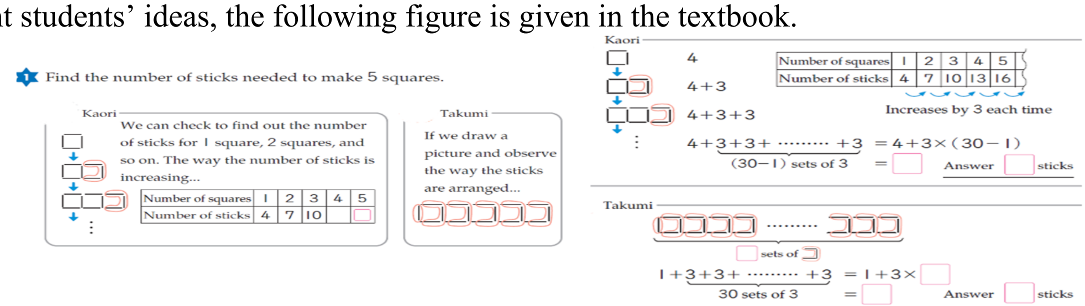


Figure 3: Students' ideas on stick and square task (Fujii and Iitaka, 2012, Grade 5, p. A103-104)

The variables are the number of squares and the number of sticks which are both discrete. A tabular representation is presented in Kaori' approach where she considers the changes between two consecutive values of the number of sticks for each square. However, Takumi examines the arrangement of sticks in the picture where his focus is on how many more sticks (i.e., 3 more sticks) is needed for each square given the first stick. Particularly, Kaori presented quantities, number of squares and number of sticks, in chunks at her table. Albeit the numerical values, her focus is on how the two quantities vary simultaneously such that the number of squares is increasing by one while the number of sticks is increasing by three each time. That is, students' attention is on the increments of 1 and 3 that quantify the increase in the number of squares and the number of sticks respectively. Also, students are expected to compare the differences (the increments of one and three) between the two quantities which might help determine the idea that for each difference (increment of one) in the number of squares, there is some other difference (increment of three) in the number of sticks. Moreover, students are expected to verbalize what the values of 3, 4, 30, and (30-1) represent given the mathematical sentence $4 + 3 \times (30 - 1)$, an important connection between algebraic and verbal representations. In addition, Kaori's table seems to illustrate the correspondence of increase in number of squares with the increase in number of sticks, and her explanation supports that she considers "the way the number of sticks increasing…" (Fujii and Iitaka, 2012, Grade 5, p. A103). Therefore, Kaori's behavior seems to underscore MA3 and indicate her covariational reasoning level as *coordination of values*. On the contrary, Takumi does not seem to recognize that for each new square there are three more sticks needed. His way of thinking suggests that he seems to focus on the number of squares and the "increase" in the number of sticks rather than coordinating the amount of change in one quantity with the change in the other as Kaori. Takumi's behavior seems to underscore MA2 which corresponds to *gross coordination of values*.

In *6th grade*, students are expected to both (i) understand and investigate direct proportional relationships and (ii) gain an awareness of inversely proportional relationships by examining the situations that involve quantities varying simultaneously (Takahashi et al., 2008). The teachers' guide states that "the quotient of two corresponding quantities remains constant" (Isoda, 2010a, p. 155) in the direct proportional relationships which represent "a straight line passing through the origin." (Isoda, 2010a, p. 155). This characteristic is explicitly stated as important to distinguish proportional relationships. Teachers are suggested to use "activities such as representing on a graph two quantities that vary simultaneously so that they [students] understand that if two quantities are in a proportional relationship, the graph representing this relationship is a straight line" (Isoda, 2010a, p. 155)

The task below from 6th grade textbook shows that *coordination of values* level of covariation (up to MA3) aimed to be deepened and *continuous covariation* (MA4) seemed to be triggered. In lieu of space, we do not provide detailed analysis but our account of how students might possibly reason.

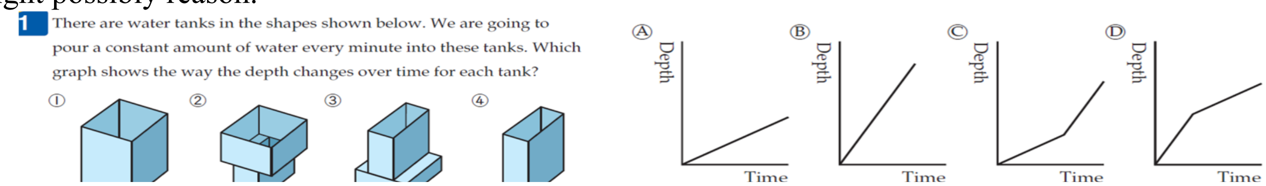


Figure 4: What Will the Graph Look Like (Fujii and Iitaka, 2012, Grade 6, p. B100)

In the task, two continuous variables, time and depth of water, are simultaneously compared through using graphical representations. Students might think that the narrower the base area of the tank the greater the amount of water poured into the tank in a minute. If a student thinks the coordination of depth with amount of time passing (M1), considers the increase in the depth of the water with regard to the time (MA2), compares the amounts of changes in depth of the water for some amount of time (MA3), and envisions the average rate of change of depth as increasing simultaneously with the time (MA4), the student might have *continuous covariation* level of covariational reasoning. However, the action of coordinating the rate of change with the uniform increments of inputs (MA4) are not examined through the numerical values of quantities. Thus, we claim that the task might be targeted to raise students' awareness about the rate of change as a scaffold for the upcoming grades rather than explicitly fostering MA4 in this task.

In *7th grade*, students are expected to further enhance their understanding of direct and inverse proportional relationships in real-life situations. Yet, differently from the 6th grade, the direct and inverse proportional relationships is planned to be re-examined through focusing on the simple forms of linear functions $y = ax$ and reciprocal functions $y = \frac{a}{x}$ respectively (Isoda, 2010b). Differently from 6th grade, students are expected to examine similarities and differences between direct and inverse proportion graphs. Moreover, the rationale of the use of proportional relationships is explicitly stated as to trigger correspondence and covariation meaning of functions in many phenomena in daily life (Isoda, 2010b).

We share an example from the MI textbook showing direct and indirect proportional relationship between two variables and how the continuous nature of their covarying relationship is represented visually (see Figure 5).

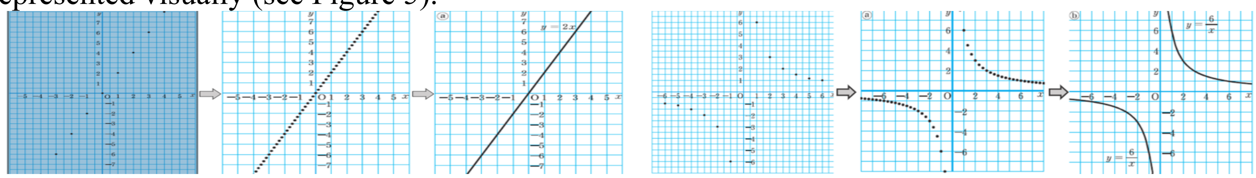


Figure 5: Graphing of y=2x (Fujii and Matano, 2012, Grade 7, p. 117-118) and graphing of $y = \frac{6}{x}$ (Fujii and Matano, 2012, Grade 7, pp. 127-129)

Different integer values of y and x are given for plotting both $y = 2x$ and $y = \frac{6}{x}$. Students are asked to explicitly think about the interval for x values and corresponding y values getting smaller and smaller producing a straight line or a curve. It seems that the continuous nature of variables is targeted through graphical representations. This way of representing might lay a foundation to smooth continuous covariation (corresponding up to MA5), although it is not explicitly introduced.

In *8th grade*, linear functions are introduced, and continuous covariation seems to be triggered and deepened. Students are expected to examine the changes and correspondences of two quantities again with a focus on graphical, algebraic, tabular, and verbal representations. In addition, the rate of change for the linear functions (i.e., in the form of $y = ax + b$) is expressed as $\frac{y_2 - y_1}{x_2 - x_1}$. Here, the rate of change which equals the constant a, is expressed as "how much y will increase when x increases by 1" (Isoda, 2010b, p. 92). All these focus on relationships between amounts of changes in variables suggested that at least MA3 is targeted. Moreover, there is an emphasis on the difference between the expressions of equations with two variables and functions. When the coordination of values of y and x is considered for an equation with two variables expressed as $ax + by + c = 0$, there is one and only one y value for every x value if $b \neq 0$. This relationship indicates that y is a function of x and the equation could be rewritten as $y = -\frac{a}{b}x - \frac{c}{b}$ to illustrate the functional relationship explicitly (Isoda, 2010b).

We present an example from the textbook (see Figure 6) in which the values of temperature are decimal numbers not increasing uniformly with uniform integer changes in time.

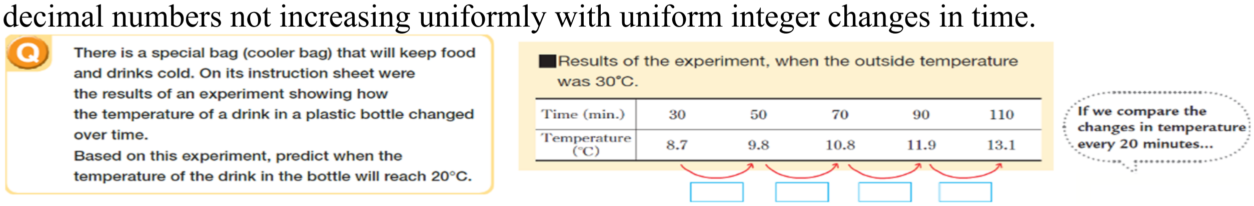


Figure 6: Temperature change of drinks task (Fujii and Matano, 2012, Grade 8, p. 70)

Students are asked to represent this situation with a graph on a coordinate system and think of the relationships between variables where temperature is a function of time. Students are expected to use tables to express the result of values of variables and placing the values of temperature for corresponding units of time on a coordinate system (MA1), verbalize the direction of change as increase or decrease (MA2), coordinate the amount of change in y with the amount of increase in x (MA3), and coordinate the average rate of change of the temperature with uniform changes in time (MA4). Thus, students are supported to reason on the *chunky continuous covariation* level. Yet, in case students consider the rate of change as an instantaneous rate of change, MA5 might also be triggered. In 8th grade, students are also asked to compare linear functions (i.e., $y = ax + b$) with proportional equations (i.e., $y = ax$). It is explicitly explained that translation of the graph of $y = ax$ on the y axis by the value of b represents $y = ax + b$, where the value of b is expressed as y-intercept. Moreover, the use of different types of representations and the relationship between them are explicitly presented.

Discussion and Conclusion

The findings showed that proportional and functional relationships are presented as intertwined with the quantitative and covariational reasoning starting from 4th grade to 8th grade in Japanese curricula. Starting from gross coordination of values, the highest level of covariational reasoning (smooth covariational reasoning) is aimed to be built in an inclusive and iterating way in Japanese curricula. Notably, attending to the developmental nature of covariational reasoning (Carlson et al., 2002) mental actions involved in each covariational reasoning level has been revisited and deepened at successive grade level. Therefore, Japanese curricula illustrate a great model for spiral curriculum, especially for developing the proportional and functional relationships throughout the elementary and lower secondary level mathematics.

The development of the concepts of proportions and linear functions in the curricula are answering the calls of research. Particularly, in Japanese curricula, the concepts of functions are developed through covariational reasoning at first, then correspondence meaning is shared in the textbook after triggering the highest covariational reasoning level iteratively. Researchers argue that covariation meaning of functions are more relevant to students' daily engagement with the topic and their use of covariational perspective can lead to the development of correspondence view of functions (Thompson, 1994b; Carlson et al., 2002; Oehrtman et al., 2008; Smith, 2003; Lloyd et al., 2010). Moreover, as suggested in literature, functions are examined through different representations and dynamic situations (e.g., Thomson & Carlson, 2017); the task variables get more complex gradually (e.g., Heinz, 2000); the difference between functions and equations are presented (e.g., Chazan & Yerushalmy, 2003); the relationship between proportional relationships and linear functions are explained (e.g., Lloyd et al., 2011); the invariant relationship between variables are introduced as rate of change in dynamic situations (e.g., Carlson, 2002); real life situations are used to study proportional relationships and functions (e.g., Carlson et al., 2002; Oehrtman et al., 2008); quantities and quantitative operations are explicitly introduced (e.g., Thompson, 1994) in Japanese curricula. That is, Japanese curricula are designed in a way to bridge the mathematics education literature with teachers' teaching. Lastly, the findings of the study pointed out that Japanese curricula attentively focus on not only conceptual understanding but also mathematical thinking in a developmental way (e.g., successive development of mental actions). Hence, we suggest curriculum developers pay attention to the spiral nature of Japanese curricula. We argue that Japanese curricula can be used by teacher educators to study and improve teachers' covariational reasoning. Our analysis also supports Thompson and Carlson (2017)'s argument that Japanese curricula exemplifies how quantitative and covariational reasoning can be integrated in standards and textbooks.

Acknowledgments

We acknowledge Prof. Tad Watanabe's guidance and support with respect to obtaining permission to use MI textbook material and for his invaluable feedback to the larger project (i.e., MS thesis study) which forms the basis for this paper.

References

Blanton, M., Levi, L., Crites, T., Dougherty, B., & Zbiek, R. M. (2011). *Developing Essential Understanding of Algebraic Thinking for Teaching Mathematics in Grades 3-5. Series in Essential Understandings*. National Council of Teachers of Mathematics. 1906 Association Drive, Reston, VA 20191-1502.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for research in mathematics education*, *33*(5), 352-378. https://doi.org/10.2307/4149958

Charalambous, C. Y., Delaney, S., Hsu, H. Y., & Mesa, V. (2010). A comparative analysis of the addition and subtraction of fractions in textbooks from three countries. *Mathematical thinking and learning*, *12*(2), 117-151. https://doi.org/10.1080/10986060903460070

Chazan, D., & Yerushalmy, M. (2003). On Appreciating the Cognitive Complexity of School Algebra: Research on Algebra Learning and Directions of Curricular Change. In J. Kilpatrick, D. Schifter, & G. Martin (Eds.), *A Research Companion to the Principles and Standards for School Mathematics* (pp. 123-135). NCTM.

Fujii, T., and Iitaka, S. (2012). *Mathematics International* (grade 1–grade 6). Tokyo Shoseki Co. Ltd., Tokyo.

Fujii, T., and Matano, H. (2012). *Mathematics International* (grade 7–grade 8). Tokyo Shoseki Co. Ltd., Tokyo.

Heinz, K. R. (2000). *Conceptions of ratio in a class of preservice and practicing teachers*. The Pennsylvania State University.

Isoda, M. (2010a). Elementary school teaching guide for the Japanese course of study: Mathematics (Grade 1–6). CRICED: University of Tsukuba, Tsukuba.

Isoda, M. (2010b). Junior high school teaching guide for the Japanese course of study: Mathematics (Grade 7–9). CRICED: University of Tsukuba, Tsukuba.

Kaput, J. J. (1994). Democratizing access to calculus: New routes to old roots. In A. H. Schoenfeld (Ed.), Mathematical thinking and problem solving (pp. 77–156). Hillsdale, NJ: Erlbaum.

Karagöz Akar, G., Watanabe, T., & Turan, N. (2022). Quantitative Reasoning as a Framework to Analyze Mathematics Textbooks. In Karagöz Akar, G., Zembat, İ.Ö., Arslan, S., Thompson, P.W. (Eds.), *Quantitative Reasoning in Mathematics and Science Education. Mathematics Education in the Digital Era*, (vol 21., pp. 107-132). Springer, Cham. https://doi.org/10.1007/978-3-031-14553-7_5

Li, Y. (2000). A comparison of problems that follow selected content presentations in American and Chinese mathematics textbooks. *Journal for Research in Mathematics Education*, *31*(2), 234-241. https://doi.org/10.2307/749754

Lloyd, G., Beckmann, S., Zbiek, R. M., & Cooney, T. (2010). *Developing Essential Understanding of Functions for Teaching Mathematics in Grades 9-12*. National Council of Teachers of Mathematics. 1906 Association Drive, Reston, VA 20191-1502.

Oehrtman, M., Carlson, M., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. *Making the connection: Research and teaching in undergraduate mathematics education*, (pp. 27-42). https://doi.org/10.5948/UPO9780883859759.004

Sağlam, R., & Alacacı, C. (2012). A comparative analysis of quadratics unit in Singaporean, Turkish and IBDP mathematics textbooks. *Turkish Journal of Computer and Mathematics Education (TURCOMAT)*, *3*(3), 131-147. https://doi.org/10.17762/turcomat.v3i3.43

Smith, E. (2003). Stasis and change: Integrating patterns, functions, and algebra throughout the K-12 curriculum. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to Principles and Standards for School Mathematics,* (pp. 136-150). Reston, VA: National Council of Teachers of Mathematics.

Son, J. W., & Diletti, J. (2017). What can we learn from textbook Analysis?. *What matters? Research trends in international comparative studies in mathematics education*, 3-32. Springer, Cham.

Son, J. W., & Kim, O. K. (2015). Teachers' selection and enactment of mathematical problems from textbooks. *Mathematics Education Research Journal*, *27*, 491-518. https://doi.org/10.1007/s13394-015-0148-9

Son, J. W., & Senk, S. L. (2010). How reform curricula in the USA and Korea present multiplication and division of fractions. *Educational Studies in Mathematics*, *74*, 117-142. https://doi.org/10.1007/s10649-010-9229-6

Takahashi, A., Watanabe, T., & Yoshida, M. (2008). English translation of the Japanese mathematics curricula in the course of study. *Global Education Resources, LLC*.

Taşova, H., Stevens, I. E., & Moore, K. C. (2018). A framework for analyzing written curriculum from a shape-thinking and (co) variational reasoning perspective. In *Proceedings of the Twenty-First Annual Conference on Research in Undergraduate Mathematics Education* (pp. 1527-1533).

Thompson, P. W. (1990). *A theoretical model of quantity-based reasoning in arithmetic and algebra* (Report No. MDR 87-51381 and MDR 89-50311). Center for Research in Mathematics & Science Education: San Diego State University.

Thompson, P. W. (1994a). The development of the concept of speed and its relationship to concepts of rate. In G. Vergnaud, G. Harel, and J. Coufrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics*, (pp. 179-234). SUNY Press. http://pat-thompson.net/PDFversions/1994ConceptSpeedRate.pdf

Thompson, P. W. (1994b). Students, functions, and the undergraduate curriculum. In E. Dubinsky, A. H. Schoenfeld, & J. J. Kaput (Eds.), Research in Collegiate Mathematics Education, 1 (Issues in Mathematics Education Vol. 4, pp. 21-44). Providence, RI: American Mathematical Society.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. *Compendium for research in mathematics education*, 421-456. National Council of Teachers of Mathematics. https://www.researchgate.net/profile/Patrick-Thompson-2/publication/302581485_Variation_covariation_and_functions_Foundational_ways_of_thinking_mathematically/links/583c389208ae502a85e3a224/Variation-covariation-and-functions-Foundational-ways-of-thinking-mathematically.pdf

Usiskin, Z. (2013). Studying textbooks in an information age—a United States perspective. *ZDM*, *45*, 713-723. https://doi.org/10.1007/s11858-013-0514-6

Valverde, G. A., Bianchi, L. J., Wolfe, R. G., Schmidt, W. H., & Houang, R. T. (2002). *According to the book: Using TIMSS to investigate the translation of policy into practice through the world of textbooks*. Springer Science & Business Media.

Wang, Y., Barmby, P., & Bolden, D. (2017). Understanding linear function: a comparison of selected textbooks from England and Shanghai. *International Journal of Science and Mathematics Education*, *15*, 131-153. https://doi.org/10.1007/s10763-015-9674-x

Watanabe, T., Lo, J. J., & Son, J. W. (2017). Intended treatment of fractions and fraction operations in mathematics curricula from Japan, Korea, and Taiwan. *What matters? Research trends in international comparative studies in mathematics education*, (pp. 33-61). Springer.

Weber, R. P. (1990). *Basic content analysis* (Vol. 49). Sage.

THE TREATMENT OF LINEAR FUNCTIONS IN JAPANESE CURRICULA THROUGH QUANTITATIVE AND COVARIATIONAL REASONING

Nurdan Turan
New York University
nt2124@nyu.edu

Gülseren Karagöz Akar
Boğaziçi University
gulseren.akar@boun.edu.tr

This study investigated how Japanese curricula represent functional relationships through the lenses of quantitative and covariational reasoning. Utilizing both macro and micro textbook analyses, we examined the tasks, questions, and representations in the Japanese elementary and lower secondary course of study, teachers' guide, and textbooks. Findings showed that, starting from the 4th grade by the end of 8th grade, Japanese curricula focus on iteratively improving learners' quantitative and covariational reasoning gradually raising up to continuous covariation level. This pointed out that Japanese curricula have a spiral nature in the learning of functional relationships involved in proportional and linear situations. We discuss the implications of findings for teaching, learning, and teacher education.

Keywords: curriculum, quantitative reasoning, covariational reasoning, linear functions

Background

School curriculum has a role in improving students' learning and acvement (Cai, 2017). The term curriculum is a broad notion including intended, implemented, attained, and potentially implemented curriculum. *Intended curriculum* is a planned set of actions for students to achieve a specific goal. *Implemented curriculum* refers to how the goals of the intended curriculum are implemented within classrooms and the *attained curriculum* is considered as what is learned by students in classrooms. Within the span of curriculum research, textbooks are considered a potentially *implemented curriculum* as they build a pathway "… between the intentions of the designers of curriculum policy and the teachers that provide instruction in classrooms" (Valverde et al., 2002, p.2). Being mediators between intended and implemented curriculum, textbooks are considered as having a strong impact on what occurs in classrooms (Valverde et al., 2002). In addition, textbooks provide information about pedagogical predispositions, nature of subject matter, arrangements of topics, and level of complexity. Therefore, textbooks have been accepted as tools to be investigated (Cai, 2017) in order to gain some possible insights about opportunities for students to learn mathematics (e.g., Son & Senk, 2010), teacher's teaching and learning (e.g., Son & Kim, 2015), and educational preferences of a particular country (e.g., Usiskin, 2013). Son and Diletti (2017) further argued that teachers' guidebooks and other supplementary materials (e.g., assessments) might be classified as potentially implemented curriculum since they also have potential impact on teaching. It is in this regard that, in this study, we examined Japanese curricula including the course of study, teachers' guide and textbooks because of their potential impact on learning of mathematics topics and classroom teaching.

Textbook analysis is mainly done either on the overall structure of textbooks, which is named as macroanalysis (i.e., horizontal analysis) or on the treatment of specific mathematics topics, which is called as microanalysis (i.e., vertical analysis). Researchers recommended to use both the macro and micro analysis to investigate the treatment of a mathematics topic so that the topic could be examined in terms of its place in specific educational system, related topics to it, and learning opportunities provided in it (Charalambous et al., 2010; Watanabe et al., 2017; Li, 2000). Regarding microanalysis, the treatment of functions is one important notion to study because functions are the backbone of the domain of algebra and lays the foundation for advanced mathematics (Kaput, 1994). Functional relationships between quantities can be modeled by linear, quadratic, polynomial, exponential functions etc. Although scarce in number (Son & Diletti, 2017), there are some studies focusing on secondary school mathematics such as quadratic functions (e.g., Sağlam & Alacacı, 2012) and linear functions (e.g., Wang et al., 2017). Particularly, Wang et al. (2017) examined mathematics textbooks in Shanghai and in England by looking at the hierarchical levels of students' understanding of linear functions through some aspects, such as dependent relationship and connecting representations. Functional relationships can mainly be expressed in two ways: (i) by a correspondence relationship between elements of two sets and (ii) by representing covariation in quantities (Blanton et al., 2011). Though, the topics of functions were suggested to be developed through quantitative and covariational reasoning and to do it starting with earlier grades rather than waiting until high school and college (e.g., Oehrtman et al., 2008). Thompson and Carlson (2017) defined:

A function, covariationally, is a conception of two quantities varying simultaneously such that there is an invariant relationship between their values that has the property that, in the person's conception, every value of one quantity determines exactly one value of the other. (p. 436)

Lloyd et al. (2010) pointed to the essentiality of understanding functions as means to outline how related quantities covary. In addition, studies showed that students who cannot reason variationally and covariationally seemed experiencing difficulties about functional and proportional relationships (e.g., Carlson et al., 2002). It is also found that undergraduate students start their collegiate study with weak understanding of functions, thus experiencing deficiency about making sense of dynamic events (e.g., Carlson et al., 2002). Therefore, research suggested designing instruction and curriculum for improving the learners' understanding of functional relationships (Thompson & Carlson, 2017; Carlson et al., 2002). Thus, in compliance with earlier research (e.g., Karagöz et al. 2022; Taşova et al., 2018), we acknowledge that quantitative and covariational reasoning might be a possible avenue for analyzing curricular materials. Similarly, Thompson and Carlson (2017) argue that Japanese curricula exemplifies how quantitative and covariational reasoning can be integrated in standards and textbooks.

It is in this regard that, in this study, we investigated the topic of functional relationships in Japanese curricula between the grades of 4 to 8 through the lenses of quantitative and covariational reasoning. We also particularly attended to the call for integrating macro and micro analyses to investigate the research questions of "How might Japanese curricula materials depicted in the topic of functional relationships potentially trigger quantitative reasoning and covariational reasoning? In what ways tasks, problem situations, questions, and the use of representations (including algebraic, graphical, tabular, and verbal) in functional relationships might potentially trigger quantitative and covariational reasoning in Grades 4 to 8?"

Theoretical Framework

Thompson (1990) defined quantitative reasoning as "the analysis of a situation into a quantitative structure—a network of quantities and quantitative relationships" (p. 12). A quantity is a measurable quality of an object coming into being with a person's conception of a situation by considering the measurable quality of an object (Thompson, 1994a). Both quantitative and covariational reasoning is about conceiving a situation where the former requires a person to conceive the situation in terms of a quantitative structure and the latter is necessary for a person to conceive the dynamic situation which involves quantities' varying values (Thompson &

Carlson, 2017). Carlson et al. (2002) defined covariational reasoning as “the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other” (p. 354). They argued that these activities (i.e., images of covariation) are developmental, meaning that “the images of covariation can be defined by level and that the levels emerge in an ordered succession” (Carlson et al., 2002, p. 354). In their covariational reasoning framework, Carlson et al. presented five mental actions originating in five covariational reasoning levels (see Figure 1).

Covariational Reasoning Level	Description of the Covariational Reasoning Level	Necessary Mental Actions for the reasoning level	Description of the highest mental action
Smooth continuous covariation	“The person envisions increases and decreases (thereafter, changes) in one quantity’s or variable’s value, and the person envisions both variables varying smoothly and continuously.” (Thompson and Carlson, 2017, p. 435)	MA5 MA4 MA3 MA2 MA1	**MA5:** “Coordinating the instantaneous rate of change of the functions with continuous changes in the independent variable for the entire domain of the function.” (Carlson *et al*., 2002, p. 357)
Chunky continuous covariation	“The person envisions changes in one variable’s value as happening simultaneously with changes in another variable’s value, and they envision both variables varying with chunky continuous variation.” (Thompson and Carlson, 2017, p. 435)	MA4 MA3 MA2 MA1	**MA4:** “Coordinating the average rate-of-change of the function with uniform increments of change in the input variable.” (Carlson *et al*., 2002, p. 357)
Coordination of values	“The person coordinates the value of one variable (x) with values of another variable (y) with the anticipation of creating a discrete collection of pairs (x,y).” (Thompson and Carlson, 2017, p. 435)	MA3 MA2 MA1	**MA3:** “Coordinating the amount of change of one variable with changes in the other variable.” (Carlson *et al*., 2002, p. 357)
Gross coordination of values	“The person forms a gross image of quantities’ values varying together, such as “this quantity increases while that quantity decreases.” The person does not envision that individual values of quantities go together. Instead, the person envisions a loose, nonmultiplicative link between the overall changes in two quantities’ values.” (Thompson and Carlson, 2017, p. 435)	MA2 MA1	**MA2:** “Coordinating the direction of change of one variable with changes in the other variable.” (Carlson *et al*., 2002, p. 357)
Precoordination of values	“The person envisions two variables’ values varying, but asynchronously- one variable changes, then the second variable changes, then the first, and so on. The person does not anticipate creating pairs of values as multiplicative objects.” (Thompson and Carlson, 2017, p. 435)	MA1	**MA1:** “Coordinating the value of one variable with changes in the other.” (Carlson *et al*., 2002, p. 357)
No coordination	“The person has no image of variables varying together. The person focuses on one or another variable’s variation with no coordination of values.” (Thompson and Carlson, 2017, p. 435)	-	-

Figure 1: Covariational reasoning levels and corresponding mental actions

Method

In this study we examined the Mathematics International (MI) textbook series published in 2011 globally by Tokyo Shoseki in collaboration with Global Educational Resources. Tokyo Shoseki is a leading textbook publisher in Japan and its textbook series are one of the six most widely used series in elementary mathematics (Watanabe et al., 2017). The MI series cover grades 1 to 9 which includes both elementary (grades 1 to 6) (Fujii & Iitaka, 2012) and lower secondary (grades 7 to 9) school mathematics (Fujii & Matano, 2012). In addition, we investigated the course of study (COS) published in 2008, which is the national curriculum standards in Japan, and teachers’ guides, which is defined as “a guidebook on Japanese curriculum standards” (Isoda, 2010a, p. i) for teachers. Both were provided by the Ministry of National Education of Japan.

To analyze the curricula materials, we utilized a content analysis method (Weber, 1990). First, we determined the grade levels in the Japanese COS where the functional relationships were covered. We found out that under the topic of functional relationships, rate, ratio, proportion, and linear functions were covered between the grades 4 and 8. In this paper we specifically focus on linear functions and proportional relationships. We conducted our analysis in the order of the course of study (COS), teachers’ guide, and the units of textbooks. Our unit(s) of analysis in all the curricular materials were a statement or a set of statements, graphs, diagrams, tables and symbols. Using both quantitative reasoning and covariational reasoning frameworks as our theoretical constructs, we examined how quantities were introduced and how the relationships between them were triggered in the statements, tasks, questions, problem situations and representations throughout curricula. As covariational reasoning enables examining quantitative reasoning in dynamic situations, using the categories in Figure 1, we examined which mental actions were targeted on the part of students to engage in covariational reasoning. This way, we determined the tasks, questions, problem situations, and representations that seemed potentially to trigger the mental actions, and eventually covariational reasoning.

Findings

Findings has shown that the level of covariational reasoning and complexity of task variables increases with the increase in grade level (see Figure 2).

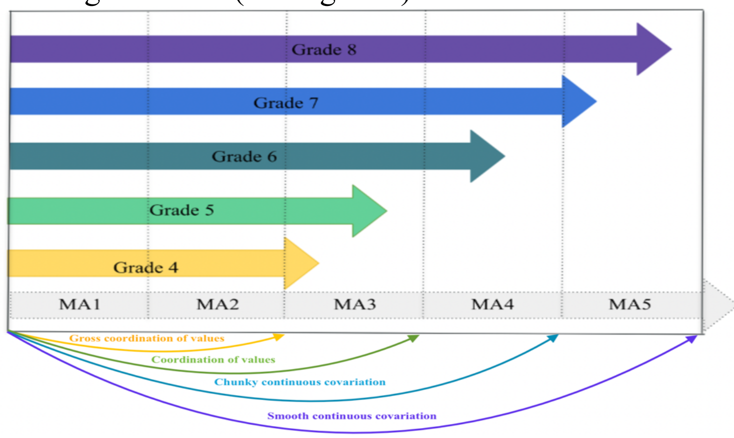


Figure 2: Spiral and iterative nature of Japanese curricula with respect to the metal actions in covariational reasoning

Particularly, in the context of functional relationships, the gross coordination level of covariational reasoning has been supported starting from the 4th grade to the 8th grade. Although the tasks and questions in each grade mostly focus on and delve into specific mental actions with the corresponding covariational reasoning level, there is a spiral nature such that the covariational reasoning level in the upcoming grade builds on and deepens on the previous ones. This suggests that mental actions in the previous covariational reasoning level are iteratively triggered to build the next level of reasoning. Particularly, *gross coordination of values* level of covariational reasoning seems to be aimed in the 4th grade by particularly triggering MA1 and MA2; *coordination of values* level seems to be targeted in the 5th grade by particularly triggering MA1, MA2, and MA3; and *continuous covariation* seems to be targeted in the 6th and 7th grades by particularly triggering MA1, MA2, MA3, and MA4; and (chunky and smooth) *continuous covariation* seems to be targeted in the 8th grade by particularly triggering MA1, MA2, MA3, MA4 and MA5. In lieu of space, we do not share detailed analysis, rather we aim to present different mental actions, reasoning levels, or representations that's not included in the previous grade.

In *4th grade*, in the COS, students are expected to "represent and investigate the relationship between two quantities as they vary simultaneously" (Takahashi et al., 2008, p. 11). Teachers are suggested to use "activities to find two quantities in everyday life that vary in proportion to each other, and to represent and investigate the relationships of numbers/ quantities in tables and graphs." (Isoda, 2010a, p. 116). Also, the Japanese curriculum explicitly differentiates quantities from numbers as stating "a quantity expresses size of an object…A number of objects can be expressed in integers, for example by counting them. On the other hand, in measuring length of strings or weight of water, the quantity can be divided infinitely and cannot always be expressed integers" (Isoda, 2010, p.34). Particularly, teachers are suggested to present activities having quantities of actual objects compared to make it easier for students "to grasp what quantity is being compared, and the meaning of the quantity will become gradually clear to students …For length, for example, "long/short"; … for speed, "fast/slow" (Isoda, 2010, p. 35). All these suggested that there is emphasis on the fact that quantities are measurable qualities of objects such that speed measures whether an object is fast or slow while length measures whether an object is long or short. In the textbook, students are given a broken-line graph to represent monthly temperature changes of Tokyo to build the gross coordination of values level of covariational reasoning. Particularly, students are expected to verbally indicate the axes and coordinate the values, such as, the highest temperature and the corresponding specific month (MA1). They are also expected to focus on the direction of change (i.e., increase, decrease, constancy) of temperature in months (MA2). Thus, the goal in this grade seems to trigger *gross coordination of values* level of covariational reasoning.

In *5th grade*, the objectives and teachers' guide explanations aim for students to (i) deepen their understanding about covarying quantities, and (ii) consider and express quantitative relationships through using different representations. The main goal seems to deepen the *gross coordination of values* level; but also *coordination of amount of changes* of variables (MA3) has seemed to be introduced. In the textbook, through a task asking to use sticks of the same length side by side to make 30 squares, students are expected to recognize the pattern of an arithmetic sequence by examining the change in quantities and their relationships. As an example of two different students' ideas, the following figure is given in the textbook.

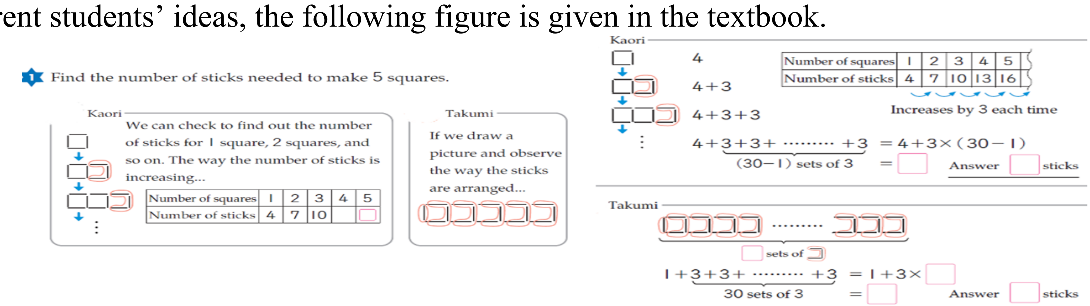


Figure 3: Students' ideas on stick and square task (Fujii and Iitaka, 2012, Grade 5, p. A103-104)

The variables are the number of squares and the number of sticks which are both discrete. A tabular representation is presented in Kaori' approach where she considers the changes between two consecutive values of the number of sticks for each square. However, Takumi examines the arrangement of sticks in the picture where his focus is on how many more sticks (i.e., 3 more sticks) is needed for each square given the first stick. Particularly, Kaori presented quantities, number of squares and number of sticks, in chunks at her table. Albeit the numerical values, her focus is on how the two quantities vary simultaneously such that the number of squares is increasing by one while the number of sticks is increasing by three each time. That is, students' attention is on the increments of 1 and 3 that quantify the increase in the number of squares and the number of sticks respectively. Also, students are expected to compare the differences (the increments of one and three) between the two quantities which might help determine the idea that for each difference (increment of one) in the number of squares, there is some other difference (increment of three) in the number of sticks. Moreover, students are expected to verbalize what the values of 3, 4, 30, and (30-1) represent given the mathematical sentence $4 + 3 \times (30 - 1)$, an important connection between algebraic and verbal representations. In addition, Kaori's table seems to illustrate the correspondence of increase in number of squares with the increase in number of sticks, and her explanation supports that she considers "the way the number of sticks increasing…" (Fujii and Iitaka, 2012, Grade 5, p. A103). Therefore, Kaori's behavior seems to underscore MA3 and indicate her covariational reasoning level as *coordination of values*. On the contrary, Takumi does not seem to recognize that for each new square there are three more sticks needed. His way of thinking suggests that he seems to focus on the number of squares and the "increase" in the number of sticks rather than coordinating the amount of change in one quantity with the change in the other as Kaori. Takumi's behavior seems to underscore MA2 which corresponds to *gross coordination of values*.

In *6th grade*, students are expected to both (i) understand and investigate direct proportional relationships and (ii) gain an awareness of inversely proportional relationships by examining the situations that involve quantities varying simultaneously (Takahashi et al., 2008). The teachers' guide states that "the quotient of two corresponding quantities remains constant" (Isoda, 2010a, p. 155) in the direct proportional relationships which represent "a straight line passing through the origin." (Isoda, 2010a, p. 155). This characteristic is explicitly stated as important to distinguish proportional relationships. Teachers are suggested to use "activities such as representing on a graph two quantities that vary simultaneously so that they [students] understand that if two quantities are in a proportional relationship, the graph representing this relationship is a straight line" (Isoda, 2010a, p. 155)

The task below from 6th grade textbook shows that *coordination of values* level of covariation (up to MA3) aimed to be deepened and *continuous covariation* (MA4) seemed to be triggered. In lieu of space, we do not provide detailed analysis but our account of how students might possibly reason.

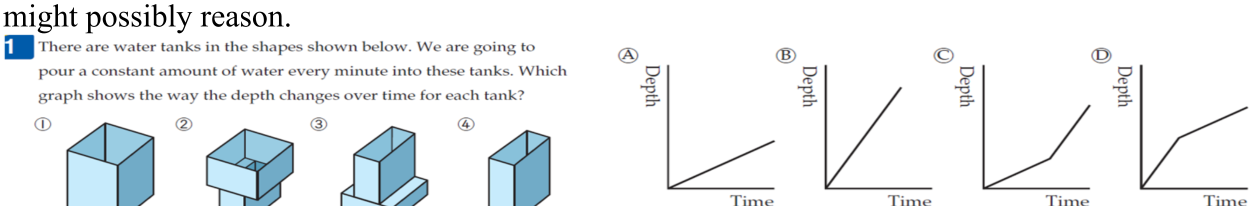


Figure 4: What Will the Graph Look Like (Fujii and Iitaka, 2012, Grade 6, p. B100)

In the task, two continuous variables, time and depth of water, are simultaneously compared through using graphical representations. Students might think that the narrower the base area of the tank the greater the amount of water poured into the tank in a minute. If a student thinks the coordination of depth with amount of time passing (M1), considers the increase in the depth of the water with regard to the time (MA2), compares the amounts of changes in depth of the water for some amount of time (MA3), and envisions the average rate of change of depth as increasing simultaneously with the time (MA4), the student might have *continuous covariation* level of covariational reasoning. However, the action of coordinating the rate of change with the uniform increments of inputs (MA4) are not examined through the numerical values of quantities. Thus, we claim that the task might be targeted to raise students' awareness about the rate of change as a scaffold for the upcoming grades rather than explicitly fostering MA4 in this task.

In *7th grade*, students are expected to further enhance their understanding of direct and inverse proportional relationships in real-life situations. Yet, differently from the 6th grade, the direct and inverse proportional relationships is planned to be re-examined through focusing on the simple forms of linear functions $y = ax$ and reciprocal functions $y = \frac{a}{x}$ respectively (Isoda, 2010b). Differently from 6th grade, students are expected to examine similarities and differences between direct and inverse proportion graphs. Moreover, the rationale of the use of proportional relationships is explicitly stated as to trigger correspondence and covariation meaning of functions in many phenomena in daily life (Isoda, 2010b).

We share an example from the MI textbook showing direct and indirect proportional relationship between two variables and how the continuous nature of their covarying relationship is represented visually (see Figure 5).

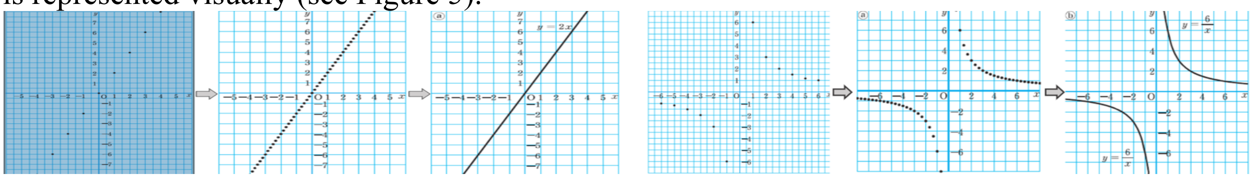


Figure 5: Graphing of y=2x (Fujii and Matano, 2012, Grade 7, p. 117-118) and graphing of $y = \frac{6}{x}$ (Fujii and Matano, 2012, Grade 7, pp. 127-129)

Different integer values of y and x are given for plotting both $y = 2x$ and $y = \frac{6}{x}$. Students are asked to explicitly think about the interval for x values and corresponding y values getting smaller and smaller producing a straight line or a curve. It seems that the continuous nature of variables is targeted through graphical representations. This way of representing might lay a foundation to smooth continuous covariation (corresponding up to MA5), although it is not explicitly introduced.

In *8th grade*, linear functions are introduced, and continuous covariation seems to be triggered and deepened. Students are expected to examine the changes and correspondences of two quantities again with a focus on graphical, algebraic, tabular, and verbal representations. In addition, the rate of change for the linear functions (i.e., in the form of $y = ax + b$) is expressed as $\frac{y_2 - y_1}{x_2 - x_1}$. Here, the rate of change which equals the constant a, is expressed as "how much y will increase when x increases by 1" (Isoda, 2010b, p. 92). All these focus on relationships between amounts of changes in variables suggested that at least MA3 is targeted. Moreover, there is an emphasis on the difference between the expressions of equations with two variables and functions. When the coordination of values of y and x is considered for an equation with two variables expressed as $ax + by + c = 0$, there is one and only one y value for every x value if $b \neq 0$. This relationship indicates that y is a function of x and the equation could be rewritten as $y = -\frac{a}{b}x - \frac{c}{b}$ to illustrate the functional relationship explicitly (Isoda, 2010b).

We present an example from the textbook (see Figure 6) in which the values of temperature are decimal numbers not increasing uniformly with uniform integer changes in time.

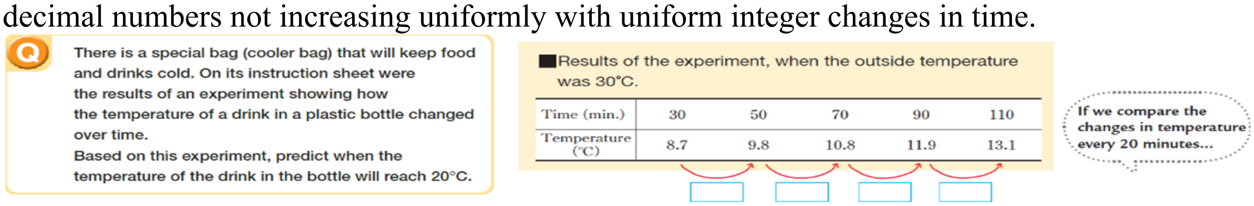


Figure 6: Temperature change of drinks task (Fujii and Matano, 2012, Grade 8, p. 70)

Students are asked to represent this situation with a graph on a coordinate system and think of the relationships between variables where temperature is a function of time. Students are expected to use tables to express the result of values of variables and placing the values of temperature for corresponding units of time on a coordinate system (MA1), verbalize the direction of change as increase or decrease (MA2), coordinate the amount of change in y with the amount of increase in x (MA3), and coordinate the average rate of change of the temperature with uniform changes in time (MA4). Thus, students are supported to reason on the *chunky continuous covariation* level. Yet, in case students consider the rate of change as an instantaneous rate of change, MA5 might also be triggered. In 8th grade, students are also asked to compare linear functions (i.e., $y = ax + b$) with proportional equations (i.e., $y = ax$). It is explicitly explained that translation of the graph of $y = ax$ on the y axis by the value of b represents $y = ax + b$, where the value of b is expressed as y-intercept. Moreover, the use of different types of representations and the relationship between them are explicitly presented.

Discussion and Conclusion

The findings showed that proportional and functional relationships are presented as intertwined with the quantitative and covariational reasoning starting from 4th grade to 8th grade in Japanese curricula. Starting from gross coordination of values, the highest level of covariational reasoning (smooth covariational reasoning) is aimed to be built in an inclusive and iterating way in Japanese curricula. Notably, attending to the developmental nature of covariational reasoning (Carlson et al., 2002) mental actions involved in each covariational reasoning level has been revisited and deepened at successive grade level. Therefore, Japanese curricula illustrate a great model for spiral curriculum, especially for developing the proportional and functional relationships throughout the elementary and lower secondary level mathematics.

The development of the concepts of proportions and linear functions in the curricula are answering the calls of research. Particularly, in Japanese curricula, the concepts of functions are developed through covariational reasoning at first, then correspondence meaning is shared in the textbook after triggering the highest covariational reasoning level iteratively. Researchers argue that covariation meaning of functions are more relevant to students' daily engagement with the topic and their use of covariational perspective can lead to the development of correspondence view of functions (Thompson, 1994b; Carlson et al., 2002; Oehrtman et al., 2008; Smith, 2003; Lloyd et al., 2010). Moreover, as suggested in literature, functions are examined through different representations and dynamic situations (e.g., Thomson & Carlson, 2017); the task variables get more complex gradually (e.g., Heinz, 2000); the difference between functions and equations are presented (e.g., Chazan & Yerushalmy, 2003); the relationship between proportional relationships and linear functions are explained (e.g., Lloyd et al., 2011); the invariant relationship between variables are introduced as rate of change in dynamic situations (e.g., Carlson, 2002); real life situations are used to study proportional relationships and functions (e.g., Carlson et al., 2002; Oehrtman et al., 2008); quantities and quantitative operations are explicitly introduced (e.g., Thompson, 1994) in Japanese curricula. That is, Japanese curricula are designed in a way to bridge the mathematics education literature with teachers' teaching. Lastly, the findings of the study pointed out that Japanese curricula attentively focus on not only conceptual understanding but also mathematical thinking in a developmental way (e.g., successive development of mental actions). Hence, we suggest curriculum developers pay attention to the spiral nature of Japanese curricula. We argue that Japanese curricula can be used by teacher educators to study and improve teachers' covariational reasoning. Our analysis also supports Thompson and Carlson (2017)'s argument that Japanese curricula exemplifies how quantitative and covariational reasoning can be integrated in standards and textbooks.

Acknowledgments

We acknowledge Prof. Tad Watanabe's guidance and support with respect to obtaining permission to use MI textbook material and for his invaluable feedback to the larger project (i.e., MS thesis study) which forms the basis for this paper.

References

Blanton, M., Levi, L., Crites, T., Dougherty, B., & Zbiek, R. M. (2011). *Developing Essential Understanding of Algebraic Thinking for Teaching Mathematics in Grades 3-5. Series in Essential Understandings*. National Council of Teachers of Mathematics. 1906 Association Drive, Reston, VA 20191-1502.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for research in mathematics education*, *33*(5), 352-378. https://doi.org/10.2307/4149958

Charalambous, C. Y., Delaney, S., Hsu, H. Y., & Mesa, V. (2010). A comparative analysis of the addition and subtraction of fractions in textbooks from three countries. *Mathematical thinking and learning*, *12*(2), 117-151. https://doi.org/10.1080/10986060903460070

Chazan, D., & Yerushalmy, M. (2003). On Appreciating the Cognitive Complexity of School Algebra: Research on Algebra Learning and Directions of Curricular Change. In J. Kilpatrick, D. Schifter, & G. Martin (Eds.), *A Research Companion to the Principles and Standards for School Mathematics* (pp. 123-135). NCTM.

Fujii, T., and Iitaka, S. (2012). *Mathematics International* (grade 1–grade 6). Tokyo Shoseki Co. Ltd., Tokyo.

Fujii, T., and Matano, H. (2012). *Mathematics International* (grade 7–grade 8). Tokyo Shoseki Co. Ltd., Tokyo.

Heinz, K. R. (2000). *Conceptions of ratio in a class of preservice and practicing teachers*. The Pennsylvania State University.

Isoda, M. (2010a). Elementary school teaching guide for the Japanese course of study: Mathematics (Grade 1–6). CRICED: University of Tsukuba, Tsukuba.

Isoda, M. (2010b). Junior high school teaching guide for the Japanese course of study: Mathematics (Grade 7–9). CRICED: University of Tsukuba, Tsukuba.

Kaput, J. J. (1994). Democratizing access to calculus: New routes to old roots. In A. H. Schoenfeld (Ed.), Mathematical thinking and problem solving (pp. 77–156). Hillsdale, NJ: Erlbaum.

Karagöz Akar, G., Watanabe, T., & Turan, N. (2022). Quantitative Reasoning as a Framework to Analyze Mathematics Textbooks. In Karagöz Akar, G., Zembat, İ.Ö., Arslan, S., Thompson, P.W. (Eds.), *Quantitative Reasoning in Mathematics and Science Education. Mathematics Education in the Digital Era*, (vol 21., pp. 107-132). Springer, Cham. https://doi.org/10.1007/978-3-031-14553-7_5

Li, Y. (2000). A comparison of problems that follow selected content presentations in American and Chinese mathematics textbooks. *Journal for Research in Mathematics Education*, *31*(2), 234-241. https://doi.org/10.2307/749754

Lloyd, G., Beckmann, S., Zbiek, R. M., & Cooney, T. (2010). *Developing Essential Understanding of Functions for Teaching Mathematics in Grades 9-12*. National Council of Teachers of Mathematics. 1906 Association Drive, Reston, VA 20191-1502.

Oehrtman, M., Carlson, M., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. *Making the connection: Research and teaching in undergraduate mathematics education*, (pp. 27-42). https://doi.org/10.5948/UPO9780883859759.004

Sağlam, R., & Alacacı, C. (2012). A comparative analysis of quadratics unit in Singaporean, Turkish and IBDP mathematics textbooks. *Turkish Journal of Computer and Mathematics Education (TURCOMAT)*, *3*(3), 131-147. https://doi.org/10.17762/turcomat.v3i3.43

Smith, E. (2003). Stasis and change: Integrating patterns, functions, and algebra throughout the K-12 curriculum. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to Principles and Standards for School Mathematics,* (pp. 136-150). Reston, VA: National Council of Teachers of Mathematics.

Son, J. W., & Diletti, J. (2017). What can we learn from textbook Analysis?. *What matters? Research trends in international comparative studies in mathematics education*, 3-32. Springer, Cham.

Son, J. W., & Kim, O. K. (2015). Teachers' selection and enactment of mathematical problems from textbooks. *Mathematics Education Research Journal*, *27*, 491-518. https://doi.org/10.1007/s13394-015-0148-9
Son, J. W., & Senk, S. L. (2010). How reform curricula in the USA and Korea present multiplication and division of fractions. *Educational Studies in Mathematics*, *74*, 117-142. https://doi.org/10.1007/s10649-010-9229-6
Takahashi, A., Watanabe, T., & Yoshida, M. (2008). English translation of the Japanese mathematics curricula in the course of study. *Global Education Resources, LLC*.
Taşova, H., Stevens, I. E., & Moore, K. C. (2018). A framework for analyzing written curriculum from a shape-thinking and (co) variational reasoning perspective. In *Proceedings of the Twenty-First Annual Conference on Research in Undergraduate Mathematics Education* (pp. 1527-1533).
Thompson, P. W. (1990). *A theoretical model of quantity-based reasoning in arithmetic and algebra* (Report No. MDR 87-51381 and MDR 89-50311). Center for Research in Mathematics & Science Education: San Diego State University.
Thompson, P. W. (1994a). The development of the concept of speed and its relationship to concepts of rate. In G. Vergnaud, G. Harel, and J. Coufrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics*, (pp. 179-234). SUNY Press. http://pat-thompson.net/PDFversions/1994ConceptSpeedRate.pdf
Thompson, P. W. (1994b). Students, functions, and the undergraduate curriculum. In E. Dubinsky, A. H. Schoenfeld, & J. J. Kaput (Eds.), Research in Collegiate Mathematics Education, 1 (Issues in Mathematics Education Vol. 4, pp. 21-44). Providence, RI: American Mathematical Society.
Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. *Compendium for research in mathematics education*, 421-456. National Council of Teachers of Mathematics. https://www.researchgate.net/profile/Patrick-Thompson-2/publication/302581485_Variation_covariation_and_functions_Foundational_ways_of_thinking_mathematically/links/583c389208ae502a85e3a224/Variation-covariation-and-functions-Foundational-ways-of-thinking-mathematically.pdf
Usiskin, Z. (2013). Studying textbooks in an information age—a United States perspective. *ZDM*, *45*, 713-723. https://doi.org/10.1007/s11858-013-0514-6
Valverde, G. A., Bianchi, L. J., Wolfe, R. G., Schmidt, W. H., & Houang, R. T. (2002). *According to the book: Using TIMSS to investigate the translation of policy into practice through the world of textbooks*. Springer Science & Business Media.
Wang, Y., Barmby, P., & Bolden, D. (2017). Understanding linear function: a comparison of selected textbooks from England and Shanghai. *International Journal of Science and Mathematics Education*, *15*, 131-153. https://doi.org/10.1007/s10763-015-9674-x
Watanabe, T., Lo, J. J., & Son, J. W. (2017). Intended treatment of fractions and fraction operations in mathematics curricula from Japan, Korea, and Taiwan. *What matters? Research trends in international comparative studies in mathematics education*, (pp. 33-61). Springer.
Weber, R. P. (1990). *Basic content analysis* (Vol. 49). Sage.

AN INTERNATIONAL COMPARISON OF PERFORMANCE ON TIMSS ELEMENTARY MATHEMATICS ITEMS WITH POTENTIAL FOR COVARIATIONAL REASONING

Julien Corven	Teo Paoletti	Allison L. Gantt
Illinois State University	University of Delaware	University of Delaware
jccorve@ilstu.edu	teop@udel.edu	agantt@udel.edu

We previously (Gantt et al., 2023; Paoletti et al., 2021) identified items from the publicly released TIMSS 2011 assessments that had potential for students to employ covariational reasoning as a solution strategy. In this report, we explore the extent to which fourth-grade students' performance on such items in mathematics differed among 26 nations. Using multi-level modeling, we conclude that, in general, fourth-grade students were less successful on mathematics items for which covariational reasoning was a viable strategy than on items for which we could not identify a possible covariational reasoning strategy. However, three countries (Finland, Sweden, and the Netherlands) did not follow this pattern.

Keywords: Assessment, Elementary School Education, Problem Solving

Coordinating two dynamically changing quantities, or reasoning covariationally (Carlson et al., 2002), is a critical skill for students to develop across mathematics and science (e.g., Gantt et al., 2023; Panorkou & Germia, 2021; Paoletti et al., 2022; Sokolowski, 2020). Covariational reasoning can be closely tied to students' construction of mathematical representations (e.g., Confrey & Smith, 1995; Moore et al., 2013; Stevens, 2018; Wilkie, 2020) and to their making sense of particular types of quantitative relationships (e.g., Ellis et al., 2015; Johnson, 2015; Thompson & Thompson, 1996). Despite the growing body of research emphasizing the importance of covariational reasoning across mathematics and science domains (see Gantt et al., 2023 for a synthesis), the majority of the research exploring K-12 students' covariational reasoning has focused on small scale qualitative studies. In this report, we address the need to explore students' covariational reasoning quantitatively using a large data set from the Trends in International Mathematics and Science Study (TIMSS).

Since 1995, the TIMSS assessment has been regularly used to measure students' achievement in mathematics and science in 90 countries (NCES, n.d.). TIMSS data has been used for international comparisons of student achievement (e.g., Archibold, 1999; Chudgar et al., 2013; Mejía-Rodríguez et al., 2021; Wang et al., 2012) and has formed the basis for government reports and decision-making about STEM education in countries around the world (e.g., NCES, 2021; Richardson et al., 2020; Thomson et al., 2020).

The TIMSS assessment is a particularly useful resource to explore students' covariational reasoning as there are indications that such reasoning may be supported differently in various countries. For example, Thompson et al. (2017) explored a large sample of US and Korean teachers' meanings related to constructing graphs. They found large differences between the two countries in terms of teachers' tendencies to exhibit covariational reasoning on tasks that could elicit such reasoning. Thompson and Carlson (2017) also described differences in Japanese and American textbooks' approaches to support students' covariational reasoning (Thompson & Carlson, 2017). Hence, by using the TIMSS assessment, we address the need to explore between-country differences in students' covariational reasoning. The research question guiding this study is: *To what extent do differences exist within and between countries in fourth-grade students' performance on TIMSS mathematics items depending on the item's potential to elicit a covariational reasoning strategy?*

Covariational Reasoning in the TIMSS Assessment

In this report, we build on and extend our prior work (Gantt et al., 2023; Paoletti et al., 2021) where we conducted a content analysis of publicly released TIMSS items from the Grade 4 Mathematics, Grade 8 Mathematics, Grade 4 Science, and Grade 8 Science assessments based on their potential to elicit students' covariational reasoning. We defined an item as having the potential to elicit covariational reasoning (PCR) if we could "1) identify a way a student might conceive two changing quantities and 2) determine some solution strategy that could reasonably entail covariational reasoning" (Gantt et al., 2023, p. 6). For example, one multiple choice Grade 4 Mathematics item prompted: "The scale on a map indicates that 1 centimeter on the map represents 4 kilometers on the land. The distance between two towns on the map is 8 centimeters. How many kilometers apart are the two towns?" (IEA, 2013, p. 12). We classified this item as a PCR item because a student could coordinate one-centimeter changes in the map with four-kilometer changes on land to determine that eight such centimeter changes would result in 32 kilometers on land. This solution entails reasoning covariationally. However, a correct response to this item does not guarantee that a student employed covariational reasoning; for example, a student could have used a memorized procedure or randomly guessed the correct answer.

In Gantt et al. (2023), we categorized approximately one-third of all publicly released items as PCR across all four assessments. Particular to this paper, we coded 27 out of the 73 (37%) publicly released Grade 4 Mathematics items as PCR. We further organized the TIMSS items into content strands across the Grade 4 and Grade 8 Mathematics assessments. Table 1 presents the number of PCR items within each Grade 4 content strand. We also reported the number of PCR items by TIMSS-identified cognitive domain (Knowing, Applying, Reasoning; Mullis et al., 2009). In Grade 4 Mathematics, we identified 6 PCR Knowing items (out of 28, 21%), 10 PCR Applying items (out of 29, 35%), and 11 PCR Reasoning items (out of 15, 73%).

Table 1: Grade 4 Mathematics PCR Items as Percentage of Total Items by Content Strand

3. Content strand	4. Total items	5. PCR items	6. % (PCR/ Total)
7. Number (Whole numbers; fractions & decimals)	8. 29	9. 14	10. 48%
11. Algebra (Patterns & relationships; number sentences)	12. 11	13. 6	14. 55%
15. Geometry (2- & 3-dimensional shapes; points, lines & angles)	16. 24	17. 1	18. 4%
19. Statistics (Reading & interpreting; organizing & representing)	20. 9	21. 6	22. 67%

Methods

Our data corpus consisted of the TIMSS 2011 publicly available fourth-grade mathematics student performance data (TIMSS 2011 International Database, 2022). We restricted our investigation to data from 26 member nations of the Organization for Economic Cooperation and Development (OECD). Each code for each student's response to each item was converted to a binary score (correct or incorrect). Because we had only coded the potential for covariational reasoning (PCR) for publicly released TIMSS assessment items (Gantt et al., 2023), we removed performance data for all other (not released) items from the dataset. Students who completed forms of the TIMSS assessment that included no publicly released items were excluded. We considered the students who remained in the data set to be randomly selected from among all students who took the assessment, since the assignment of forms to students was randomized within each classroom. Due to the TIMSS sampling methodology and the distribution of publicly released fourth-grade mathematics items within forms of the assessment, students included in the data set contained performance data for at least half of the items presented to them.

We created multi-level (items nested within students) logistic regression models separately for each country. The unconditional (intercept-only) model gave the log-odds of an average student getting an average item correct. The conditional model incorporated the binary predictor of PCR for each item. In this second model, the intercept represented the log-odds of an average student getting an average non-PCR item correct, and the sum of the intercept and the coefficient for PCR represented the log-odds of an average student getting an average PCR-coded item correct. We converted all log-odds to probabilities in our results for ease of interpretation. Although a correct answer to a PCR item does not imply that a student employed covariational reasoning, we interpret a smaller probability within the PCR category as possibly indicating that an average student from that country was less likely to reason covariationally for such a problem.

Finally, for each country, we conducted an ANOVA test to compare the conditional model to the unconditional model to see for which countries the conditional model was a statistically significant improvement over the unconditional model. A statistically significant ANOVA test indicated that the conditional model significantly reduced the variance of the residuals from the data over the unconditional model, meaning that adding the PCR predictor significantly improved the model's predictive capabilities.

Results

Table 2 contains conditional model results (as probabilities) for 26 OECD member nations.

Table 2: Probability Results of Conditional Models for 26 OECD Member Nations

23. ountry	24. on-PCR	25. CR	26. iff.	27. ountry	28. on-PCR	29. CR	30. iff.
31. ustralia	32. 575	33. 525	34. .050*	35. orea	36. 794	37. 773	38. .021*

39. ustria	40. 574	41. 493	42. .081*	43. etherlands	44. 613	45. 642	46. 029*
47. elgium	48. 660	49. 606	50. .054*	51. ew Zealand	52. 493	53. 469	54. .024*
55. hile	56. 488	57. 421	58. .067*	59. orthern Ireland	60. 734	61. 679	62. .055*
63. zech Republic	64. 575	65. 540	66. .035*	67. orway	68. 529	69. 466	70. .063*
71. enmark	72. 617	73. 593	74. .024*	75. oland	76. 522	77. 471	78. .051*
79. ngland	80. 667	81. 614	82. .053*	83. ortugal	84. 664	85. 579	86. .085*
87. inland	88. 628	89. 624	90. .004	91. lovak Republic	92. 563	93. 485	94. .078*
95. ermany	96. 614	97. 572	98. .042*	99. lovenia	100. 581	101 480	102. .101*
103. ungary	104. 626	105 502	106 .124*	107. pain	108. 523	109 446	110. .077*
111. reland	112. 611	113 564	114 .047*	115. weden	116. 528	117 520	118. .008
119. taly	120. 580	121 474	122 .106*	123. urkey	124. 494	125 401	126. .093*

127. apa n	128 7 3 6	129 7 0 5	130 .0 3 1 *	131. nited State s	132. 6 7 2	133 5 9 2	134. .0 80 *

135. *Note:* * represents a statistically significant difference at $p < .001$. Lack of a * indicates the difference was not statistically significant at $p < .05$.

As Table 2 shows, for all the countries in our sample except Finland, the Netherlands, and Sweden, the probability of an average student getting a non-PCR item correct exceeds the probability of an average student getting a PCR item correct. For nearly all countries in the sample, the conditional model is an improvement over the unconditional model and the coefficient for PCR in the model is statistically significant ($p < .001$) and negative. The first result shows that analyzing performance data with the PCR status of an item better predicts student performance than analyzing the data without this distinction, whereas the second result demonstrates that students performed significantly better on non-PCR than PCR items.

As noted above, three countries do not follow this pattern. For Finland and Sweden, the PCR coefficients in the conditional models are not statistically significant, meaning there was no significant difference in average fourth-grade student performance on non-PCR and PCR items on the mathematics assessment. Unsurprisingly, the conditional model is also not a significant improvement over the unconditional model for Finland ($p = .377$) and Sweden ($p = .106$). Finally, for the Netherlands, the conditional model is a significant improvement over the unconditional model ($p < .001$), but the sign of the PCR coefficient is positive instead of negative. The positive coefficient indicates that the average fourth-grade student in the Netherlands was more likely to answer a PCR mathematics item correctly than a non-PCR item.

Discussion

Our results suggest that, in most of the countries for which we analyzed performance data, the average student had a lower probability of getting an average PCR item correct compared to a non-PCR item. Although prior research indicates that emphases on covariational reasoning in teaching and curricula vary from country to country (e.g., Japan and Korea compared to the US; Thompson & Carlson, 2017; Thompson et al., 2017), we found similar differences in how students in these three countries performed on PCR and non-PCR items. In other words, although the average student in Japan and Korea outperformed the average student in the US on both PCR and non-PCR items from the fourth-grade mathematics TIMSS assessment, there are significant gaps in performance between these item groups in all three countries. We notice, however, that the gaps are smaller in Japan and Korea than in the US, and we intend to conduct follow up analyses to determine whether the difference in gaps is statistically significant.

We note that Finland, Sweden, and the Netherlands, the three countries for which we found either no difference in performance on PCR and non-PCR items or better performance on PCR items than non-PCR items, are geographically close to each other. We hypothesize that differences in written curricula, teacher knowledge, or instructional practices might explain our findings. However, an important limitation of this investigation is that the quantitative analyses we conducted cannot explain differences, only document them. Further qualitative research will be needed to contextualize our findings and possibly explain why these differences occurred.

We emphasize that the analyses we present here are preliminary, and there are other possible explanations for our findings that are not yet addressed. For example, it may be that differences in performance in different content areas or in different cognitive domains are confounded with differences in performance on PCR items. We intend to conduct follow-up analyses using just the data from items in the Number and Algebra strands and incorporating cognitive domain as a separate predictor to address this question. We also will extend these analyses to Grade 4 Science and Grade 8 Mathematics and Science items to see if the differences we found persist. Finally, we focus on within-country differences in this report and only began to explore relative performance differences between countries by noting that some countries do not follow the expected pattern. Future analyses will extend and better contextualize our findings. In doing so, we hope to determine where students are succeeding in learning to reason covariationally and identify best practices from qualitative analyses of curricula and teaching.

References

Archibold, R. C. (1999). 10 states plan new math curriculum and exams. *New York Times*. Retrieved February 8, 2023 from https://archive.nytimes.com/www.nytimes.com/library/national/050699math-exams-edu.html

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, *33*(5), 352–378. https://doi.org/10.2307/4149958

Chudgar, A., Luschei, T. F., & Zhou, Y. (2013). Science and mathematics achievement and the importance of classroom composition: Multicountry analysis using TIMSS 2007. *American Journal of Education*, *119*(2), 295–316. https://doi.org/10.1086/668764

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education*, *26*(1), 66–86. https://doi.org/10.2307/749228

Ellis, A. B., Özgür, Z., Kulow, T., Williams, C. C., & Amidon, J. (2015). Quantifying exponential growth: Three conceptual shifts in coordinating multiplicative and additive growth. *The Journal of Mathematical Behavior*, *39*, 135–155. https://doi.org/10.1016/j.jmathb.2015.06.004

Gantt, A. L., Paoletti, T., & Corven, J. (2023). Exploring the prevalence of covariational reasoning across mathematics and science using TIMSS 2011 assessment items. *International Journal of Science and Mathematics Education*. https://doi.org/10.1007/s10763-023-10353-2

International Association for the Evaluation of Educational Achievement [IEA]. (2013). *TIMSS 2011 grade 4 released mathematics items*. TIMSS & PIRLS International Study Center, Lynch School of Education, Boston College, Chestnut Hill, MA, and International Association for the Evaluation of Educational Achievement (IEA), IEA Secretariat, Amsterdam, the Netherlands. https://nces.ed.gov/timss/pdf/TIMSS2011_G4_Math.pdf

Johnson, H. (2015). Secondary students' quantification of ratio and rate: A framework for reasoning about change in covarying quantities. *Mathematical Thinking and Learning*, *17*(1), 64–90. https://doi.org/10.1080/10986065.2015.981946

Mejía-Rodríguez, A. M., Luyten, H., & Meelissen, M. R. M. (2021). Gender differences in mathematics self-concept across the world: An exploration of student and parent data of TIMSS 2015. *International Journal of Science and Mathematics Education, 19*(6), 1229–1250. https://doi.org/10.1007/s10763-020-10100-x

Moore, K, C., Paoletti, T., & Musgrave, S. (2013). Covariational reasoning and invariance among coordinate systems. *Journal of Mathematical Behavior*, *32*(3), 461–473. https://doi.org/10.1016/j.jmathb.2013.05.002

Mullis, I. V. S., Martin, M. O., Ruddock, G. J., O'Sullivan, C. Y., & Preuschoff, C. (2009). *TIMSS 2011 assessment frameworks*. TIMSS & PIRLS International Study Center Lynch School of Education, Boston College. https://timssandpirls.bc.edu/timss2011/downloads/TIMSS2011_Frameworks.pdf

National Center for Education Statistics [NCES]. (n.d.). *Participating countries: Trends in International Mathematics and Science Study (TIMSS)*. Retrieved February 20, 2023, from https://nces.ed.gov/timss/participation.asp

National Center for Education Statistics [NCES]. (2021). *TIMSS 2019 U.S. highlights web report*. U.S. Department of Education, Institute of Education Sciences. https://nces.ed.gov/timss/results19/index.asp

Panorkou, N., & Germia, E. F. (2021). Integrating math and science content through covariational reasoning: The case of gravity. *Mathematical Thinking and Learning*, *23*(4), 318–343. https://doi.org/10.1080/10986065.2020.1814977

Paoletti, T., Gantt, A., & Corven, J. (2021). Items with potential for covariational reasoning across mathematics and science TIMSS assessments. In D. Olanoff, K. Johnson, & S. Spitzer (Eds.), *Proceedings of the forty-third*

annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (pp. 88–93). Philadelphia, PA.
Paoletti, T., Lee, H. Y., Rahman, Z., Vishnubhotla, M., & Basu, D. (2022). Comparing graphical representations in mathematics, science, and engineering textbooks and practitioner journals. *International Journal of Mathematical Education in Science and Technology*, *53*(7), 1815–1834. https://doi.org/10.1080/0020739X.2020.1847336
Richardson, M., Isaacs, T., Barnes, I., Swensson, C., Wilkinson, D., & Golding, J. (2020). *Trends in International Mathematics and Science Study (TIMSS) 2019: National report for England*. UCL Institute of Education. https://assets.publishing.service.gov.uk/government/uploads/system/uploads/attachment_data/file/941351/TIMSS_2019_National_Report.pdf
Sokolowski, A. (2020). Developing covariational reasoning among students using contexts of formulas: (Are current formula notations in physics aiding graph sketching?). *The Physics Educator*, *2*(4), 2050016. https://doi.org/10.1142/S266133952050016X
Stevens, I. E. (2018). The parallelogram problem: Supporting covariational reasoning in the construction of formulas. In T. E. Hodges, G. J. Roy, & A. M. Tyminski (Eds.), *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 624–627). University of South Carolina & Clemson University.
Thompson, A. G., & Thompson, P. W. (1996). Talking about rates conceptually, Part II: Mathematical knowledge for teaching. *Journal for Research in Mathematics Education*, *27*(1), 2–24. https://doi.org/10.2307/749194
Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421–456). National Council of Teachers of Mathematics.
Thompson, P. W., Hatfield, N. J., Yoon, H., Joshua, S., & Byerley, C. (2017). Covariational reasoning among US and South Korean secondary mathematics teachers. *Journal of Mathematical Behavior*, *48*, 95–111. https://doi.org/10.1016/j.jmathb.2017.08.001
Thomson, S., Wernert, N., Rodrigues, S., & O'Grady, E. (2020). *TIMSS 2019 Australia highlights*. Australian Council for Educational Research. https://research.acer.edu.au/cgi/viewcontent.cgi?article=1000&context=timss_2019
TIMSS 2011 International Database. (2022). Retrieved February 18, 2023 from https://timssandpirls.bc.edu/timss2011/international-database.html
Wang, Z., Osterlind, S. J., & Bergin, D. A. (2012). Building mathematics achievement models in four countries using TIMSS 2003. *International Journal of Science and Mathematics Education, 10*(5), 1215–1242. https://doi.org/10.1007/s10763-011-9328-6
Wilkie, K. J. (2020). Investigating students' attention to covariation features of their constructed graphs in a figural pattern generalisation context. *International Journal of Science and Mathematics Education, 18*(2), 315–336. https://doi.org/10.1007/s10763-019-09955-6

A TRIVIUM CURRICULUM FOR STUDENTS' MATHEMATICS COMPETENCY WITH REGARD TO INDIGENOUS MATHEMATICS KNOWLEDGE

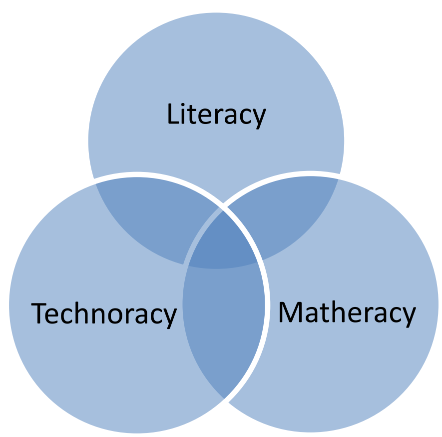
Emmanuel Chukwuebuka Umeh
University of Missouri
ecung6@missouri.edu

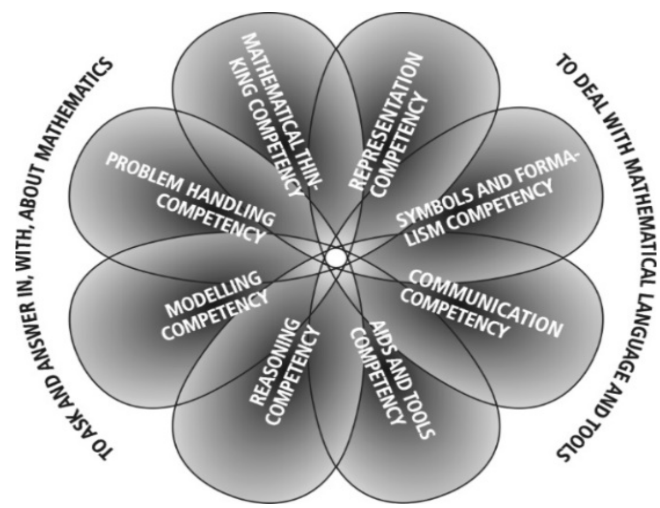
Samuel Otten
University of Missouri
ottensa@missouri.edu

Keywords: Trivium Curriculum, Mathematics Competency, Indigenous Mathematics Knowledge

Drawing on indigenous knowledge that can foster critical thinking and help students become active participants in a democratic society, the Trivium Curriculum (D'Ambrosio, 1999) entails three main components: literacy (communicative instruments), matheracy (analytical instruments), and technoracy (material and technological instruments) aim for the transformation of mathematics into a living subject, lying within a broader perspective of learning as a product of the interaction and integration among teachers, students, mathematical content, and environmental context (Rosa, 2010). Through learning activities involving literacy, matheracy, and technoracy, the trivium curriculum can incorporate the students' values, cultural knowledge, and perspectives (Niss & Højgaard, 2011). For instance, the trivium framework can be applied to teaching mathematics while incorporating conventional stories, customs, and viewpoints related to indigenous knowledge. It can be made more relevant and approachable by connecting it to students' cultural experiences.

This poster presentation juxtaposes literacy, matheracy, and technoracy (Rosa & Orey, 2010) with a mathematics competency framework (Niss & Højgaard, 2011). The poster will contain examples of activities that are rooted in the Trivium and also develop specific competencies such as communication competency (literacy), problem handling competency (matheracy), and modeling competency (literacy, matheracy, and technoracy).





136.

Figure 1: Connections between trivium curriculum and mathematics competency, adapted from Niss & Højgaard (2011) and Rosa & Orey (2015).

This work blends broad categories of indigenous knowledge with specific forms of mathematics competency with the goal of encouraging students to examine mathematical activities in their sociocultural contexts (Rosa & Orey, 2007) and offer respect to the diverse cultural backgrounds of students (Bassanezi, 2000). We wish to bring more attention to the Trivium Curriculum's potential in developing a deep understanding of mathematical concepts and operations and applying indigenous mathematical knowledge and reasoning to solve problems.

Acknowledgments

Our sincere gratitude goes to the faculty members in the University of Missouri Columbia mathematics education program. It has been an honor to learn from their experience and expertise.

References

Bassanezi, R. C. (2000). *Ensino-aprendizagem com modelagem matemática*. Editora Contexto.

D'Ambrosio, U. (1999). Literacy, matheracy, and technoracy: A trivium for today. *For Learning Mathematics*, *1*(2), 131–153.

Ladson-Billing, G. (1995). Toward a theory of culturally relevant pedagogy. *American Educational Research Journal*, *32*(3), 465–491.

Niss, M., & Højgaard, T. (2011). *Competencies and mathematical learning: Ideas and inspiration for the development of mathematics teaching and learning in Denmark*. Roskilde University.

Rosa, M., & Orey, D. C. (2007). Cultural assertions and challenges towards pedagogical action of an ethnomathematics program. *For Learning Mathematics*, *27*(1), 10–16.

Rosa, M., & Orey, D. C. (2015). A trivium curriculum for mathematics based on literacy, matheracy, and technoracy: An ethnomathematics perspective. *ZDM*, *47*(4), 587–598.

CLASSIFYING CURRICULAR REASONING:
WAYS FOR CAPTURING TEACHERS' CURRICULAR DECISIONS

Shannon Dingman
University of Arkansas
sdingman@uark.edu

Dawn Teuscher
Brigham Young University
dawn.teuscher@byu.edu

Travis Olson
University of Nevada Las Vegas
travis.olson@unlv.edu

Amy Roth-McDuffie
Washington State University
mcduffie@wsu.edu

Mathematics teachers make numerous decisions that form lessons that in turn greatly influence what students learn. In making these decisions, teachers rely on their curricular reasoning (CR) to decide on what mathematics to teach, how to structure their lesson, and what problems or tasks to use to achieve their lesson goals. However, teachers differ with respect to the sophistication of their CR and the diversity of CR aspects used in their reasoning. In this paper, we detail two ways to classify teachers' CR: a leveled approach to capture the increasing sophistication of teachers' CR, and a heat map approach that highlights the extent to which teacher use various CR aspects in their planning. These methods provide stakeholders avenues by which CR can be studied and that teachers' CR abilities can be further developed.

Keywords: Curriculum; Instructional Activities and Practices; Instructional Vision

Mathematics teachers make innumerable decisions that shape their lessons and impact students' opportunity to learn. Past research has often focused on teachers, students, and the mathematical content as key classroom elements that drive classroom interactions (Cohen & Ball, 1999; Cohen et al., 2003), yet research has illustrated the essential role curriculum plays in influencing instruction (Stein et al., 2007; Rezat, 2006). Over the past 25 years, mathematics education researchers have extensively studied mathematics curriculum and how teachers use it, including how curriculum can be educative for both teachers and students. Lloyd et al. (2017) defines curriculum as the "written curriculum materials and textbooks...and the resources with which students and teachers work most closely in the mathematics classroom" (p. 824). We extend the definition of curriculum to include any materials (e.g., written or digital resources) that teachers use to plan and enact lessons that support students' mathematics learning.

Following current trends in the field, we have come to view teachers as designers who engage in a participatory relationship as they work with curriculum (Brown, 2009; Remillard, 2005). From this perspective, Roth McDuffie and Mather (2009) expanded the work of Shulman (1986) on curricular knowledge by exploring the cognitive work teachers engage in while working with curriculum. This cognitive work, termed as curricular reasoning (CR), encompasses the "thinking processes that teachers engage in as they work with curriculum materials to plan, implement, and reflect on instruction" (Breyfogle et al., 2010, p. 308). CR is heavily informed by teachers' knowledge, background, and teaching experience, and is used by teachers as they operationalize curriculum and enact decisions with different types of curricula. Building on Roth McDuffie and Mather's (2009) research, Dingman et al. (2021) identified five

CR aspects that teachers reason with as they use curriculum to plan and enact mathematics lessons. The five empirically identified CR aspects identified in our research are: 1) *Viewing Mathematics from the Learner's Perspective* (teachers' reasoning about the assessment and anticipation of student thinking and reasoning about the purpose of the task, given their prior knowledge about students' backgrounds and experience); 2) *Mapping Learning Trajectories* (teachers' reasoning about connections across content in the unit, across units, or across grades); 3) *Considering Mathematical Meanings* (teachers' reasoning about the mathematics for themselves or for the student); 4) *Analyzing Curriculum Materials* (teachers' reasoning about the curriculum, identifying strengths and limitations); and 5) *Revising Curriculum Materials* (teachers' reasoning about past teaching experiences to make changes to the task or the curriculum). Teachers vary in terms of how many CR aspects they reason with while planning and enacting lessons. Qualitative findings indicate that these differences in teachers' reasoning influence students' learning of mathematics (Dingman et al., 2021).

Much research on teachers' decisions is based on the Instructional Triangle introduced by Cohen et al. (2003), which is used to study interactions among teachers, students, and the content under study. However, some researchers (Rezat, 2006; Rezat & Sträber, 2012; Tall, 1986) suggest that this focus is too narrow, neglecting other resources that inform classroom interactions such as technology and curricula. Our findings demonstrate that curriculum is a critical element of teachers' practice, and that CR is inherent to teachers' work as they plan and enact lessons (Choppin et al., 2018, 2020, 2021; Dingman et al., 2021; Roth McDuffie et al., 2018). Given the diverse approaches to mathematics provided by various curricula, these findings make sense. Our findings also suggests that teachers' CR influences their decisions and students' learning opportunities. Further investigations should consider curriculum as well as the teachers, students, and the content under study.

To that point, Dingman et al. (2021) propose the Instructional Pyramid displayed in Figure 1 that expands the Instructional Triangle (Cohen et al., 2003) in order to represent and capture the myriad interactions that occur in the instructional environment. Figure 1 illustrates the interplay among these four key classroom elements and highlights the aspects of CR teachers employ as they reason with these elements. Teachers' curricular decisions are based upon the classroom *elements* (vertices) and the CR *aspects* (edges and faces) found in the Instructional Pyramid.

How and why teachers make these curricular decisions is important to understand, given the potentially limited role of written textbooks in determining what is taught, in favor of open-source and teacher-developed activities aligned (or purported to align) to standards (Banilower et al., 2013). Teachers' decisions regarding the use of these various resources holds considerable influence over students' opportunity to learn mathematics (Stein et al., 2007).

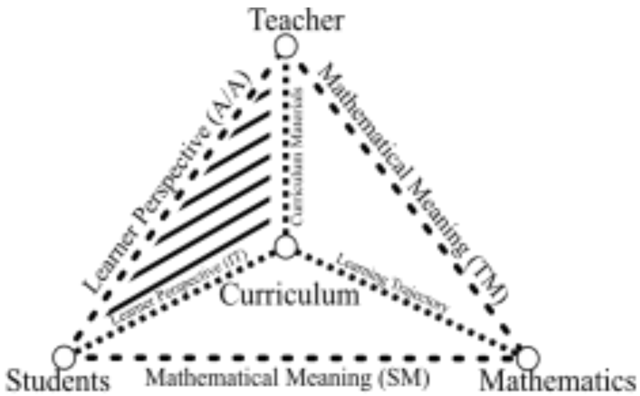


Figure 1: Instructional Pyramid for Curricular Reasoning

The importance of viewing teachers' decisions and reasoning through the Instructional Pyramid model for CR is based on data from our project that suggests that oftentimes teachers reason on more than one edge of the model. Investigating teachers' CR can elucidate why teachers make specific mathematical decisions, such as skipping or reordering lessons, revising tasks, or modifying definitions. While some decisions may appear to be inappropriate for a given situation, these decisions are likely multi-faceted and nuanced in ways that elude clear notions of correct and incorrect, or right and wrong. By determining how and why teachers come to these decisions and identifying the aspects of CR teachers reason with most frequently, professional developers, teacher leaders, and mathematics teacher educators can build teachers' capacity to reason differently and more robustly about these decisions.

Our working hypothesis is that teachers who coordinate multiple CR aspects in their decision-making provide different learning opportunities for students than those who reason with only one or two CR aspects. In the preliminary analysis of data from teachers within our project in relation to the Instructional Pyramid, it is apparent that some teachers tend to reason with certain CR aspects more than others. In fact, many teachers reasoned with *Viewing Mathematics from the Learner's Perspective* (Anticipating/Assessing) and *Considering Mathematical Meanings* (Teacher and Student Mathematics) yet used the other CR aspects less often.

However, some teachers reasoned with multiple CR aspects as they made individual decisions. This suggests that some teachers may need support to recognize ways in which they are reasoning, to understand what they are not attentive to, and to develop productive ways to learn how to reason with different CR aspects so as to create different learning opportunities for their students.

To this point, we propose two potential ways to characterize teachers' CR in terms of its sophistication and its diversity. In this paper, we detail these approaches and provide data from our work with middle grades mathematics teachers to illustrate ways to capture differences in teachers' CR as well as highlight CR aspects that are most/least used. Data discussed in this paper derive from our research question under investigation: What CR aspects do middle school teachers reason with as they plan and enact mathematical lessons? See Dingman et al. (2021) for detailed discussion of our overall research project.

A Leveled Approach to Characterizing CR

Our first approach to capturing differences in teachers' CR is a six-level framework that works to classify the varying levels of sophistication teachers incorporate as they reason about curriculum. This framework aims to capture the degree to which teachers use the various vertices and edges depicted in the Instructional Pyramid (see Figure 1) when making curricular decision. These six levels are:

- **Level 0:** A teacher reasons only with the Reflect and Revise CR aspect. In this case, *no vertices or edges* are used in the reasoning. In these instances, the teacher is implicitly reasoning with one or more elements but the reasoning is not explicit.
- **Level 1:** A teacher reasons with *any single edge* of the Instructional Pyramid. In doing so, the teacher reasons with one CR aspect and uses only <u>two elements</u> in the decision-making process. For example, a teacher reasons only with the Mapping Learning Trajectory CR aspect, which connects the two elements Curriculum and Mathematics.
- **Level 2:** A teacher reasons with *two edges connected by a common vertex* on the Instructional Pyramid. In doing so, the teacher reasons with two CR aspects that connect three elements in their decisions. For example, a teacher reasons with the Considering
- Mathematical Meaning (TM) and Mapping Learning Trajectory CR aspects, which incorporates the elements of Teacher, Mathematics and Curriculum (but no discussion of Students).
- **Level 3:** A teacher reasons with *three edges that form a face* on the Instructional Pyramid. In doing so, the teacher reasons with three CR aspects that connect three elements but does not incorporate the fourth element in their reasoning. For example, a teacher reasons with the Considering Mathematical Meaning (TM), Mapping Learning Trajectory, and Analyzing Curriculum Materials CR aspects, which connect the three elements Teacher, Mathematics, and Curriculum on a complete face.
- **Level 4:** A teacher reasons with *two unconnected edges* on the Instructional Pyramid. In doing so, the teacher incorporates all four elements that form the Instructional Pyramid but in a manner in which the two edges are not connected. For example, a teacher uses the Viewing Mathematics from the Learner Perspective-Anticipating/Assessing (A/A) and the Mapping Learning Trajectory CR aspects, which uses all four elements on two edges of the pyramid that are unconnected. Note that, even though a teacher is reasoning with fewer CR aspects in Level 4 in comparison to Level 3, all four elements are used in Level 4 reasoning, as opposed to only three of the four elements used in Level 3.
- **Level 5:** A teacher reasons with *three or more connected edges* on the Instructional Pyramid. In doing so, the teacher reasons in a manner that connects all four elements of the Instructional Pyramid. For example, a teacher reasons with the Viewing Mathematics from the Learner Perspective (A/A) the Viewing Mathematics from the Learner Perspective - Intentionality of Task (IT), and the Mapping Learning Trajectories CR aspects, connecting all four elements in a path around the edges of the pyramid.

As part of our research, we collected interview data from 15 middle grades teachers as they planned instruction for a unit on geometric transformations. These teachers were given the geometric transformations unit from the *UCSMP* series (Benson et al., 2009) and used this curriculum as the basis for planning their grade 8 unit pertaining to geometric transformations (reflections, rotations, translations, and sequences of transformations). This topic was chosen as a content area that had traditionally appeared in secondary mathematics but that had now emerged in the grade 8 curriculum after the widespread adoption of the *Common Core State Standards for Mathematics (CCSSM)*. The *UCSMP* unit on geometric transformations was chosen because of its unique approach to geometric transformations to construct the definition of congruence, which was rarely seen in past state standards but is now used in *CCSSM.* Teachers were interviewed before and after teaching lessons with the *UCSMP* curriculum. These interviews were partitioned in initial pass coding according to teachers' mathematical decisions that emerged during the pre- and post-interviews and then coded for the various CR aspects seen in Figure 1. The codes for each decision (N) were then analyzed and classified by the levels described above. Our analysis illustrated considerable differences in teachers' levels of reasoning when planning and reflecting upon their curricular decisions. We share the results from two teachers—Jill with 12 years of teaching experience, and Cathy with 8 years of teaching experience—to highlight these differences in Table 1.

Table 1: Breakdown by level of teachers' CR

Teacher	N	Level 0	Level 1	Level 2	Level 3	Level 4	Level 5
Jill	161	10 (6.2%)	104 (64.6%)	36 (22.4%)	3 (1.9%)	1 (0.6%)	7 (4.4%)
Cathy	281	3 (1.1%)	120 (42.7%)	91 (32.4%)	6 (2.1%)	15 (5.3%)	46 (16.4%)

As seen in table 1, Cathy used a greater percentage of higher-level CR (levels 4 & 5) when making decisions than Jill. Further analysis of the data revealed trends on which vertices and edges each teacher uses most when making decisions. To provide an illustration for one of these teachers, 58% of Cathy's Level 1 instances were on the *Viewing Mathematics from the Learner Perspective (A/A)* edge connecting the Student and Teacher elements, while nearly 70% of Cathy's Level 2 codes involved either the Student or Teacher element as *the connector* between the two CR aspects used in her reasoning. However, Cathy's Levels 3 and 4 instances saw an even balance across the four elements (student, teacher, mathematics and curriculum), while her Level 5 instances contained a heavy emphasis on either Mathematics or Curriculum. This led us to conclude that Cathy tended to reason more about the Student and Teacher elements in her instructional decisions, while her limited higher level reasoning involved her expanding her focus to the Mathematics and Curriculum elements.

Using Heat Maps to Identify Use of CR Aspects

Our second approach to analyzing teachers' use of CR aspects involved the utilization of heat maps. After collecting and analyzing the pre- and post-interview data, we compared the frequency of reasoning with each CR aspect to the total number of decisions made by individual teachers to calculate a percentage of use for each CR aspect by teacher. Because teachers used multiple CR aspects to make decisions or teachers did not provide reasoning for some decisions, the percentages do not add to 100%. These CR percentages were used to create heat-map models - a graphical representation where data values are labeled with cool and warm colors. Warmer colors signify that the teacher reasoned with the CR aspect often, while cooler colors signify that the teacher reasoned with the CR aspect less often. Table 2 displays the scale we used to design the data-generated heat-maps. We determined that five colors allowed us to differentiate among teachers more easily than three colors. We also note that green is the optimal range because if teachers reasoned with a CR aspect over 45% of their decisions, this often limited their use of other CR aspects. Using the Instructional Pyramid model of CR we developed a data-generated model for each teacher.

Table 2: Color Scale for Heat-Map Models

	Curricular Reasoning Percentage
Purple	0-15%
Blue	15.01-25%
Green	25.01-45%
Orange	45.01-55%
Red	55.01%+

Data-Generated Models

Our goal in creating the data-generated models for the teachers participating in our study was to highlight similarities and differences in reasoning and identify CR aspects that may not be used as often as others. This led to implications for teachers, teacher educators, and professional developers. A few notable patterns emerged when comparing the data-generated models across all teachers. First, we found that for 91% of the teachers in our study, the CR aspect *Viewing Mathematics from the Learner Perspective (IT)* was used the least-often in their decisions regarding their planning and enacting of mathematics lessons. These decisions connect the student and curriculum aspects of the Instructional Pyramid model of CR. Secondly, we found that 74% of our teachers used the CR aspect *Viewing Mathematics from the Learner Perspective (A/A)* the most often in their reasoning, with the remaining 26% of teachers using the CR aspect *Considering Mathematical Meaning (TM)* the most often. Both of these CR aspects are represented on the front face of the Instructional Pyramid of CR, connecting the elements teachers and students, and teachers and mathematics respectively. Importantly, the CR aspects with the highest use do not connect the elements of teacher, student, or mathematics to the curriculum while the CR aspect with the lowest use has the curriculum connection. In fact, across all these data, the majority of teachers reasoned with CR aspects on the front face of the Instructional Pyramid of CR (i.e., *Viewing Mathematics from the Learner Perspective (A/A)* and *Considering Mathematical Meaning (TM/SM)*) with greater frequency than the CR aspects connected to curriculum (i.e., *Viewing Mathematics from the Learner Perspective (IT)*, *Analyzing Curriculum Materials*, and *Mapping Learning Trajectory*). These patterns have potential implications for professional development of teachers and pre-service teachers.

Figure 3 displays data-generated models for two teachers in our study that highlight extremes to show the range in our data. Helen (left) was an example of a teacher in the mid-range with five of the six CR aspects in the green and the sixth being close to the less extreme mid-range (blue). Vance (right) was an example of a teacher who reasoned with fewer CR aspects. The majority of teachers' models were more similar to Vance than Helen, with more variety in the colors on the model. In fact, only 17% of the teachers' models had four or more CR aspects in the green mid-range. However, with all but one CR aspect outside of the mid-range, Vance displayed a more varied CR use with the more extreme-ranges (i.e., purple, red) present than many of the other models. We found that the majority of teachers (52%) had at least half of the six CR aspects in the green mid-range. The data were reasonably centered around the green mid-range.

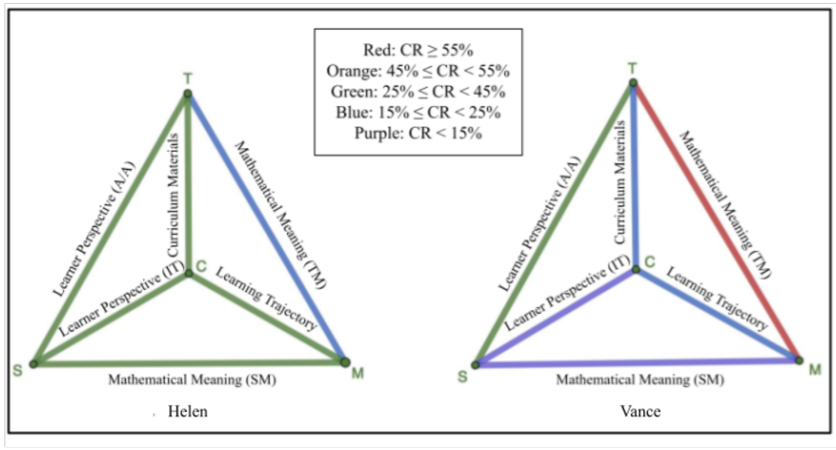


Figure 2: Two Teachers' Heat-Maps to Represent Their Curricular Reasoning

Implications/Conclusion

Using our two ways we can see how teachers are using their CR in two different ways. In the leveled approach, we found that teachers reasoned most often with CR aspects connecting the elements of teacher, student, and mathematics on the Instructional Pyramid, but reasoned much less frequently with those connected to the curriculum vertex on the pyramid. In the heat-map approach, we found that teachers reasoned most often with the *Viewing Mathematics from the Learner Perspective (A/A)* and *Considering Mathematical Meaning (TM)* as they planned and enacted lessons.

Both ways highlighted above provide different and significant approaches to characterize teachers' CR. The leveled approach allowed for the examination of the sophistication of teachers' reasoning. As the levels of reasoning increased, teachers incorporated more of the four classroom elements (Teacher, Student, Mathematics, Curriculum) represented as the vertices on the Instructional Pyramid as well as more of the CR aspects represented as the edges on the Instructional Pyramid. As stated previously, our working hypothesis is that teachers who coordinate multiple CR aspects and subsequently focus their attention on greater numbers of elements provide different learning opportunities for students than those who reason with only one or two CR aspects and subsequently fewer elements. The leveled approach provides a method to examine how often teachers reason with greater sophistication (more CR aspects and elements). On the other hand, the heat map approach allows for a detailed analysis of how often teachers use different CR aspects. In the process, teachers can see the CR aspects that figure most prominently in their reasoning as well as the CR aspects used least often. To that point, the heat map approach can be used to provide support to teachers in developing their abilities to diversify their reasoning in order to coordinate more CR aspects into their decision making.

While both approaches provide more information about a single teacher, these findings have implications for teachers, teacher educators, and professional developers. The first is that current professional development and pre-service education appear to be developing teachers' ability to reason with *Viewing Mathematics from the Learner Perspective (A/A*) and *Considering Mathematical Meaning (TM)* as those two CR aspects are more widely used by teachers in our study. In the heat-map models these were the CR aspects that were most often in the warm colors (i.e., red and orange) on teachers' models. This suggests that teachers are reasoning with these aspects often. In our leveled model, we also found that these CR aspects are the ones that connect the elements of teacher, student and mathematics. Second, teachers' reasoning with the other CR aspects that connect curriculum in the Instructional Pyramid: *Analyzing Curriculum Materials*, *Mapping Learning Trajectories*, and *Viewing the Learner Perspective (IT*) were not as widely used. This has implications for students as their teachers are generally reasoning less with the element of curriculum as they prepare and enact lessons. These results may not be surprising because professional developments have primarily focused on making decisions and reasoning from the front face of the Instructional Pyramid. We believe these ways of reasoning are important and should remain a key part of teacher education and professional development.

Therefore, we recommend that teacher educators and professional developers include explicit activities and task that encourage teachers to make decisions and reason with the other CR aspects that include curriculum as all CR aspects are important to provide the best possible learning opportunities for students.

These findings also have implications for teacher educators and professional developers, informing content to be taught. Based on initial work, we propose that the Instructional Pyramid model can be used to examine how teachers reason with the classroom elements and CR aspects discussed previously. This can allow stakeholders to analyze the factors and reasons that shape teachers' decisions as they plan and implement mathematics lessons. The Instructional Pyramid model can also be used as a self-assessment for teachers to identify strengths and weaknesses in their CR. Through the use of both the leveled approach and the heat map approach, teacher educators can support teachers' continuing development of the sophistication and diversity of their CR.

Acknowledgments

This material is based upon work supported by the National Science Foundation under Grant Nos. 1561542, 1561154, 1561569, 1561617. Any opinions, findings, and conclusions are recommendations expressed in this material are those of the authors and do not necessarily reflect the view of the National Science Foundation.

References

Banilower, E. R., Smith, P. S., Weiss, I. R., Malzahn, K. A., Campbell, K. M., & Weis, A. M. (2013). Report of the 2012 national survey of science and mathematics education. Horizon Research, Inc.

Breyfogle, M. L., Roth McDuffie, A., & Wohlhuter, K. A. (2010). Developing curricular reasoning for Grades pre-K-12 mathematics instruction. In B. Reys, R. Reys, & R. Rubenstein (Eds.), Mathematics Curriculum: Issues, Trends and Future Direction, 72nd Yearbook (pp. 307-320). Reston, VA: National Council of Teachers of Mathematics.

Brown, M. W. (2009). The teacher-tool relationship: Theorizing the design and use of curriculum materials. In J. T. Remillard, B. A. Herbel-Eisenmann & G. M. Lloyd (Eds.), Mathematics teachers at work: Connecting curriculum materials and classroom instruction (pp. 17-36). New York: Routledge.

Choppin, J., Roth McDuffie, A., Drake, C., & Davis, J. (2018). Curriculum ergonomics: Conceptualizing the interactions between curriculum design and use. International Journal of Educational Research, 92, 75-85.

Choppin, J., Roth McDuffie, A., Drake, C., & Davis, J. (2020). The role of instructional materials in the relationship between the official curriculum and the enacted curriculum. Mathematics Thinking and Learning. DOI: 10.1080/10986065.2020.1855376
Choppin, J., Davis, J., Roth McDuffie, A., & Drake, C. (2021). The influence of features of curriculum materials on the planned curriculum. ZDM Mathematics Education, 53(6), 1249-1263.
Cohen, D., & Ball, D. L. (1999). Instruction, capacity, and improvement (Research Report Series No. RR-43). Philadelphia: University of Pennsylvania, Consortium for Policy Research in Education.
Cohen, D., Raudenbush, S., & Ball, D. (2003). Resources, instruction and research. Educational Evaluation and Policy Analysis, 25(2), 119-142.
Dingman, S. W., Teuscher, D., Olson, T. A., & Kasmer, L. (2021). Conceptualizing curricular reasoning: A framework for examining mathematics teachers' curricular decisions. Investigations in Mathematics Learning, 13(4), 267-286.
Lloyd, G. M., Cai, J., & Tarr, J. (2017). Issues in curriculum studies: Evidence-based insights and future directions. In J. Cai (Ed.), First compendium of research on mathematics teaching and learning (pp. 824-852). Reston, VA: National Council of Teachers of Mathematics.
Remillard, J. (2005). Examining key concepts in research on teachers' use of mathematics curricula. Review of Educational Research, 75(2), 211-246.
Rezat. S. (2006). A model of textbook use. In J. Novotna, H. Moraova, M. Kratka, & N. A. Stehlikova (Eds.) Proceedings of the 30th conference of the international group of the psychology of mathematics education (Vol. 4, pp. 409-416). Prague: Charles University, Faculty of Education.
Rezat, S. & Sträber, R.(2012). From the didactical triangle to the socio-didactical tetrahedron: Artifacts as fundamental constituents of the didactical situation. ZDM Mathematics Education, 44(5), 641-651.
Roth McDuffie, A., & Mather, M. (2009). Middle school mathematics teachers' use of curricular reasoning in a collaborative professional development project. In J. T. Remillard, B. A. Herbel-Eisenmann & G. M. Lloyd (Eds.), Mathematics teachers at work: Connecting curriculum materials and classroom instruction (pp. 302-320). New York: Routledge.
Roth McDuffie, A., Choppin, J., Drake, C., & Davis, J. (2018). Middle school teachers' orientations and noticing of features of mathematics curriculum materials. International Journal of Educational Research, 92, 173-187.
Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. Educational Researcher, 15(2), 4-14. Stein, M. K., Remillard, J. T., & Smith, M. S. (2007). How curriculum influences student learning. In F. Lester Jr. (Ed.), Second Handbook of Research on Mathematics Teaching and Learning (Vol. 1, pp. 319-369). Reston, VA: National Council of Teachers of Mathematics.
Tall, D. (1986). Using the computer as an environment for building and testing mathematical concepts. In Papers in Honour of Richard Skemp (pp. 21-36). Warwick.

COGNITIVE COMPLEXITY OF ELEMENTARY MATH TEXTBOOK LANGUAGE IN LOW- AND HIGH-INCOME SCHOOL DISTRICTS

Mike Pacheco
Nevada Department of Education
mpacheco@doe.nv.gov

Lynda R. Wiest
University of Nevada, Reno
wiest@unr.edu

Fifth-grade mathematics textbooks from the five highest- and five lowest-income school districts in the United States were analyzed for cognitive demand in terms of level of verbs used according to Bloom's Revised Taxonomy. Verbs that appeared most often were the three lowest levels of the six verb categories. The results are encouraging from an equity standpoint but discouraging in terms of cognitive expectations placed on students in school mathematics.

Keywords: Elementary School Education; Curriculum; Equity, Inclusion, and Diversity; Math Textbooks; Cognitive Complexity; Quantitative Text Analysis

The purpose of this study was to investigate the relationship of school districts' economic status to the cognitive complexity of adopted fifth-grade mathematics textbooks. Teachers tend to associate students from low socioeconomic backgrounds with lower intelligence (Rist, 2000), and teachers' beliefs can shape their instructional practices in mathematics (e.g., Beswick, 2012; Francis, 2015; Stipek, 2012). In many cases textbook selection is done by committees comprised of classroom educators and community members often representing particular interest groups (Watt, 2009). Textbook selection is thus intentional and complex.

Methods

In line with other studies (e.g., Al-hasanat & AbdRabbeh, 2016; Assaly & Smadi, 2015; Upahi & Jimoh, 2016), a Bloom's-type taxonomy was used to analyze the cognitive complexity of mathematics textbooks. Fifth-grade mathematics texts were obtained from the five most and five least economically affluent U.S. school districts, as defined by median household income. They were scanned to PDF and optimized using Adobe Acrobat Pro DC software. Verbs in the sample texts were quantified Using NVivo qualitative data analysis software and coded using Stanny's (2016) classification system that aligns with the categories in Bloom's Revised Taxonomy: Remember, Understand, Apply, Analyze, Evaluate, and Create. A chi-square test was used to determine whether elementary mathematics textbooks used by lower- and higher-income school districts differ in level of cognitive demand, as measured by verb choice.

Results and Implications

Results indicate that a significant relationship does not exist between school district economic status and cognitive complexity of adopted mathematics textbooks. This might indicate limits in the effect of median household income on textbook selection or of the small study sample. Analysis showed that Application (Apply) verbs tended to appear most often in the texts, followed by Understand or Knowledge (Remember) Verbs. As these are the three lower-cognitive demand verbs, greater cause for concern might lie in cognitive demand of elementary mathematics texts in general, rather than in differences by school district economic status.

References

Al-hasanat, H. A. A. (2016). Analyzing assessment questions in an Arabic textbook (communication skills) for eighth grade in Jordan according to Bloom's Taxonomy of levels of knowledge aims. *World Journal of Education*, *6*(2), 68–81. https://doi.org/10.5430/wje.v6n2p68

Assaly, I. R., & Smadi, O. M. (2015). Using Bloom's Taxonomy to evaluate the cognitive levels of master class textbook's questions. *English Language Teaching*, *8*(5), 100–110.

Becker, H. (1952). Letter from Howard Becker. *The Annals of the American Academy of Political and Social Science, 282*, 190-190.

Beswick, K. (2012). Teachers' beliefs about school mathematics and mathematicians' mathematics and their relationship to practice. *Educational Studies in Mathematics,* 79(1), 127-147. Bickel, R., & Howley, C. (2003). Math achievement and the development of poor, rural areas: effects of contextual factors intrinsic to the modern world. *Educational Foundations*, *17*(4), 83–105.

Francis, D. I. C. (2015). Dispelling the notion of inconsistencies in teachers' mathematics beliefs and practices: A 3-year case study. *Journal of Mathematics Teacher Education*, 18(2), 173- 201.

Rist, R. C. (2000). Student social class and teacher expectations. The self-fulfilling prophecy in ghetto education. Harvard Educational Review, 70(3), 266-301. doi: 10.17763/haer.70.3. 1k062416102u2725.

Stanny, C. (2016). Reevaluating Bloom's Taxonomy: What measurable verbs can and cannot cay about student learning. *Education Sciences*, *6*(4), 37. https://doi.org/10.3390/educsci6040037

Stipek, D. (2012). Context matters: Effects of student characteristics and perceived support from administrators and parents on teacher self-efficacy. The Elementary School Journal, 112(4), 590-606. Retrieved from http://www.jstor.org/discover/10.1086/664489.

Upahi JE, Jimoh M. Classification of end-of-chapter questions in senior school chemistry textbooks used in Nigeria. *European Journal of Science and Mathematics Education*. 2016;4(1):90-102.

Watt, M. G. (2009). Research on the textbook selection process in the United States of America. *Online Submission*. Retrieved from https://search-ebscohost-com.unr.idm.oclc.org/login.aspx?direct=true&db=eric&AN=ED506523&site=ehost-live&scope=site.

DILATING PERSPECTIVES: A COMPARISON OF TEACHER- AND STUDENT-FACING TEXTS ON A UNIT ON SIMILARITY

Soobin Jeon
University of Michigan
sbjeon@umich.edu

Patricio G. Herbst
University of Michigan
pgherbst@umich.edu

Mathematics curricula are often the primary resource used in the teaching and learning of mathematics–in particular, they represent mathematical knowledge for teachers and students. Yet, there seems to be a discrepancy between the curricular materials written for teachers and students (e.g., more variety in the semiotic choices in text provided for teachers than for students). This paper illustrates a way to compare the language written for the teacher and the student in curricula. Drawing on the social semiotic theory known as systemic functional linguistics (SFL), we examine what meaning potentials can be associated with the exchanges of knowledge proposed in mathematical text. Examining a Unit on Similarity in an online mathematics curriculum widely used in the United States, the paper explores how the semiotic systems in text presented for the teacher and student may impact the students' opportunity to learn and pose challenges in the instructional exchange between teacher and student.

Keywords: Curriculum, Geometry and Spatial Reasoning, Instructional Activities and Practices, Communication

Background and Theoretical Framing

Brousseau (1997) proposed that the relationship among teacher, students, and mathematics is mediated by a didactical contract, a set of tacit responsibilities that bind the teacher, students, and content. For example, the contract makes the teacher responsible for students' acquisition of knowledge at stake and the students responsible for engaging in the work the teacher organizes for them to acquire that knowledge (Herbst & Chazan, 2012). Similarly, in an instructional system, "a teacher has the responsibility to organize and sustain activities for students in which the work that students do can be described by appealing to a culturally current representation of the knowledge at stake" (Herbst & Chazan, 2012, p. 605). Among the work a teacher needs to do is manage instructional exchanges, which involves equating or reconciling seemingly unequal representations of knowledge, such as the work the students did on one or more tasks and the knowledge at stake in those tasks. The management of an instructional exchange requires teachers not only to apply the didactical contract to the task at hand but also identify what the students need to do and figure out what aspects of the mathematical task point to the knowledge at stake (Herbst & Chazan, 2012). The statement of the knowledge at stake in an instructional exchange makes use of various elements of the mathematical register but managing instructional exchanges also "requires the teacher to engage in serious interpretation of much more opaque signs in the realm of the students' work" (Herbst & Chazan, 2012, p. 606). Mathematics curricula are one important medium that guides teachers on how to engage in instructional exchanges.

Mathematics curricula play a crucial role in the teaching and learning of mathematics. Textbooks are often the primary resource used by teachers to plan and deliver instruction and students also rely on textbooks as a reference and study guide (Reys et al., 2004). Mathematical concepts in textbooks are written in a voice that is appropriate for students (Remillard, 2000) and teachers are often regarded as mediators of the text (Love & Pimm, 1996). Given the prominent role of mathematics curricula, "making a wise selection is crucial because it determines the scope of mathematics that students experience and, to some extent, how teachers present the material and how students learn" (Reys et al., 2004, p. 65). While studies on the use of textbooks or curricula focus on the teacher or student, there is a lack of attention to the interactions between the teacher and student when using mathematical texts (Rezat, 2011) and the meanings construed in texts addressed to the teacher and students. As mathematical curricula often provide separate instructions for the teacher and student, semiotic resources in the teacher-facing and student-facing instructions may differ, allowing for different experiences, meaning-making, and shaping how the teacher enacts their responsibility in an instructional exchange. Specifically, the language choices made in the text addressed to students may differ from those made in the text addressed to teachers and both of them may mediate when and how the teacher affects instructional exchanges. An analysis of the different language choices in these texts can provide us a first approximation at what work might exchange for what knowledge claims in instruction. The goal of this paper is to illustrate this way of using curriculum resources to analyze a priori the instructional exchanges that might take place in classes in which these resources are used.

Methods

Context

We analyze a Geometry Unit on Similarity from the Illustrative Mathematics (IM) curriculum for grades 9-12 (Illustrative Mathematics Certified, 2023). Illustrative Mathematics is an independent nonprofit that provides high-quality K-12 instructional materials, rigorous, standards-aligned content, and engagement in mathematical discussion. The Unit on Similarity has students learn the definition of similarity in terms of dilations and rigid transformations. Students prove that if triangles have three pairs of congruent corresponding angles and three pairs of corresponding sides in a proportional relationship, the triangles are similar. They also draw conclusions about figures they have proven to be similar, such as, corresponding angles are congruent and corresponding sides are in a proportional relationship.

In terms of structure, the unit consists of 16 lessons organized into four sub-sections: (A) properties of dilations, (B) similarity transformations and proportional reasoning, (C) similarity in right triangles, and (D) putting it all together. There is also a modeling prompt, called Scaling a Playground, that follows the first lesson. The modeling prompt provides students with an opportunity to choose and use appropriate mathematics and statistics to analyze empirical situations, helping students understand that they can use mathematics to better understand things they are interested in. During the task, students formulate a model, which can be a geometric, graphical, tabular, algebraic, or statistical representation that describes the relationships between variables in the situation.

The IM curriculum provides text that is for the teacher (e.g., labeled under "teacher-facing") and text that is for the student (e.g., labeled under "student-facing"). Each of the 16 lessons has a set of teacher-facing learning goals and student-facing learning goals. The teacher-facing learning goals describe mathematical, pedagogical, and language goals of the lesson for a teacher audience. While these goals are intended to be read by teachers, the actor they allude to is the students. For example, if a goal reads, "Comprehend that dilations take angles to congruent angles" (see Figure 1a), it construes for the teacher the meaning that students are ultimately to be able to comprehend that dilations take angles to congruent angles. In contrast, the student-facing learning goals are written in the first person, often starting with "let's" and "I can," and invite students to the lesson of the day. Likewise, the modeling prompt distinguishes sections for the intended audience, calling sections “teacher instruction” or “student facing statements.”

Data Analysis

We conducted an analysis of discourse that seeks patterns in linguistic data by drawing on a meaning-focused theory of language, *systemic functional linguistics*. In particular, we draw on Halliday and Matthiessen’s (2004) system of Transitivity in lexicogrammar, that states a clause, as a token of ideational meanings (i.e., goings-on in the world) can be analyzed by identifying processes, participants, and circumstances, which are “organized in configurations that provide the models or schemata for construing our experience of what goes on” (p. 175). The ideational metafunction of text construes meaning through the different types of processes within the Transitivity system. Analyzing the domain of experiential meaning in mathematics is helpful in understanding how teachers and students can be engaged with and connect to the learning experience (O'Halloran, 2005).

Ideational meanings are construed through choices of processes, which can be *Mental*, *Material*, *Relational*, *Existential*, *Verbal*, *Behavioral*, or *Operational*. Processes are typically realized with verb choices in simple clauses but in more complex clauses they may be nominalized. *Mental* (sensing) processes may be realized with verb choices such as consider or think, material (doing) processes with verb choices such as draw or project. *Relational* clauses relate two separate entities, they “characterize and identify” (Dimmel & Herbst, 2015, p. 166) as in the statement of a mathematical definition. *Operational* processes, a process type specific to mathematical discourse identified by O'Halloran (2005) are often realized by choices of arithmetic and algebraic operation symbols or by the “words that in mathematics mean an operation” (Dimmel & Herbst, 2015, p. 167) or symbols that represent a mathematical action or function (e.g., a raised number or letter next to a letter or number indicating the operation “raised to the … power”). *Existential* processes show that “phenomena of all kinds are simply recognized to ‘be’” (Halliday & Matthiessen, 2004, p. 171). *Existential* processes stipulate the existence of something (Dimmel & Herbst, 2015). For example, in a proof context, the word ‘let’ in “let O be the center of a dilation” tells the student that there exists an entity, O, that they can utilize in doing the proof.

The system of Transitivity “offers a range of options for ideational (content) meaning that is comprehensive of the ways language varies in presenting experience: as doing, sensing, saying, or being” (Schleppegrell, 2013, p. 22). For example, *Verbal* processes are enacted in the form of ‘saying.’ *Behavioral* processes are enacted in the form of partly ‘doing’ and partly ‘sensing,’ projecting the outer reflection of the consciousness (e.g., observe). Identifying the ideational meaning that can be represented in the form of processes, we compare teacher-facing text and student-facing text presented in (1) the lesson goals and (2) the unit’s modeling prompt. The lesson goals span across 16 lessons, allowing us to see patterns in the text across lessons that vary in the ways they position the intended reader (the teacher or the student). Looking at the modeling prompt offers another perspective on how text is presented to the teacher and the student, how their interaction is supported, and how printed language may stand on its own or be supported by other modalities (e.g., visual, technological). In particular, we identify meanings in the text construed through choices from the transitivity system. In our analysis, we first determine the form and quantity of processes in the teacher-facing text and student-facing text within the lesson goals and modeling prompt. Subsequently, we compare the processes in each lesson or prompt between the teacher- and student-facing texts, determining whether the processes are more or less aligned with similar meanings. Through the analysis of transitivity patterns, we consider the following questions:

1. To what extent do the teacher-facing text and student-facing text draw on processes with similar meanings?
2. What patterns can we observe in the processes used within and across the teacher-facing and student-facing texts?

Findings

The analysis of transitivity patterns displayed a contrast between the teacher- and student-facing texts in the learning goals: More use of the *mental* (cognizing) processes in the teacher-facing goals and more use of the *material* and *verbal* processes in the student-facing goals. The teacher-facing goals are represented in statements that start with verbs that indicate what the students should be able to know by the end of the lesson. In terms of instructional exchanges, these instructional goals refer to the claim that the teacher may be able to make on behalf of the student based on what the teacher sees the students doing. If a teacher-facing goal says, "Comprehend that dilations take angles to congruent angles" (Figure 1a), the teacher is expected to ensure that students mentally understand this idea, as comprehend realizes a *mental* process. At the same time, the corresponding student-facing goal says, "Let's dilate lines and angles" (Figure 1b), representing the work students will do as taking action in the *material* world, doing the dilation. The distinction between the *mental* process in the teacher-facing goal and the *material* process in the student-facing goal is also highlighted in the difference between the words "dilation" and "dilate". These are key technical terms in this unit that identify an important concept students will use to justify that triangles are similar. We can see a distinction between "dilation" in the teacher-facing goal and "dilate" in the student-facing goal by how often each form (verb or its nominalization) is used and which transitivity process co-occur with their use.

Comprehend that *dilations* take angles to congruent angles. **Prove** that a *dilation* takes a line not passing through the center of the *dilation* to a parallel line, and leaves a line passing through the center unchanged.	Let's **dilate** lines and angles. I can **explain** what happens to lines and angles in a *dilation.*
Figure 1a: Teacher-facing Goals	Figure 1b: Student-facing Goals

Figure 1: Learning Goals for Lesson 4: Dilating Lines and Angles

"Dilation" appears five times in the teacher-facing goals across the unit. For all five clauses, "dilation" is introduced by a *mental* process (i.e., comprehend, prove, Figure 1a), emphasizing to the teacher that they should be supporting students' understanding of and perception on the concept of dilation. In contrast, "dilation" appears only once in the student-facing goal when students are asked to verbalize (i.e., explain, Figure 1b) what happens in a dilation. "Dilate," the verb form and *material* process, is more commonly shown in the student-facing goals, appearing four times while appearing not once in the teacher-facing goals. In a general sense, the teacher is being asked to focus on having students sense what a dilation is while the student is being asked to do the dilation, or to dilate, or to narrate what they did (or verbalize). It then becomes the teacher's responsibility to not only ensure that the students dilate different properties of figures but also to interpret such actions as comprehending the properties of dilation what it means for dilations to work. Whereasthis is evidence of how nominalization supports the development of mathematical knowledge that is more concise and advanced, it packs within it material actions that may but also may not always entail all the properties attributed to dilations.

Relational processes play the important role of describing the qualities and characteristics of entities. In the context of a geometry curriculum, relational clauses are essential for making meaning as they provide a way to describe and establish the properties of and relationships among geometric objects. The *relational* processes in the teacher-facing and student-facing goals convey similar meaning but are represented differently. In particular, the *relational* clauses in the teacher-facing goal are presented as complex noun phrases in the student-facing goals. For example, a *relational* clause in the teacher-facing goal has an embedded clause that is linked by 'are similar to' the noun group, the original right triangle: "the two smaller right triangles formed when a right triangle has an altitude drawn to its hypotenuse are similar to the original right triangle" (Figure 2a). In the student-facing goal, two complex noun phrases describe what is to be explored and found: "right triangles with altitudes drawn to the hypotenuse" and "similar triangles formed by the altitude to the hypotenuse in a right triangle" (Figure 2b). We see that the complex noun groups in the student-facing goals are more concise and simplified. Yet, they are more complicated in parsing the mathematical meaning, requiring more work from the students to unpack them.

The *material* processes in the student-facing goals (i.e., explore, find, Figure 2b) provide a concrete basis for understanding what is presented to the teacher in a *relational* process that would otherwise be difficult without an example or a visual support. Through exploration, students can physically draw altitudes on right triangles and see how they relate to the hypotenuse. In addition, by finding similar triangles, students can see how similar triangles are formed by right triangles with altitudes drawn to the hypotenuse. As the *material* processes support students' concept of similar triangles in a more tangible way, the teacher-facing goal presents the same meaning but highlights the cognitive action through the use of the *mental* process (i.e., justify, Figure 2a). The *mental* process is important because it involves cognitive processes such as critical thinking and reasoning, deepening the students' understanding of the properties of similar triangles. The *mental* process also shows that the teacher supports students' learning from doing to sensing the relationship between the original right triangle and the two smaller triangles.

Justify that the two smaller right triangles formed when a right triangle has an altitude drawn to its hypotenuse are similar to the original right triangle.	Let's **explore** right triangles with altitudes drawn to the hypotenuse. I can **find** similar triangles formed by the altitude to the hypotenuse in a right triangle.
Figure 2a: Teacher-facing Goals	Figure 2b: Student-facing Goals

Figure 2: Learning Goals for Lesson 13: Using the Pythagorean Theorem and Similarity

Operational processes are helpful in elaborating the *material* process because the *operational* processes articulate steps to carry out a particular action. For example, the *operational* process of multiplication in the teacher- and student-facing goals (Figure 3) supports the elaboration of the *material* process of dilating. While the relationship between the *operational* process of multiplication and *material* process of dilation remains the same, it is represented differently in the teacher-facing and student-facing goals. In the teacher-facing goal, the *operational* process of multiplication is directive; the teacher may instruct the students to determine that dilating results in multiplying all lengths (Figure 3a). The focus is on the process of understanding how dilating works and recognizing that dilation is equivalent to the sequences of operations. On the other hand, in the student-facing goal, the operational process of multiplication will come to be known and the student will understand the concept, knowing that when figures are dilated by a scale factor of k, all lengths in the figure are multiplied by k (Figure 3b). The operational process in both teacher- and student-facing goals is projected by the *mental* process (i.e., determine, Figure 3a; know, Figure 3b), comprehending that dilate is multiply. The *mental* process reports that students are able to apply multiplication when thinking of dilations. For example, students can understand that by multiplying all lengths in the figure by a scale factor of k, dilating is able to change the size and dimensions of the figure but maintain its shape and proportions.

Determine that *dilating* by a scale factor of k *multiplies* all lengths by k	I **know** that when figures are *dilated* by a scale factor of k, all lengths in the figure are *multiplied* by k
Figure 3a: Teacher-facing Goals	Figure 3b: Student-facing Goals

Figure 3: Learning Goals for Lesson 3: Measuring Dilations

In the modeling prompt, the *existential* process especially plays an important role in setting the context. The *existential* processes in the teacher- and student-facing sections introduce the diagram (teacher is to tell students “this is a diagram of a playground”, Figure 4a; students are given “Here’s a playground for a school of 360 children in Springfield”, Figure 4b). The purpose of the *existential* process here is to introduce the diagram as a model of the actual playground (Figure 4a).

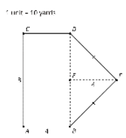
In the student-facing section, there is no specific goal set for students that indicates the processes they will engage in. Instead, they are given a scenario about the playground with the diagram and asked to answer questions (i.e., “how many?”, “how much?”, Figure 4b) that prompt their solving the problems and making sense of the modeling task. The teacher-facing section (Figure 4a) guides teachers to engage in *material* and *verbal* process (i.e., display, ask, invite, tell, clarify) so that students can engage in the *material* process (i.e., use) and ultimately the *mental* processes (i.e., wonder, notice) as they “share” (*verbal* process). Teaching-facing components also emphasize the *mental* processes, such as considering how students will experience something when it happens (e.g., “notice”). In the student-facing instruction, details on the hypothetical context are provided, placing more emphasis on the *existential* processes.

<table>
<tr>
<td>

Display this image of the original playground for all to see:

Ask, “What do you notice? What do you wonder?” After some quiet think time to **notice** and **wonder**, ask students to **share** with a partner. Then **invite** students to **share** what they noticed and wondered with the class, and record the responses for all to see.

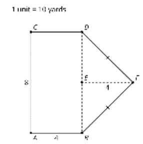
Then **display** this diagram:



Tell students that *this is a diagram of a playground,* and they will **use** it to design a new playground to fit different constraints. If needed, **clarify** that the fence of the playground only goes around the perimeter, not along the segments.

</td>
<td>

Here’s a playground for a school of 360 children in Springfield.



1. The fence around the playground costs 90 dollars per 50-foot roll. Laying the grass across the area costs 400 dollars per 500 square feet. How much did the fencing and grass for this playground cost to build?

2. How many kids per square yard can Springfield’s playground hold?

3. There’s a new playground going up in nearby Wintermeadow. Wintermeadow has a budget about 3 times greater than Springfield’s. Recommend a playground shape and size that would fit Wintermeadow’s budget and hold at least 3 times as many kids as Springfield’s playground at the same density of kids per square yard. How many kids can your playground hold?

</td>
</tr>
<tr>
<td>Figure 4a: Teacher Instruction</td>
<td>Figure 4b: Student-facing Statement</td>
</tr>
</table>

Figure 4: Modeling Prompt: Scaling a Playground

Discussion

The different semiotic resources used in the teacher- and student-facing components of curricula provide opportunities for researchers to anticipate how teachers and students make meaning of mathematical concepts and whether any differences in the meanings construed can be expected. The Illustrative Mathematics curricular texts showcase an intriguing relationship between the articulation of goals for teachers and students. While teachers are directed to approach these lessons with the *mental* processes students are to develop through the unit foregrounded, students are focused on the practical aspect of doing, from a *material* standpoint. Even when the student’s goal is to know something, the expression of what they are to know focuses on the *material* processes involved in dilation. The focus on the *doing* process, although it is emphasized in the students’ learning goals, is only a partial understanding of their learning experience. This raises questions about whether the objectives for students’ learning should involve more reflection around the concepts of dilation, and whether teachers require a better understanding of how they can connect the *material* processes students engage in with the cognitive outcomes that are expected. The discourse facing students often fails to represent their *mental* activities, making it challenging for them to explain their approach to a task. This issue is especially important in current approaches to mathematics, where a focus on developing students’ thinking is critical for advanced learning. While the field recognizes the importance of justification and proof in mathematics, mathematics teachers need guidance in supporting students to articulate the ways they are thinking through meaningful classroom discourse.

Effective mathematics instruction requires a continuum of modes, including spoken, written, stand-alone, and visual support, to construct activities that meet students' diverse needs. Collaborative interaction, allowing for meaning to be construed between individuals rather than within, is a key way teachers can support students' movement between doing and thinking through talk (Gibbons, 2003). Teachers can model this by thinking aloud themselves to show how *material* processes can be articulated in what students achieve in more abstract language, as implying a cognition of sorts. This movement between the observed and the inferred, encoded in language by the choice of *material* and *mental* processes can support students to communicate their thinking for others to engage with (Herbel-Eisenmann et al., 2013). Through such efforts, teachers can help their students develop a deeper understanding of mathematical concepts and communicate their thinking effectively. The instructional exchange between the teacher and students involves building on the language students bring and adding the technical and abstract languages unique to mathematics to enhance students' understanding of mathematical concepts.

Addressing the discrepancy in texts between the teacher- and student-facing goals can improve the quality and specificity of curriculum materials and equip teachers to better support students. The teacher can report what the student is doing using the *material* processes and align instructional goals with *mental* processes, enabling the teacher to manage the instructional exchange. The management of an instructional exchange not only complies with the didactical contract but also highlights how students are expected to demonstrate the specific knowledge at stake. The teacher has the responsibility to engage students in actions that instantiate the processes in the instructional goals, regardless of what students actually think, and help develop concrete understanding of abstract mathematical concepts, but then the articulation of what students have learned needs to be further supported. The next logical steps in capitalizing on this understanding would be to conduct research that addresses how teachers can become more aware of the role of language in positioning them and their students to engage in discourse. By conducting such research, researchers can work towards ensuring that curricular texts effectively support instructional exchanges and high-quality mathematics education that meets students' individual needs and builds on their languages.

References

Brousseau, G. (1997). Theory of didactical situations in mathematics: Didactique des mathematiques 1970–1990 (N. Balacheff, M. Cooper, R. Sutherland, & V. Warfield, Eds. and Trans.). Dordrecht: Kluwer.

Dimmel, J. K., & Herbst, P. G. (2015). The semiotic structure of geometry diagrams: How textbook diagrams convey meaning. Journal for Research in Mathematics Education, 46(2), 147-195. https://doi.org/10.5951/jresematheduc.46.2.0147

Gibbons, P. (2003). Mediating language learning: Teacher interactions with ESL students in a content-based classroom. TESOL Quarterly, 37(2), 247-273. https://doi.org/10.2307/3588504

Halliday, M. A. K., & Matthiessen, C. M. I .M. (2004). An introduction to functional grammar (3rd ed.). Hodder Arnold.

Herbel-Eisenmann, B., Steele, M. D., & Cirillo, M. (2013). (Developing) teacher discourse moves: A framework for professional development. Mathematics Teacher Educator, 1(2), 181-196. https://doi.org/10.5951/mathteaceduc.1.2.0181

Herbst, P., & Chazan, D. (2012). On the instructional triangle and sources of justification for actions in mathematics teaching. ZDM Mathematics Education, 44, 601-612. https://doi.org/10.1007/s11858-012-0438-6

Illustrative Mathematics Certified. (2023). IM Illustrative Mathematics. Illustrative Mathematics Learn Math For Life. https://curriculum.illustrativemathematics.org/

Love, E., & Pimm, D. (1996). This is so': a text on texts. International handbook of mathematics education, 1, 371-409.
O'Halloran, K. (2005). Mathematical discourse: Language, symbolism and visual images. Continuum.
Remillard, J. T. (2000). Can curriculum materials support teachers' learning? Two fourth-grade teachers' use of a new mathematics text. The elementary school journal, 100(4), 331-350. https://doi.org/10.1086/499645
Reys, B. J., Reys, R. E., & Chavez, O. (2004). Why mathematics textbooks matter. Educational Leadership, 61(5), 61-66. https://www.ascd.org/el/articles/why-mathematics-textbooks-matter
Rezat, S., Fan, L., & Pepin, B. (2021). Mathematics textbooks and curriculum resources as instruments for change. ZDM Mathematics Education, 53, 1189-1206. https://doi.org/10.1007/s11858-021-01309-3
Schleppegrell, M. J. (2013). Systemic functional linguistics. In M. Handford, & J. P. Gee (Eds.), The Routledge handbook of discourse analysis (pp. 21-34). Routledge. https://doi.org/10.4324/9780203809068

ELEMENTARY MATHEMATICS CURRICULUM: STATE POLICY, COVID-19, AND TEACHERS' CONTROL

Mona Baniahmadi
University of Texas Austin
mbaniahmadi@tacc.utexas.edu

Bima Sapkota
Ball State University
bima.sapkota@bsu.edu

Amy M. Olson
Duquesne University
olsona@duq.edu

In the U.S., state guidance to schools in response to the COVID-19 pandemic was politicized. We used state-level political affiliation to explore whether access to curricular resources differed pre-pandemic or during pandemic remote teaching and teachers' reported control over curricular resources during pandemic teaching. We found that pre-pandemic the percentage of teachers in Republican states reported higher levels of resources overall, and use of core and teacher-created curricular resources in particular. They also reported having greater control over their curricular decision-making during the pandemic. There were no state-level differences in teachers' level of preparation for pandemic teaching, but teachers in Democrat states reported a greater proportion of their students had sufficient resources for online learning. We discuss the implications of these findings in terms of teacher control and state policies.

Keywords: Policy, Curriculum, Elementary School Education

Purpose of the Study

The purpose of this study was to report the curriculum resources used by elementary mathematics teachers before and during the COVID-19 pandemic as well as perceived teacher control related to making several curricular decisions with the onset of remote learning in March 2020. The following research questions guided this study:

1. Pre-pandemic, were there Republican and Democrat state-level differences in elementary mathematics teachers' reported use of core, supplemental, and teacher-created curricular resources, or in the overall number of resources teachers used?
2. During pandemic remote teaching, were there Republican and Democrat state-level differences in elementary mathematics teachers' reported use of core, supplemental, and teacher-created curricular resources, or in the overall number of resources teachers used?
3. During pandemic remote teaching, were there Republican and Democrat state-level differences in elementary mathematics teachers' reported level of control over their curricular decision making, preparation to engage in remote teaching, or students' access to sufficient technological resources to engage in remote teaching?

Theoretical Perspectives

We use Wahlström's (2023) following conceptualization of curriculum: "Curriculum is not only a stable body of knowledge in accordance with a discipline. It is also about meaning making and negotiation among different actors in society and the education system, as politicians, national education authorities, teachers, etc." (pp. 260-261). Thus, how teachers interact with and implement curriculum is influenced by the context specific events (Doyle, 1992). In our study, we investigated how, if at all, the teachers' curricular decision-making processes differed by state policy.

Politics of Curriculum: Teacher Control

In the U.S., there is historical precedent for the impact of politics on curricular resources and how stakeholders make decisions about those resources. For example, the practice of textbook adoption refers to the use of state funds to purchase selected textbooks for use by teachers in a given state whereas local funds may be needed to purchase textbooks and curricular resources not adopted by the state as a whole (Silver, 2021). Teachers often rely heavily on a core curriculum resource or textbook to select and implement content and instructional strategies; and thus, selecting textbooks from the abundance of available textbooks in the market becomes a political decision (Polikoff et al., 2020). Moreover, states often follow different procedures to adopt textbooks. For example, some states allow teachers to make suggestions on which textbooks they want to use while other states include parents and school boards on the textbook adoption committees. The degree to which materials purchased with state and local funds are mandated for use by teachers may also be politicized (Cohen & Ball, 1990) as can the degree to which state tests and teacher evaluation metrics are aligned with materials adopted by the state (Mathison & Freeman, 2003). Thus, how textbooks are selected, adopted, mandated, and evaluated at the state level has a direct impact on how much control teachers have over the implementation of their curricular resources. It is therefore important to consider external and sometimes political factors in addition to the many internal factors that influence teachers' perceived control of curriculum (Emirbayer, 1992).

Teacher's control of the curriculum is one aspect of teacher autonomy. Autonomy may be defined as having two components, (a) general autonomy, including classroom standards of conduct, and (b) curricular autonomy, including selecting curricular materials (Pearson & Hall, 1993). In this study, we focused on teacher curricular control, specifically the independent decisions teachers needed to make on selecting and using curricular resources and the degree to which teachers felt empowered to make curricular decisions in a way that benefits students (Dampson et al., 2019).

Politics of Curriculum: The Context of COVID-19

The need to shift to remote learning in the U.S. during the novel coronavirus (COVID-19) pandemic introduced a new and complex situation for K-12 teachers as they had to adjust many aspects of their curricular decisions. While all 50 states announced the closure of schools in March 2020, there was significant variation in terms of the guidelines for remote teaching and the resources teachers were provided with when schools were closed (Reich et al., 2020). For example, some states (e.g., Massachusetts, Illinois, and Kansas) published lists of resources, websites, and subscription applications to facilitate teachers' need to address learning needs across grade levels and content areas as soon as schools were closed while others left teachers with both more freedom and less scaffolding for the unique teaching context of the pandemic. As the pandemic continued past the first few weeks of closure, states and districts continued to develop policies related to levels and types of controls on remote teaching and to the return to in-person instruction. In order to adapt to shifting state policies and evolving understandings of the pandemic, teachers needed to be responsive in their decision-making (Baniahmadi et al., 2021). While we recognize that every state instituted different policies and procedures during the pandemic, historically those decisions have been influenced by state policies (Placier et al., 2002). Additionally, evidence suggests the pandemic response in the U.S. was highly politicized with Republican policymakers more likely to endorse a swift return to "normal" while Democrat policymakers were more likely to endorse mask wearing and social distancing mandates (Grossmann et al., 2021). In this study, we explore the curricular resources used by teachers in

Republican and Democrat states prior to the COVID-19 pandemic and whether those curricular resources changed during the pandemic remote teaching.

Teacher Control During the COVID-19 Pandemic

Teachers all over the globe faced unique opportunities and challenges during the COVID-19 pandemic. The transition to remote learning increased variability in terms of class locations, class times, types of curricula to be selected for students, course content to cover, instructional approaches, learning resources and location, technology use, the requirements for entry/completion dates, and communication medium (Huang et al., 2020). Such variability suggests that teachers may have had the potential to experience more control in terms of curricular use during remote teaching. However, when teachers have more control over curricular choices, they face several challenges in terms of curricular decisions. For example, teachers often struggle to select curricular resources that are appropriate to achieve the assessment requirements while also addressing students' needs (Ormond, 2017), and these effects were likely magnified during the pandemic when student needs and access to resources varied across households (Baniahmadi et al., 2021). Additionally, how teachers use and perceive their control is influenced to some extent by teacher characteristics such as gender, age, teaching experiences, and contextual factors (Dampson et al., 2019).

The COVID-19 pandemic also changed how curriculum was implemented and evaluated. During the transition from face-to-face to remote modalities, many teachers, administrators, and policymakers attempted to create appropriate and equitable learning environments for all learners (Aguliera & Nightengale-Lee, 2020; Huck & Zhang, 2021). Schools, especially those with fewer resources, were challenged to provide access to functioning technology and even to develop materials that were easily accessible to students who had limited access to technology (Doherty et al., 2022; Gross & Opalka, 2020). Teachers needed to use new teaching models that met requirements imposed by schools under the guidance of district and state policymakers while also exploring new online teaching resources. Teachers also needed to consider the resources that students have access to as well as other adjustments as they switched teaching modalities (Aguliera & Nightengale-Lee, 2020). During this period, teachers had to make many curricular decisions to optimize educational resources and promote educational equity (Zhou, 2020).

Methods

Teachers

This study is part of a larger project, *Project C3T2*. The primary aim of the project is to understand and support the curricular decisions that teachers need to make in diverse curricular contexts to provide a coherent sequence of activities while being responsive to the needs of their students. The project was conceptualized prior to COVID-19 and was thus positioned to explore curriculum as schools shifted in relation to the pandemic. The data presented here are drawn from a survey administered in September 2020 to 524 third, fourth, and fifth-grade teachers from 46 states. The majority of the teachers taught in public schools (90%) located in suburban (55%), urban (28%), and rural districts (17%).

Survey

Teachers were asked about their use of mathematics curricular resources in Spring 2020 prior to emergency closures of schools and during the pandemic remote instruction. For example, we asked the teachers why they started or stopped using a certain type of mathematics curricular resources when they moved to remote teaching. They were also asked about their level of control over the usage of their curricular resources and about supports and barriers they experienced to successful remote teaching.

State Politics

We identified the 46 states represented in the data set as Republican, Democratic, and mixed based on how the state voted in the 2016 and 2020 presidential elections. That is, a state that voted for Former Secretary of State Clinton in 2016 and President Biden in 2020 was coded as Democrat, and a state that voted for President Trump in 2016 and 2020 was coded as Republican. States that voted Democrat in one election and Republican in the other were coded as mixed/purple. The analysis presented here focuses on Republican and Democrat states. See Table 1 for state politics coding.

Table 1: State Politics Coding

Politics	States
Democrat	California, Connecticut, Delaware, Hawaii, Illinois, Maine, Maryland, Massachusetts, New Jersey, New York, Oregon, Rhode Island, Vermont, Virginia, Washington, and the District of Columbia.
Republican	Alabama, Arizona, Alaska, Arkansas, Florida, Idaho, Indiana, Kansas, Kentucky, Louisiana, Mississippi, Missouri, Montana, Nebraska, North Carolina, North Dakota, Ohio, Oklahoma, South Dakota, South Carolina, Tennessee, Texas, Utah and Wyoming.

Defining Curricular Resources and Their Types

Curricular resources were defined as including any curricular materials used by teachers for the purposes of planning, teaching, and/or assessment, such as (a) packaged curriculum; (b) individual lesson plans, activities, and materials; and (c) electronic and online resources and apps. In this study, examples were provided to teachers, including textbooks and teacher guides purchased by schools and districts (such as Everyday Mathematics or Houghton Mifflin Math), online curricula (such as EngageNY), online resources (such as IXL, Zearn, or BrainPop), materials downloaded from websites (such as TeachersPayTeachers or Pinterest), or materials teachers create for themselves. We categorized these resources into three categories, depending on how those resources are created and the purpose of the curricular resources, (a) core, (b) supplemental, and (c) teacher created. We categorized a curricular resource as a core resource if it addresses all or nearly all of the given standards for the grade level. Supplemental curricular resources were identified as those used as secondary resources along with the core curriculum. The goal of supplemental curricular resources was to extend activities from core curricular resources in order to meet the needs of individual learners, provide practice opportunities, and make the classroom more engaging. Thus, supplementary curricular resources might align with a subset of standards. Teachers-created curricular resources are the ones created by the teachers and shared with their colleagues personally or through online repositories. These resources are also used to extend the core curricular resources.

Analysis

Independent t-tests were used to compare Republican and Democrat states to answer the research questions. Data reflect two time points: Spring 2020 before the pandemic and remote pandemic teaching after March 2020.

Findings

Teacher Use of Curricular Resources Pre-pandemic

More than half of teachers in Republican states reported using core curricular resources while only about a third of teachers in Democrat states did. The 224 teachers sampled in Republican states (M = .52, SD = .50) compared to the 167 teachers sampled in Democrat states (M = .32, SD = .47) were significantly more likely to report they used a core curriculum pre-pandemic, $t(370.128) = 4.163$, $p < .001$.

The teachers from Republican states (M = .56, SD = .50) compared to the teachers from Democrat states (M = .43, SD = .50) were significantly more likely to report they used teacher-created curricular resources pre-pandemic, $t(389) = 2.496$, $p = .006$. There was no significant difference between teachers' reported use of supplemental curricular resources in Republican (M = .47, SD = .50) and Democrat (M = .41, SD = .49) states pre-pandemic, $t(360.567) = 1.214$, $p = .225$. Overall, the teachers from Republican states (M = 3.05, SD = 1.75) compared to the teachers from Democratic states (M = 2.54, SD = 1.46) reported using significantly more curricular resources than their peers pre-pandemic, $t(389) = 3.083$, $p = .001$.

Summary. Teachers from Republican states were using more curricular resources overall, more likely to have a core curriculum, and more likely to find teacher-created curricular resources (e.g., downloading from Teachers Pay Teachers or designing their own) to support student learning in their classrooms. There was no difference in the use of online supplementary resources such as IXL and BrainPop across Republican and Democrat states.

Teacher Use of Curricular Resources During Pandemic Remote Teaching

Although the use of core curricular resources declined during pandemic remote teaching across both Republican and Democrat states, the teachers sampled in Republican states (M = .44, SD = .50) continued to be significantly more likely to report using core curricular resources compared to teachers sampled in Democrat states (M = .28, SD = .45) during pandemic remote teaching, $t(374.616) = 3.421$, $p = .001$. There was again no difference between teachers' reported use of supplemental curricular resources in Republican (M= .43, SD = .50) and Democrat (M = .43, SD = .50) states during pandemic remote teaching, $t(389) = .473$, $p = .946$. There was also no significant difference between teachers' reported use of teacher-created curricular resources in Republican (M = .46, SD = .50) and Democrat (M = .41, SD = .50) states during pandemic remote teaching, $t(389) = .918$, $p = .180$. Overall, during pandemic remote teaching, there was no significant difference between the total number of curricular resources teachers reported using in Republican (M = 2.79, SD = 1.63) and Democrat (M = 2.55, SD = 1.49) states, $t(389) = .868$, $p = .193$.

Summary. Despite expectations that teachers may need to find new and varied curricular resources to meet the needs of their students during pandemic remote teaching, the number of curricular resources teachers were using declined in Republican states and remained about the same in Democrat states. The decline in Republican states occurred in core, supplemental, and teacher-created resources, resulting in no state-level differences in any curricular category and no state-level differences in the overall number of resources used in Republican and Democrat states.

Table 2: Total Number of Curricular Resources

State Politics	Time Point	Minimum	Maximum	M	SD
Republican	Pre-Pandemic	0	10	3.05	1.75

	Pandemic Remote	0	11	2.70	1.63
Democrat	Pre-Pandemic	0	7	2.54	1.45
	Pandemic Remote	0	7	2.56	1.49

Teacher Control During Pandemic Remote Teaching

On a scale from 0 (“none at all”) to 3 (“complete”) control, the teachers from Republican states (M = 1.42, SD = .91) compared to the teachers from Democrat states (M = 1.15, SD = .94) reported having significantly more control over their curricular resources than their peers, $t(389) = 2.894$, $p = .002$. In fact, the majority of teachers in Republican states (54%) said they had "complete" control over their curricular use. Complete control was much less common in Democrat (27%) states. In contrast, the largest percentage of teachers indicated they had no control in Democrat (41%) as compared to 31% in Republican states.

Table 3: Teacher Control over Curricular Resources in States

Teacher Control	Republican	Democrat
Complete	54%	27%
A lot	48%	29%
A bit	43%	30%
None at all	31%	41%

Student Access to Technology During Pandemic

On a scale from 1 (“25% or less of students had sufficient access to technology”) to 4 (“75% or more of students had sufficient access to technology”) during pandemic remote teaching, 224 teachers from Republican states (M = 2.92, SD = 1.00) compared to 167 teachers from Democrat states (M = 3.17, SD = .93) reported having significantly fewer students who had sufficient access to technology, $t(328) = -2.354$, $p = .019$.

Table 4: Student Access to Technology

Sufficient Access to Technology	Republican	Democrat
76% or more	35%	45%
41-75%	33%	34%
26-50%	21%	13%
25% or less	11%	8%

Teacher Preparedness for Pandemic Remote Teaching

On a scale from 1 (“not at all”) to 4 (“completely”) of how well teachers felt prepared for pandemic remote teaching, there was no significant difference between 222 teachers from Republican states (M = 1.51, SD = .65) compared to 167 teachers from Democrat states (M = 1.51, SD = .65), $t(360.248) = .500$, $p = 1.00$.

Table 5: Teacher Preparedness

Teacher Preparedness	Republican	Democrat

Completely	0%	1%
A lot	9%	7%
A bit	33%	36%
Not at all	58%	57%

Discussion

Our pre-pandemic findings support that state-level political differences may be worth exploring in curricular policy. In particular, these findings suggest that teachers in Republican states were more likely to use core curricular resources and more likely to use teacher-created curricular resources. In addition, teachers in Republican states were using more curricular resources than their peers in Democrat states pre-pandemic. There were no significant differences between Republican and Democrat state teachers in terms of supplemental curricular resource usage.

During the initial remote teaching response to the pandemic, the number of resources used by teachers decreased somewhat for both Republican and Democrat state teachers. Although Republican state teachers continued to be significantly more likely to use core curricular resources than their peers in Democrat states, no differences were found between states for supplemental and teacher-created curricular resources and overall, there were no longer any differences in the number of resources used by teachers in Republican and Democrat states.

Limitations in this study included that the data were generated from teacher self-report and that while there was an attempt to secure a sample across states, there were no structures in place to ensure that teachers were sampled equitably across Republican and Democrat political boundaries. Additionally, the timing of the survey (during the first fall semester after the initial onset of the pandemic) may have influenced teachers' time and ability to respond as they were still engaging in rapid instructional shifts due to pandemic policies that differed across state lines.

Educators and policymakers were concerned that teachers had to create new curricular resources when they moved to remote teaching (Huang et al., 2020), but the findings presented here suggest that many teachers were not focused on creating new materials during the initial pandemic push to remote instruction, and instead the findings related to teacher-created curricular resources are consistent with other research (e.g., Aguliera & Nightengale-Lee, 2020; Gross & Opalka, 2020) found that since teachers had to quickly pivot to new teaching models, they focused less on creating new curricular materials during the pandemic. Another potential reason for the decline in the overall number of resources teachers were using and in the likelihood for teachers to use teacher-created resources could be because teachers did not have as much experience creating curricular resources for fully remote/online teaching. Teachers continued use of core curriculum before and during the pandemic in Republican states also support the claim that teachers tended to use ready-made curricular resources instead of investing time in creating new curricular resources while dealing with other challenges.

Research has shown that how rapidly states resumed face-to-face instruction was associated with political interests (Valant, 2021). In addition, the finding that Republican state teachers reported having significantly fewer students who had access to sufficient technology to engage in pandemic remote teaching may have influenced those states' decisions to reopen schools even beyond politics. However, we do not have sufficient evidence if this factor influenced the findings of our study.

There is no doubt that teachers were asked to make rapid instructional changes as schools shut down and remote teaching was instituted across the country. These findings suggest that teachers in Republican states felt like they had more curricular control in responding to emergency shifts. However, there was no difference in how well-prepared teachers in Republican and Democrat states felt to meet the needs of remote instruction, with teachers across the country responding they felt, on average, between "not at all" and "a bit" prepared to take on the challenges. This finding suggests that though teachers may have control over their curricular resources, there may be other barriers that prevent them from taking advantage of their autonomy (e.g., lack of time, preparation, and technological support for online instruction).

As suggested by Dampson and colleagues (2019), how teachers perceive and use their curricular control depends on contextual factors, including the modality of teaching and likely the impacts of the pandemic in their community. As a practical implication, how teachers benefit from and make use of teacher control is largely influenced by external factors, thus, we recommend policymakers consider providing teachers with the opportunities to learn about curricular decision-making and where they can find resources to make those curricular decisions. An additional recommendation may be to ensure that teacher's voice is included in state legislatures and school board debates about curriculum when enacting policies that impact teachers' usage of curriculum. Our research team is currently conducting a series of individual and focus-group interviews with teachers to explore how teachers' curricular decisions are influenced by several factors such as teacher control in less extreme circumstances.

Acknowledgments

This material is based upon work funded by a grant from the National Science Foundation, Grant Number: 1908165.

References

Aguliera, E., & Nightingale-Lee, B. (2020). Emergency remote teaching across urban and rural contexts: Perspectives on educational equity. *Information and Learning Sciences, 121*(5-6), 471-478. https://doi.org/10.1108/ILS-04-2020-0100

Baniahmadi, M., Olson, A.M., Wood, M. B., Giorgio-Doherty, K., Newton, J., Drake, C., & Sapkota, B. K. (2021). Curriculum Resources and Decision-making: Findings from a National Survey of Elementary Mathematics Teachers. *Association of Mathematics Teacher Educators (AMTE) Connections Journal*. Available at: https://amte.net/connections/2021/12/curriculum-resources-and-decision-making-findings-national-survey-elementary

Cohen, D. K., & Ball, D. L. (1990). Policy and practice: An overview. *Educational Evaluation and Policy Analysis, 12*(3), 233-239.

Dampson, D. G., Apau, S. K., & Amuah, U. (2019). Freedom to choose within limits: Teacher autonomy from the perspectives of basic school teachers in Ghana. *European Journal of Educational Management, 2*(1), 35-44. https://doi.org/10.12973/eujem.2.1.35

Doherty, K., Baniahmadi, M., Sapkota, B. K., Newton, J., Olson, A. M., Wood, M., & Drake, C. (2022). *Exploring Dissonance and Harmony in Elementary Mathematics Teachers' Curricular Use, Autonomy, Decision-Making, and Coherence.* In D. Lischka, A. E., Dyer, E. B., Jones, R. S., Lovett, J. N., Strayer, J., & Drown, S. (Eds.) The forty-fourth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Nashville, TN.

Doyle, W. (1992). *Curriculum and pedagogy*. In P. W. Jackson (Ed.), Handbook of research on curriculum (pp. 486–516). Macmillan.

Emirbayer, M. (1992). Beyond structuralism and voluntarism: The politics and discourse of progressive school reform. *Theory and Society, 21*(5), 621-664. http://www.jstor.org/stable/657804

Gross, B., & Opalka, A. (2020). Too many schools leave learning to chance during the pandemic. *Center for Reinventing Public Education*. https://www.crpe.org/sites/ default/files/final_national_sample_brief_2020.pdf

Grossmann, M., Reckhow, S., Strunk, K. O., & Turner, M. (2021). All states close but red districts reopen: The politics of in-person schooling during the COVID-19 pandemic. *Educational Researcher, 50*(9), https://doi.org/637-648.10.3102/0013189X211048840

Huang, R. H., Liu, D. J., Tlili, A., Yang, J. F., & Wang, H. H. (2020). Handbook on facilitating flexible learning during educational disruption: The Chinese experience in maintaining undisrupted learning in COVID-19 Outbreak. *Beijing: Smart Learning Institute of Beijing Normal University, 46.*

Huck, C., & Zhang, J. (2021). Effects of the COVID-19 pandemic on K-12 education: A systematic literature review. *New Waves-Educational Research and Development Journal, 24*(1), 53-84.

Mathison, S., & Freeman, M. (2003). Constraining elementary teachers' work: Dilemmas and paradoxes created by state mandated testing. *Education Policy Analysis Archives, 11*, 34-34.

Ormond, B. M. (2017). Curriculum decisions - The challenges of teacher autonomy over knowledge selection for history. *Journal of Curriculum Studies, 49*(5), 599-619. https://doi.org/10.1080/00220272.2016.1149225

Pearson, L. C., & Hall, B. W. (1993). Initial construct validation of the teaching autonomy scale. *The Journal of Educational Research, 86*(3), 172–178. https://doi.org/10.1080/00220671.1993.9941155

Placier, M., Walker, M., & Foster, B. (2002). Writing the “Show–Me” Standards: Teacher Professionalism and Political Control in US State Curriculum Policy. *Curriculum Inquiry, 32*(3), 281-310.

Polikoff, M. S., Campbell, S. R., Koedel, C., Le, Q. T., Hardaway, T., & Gasparian, H. (2020). The formalized processes districts use to evaluate mathematics textbooks. *Journal of Curriculum Studies, 52*(4), 451-477. https://doi.org/10.1080/00220272.2020.1747116

Reich, J., Buttimer, C. J., Fang, A., Hillaire, G., Hirsch, K., Larke, L. R., … Slama, R. (2020, April 2). *Remote Learning Guidance from State Education Agencies During the COVID-19 Pandemic: A First Look.* Massachusetts Institute of Technology teaching. https://doi.org/10.35542/osf.io/437e2

Savin-Baden, M., & Tombs, G. (2017). *Research methods for education in the digital age*. Bloomsbury.

Silver, D. (2022). A theoretical framework for studying teachers’ curriculum supplementation. *Review of Educational Research, 92*(3), 455-489. https://doi.org/10.3102/00346543211063930

Valant, J. (2020). *School reopening plans linked to politics rather than public health.* Brown Center Chalkboard. https://www.brookings.edu/blog/brown-center-chalkboard/2020/07/29/school-reopening-plans-linked-to-politics-rather-than-public-health/

Wahlström, N. (2023). Curriculum policy, teaching, and knowledge orientations. *Teachers and Teaching, 29*(3), 259-275, https://doi.org/10.1080/13540602.2022.2159364.

Zhou, Z. (2020). On the lesson design of online college English class during the COVID-19 pandemic. *Theory and Practice in Language Studies, 10* (11), 1484-1488. http://dx.doi.org/10.17507/tpls.1011.21

ELEMENTARY TEACHERS' USE OF MATHEMATICS CURRICULAR MATERIALS: FOCUS ON TEACHERS PAY TEACHERS AND PINTEREST

Bima Sapkota
Ball State University
bima.sapkota@bsu.edu

Kristin Doherty
Michigan State University
kdoherthy@msu.edu

Marcy B Wood
University of Arizona
mbwood@email.arizona.edu

Jill Newton
Purdue University
janewton@purdue.edu

Keywords: Curriculum, Elementary School Education, Instructional Activities and Practices

Perspectives, Objectives, and Method of Inquiry

In a 2021 survey of 500 elementary teachers, most respondents (85%) reported using at least one online mathematics curricular resource. Teachers Pay Teachers (TPT) was the most widely used resource, with at least 40% of the teachers using it; Pinterest, another website offering teacher-made resources, was used by approximately 15% of the teachers (Baniahmadi et al., 2021). Given this widespread use along with reports of use from the five case study teachers, we further investigated their use of TPT and Pinterest with a series of questions, asking them (a) which types of mathematics curricular materials from these sources they find most useful, (b) to provide examples of resources used from each website along with a rationale for their use, and (c) to share their strategies for choosing and evaluating the resources.

Findings

Although all five teachers reported using TPT and four teachers used Pinterest, the frequency of their use of these resources varied significantly. Mary, the teacher who reported rarely using TPT, only turned to these online resources when a standard was not addressed in the primary curriculum. Audrey and Sarah, second-grade teachers, reported using TPT once a week to

enhance students' learning. In contrast, Jamie reported using TPT every day, saying "I'd say materials I found on Teachers Pay Teachers or other sites like that is probably my primary

source." The teachers reported having complete control of their use of Pinterest and TPT; that is, the district did not require or monitor their use of these resources. Teachers, however, mentioned that they usually had to pay for these materials themselves. They reported using TPT for multidisciplinary project- and theme-based lessons. Audrey planned a week-long baseball unit in which students investigated how to make the Commissioner's Cup hold a baseball. Sarah, a self- proclaimed lover of science, found a STEM unit focused on plant life cycles and pollination that highlighted cross-curricular connections, and students in Kasie's class learned about the concept of area in their design of a leprechaun village for St. Patrick's Day. The teachers reportedly used Pinterest less often than TPT, but they expressed appreciation for "creative" materials they could find for anchor charts and bulletin boards, as well as "ready-made" resources. When asked how they made decisions about the materials to use from TPT and Pinterest, teachers reported multiple criteria and processes utilized for their selection, including search filters; checking previews of particular materials to establish an appropriate level of mathematical "rigor," and reading teachers' reviews. They expressed appreciation for the community of teachers engaged on TPT and Pinterest and the evaluations of resources written by current elementary teachers.

References

Baniahmadi, M., Olson, A.M., Wood, M. B., Doherty, K., Newton, J., Drake, C., & Sapkota, B. K. (2021). Curriculum Resources and Decision-making: Findings from a National Survey of Elementary Mathematics Teachers. Association of Mathematics Teacher Educators (AMTE) Connections Journal, 31(2). Available at: https://amte.net/connections/2021/12/curriculum-resources-and-decision-making-findings-national-survey-elementary

ESTABLISHING STATISTICAL SIGNIFICANCE FOR COMPARISONS USING PATTERN-BASED ITEMS: CHANGE AT SCALE

Walter M. Stroup
University of Massachusetts, Dartmouth
wstroup@umassd.edu

Anthony Petrosino
Southern Methodist University
apetrosino@smu.edu

Corey Brady
Southern Methodist University
corey.brady@smu.edu

Karen Duseau
Bridgewater State University
kduseau@bridgew.edu

Tests of statistical significance often play a decisive role in establishing the empirical warrant of evidence-based research in education. The results from pattern-based assessment items, as introduced in this paper, are categorical and multimodal and do not immediately support the use of measures of central tendency as typically related to interpretations of measures of statistical significance. Responses from the duplicate implementation of selected pattern-based items (PBIs) in successive grades (3-8) as part of the statewide Interim Assessment Program in Texas are used to illustrate how non-parametric methods can be used to establish statistically significant comparisons of student results. Not all the repeat-item results improved across years.

Keywords: Assessment, Policy, Research Methods, Systemic Change

During an assessment window from August 31, 2018 through March 31, 2019 a series of non-dichotomous, or "pattern-based," items, focused primarily on mathematics but also including reading items, were implemented with more than 400,000 students in grades three through eight on a statewide assessment program. The items were part of an opt-in Interim Assessment Program made available to districts by the Texas Education Agency (TEA). A central question addressed in a report to TEA (Stroup, 2020) was whether the greater information density of pattern-based items (PBIs) would provide significant, actionable insights into student learning outcomes. A central focus of this paper, however, is to use results from the statewide implementation to outline how measures of statistical significance can be established for PBIs.

Results from pattern-based assessment items, as presented below, are categorical and multimodal and do not immediately support the use of measures of central tendency, as typically related to interpretations of measures of statistical significance. As will be illustrated with across-grade results, using either a Fisher's Exact Test or a Pearson's Chi-squared Test we can evaluate the statistical significance of differences in the multimodal patterns of responses. Then, once a level of statistical significance is arrived at, projections onto axes of relative performance – e.g., assigning "partial credit" or using exact full credit scoring – can be used for comparisons of achievement-related outcomes for groups of students.

Pattern-based items were developed by teachers and classroom-focused researchers to provide colleagues -- as part of supporting ongoing instruction -- with a more detailed "picture" of students' understandings (Stroup, 2020). Consistent with this year's conference theme, classroom assessments, support by shared display capabilities, immediate forms of feedback, and suggestions for further inquiry, should center on engaging *all* learners. Accordingly, pattern-based approaches to assessment are meant to support students' full participation in ongoing classroom-based teaching and learning. However, educator reports of the effectiveness of pattern-based approaches in supporting ongoing instruction may not be enough. We live at a time when phrases like “evidence-based practice” are assumed to denote a highly constrained set of methodological commitments. Absent ways of engaging in forms of hypothesis testing, centered on measures of statistical significance, it may be too facile for critics of understandings-focused classroom assessment (*cf.*, Bennett, 2011) to attempt to limit the reach of educator-driven, pattern-based approaches to assessment. In response, the focus of this paper is on the use of PBIs to support making empirically warranted claims that might, for example, be plausibly related to differences in instruction between individual classrooms, schools, districts, or states and/or to differences in instruction organized in terms of temporal change, or “growth”, in student achievement.

What is a Pattern-Based Item?

Pattern-based items are developed to provide a significant, and fully scalable, alternative to current, dichotomously scored and analyzed, items. While PBIs may appear similar to standard multiselect multiple-choice items, they should be seen as fundamentally distinct, both in terms of how they are developed and in terms of how they are analyzed.

For an individual assessment question with four responses – A, B, C, or D – both multiselect and pattern-based items allow the students to select, or endorse, more than one response (e.g., “A and C” or “B, C, and D”). While Rasch-based partial credit models have been explored since the early 1980’s (Masters, 1982), it remains the case that standard multiselect items have only one combination, the “exact match”, be scored as correct (“1”). All other combinations are typically scored as incorrect (“0”). While these “legacy” multiselect multiple-choice items might be viewed as an improvement over the more commonly used single-select multiple-choice items, the subsequent (non-polytomous) analyses, for both single-select and multiselect items, continue to be based, in most cases, on only two “states”, either 1 or 0.

In contrast, the pattern-based items appearing on the Interim Assessments in Texas were developed to work with the full combinatoric space of student responses. This means that for the PBIs discussed below, there are sixteen possible selection combinations of the four available responses (or fifteen, if “no response” is not included). Rather than continue the current practice of reducing these states to only two states (i.e., correct/incorrect), the greater information density of pattern-based items (sixteen states versus two states) is meant to allow for more detailed, actionable insights related to student learning outcomes. More information about their students supports teachers in improving student outcomes. Pattern-based assessments can be shorter and more informative than legacy assessments in ways that are meant to directly support better instruction. Although not discussed in this paper, pattern-based items or tasks are not limited to interactions where the student selects among given responses (*cf.* Stroup [1996], Stroup & Wilensky [2000]).

Differences in the patterns in responses of student groups to PBIs can be analyzed across scale and also over time, allowing for more detailed longitudinal evaluation of teaching and learning at the class, school, district, and state levels. Based on these uses of PBIs, our task is to provide a framework for establishing measures of item-level statistical significance as illustrated using observed differences in student results across grade levels. The framework is intended to serve as a widely-applicable, principled, approach to the determination of significance for comparisons using pattern-based items and tasks.

Implementation

The students participating in the state-sponsored Interim Assessment Program were required to complete a series of single-select multiple-choice State of Texas Assessments of Academic Readiness (STAAR) items prior to then having the option to complete the pattern-based items. The directions stated: “The next set of questions is optional. These questions will not be counted as part of your score.” Moreover, due to terms agreed to by the Texas Education Agency (TEA) and the schools participating in the Interim Assessment Program, no datasets containing student-, school-, or district-identifiable data would be provided by the vendor to the Agency.

Consistent with these terms, we received from the TEA two, large, fully anonymized, datasets containing only each student’s selections for the pattern-based items. As a result, the data cannot be used to provide an account of the makeup of the students who participated in the overall Interim Assessment Program or of those who then elected to complete the optional items. Comparisons between results for dichotomous legacy items and results for PBIs on the Interim Assessments are also precluded. To be able to illustrate how PBIs support comparisons across scale – e.g., comparing individual classroom results or school-wide results with statewide results – in the Report to TEA (Stroup, 2020) we augmented the datasets from the Interim Formative Assessment Program with datasets from a December 2018 pilot implementation of the same items in two elementary schools in central Texas.

The average completion rate for the last item was more than 98% of the average completion rate for the first item. If there were difficulties with the implementation of items at scale, then one would expect much less consistency in the levels of participation. No issues with the statewide implementation were reported.

Comparing changes across grades

Even absent information about student, school, district, or temporal implementation (within the seven-month window) in the Interim Assessment Program dataset provided by the Texas Education Agency, the sensitivity of pattern-based items to one kind of overall student growth can be assessed by comparing results for items deployed across grade levels. The pattern-based item shown in Figure 1 assesses student understanding of equivalent ratios in relation to adding drops of blue food coloring to water to create a blue solution. As also shown in Figure 1, the fifteen letter combinations (“no response” is not included) can be shaded and sorted from lowest partial credit score (light shading) on the left, up to the full credit score (dark shading) on the right.

Partial credit, in this context, is assigned in terms of the degree of match with the full-credit response. With PBIs, not selecting an incorrect response is typically assigned the same partial credit as selecting a correct response. For this item, then, full credit is assigned to selecting B, C, and D and not selecting A. The histograms of results shown in Figure 1 are sorted from zero partial credit for only selecting the one incorrect response, A, up to the full-credit response of BCD (where not selecting A is implicit in A not appearing in labelling of the histogram bin). The assigning of credit in this way is treated as a projection from the combinatoric space of the students’ actual selected responses onto a single axis of relative performance.

Of course, other partial credit projections are also available. The most widely deployed among these alternatives might be the partial credit models used by most learning management systems (*cf.* Jones, [2016]). Assuming that whatever model is deployed is consistent in how the multimodal results from pattern-based items are projected onto a partial credit axis, if statistical significance is established at the level required, then comparisons of partial credit results may be used to represent relative changes in student outcomes or as an overall measure of effect.

A key curricular transition in moving from late elementary grades to middle school grades is extending emergent multiplicative forms of reasoning about fractions to comparisons of ratios and proportions. To be able to assess the development of students' ratio and proportional reasoning as they enter, and then begin to move through, middle school, this pattern-based item was included on both the grade 6 (N=58,947) and grade 7 (N=46,483) Interim Assessments. To further situate how it is that the use of the same item across grade levels can be informative at both levels, we begin with a brief discussion of some of the state's related curriculum standards.

A Grade 6 mathematics Texas Essential Knowledge and Skills (TEKS) standard requires students to be able to "apply qualitative and quantitative reasoning to solve prediction and comparison of real-world problems involving ratios and rates" (6.b.4.B) and "give examples of ratios as multiplicative comparisons of two quantities describing the same attribute" (6.b.4.C). To satisfy the Grade 7 TEKS, students must be able to "solve problems involving ratios" framed more explicitly in terms of proportional reasoning (7.b.4.D). With the item shown in Figure 1, students are assessed on their ability to use "multiplicative comparisons of two quantities" (drops of food coloring and amount of water) in a real-world context to describe the "same attribute" of what would be the blueness of the resultant solutions.

At both grade levels, students will often attempt to extend additive forms of reasoning to a task that requires multiplicative comparisons. Response A is intended to assess whether students are attempting to reason additively – adding 10 to both the number of drops and to the amount of water – about a task requiring multiplicative reasoning. The difference graph shown in Figure 1 reflects a 6.8% decrease in this pattern of reasoning: moving from 16.4% of the grade 6 students selecting only A to 9.6% of the grade 7 students selecting only A.

In the context of the design of pattern-based items, this decrease illustrates the significance of students' *not* selecting a response in contributing to an overall assessment of the depth of their understanding. In contrast to this additive, incorrect, answer, response B correctly multiplies the 5 drops of food coloring by 2 and the 40 milliliters of water by 2. Response D requires students to simplify the given ratio of 5 drops of food coloring to 40 milliliters of water to the equivalent "unit ratio" of 1 drop of food coloring to 8 milliliters of water, and then correctly multiply each quantity by 6. Response C also requires this simplification of the original ratio and then correctly multiplying each quantity by 0.2. Increasing depth of understanding is assessed in this pattern-based item in moving from not selecting A, to selecting B, to selecting D and then, at the highest level of understanding, selecting C.

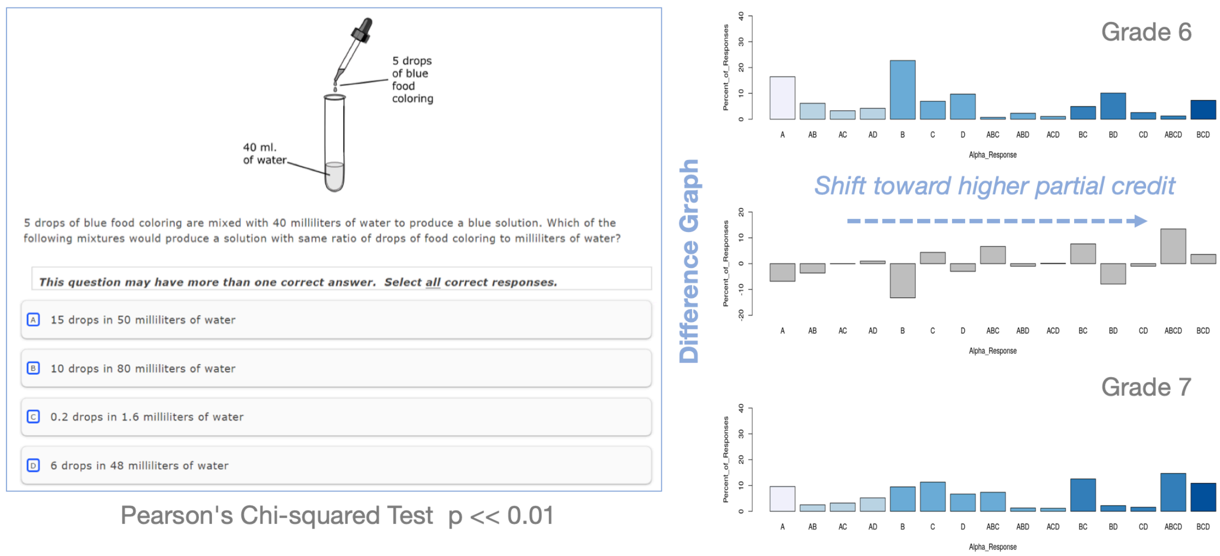


Figure 1. Analysis of differences in item results at scale.

The graph in the middle of Figure 1 shows all the relative changes in the percentages for each of the combinations of responses. Although the shifts can appear complex across the various combinations, there remains general movement from lower partial credit responses to higher partial credit responses. The respective percentages receiving full credit are 7% for grade 6 and 11% for grade 7 and the respective average partial credit scores are 47% and 56%.

Table 1 shows the contingency table for the 15 combinations of responses for each grade level.

Table 1: Observed frequencies of responses for IAP implementations of the same pattern-based mathematics item in grades 6 & 7

Responses	A	AB	AC	AD	B	C	D	ABC	ABD	ACD	BC	BD	CD	ABCD	BCD
Students															
IAP Grade 6	9625	3589	1919	2470	13297	4072	5684	427	1350	615	2871	5919	1515	736	4275
IAP Grade 7	4466	1164	1495	2418	4409	5255	3113	3431	601	553	5830	1033	738	6828	5054

The selection of the respective combinations and the result across grades (no students took both grade-level assessments) can be reasonably assumed to be independent. Then, for this comparison, the substantial number of students in all the cells at each grade level supports the use of a non-parametric Pearson's Chi-squared test with simulated p-value to evaluate statistical significance. With the p-value being 0.0005, these results are highly statistically significant.

The very low p-value supports the subsequent claim that positive differences in the projected outcomes on this item are of high statistical significance. Specifically, the positive differences in full credit and partial credit results for grade 7 compared to grade 6 can be represented as statistically highly significant. This ability to show relative improvement over time from grade 6 to grade 7, using any one of a number of possible projected metrics, illustrates the ability of pattern-based items to be used to assess changes in scores at scale. We would even argue that this logical transfer of statistical significance would extend to the 6.8% decrease in moving from 16.4% of the grade 6 students selecting only A to 9.6% of the grade 7 students selecting only A. Attending to students' not choosing certain responses is important to the development, analysis, and pedagogical utility of PBIs.

Extending the Utility of Reporting Effect in Terms of Partial Credit Projections

The evaluation of variations in outcomes for specific within-year treatments – e.g., a new curriculum – or comparisons of longitudinal growth for given groups of students would require, as would be the case with existing legacy items, a more sophisticated research design than was possible within the structure of the Interim Assessment Program in Texas. We can, however, illustrate how the evaluation of statistical significance, when combined with partial credit projections, can provide an account of overall levels of standards-specific achievement across years.

Five PBIs were repeated across grade levels. Figure 2 depicts the partial credit results for each of these items by grade level. Each point represents between forty and sixty thousand students in each grade. Scores for three of the items increased in ways suggesting improvement in the levels of standard-specific attainment across the respective grades.

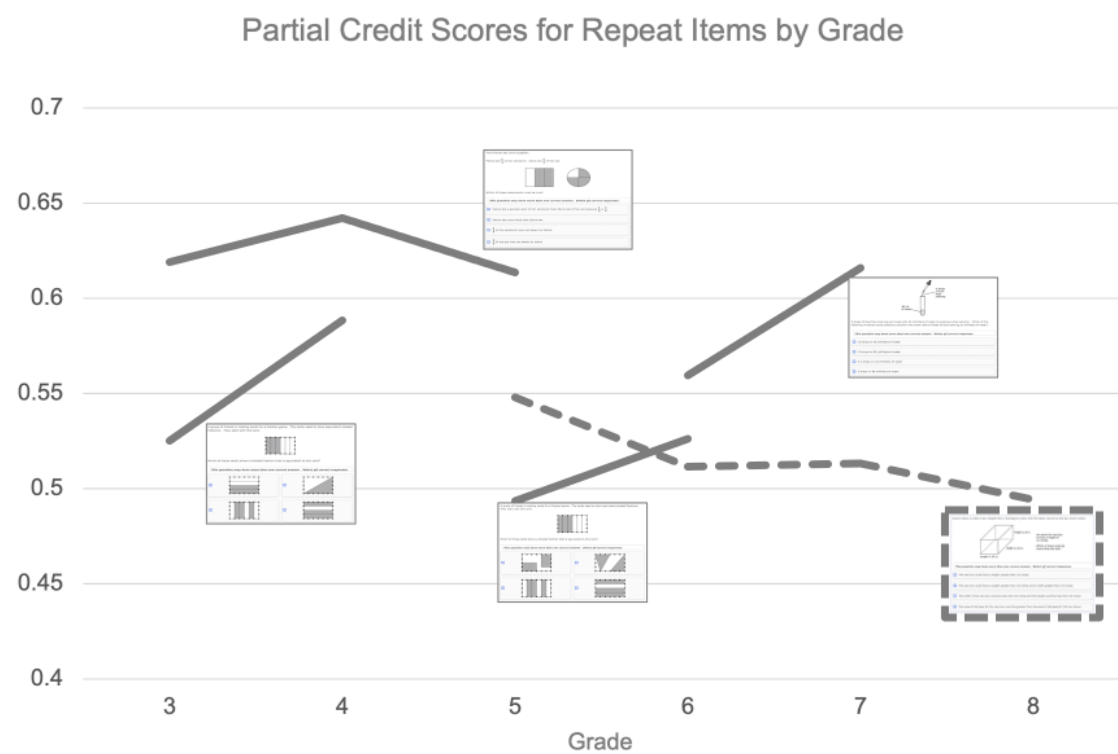


Figure 2. Partial credit results for repeated pattern-based items deployed across the grade levels shown.

Results for one repeated fractional-comparison question went up and then down. Of greater concern, however, is how the partial credit score for a volume item (the dashed line in Fig. 2) went down from grade five to grade eight. It considers a new box with the same volume and one known dimension.

Volume appears in the state curriculum standards for each grade assessed. As shown in Figure 3, the question starts with a given rectangular box and considers a new box with the same volume and one known dimension.

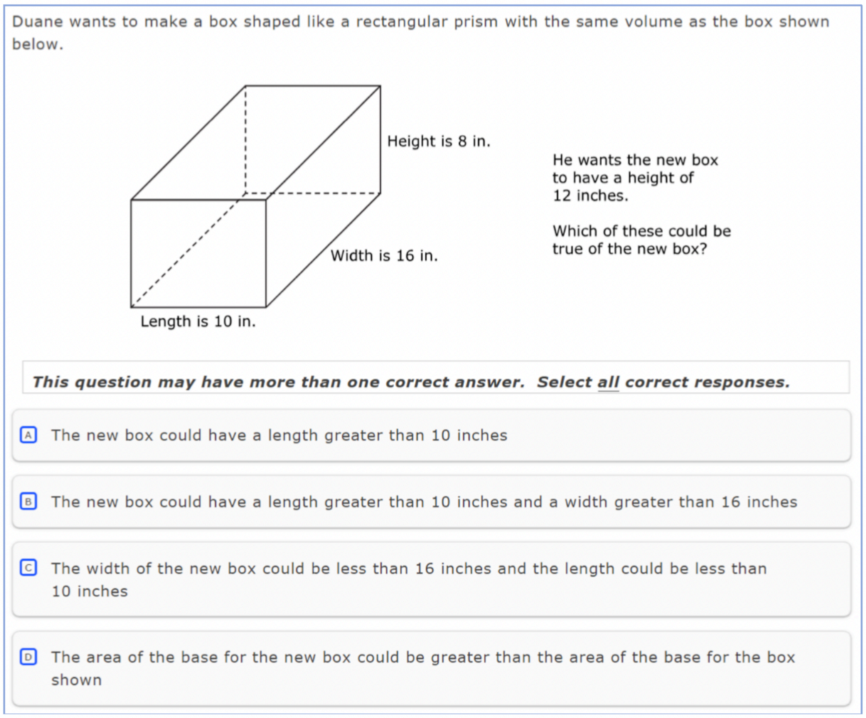


Figure 3. Volume item for which partial credit scores trended downward.

The student is asked about various possible changes to the box that could keep the volume the same (Stroup, 2020). When we've asked educators what might explain the decrease in scores for the volume item, some have suggested there may be an over-reliance in upper grades on formalisms to give "the answer." Students may come to rely on "the formula" (Volume = length*width*height) in a way that leaves them less able to think conceptually about how a specified change in one dimension could be compensated for with possible changes in the remaining dimensions.

Independent of what the explanations or the possible pedagogical responses might be for specific scores on a given item going up or down, the larger point, illustrated by the graph shown in Figure 2, is that the use of partial credit projections is a relatively transparent way to interpret results having statistical significance. Partial credit scores going up, or down, by some percent is likely to be more widely understood than, for example, using Cramer's V as a way to discuss effect size. The generation of a Cramer's V value between 0 and 1 can be considered relatively opaque, certainly in comparison to calculating partial credit. Moreover, the values used to distinguish between "weak" (0.1-0.3), "medium" (0.4 to 0.5), and "strong" (>0.5) associations using Cramer's V are simply conventions that have evolved to become accepted standards in practice. As is illustrated in Figure 2 and in an earlier example, we would suggest using comparisons of partial credit results as a preferred way of characterizing relative differences in student outcomes.

Summary and Conclusion

In order for assessments based on the use of pattern-based approaches to be more widely deployed as a significant, and fully scalable, alternative to current assessments using dichotomously scored and analyzed items, the results from PBIs must be useful to educators' efforts to engage and support all learners. This, however, may not be enough. Results from assessments are frequently used to make statistically warranted claims about comparisons in outcomes. There need to be methods for evaluating the statistical significance of differences in PBI results for groups of students. The utility of PBIs for educators is addressed more directly elsewhere. This paper, instead, has focused on the issue of evaluating statistical significance.

Acknowledgments

This work was supported by funding from the National Science Foundation under Grant No. DRL-16152 (Group-based Cloud Computing for STEM Education). The views expressed are solely those of the authors.

References

Bennett, R. E. (2011). Formative assessment: A critical review. *Assessment in Education: Principles, Policy & Practice, 18*(1), 5-25. https://doi.org/10.1080/0969594X.2010.513678

Jones, J. (2016, April 21). *Understanding multiple answers questions*. Instructure. https://community.canvaslms.com/t5/Higher-Ed-Canvas-Users/Understanding-Multiple-Answers-Questions/ba-p/263903

Masters, G. N. (1982). A Rasch model for partial credit scoring. *Psychometrika*, 47, 149–174. https://doi.org/10.1007/BF02296272

Stroup, W. (1996). *Embodying a nominalist constructivism: Making graphical sense of learning the calculus of how much and how fast* (Unpublished doctoral dissertation). Harvard University, Cambridge, MA.

Stroup, W. M., & Wilensky, U. (2000). Assessing learning as emergent phenomena: Moving constructivist statistics beyond the bell curve. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research in mathematics and science education*. Mahwah, NJ: Lawrence Erlbaum Associates.

Stroup, W. M. (2020). *Interim Assessment Pattern-Based Item Report* (Unpublished report). Submitted to the Texas Commissioner of Education on December 12, 2019 and approved for release on June 12, 2020. Retrieved from https://drive.google.com/file/d/1krNGNe97YIe1GVCo0g-vz9zEnX0r09Ie/view?usp=sharing

EXAMINING THE VALIDITY OF A GENERATIVE EDUCATION PATTERN BASED QUESTION

Karen Leary Duseau
University of Massachusetts, Dartmouth
kduseau@umassd.edu

Assessment is a topic of concern to all stakeholders in our educational system. Pattern Based Questions are an assessment tool which is an alternative to the standardized assessment tool, and they are based on generative learning pedagogy, which shows promise in engaging all learners and usefulness in teaching and learning but validity has not yet been empirically established. Pattern-based questions seek to provide a qualitative expression of student understanding. It is the purpose of this research to empirically explore a correspondence between student response patterns and the students' expressed ways of thinking using a grounded theory approach and clinical interviews. Findings include rich descriptions of participants' ways of thinking about equivalent fractions as written in the PBQ. One response pattern was clearly differentiated from the others. Future research is discussed.

Keywords: Assessment, Rational Numbers & Proportional Reasoning, Undergraduate Education.

Introduction

Assessment is a concern for all stakeholders in education. Current quantitative standardized assessment has been critiqued for focusing on procedural understanding instead of conceptual understanding (Herman et al., 1992; Kohn, 2000; Schwartz, 1991; Stake, 1995), for promoting inequities in education (Knoester & Au, 2017; Noble et al., 2012; Stroup, 2009), for not accurately representing student knowledge (Noble et al., 2012; Stake, 1995), and for promoting teaching to the tests (Riffert, 2005). Pattern-based questions (PBQs) are a qualitative alternative which can be used at scale. PBQs are an assessment tool associated with a larger pedagogical theory called generative education (Stroup et al., 2004; Wittrock, 1974). (in prior work PBQs have been referred to as PBIs – pattern-based items and NDMC – non dichotomous multiple choice.) Learning environments designed using generative education pedagogy can provide students with opportunities to engage in activities and gain a deep understanding of meaningful mathematics (Duseau, 2019). As a tool of generative education, PBQs promote engaging *all* students by utilizing a "low threshold, high ceiling" (Papert, 1980/1993) approach. Students may participate at a low level of understanding (low threshold) or at a high level of understanding (high ceiling). Pattern-based questions seek to provide a qualitative expression of student understanding which does not reduce student understanding down to a single number. Some Educators believe that the response patterns of PBQs identify patterns in student thinking. This allows students to show what they do know instead of focusing on their deficits. Educators have created PBQs using what they know about students' common understandings of mathematical material. To date, however, there has been no systematic investigation of the empirical warrant for the correspondence between students' response patterns on PBQs and students' verbalized understanding of mathematical concepts. Therefore, my research questions are:

- Can this pattern-based question identify students' ways of understanding equivalent fractions? If so, what are these ways of thinking?

- Can some correspondence be established between students' response patterns and students' verbalized understanding of mathematical concepts?

Theoretical Framework

PBQs are an assessment tool associated with the larger pedagogical theory of Generative Design. Generative design is a pedagogical theory of learning and teaching in which students generate artifacts in order to actively engage in the mathematics, make sense of the mathematics, make connections to other mathematics, and to be creative (Stroup et al., 2004). Stroup, Ares, and Hurford (2004) describe generative design as a "space creating" pedagogy in which students create "a space - a coordinated collection – of expressive artifacts and actions in relation to some shared task or set of rules" (p. 839). This means the students are creating artifacts such as equations, functions, or expressions which share some property (or rule) and the collection of those artifacts should enlighten the students to some structure in mathematics. PBQs can be developed using that same space, or collection of student created artifacts because they represent the students own conceptions of a mathematical idea. PBQs are intended to display the current state of students' understanding and support educators' analyses of those understandings.

This study framed the research through a constructivist lens and used grounded theory (Charmaz, 2014) and clinical interviews (Ginsburg, 1997) as tools. Charmaz's (2014) perspective on grounded theory is appropriate because she claims a constructivist view of grounded theory as opposed to Glaser and Strauss who have a more post-positivistic view. Generative design, as discussed previously and as linked to the development of PBQs, has emerged from the constructivist tradition.

Piaget is credited for introducing the clinical interview method to contrast with the standardized administration of tests. Piaget (1954) believed that children construct their own knowledge or understanding. His methodology of the clinical interview is aligned with his understanding of the constructivist perspective. Ginsburg (1997) wrote a more detailed description of the clinical interview and expounded upon the rationale for using the clinical interview. Ginsburg (1997) gave many reasons to use clinical interview methods, including 1) constructivist theory requires a method "that attempts to capture the distinctive nature of the child's thought" (p. 58), 2) clinical interviews allow us to deal with fluidity of thinking, 3) clinical interviews work for the individual or the general, and 4) clinical interviews "embody a special kind of methodological fairness, especially appropriate in a multicultural society" (p. 58). The clinical interview allowed us to use questioning to probe the understandings associated with participants' responses.

Methodology

For this study, participants were given an online assessment item via google forms. The online assessment was a "choose all" PBQ. After the assessment results were organized and examined using google sheets to organize and display response patterns, some participants were invited to participate in a clinical interview via Zoom. A retrospective protocol was implemented for these task-based interviews. The data was analyzed using Charmaz's (2014) three types of coding.

Participants

I solicited participants from a local State University in Massachusetts. This is an important population to study and understand because this large population of students are admitted into college and university and are not prepared to take college level courses in mathematics. The current Massachusetts Curriculum Framework for Mathematics (DESE, 2017) proposes to prepare students to be college and career ready in mathematics. Specifically, they state "Students who are college and career ready in mathematics will at a minimum demonstrate the academic knowledge, skills, and practices necessary to enter into and succeed in entry-level, credit bearing courses in College Algebra, Introductory College Statistics, or technical courses" (DESE, 2017, p. 9). However, many students exit high school not yet prepared and ready to succeed in entry-level, credit bearing courses at the public college level. This is a current concern of many colleges, universities, and the students themselves.

The Assessment Item

The assessment item was selected for this proposed research for three reasons. First, this item has been refined and implemented at a statewide level prior to this research (Stroup, 2019). Second, the participants have been previously exposed to the fraction concepts. I chose fractional equivalence which appears much earlier in the typical mathematics sequence, to ensure that students would have experience with the topic and would be able to engage in the concepts. By the time students reach the undergraduate level, one would expect that they have years of experience with fraction concepts and will have some fairly stable understanding of those concepts. Lastly, the fractional equivalence represents important ideas that impact the students' success in their current courses. Fraction concepts continue to create obstacles to success for developmental mathematics students. Undergraduate developmental mathematics students have seen this material and this material still contributes to student difficulties in their current (and future) mathematics classes.

The assessment item discussed in this research is shown in Figure 1.

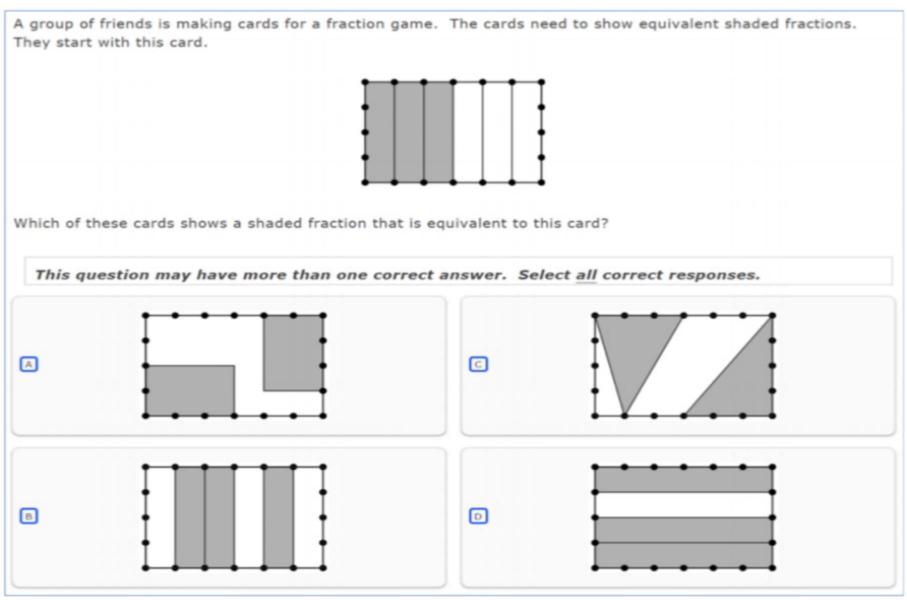


Figure 1: The assessment item, Equivalent Fractions

The "low threshold" option for this item is option B. Students need only a foundational understanding of fraction equivalence to identify that option B is equivalent to the original fraction card. Options A and C give students some steps towards the "high ceiling" in that a deeper understanding of fractional equivalence is needed to understand that A and C are also equivalent to the original. Option D, which is not equivalent to the original card, introduces an alternative that can reveal important features of student thinking. For the response patterns of BD, teachers assume that the students are counting bars and for that way of thinking, any option that has three shaded bars will represent an equivalent fraction. This combination (BD) of one correct response and one incorrect response potentially gives us insight into how the student is thinking about the mathematics. This is an example of how a "correct" answer or an "incorrect" answer do not give the full picture, but the combination of response options makes clear the student thinking.

The Interview

Students with common response patterns were asked to participate in a virtual Zoom interview. The common response patterns will be informed by Stroup's (2019) *Interim Assessment Pattern-Based Item Report* and the response pattern diagram created in Google Sheets.

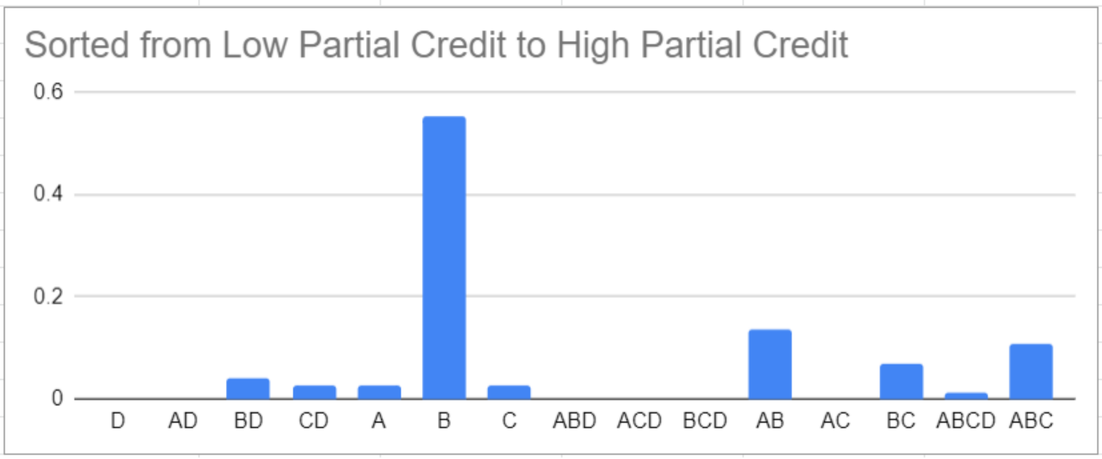


Figure 2: The Response Patterns Diagram from the Assessment Item

Common response patterns are seen as "bumps" or modes in the response pattern diagrams because they are response patterns which are selected by students more frequently than other possible response patterns. Specifically, I chose participants who chose the B combination because there were such a high percentage of participants who chose that option. Additionally, I chose participants who chose any combination include the D option to examine the "BD" type of thinking. I also chose an arbitrary number of other response patterns to open up possibilities I may not have foreseen. Unfortunately, not all students who were invited to participate in an interview were agreeable. I was not able to get as much "BD" data as I wanted to. Additionally, some of the students who began with the "BD" combination changed their answers once the interview began.

IRB consent forms were collected from participants before each interview. Interviews were conducted via Zoom to ensure safe COVID19 protocols. The zoom interview was recorded and automatically transcribed.

During the interview, I asked participants to walk me through their thinking, explain how they approached the question, or how they made the decision to choose their specific response pattern. I attempted to avoid the question "why" because this can be interpreted as a judgmental question and I was attempting to establish a comfortable, "no judgement" zone atmosphere during the interviews. I also avoided any use of the words "correct" or "incorrect". I was not interested in finding "right" or "wrong" answers, I was only looking for patterns in participants' thinking.

Data Analysis

Data analysis was done using Charmaz's (2014) three types of iterative coding and memo writing. Initial coding was done by using "in vivo" (Charmaz, 2014) codes which are special words or terms used by the participants that appeared to represent a way of thinking used by the participant. Focused coding examined the initial codes (and the text data) to see if two codes could be combined or if one code could subsume a second code. Charmaz (2014) tells us to "use focused coding to pinpoint and develop the most salient codes and then put them to the test with large batches of data" (p. 114). As the focused codes were developed the text data was referred to constantly. The codes were examined and considered iteratively and at length. Then the text data was again coded using the focused codes to ensure that the "ways of thinking" identified real patterns in participants thinking and the focus codes were not an artificial creation of manipulating the initial codes. This iterative step took quite back and forth work because in some instances, participants used similar words when they had different ways of thinking and in other instances participants used different words to describe similar ways of thinking. The participants had difficulties explaining their thinking, which made analyzing the text data challenging.

Lastly, theoretical coding can adopt concepts from the literature on the mathematical concepts to help structure emerging theories. For this research, there was limited structure. The concepts from the mathematics education literature will be compared to the ways of thinking (identified in the findings) in the discussion section.

Memo writing was used throughout the analysis process to identify and record anything which stood out to me or to identify ideas that emerged during the entire process. Most of the memo writing reflected participants affect, or feelings about doing and discussing mathematical thinking and does not directly reflect students' ways of thinking.

Findings

Students' ways of thinking about this question varied considerably. I have identified six different ways of thinking about fractional equivalence. I have restricted myself to ways of thinking that the participants gave explicit verbal evidence of. If they did not say it, I did not assume it. The following list describes the participants' ways of thinking.

1. Visualize – Most students visually inspected the cards and tried to "match" to the original. Sometimes this involved "moving" and "fitting" pieces together. Some students were unable to visualize moving and fitting pieces together but could identify that "three rectangles shaded and three rectangles not shaded", regardless of order, were equivalent.
2. "3 shaded and 3 not shaded" is different than "3 shaded and 1 not shaded". After visually matching up the original with response selection B, I often asked about response selection D because it also had three shaded rectangles. Many of the students were able to differentiate between the two cards saying that three shaded with three unshaded was different than three shaded with one unshaded. In some cases, this may have been an indication of understanding a fractional ratio, but some participants were unable to verbalize this. In one case, the participant simply noticed that the original was "50 – 50",

and three shaded with one not shaded was “more shaded than not” and so it is not equivalent to the original.

- ¾ not ½ - A number of students mentioned that three shaded with one unshaded was ¾ and that the original was ½ and that those were not the same. This might indicate the understanding of the fractional shading.
- Counting pieces – there were two different types of counting pieces.

  1. One instance (Interview 6) of counting pieces was counting the number of shaded rectangles and matching three shaded rectangles to three shaded rectangles.
  2. In two instances (Interview 4 and 8) the participants tried to count pieces of different shapes and sizes and force the shaded number of pieces into a fractional ratio with the total number of pieces.

- Rectangles only – In two instances (Interview 7 and 9) the participants considered only responses which were rectangles. They discounted response options A and C because there were shapes other than rectangles involved.
- Area – One participant (Interview 10) calculated the area of the shaded region of the original and compared that to the calculated area of each of the options. This participant missed option A because of a small mathematical error.

The following table indicates which interviews provided evidence of each type of reasoning. Additionally, this table gives their final response pattern and types of changes, if any.

Table 1: Summary Table of Findings. This table indicates the interview number, the response pattern, the types of reasoning evidenced, and the type of change, if any.

Int #	Res p.	Vis ual	Sha de vs not	¾ not ½	Cou nt pieces	Rec t only	Are a	Change
1	AB	*	*					Stable
2	BC	*		*				Stable
3	AB	*	*					Learni
4	B	*	*		*			Unstab
5	B	*	*					Stable
6	BD				*			Stable
7	B	*	*			*		Stable
8	B	*	*	*	*			Stable
9	B	*	*			*		Stable
10	BC	*					*	Learni

Changing Answers

Half the participants changed their responses during the interview. From my earliest memoing, students’ changing their answers was a salient behavior, and it seemed to be relevant to the validity of these types of questions. The PBQ questions allowed students to modify their ways of thinking. I believe educators would be interested in knowing if a student was committed to a solution or changed their answer and in what ways they may have made changes. All who changed answers moved from exhibiting a less “thought through” reasoning in the original response pattern to a more complete expression of understanding. There appear to be three types of changes involved. First is a stable change. Interview 1 changed from CD to ABC and interview 2 changed from AB to BC. Both of these changes were made immediately, and they could not describe why they had chosen the original response pattern. This scenario presents as though the participant was not initially motivated to think carefully about their response pattern. Once the change was made, the participants were stable in their thinking. The second type of change was called “active learning”. Interview 3 changed from B to ABC and Interview 10 changed from C to BC. In these instances, participants made verbal indications that they were thinking about the questions as they were explaining their responses to me. Interview 3 made the comment “wait a minute, let me check ….” and Interview 10 used the phrase “it’s going to be like, wait, 1234, wait … Oh”. These utterances seem to imply the participant was activing thinking about the problem. I called the last type of change unstable because the participant changed repeatedly and did not appear to understand why they were changing. Interview 4 changed from CD to B and used the phrases “It kind of would look the same, but, …”, “honesty, I don’t even know”, “I think it might be B”, and “I didn’t think so, but now I feel like it does”. These utterances do not indicate stability in their ways of thinking.

Discussion

The participants’ ways of thinking about equivalent fractions is consistent with the literature on rational number concepts. The participants changing answers is a healthy part of learning and does not challenge the validity in respect to the goals of a qualitative PBQ. Correspondence between the BD response pattern and the “counting pieces” thinking was evidenced and the correspondence between the other response patterns and participants ways of thinking were a bit blurred.

Ways of Thinking and Rational Number Literature

The participants use of the “visualize” way of thinking where the participant matches up the three shaded rectangles and three not shaded rectangles is consistent with the literature on “part – whole” conception of fractions (Behr et al., 1983; Kieren, 1980; Lamon, 2007; Pederson & Bjerre, 2021). This conception frames the fraction equivalence as “The part-whole interpretation of rational number depends directly on the ability to partition either a continuous quantity or a set of discrete objects into equal-sized subparts or sets” (Behr et al., 1983, p. 93). This is seen clearly when the participants visualize three shaded and three not shaded rectangles. Furthermore, Behr et al. (1983) also refer to the part-whole conception as geometric regions and the participants certainly looked at the rectangles as geometric shapes.

The “three shaded and three not is different from three shaded and one not” way of thinking about fractional equivalence is consistent with ratio as a relationship between two quantities (Behr et al., 1983; Keiren, 1980; Lamon, 2007; Pederson & Bjerre, 2021) because the participants were able to see that the relationship between the number of shaded rectangles and the number of not shaded rectangles was different and therefor the fractions were not equivalent. What is not clear from the participants responses, and perhaps because of the assessment item itself is if the participants can differentiate the equivalence of “relative magnitudes” (Pedersen & Bjerre, 2021, p. 144) because all the cards in the assessment item were the same size.

The way of thinking which I called “¾ is not equal to ½” is consistent with the rational number concept of “fraction as a number” (Lamon, 2007, p. 635) because the participants were able to name three shaded rectangles and three not shaded rectangles as the number ½ and they named three shaded rectangles and one not shaded rectangles as ¾ . Additionally, they understood that those two numbers were not an absolute equal amount. We cannot be sure if they understood if ¾ and ½ were not equal relative amounts.

The "calculate area" way of thinking is consistent with Behr et al.'s (1980) reference to geometric regions. Using the dots on the cards as unit lengths, this participant found the geometrical area of the original card and the four solution option cards and compared the absolute area to the absolute area.

"Counting pieces" and "Only rectangles" are limited conceptions of fractional equivalence. In both cases these ways of thinking show participants difficulties with rational number concepts. These ways of thinking are still very real to the participants and important for the educator to be aware of to help students grow and develop better understandings of rational number concepts.

Changing Answers

Changing one's response pattern is a natural extension of learning and as such it is encouraging that PBQs can have a role in providing a pathway, or context, for students to extend their understandings. There are a number of possible reasons for shifts in students' understandings. Two shifts discussed were the stable change when a student changes immediately because they were not motivated to think carefully about the item when they first attempted it, but are now more motivated because they need to explain their thinking to the interviewer. This also lends credence to the idea that explaining one's thinking is valuable both to the student to organize their thoughts but also to the assessor to have a clearer idea of what the students' thinking is. Another type of change was called "active learning". In this change, students were willing to think while explaining and the student had an "ah-ha" moment during the interview. Both forms of changing answers are seen as corresponding to expressed shifts, or clarification, in students understanding. These shifts would not make the items invalid because they no longer represent the students' thinking, validity is not fixed in relation to the initial responses but instead can include changes or updates that are what come to be seen as linked to selection a new combination of responses. Students whose understanding of a problem shifts from one way of thinking to another way of thinking can still find a response pattern which can represent the new way of thinking. Education involves developing more-fully integrated and robust forms of understanding. Assessment should be able to accommodate these shifts in understanding.

Correspondence between Response Patterns and Ways of thinking

There was a clear correspondence between the response pattern of BD and the counting shaded rectangles. The other response patterns of B, BC, and ABC were not clearly aligned or corresponding with one particular way of thinking. I believe this is because there is overlap between the different types of thinking displayed by "visualization", "3 shaded and 3 not shaded", and "¾ is not ½". These other types of thinking seem to stem from the part-whole conception. In a number of instances, the only difference between participants response patterns came down to small differences such as the ability to visualize if the shapes could "fit" together to create the same shaded shape as in the original card.

Conclusion

This PBQ does identify students' ways of thinking and in at least one instance there was a clear correspondence between the response pattern and the ways of thinking about fractional equivalence. Ideally, every PBQ would be researched and refined to empirically establish validity. With validity established, PBQs could operate as an alternative to standardized assessments. Research should also be continued in the area of marginalized populations to establish if PBQs, which provide an opportunity for equity, can empirically be shown to provide a more equitable assessment tool.

References

Behr, M. J., Lesh, R., Post, T., & Silver, E. A. (1983). Rational number concepts. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 91 – 125). New York: Academic Press.

Charmaz, K. (2014). *Constructing grounded theory* (2nd edition). Sage Publications.

Duseau, K. (2019, April). Generative Design: The Teachers' Experience. [Paper Presentation]. The annual conference of the American Educational Research Association (AERA), Toronto, Canada.

Ginsburg, H. (1997). *Entering the child's mind: The clinical interview in psychological research and practice*. Cambridge University Press.

Herman, J. L., Aschbacher, P. R., Winters, L. (1992). *A practical guide to alternative assessment*. Association for Supervision and Curriculum Development, Alexandria, VA.

Kieren, T. E. (1980). The rational number construct: Its elements and mechanisms. In T. E. Kieren (Ed.), *Recent research on number learning* (pp 125 – 150). Columbus, OH: ERIC Clearinghouse for Science, Mathematics and Environmental Education.

Knoester, M., & Au, W. (2017). Standardized testing and school segregation: like tinder for fire? *Race Ethnicity and Education*, *20*(1), 1-14.

Kohn, A. (2000). Standardized testing and its victims. *Education Week*, *20*(4), 46-47.

Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 629-667). Reston, VA: Lawrence Erlbaum.

Massachusetts Department of Elementary & Secondary Education (DESE), (2017). *Massachusetts Curriculum Framework – 2017: Mathematics grades Pre-Kindergarten to 12*, Massachusetts Department of Elementary & Secondary Education. https://www.doe.mass.edu/frameworks/math/2017-06.pdf

Noble, T., Suarez, C., Rosebery, A., O'Connor, M. C., Warren, B., & Hudicourt-Barnes, J. (2012). "I never thought of it as freezing": How students answer questions on large-scale science tests and what they know about science. Journal of Research in Science Teaching, 49(6), 778-803.

Papert, S. A. (1993). Mindstorms: Children, computers, and powerful ideas, 2nd edition. Basic books. (Original work published in 1980)

Pedersen, P. L., & Bjerre, M. (2021). Two conceptions of fraction equivalence. Educational Studies in Mathematics, 107(1), 135-157. https://doi.org/10.1007/s10649-021-10030-7

Piaget, J. (1954). *The construction of reality in the child* (M. Cook, Trans.). Basic Book, Inc. publishers, New York.

Riffert, F. (2005). The use and misuse of standardized testing: A Whiteheadian point of view. *Interchange*, *36*(1-2), 231-252.

Schwartz, J. L. (1991). The intellectual costs of secrecy in mathematics assessment. In V. Perrone (Ed.), *Expanding student assessment*, (pp. 132-141). The Association for Supervision and Curriculum Development (ASCD).

Stake, R. E. (1995). The invalidity of standardized testing for measuring mathematics achievement. In T. A. Romberg (Ed.), *Reform in school mathematics and authentic assessment*. (pp. 173 – 235). State University of New York Press.

Stroup, W. M., Ares, N. M., & Hurford, A. (2004). A taxonomy of generative activity design supported by next-generation classroom networks. In Proceedings of the twenty-sixth annual meeting of the Psychology of Mathematics Education, (pp. 1401-1410). Toronto: OISE/UT.

Stroup, W. (2009). What Bernie Madoff can teach us about accountability in education. Education Weekly, 28(25), 22-23.

Stroup, W. M. (2019, October). Interim Assessment Pattern-Based Item Report. GenEd Corporation.

Wittrock, M. C. (1974). A generative model of mathematics learning. *Journal for Research in Mathematics Education*, 181-196.

IMPLEMENTATION OF A NEW MATHEMATICS CURRICULUM: THE CASE OF INQUIRY AND NORWEGIAN TEACHERS

Suela Kacerja
University of South-Eastern Norway
suela.kacerja@usn.no

Trond Stølen Gustavsen
University of South-Eastern Norway
trond.gustavsen@usn.no

Teachers are important actors in implementing a new curriculum as they are responsible for interpreting and enacting it in the classrooms. In our study, we focus on how inquiry becomes visible when mathematics teachers (grades 1-7) work with the implementation of the new Norwegian curriculum. In a preliminary analysis of data from interviews with teachers and observation of their teaching, we found that some elements of inquiry have become prominent in the teacher intended and enacted curriculum and we identified some challenges.

Keywords: curriculum; inquiry; mathematics teachers

Introduction

In the present paper, we introduce an ongoing four-year study of the implementation of the new mathematics curriculum in Norway focusing on teachers for grades 1-7. Teachers are the ones who need to translate curriculum into teaching to achieve its objectives, which is not a straightforward task (Zumwalt, 2004; Remillard & Heck, 2014). As Remillard (2005) argued, one of the reasons for the failure of previous reforms in the USA is a failure to acknowledge the role of the teacher, and that the curriculum depends on the teachers' ability and willingness to implement it. In addition to teachers' self-reporting of curriculum implementation, there is a need to also look at classroom teaching (Roehrig et al., 2007). Given the central role of the teacher, it is relevant to focus on their interpretation of the curriculum and how they work with it.

The current mathematics curriculum in Norway for grades 1-10 is in use since 2020 (Ministry of Education and Research, 2019) with its constituting elements, the subjects' relevance, six core elements, the competence aims and assessment for each year and schedule. The new curriculum focuses on students as good problem solvers, on discovering relationships within mathematics and its connections with other subjects (Norwegian Directorate of Education, 2020). It emphasizes student active ways of working with the subject, exploring and communicating about it. The core curriculum (Ministry of Education and Research, 2017), with the core values and principles for education, is also relevant for mathematics.

In this paper we focus on exploration ("utforsking" in Norwegian) as one of the six core elements (together with problem solving) that appears around 32 times in the competence aims. In mathematics, the term "exploration" is described as searching for patterns and relationships, focusing on strategies and approaches on how to solve problems, and discussing to achieve a common understanding (Ministry of Education and Research, 2019). The verb to explore is extensively used in all the subjects' curricula, and a broader definition emphasizes elements such as to experience and experiment to foster curiosity and wonder, sense, search, discover, observe and scrutinize, to examine different sides of an issue through open and critical discussion, and to test or try out and evaluate work methods, products or equipment (Ministry of Education and Research, 2018, p. 16). This latter definition is close to the inquiry concept (Artigue & Blomhøj, 2013), as we will argue for in the theoretical part. Given its visibility, in our paper, we focus on inquiry and investigate the research question: how is inquiry visible in Norwegian mathematics teachers' intended and enacted curriculum?

Theoretical perspectives

We use a theoretical framework proposed by Remillard & Heck (2014) to place our study in the mathematics curriculum studies. We distinguish between the official part of the curriculum or the intended, what the students should learn, and the operational part or what is implemented in school. In the *official* part of the curriculum, there are all the elements of the curriculum explained in the introduction, in addition to *national assessments* and to the *designated curriculum* consisting of instructional packages about the curriculum, district instructional plans and annual school plans. In line with Pepin et al. (2013) who argued for values and culture being part of the curriculum, we argue for the *core Norwegian curriculum* to be part of the official mathematics curriculum. The *operational* curriculum consists of the teacher-intended curriculum, the enacted curriculum and student outcomes. In the *teacher intended* curriculum the focus is on the individual teacher, how he/she interprets the official curriculum and makes lesson plans. We consider part of the teacher-intended curriculum also the teachers' collective work with resources which is missing in Remillard and Heck (2014). The *enacted curriculum* is what happens in the mathematics classroom, how the planned lesson unfolds as the teacher interacts with the students and with the tasks.

Previous research has identified different factors that can support or hinder the implementation of a curriculum. The teachers' commitment to curriculum reform (OECD, 2019; Smith & Their, 2017), their alignment of beliefs about teaching and learning to curriculum reform (OECD, 2019; Gujarati, 2011; Roehrig et al., 2007), students' and teachers' competence and teachers' professional networks (Alrichter, 2005; Gujarati, 2011), clarity of the innovation (Alrichter, 2005), supporting leadership and available resources (OECD, 2019; Smith & Thier, 2017; Priestley et al., 2014; Roehrig et al., 2007) are among the pinpointed supporting factors. On the other hand, teacher misunderstanding or misinterpretation of content and pedagogy (OECD, 2019), teacher workload and insufficient time (OECD, 2019; Gujarati, 2011; Priestley et al., 2014), complexity of the innovation (Alrichter, 2005) and abstruse documents with unclear guidelines (OECD, 2019; Smith & Thier, 2017; Priestley et al., 2014), primacy (fear of undermining) of traditional subject areas (OECD, 2019; Priestley et al., 2014) and tensions between child-centered reforms and high-stakes exams (OECD, 2019), classroom norms, lack of time and lack of materials (Gujarati, 2011) are all hindering factors for a successful implementation. In a study of a science curriculum implementation, Rogan and Aldous (2005) identified a tendency of teachers to interpret the curriculum in superficial ways, or to use new curriculum concepts on the same old content or methods similarly to Priestley et al. (2014).

Artigue and Blomhøj (2013) discuss inquiry as a pedagogy derived from the work of Dewey. Amongst the ten identified concerns of inquiry-based mathematics education there are: working with relevant problems, giving students autonomy and responsibility to explore problems in collective collaboration and the teacher to guide them, and a focus on developing problem solving skills and habits of mind (p. 809). There is a focus on the teacher building upon students' reasoning and connecting on their experiences, a dialogic classroom culture sharing a common purpose, and on students posing questions, collaborating, and inquiring (engage, explore, explain, extend, evaluate) open questions with many solution strategies. Artigue and Blomhøj emphasize the inquiry process as a way of being, and real-life situations, hands-on activities, and learners' experience, accompanied by reflective inquiry, as important elements in Dewey's theory to question the reduced view of inquiry often found in curricula. Skovsmose (2001) distinguished between mathematics teaching in traditional ways with an authoritative teacher and one right answer, as seen in the exercise paradigm, and mathematics teaching in a landscape of investigation where students have a more active role and there is not a one-to-one correspondence between a task and an answer to it. One can see the characteristics of the landscapes of investigation as connected to the inquiry-based teaching of mathematics.

In the Norwegian curriculum as discussed in the introduction, the core element of exploration is presented together with problem solving. Karseth et al. (2020) connect the widespread use of the verb "to explore" in the curriculum to an active role of the students in teaching and learning. As presented in the introduction, the concept includes a range of meanings from exploring and searching for patterns, experimenting, observing, sensing and scrutinizing etc. Because of these characteristics, we see the core element of exploration as being about inquiry in mathematics.

In our study, we focus on the teachers' intended mathematics curriculum and its enactment in their teaching as in the framework by Remillard and Heck (2014). Inside that part of the framework, we identify inquiry by using Artigue & Blomhøj's (2013) elements.

Methods

We analyze data from the first of four yearly qualitative data collections in a project investigating the implementation of a new National curriculum (from 2020) focusing on Mathematics and three other subjects. Four primary schools (grades 1-7) were selected based on factors that could impact curriculum implementation, such as economy, urban/rural location, and school size. In the mathematics part of the study, nine mathematics teachers participated, and data were collected through four semi-structured group interviews and six observed lessons. Ethical guidelines were followed. The authors took field notes from the non-participatory observations and a pre- and post-interview in connection with each observation. An interview guide focusing on identifying teachers' experienced changes in the curriculum and changes in their practice because of the new curriculum was designed. We looked for elements of inquiry in the interviews and observations and identified teacher's expressed challenges. In our preliminary analysis, we focus on those pieces of data where inquiry is involved and identify its characteristics by referring to the theoretical perspective.

Results and discussion

To answer the research question, how is inquiry visible in Norwegian mathematics teachers' intended and enacted curriculum, we have focused on different elements of the curriculum in the Remillard and Heck (2014) framework. By talking to the teachers, we have focused on the teacher intended curriculum, and by looking at the classroom teaching we have focused on the teacher enacted curriculum. In the interviews, teachers described the emphasis on core elements in mathematics as a change. As one teacher puts it, "the biggest change is …a methodological freedom and problem solving, a greater focus on the path to the goal than on the goal" meaning a focus on the working processes with mathematics tasks and less on the answer itself. Many teachers highlighted this together with mathematical talk, as changes in the curriculum. Both are seen as connected to inquiry, to discussing different solution methods, and to students being prepared to explain their own work. The teachers pointed thus to more active students, who need to find out and make things themselves and work practically, and to the teacher as a supervisor. This fits with the definition of exploration in the curriculum according to Kvamme et al. (2020), but also with characteristics of inquiry pointed out by Artigue and Blomhøj (2013). Other characteristics of inquiry in the interviews were the open tasks, problem solving, where there is not just one solution and where one is "forced [to think] outside the box" and see connections, focusing on different strategies and ways of solving tasks, which is in line with a landscape of investigation (Skovsmose, 2001).

Our findings suggest that inquiry has become more prominent in the teacher-intended curriculum. However, teachers also reported challenges in implementing it. Many emphasized that students are not used to exploring and inquiry, “it’s a totally new way of thinking, and I love it, but the kids aren’t always there” as a 3rd grade teacher said. Similarly, a teacher for grades 5-7 also mentioned that students are used to find a right answer to a mathematical task and not to explore. This corresponds to the traditional mathematics classroom or the exercise paradigm by Skovsmose (2001). The students’ knowledge and experiences, as factors in implementing the curriculum are also pinpointed by Alrichter (2005) and Gujarati (2011) in addition to classroom norms. Since inquiry means a higher focus on student activity, teachers need to “give up control in one way or another” and this can be a challenge for first grade students.

Inquiry, and exploration, is seen as depending on the group of students, the grade, their school experience, and their previous knowledge. One teacher claimed that students should have a basic understanding in mathematics before they can do the rest. This reinforces suggestions from previous studies (Alrichter, 2005; Gujarati, 2011; Roehrig et al., 2007) that teacher knowledge and beliefs about teaching and learning are important for curriculum implementation. It is experienced as challenging not having textbooks that correspond to the new curriculum, the teachers feel they are left alone to interpret the curriculum, and it takes time to find and adapt resources. A last challenge in connection with inquiry is the evaluation of the competence aims, “you have to decide on something more concrete that you consider, whether they have gained anything from this inquiry”, since as another teacher similarly pointed out “it is very big things”.

In the observed mathematics lessons, we found that inquiry-based teaching was implemented in different ways. While some teachers emphasized open-ended tasks and encouraged problem-solving skills and habits of mind, others relied on closed, structured tasks. This latter is in contrast with a definition of inquiry (Artigue & Blomhøj, 2013). In many cases, teachers used collaborative activities allowing students to work together to find solutions. However, in some cases the group work was not purposeful in terms of students exploring, but it was more an example of superficial interpretation of the curriculum as in Rogan and Aldous (2005). In many cases, teachers emphasized multiple strategies, mathematical conversations, and student autonomy, and took a guiding role, as in inquiry-based mathematics teaching (Artigue & Blomhøj, 2013). In our observations, students suggested strategies or asked questions around different strategies in solving problems posed by the teacher. But we did not observe examples of students posing their own questions out of curiosity or extending problems and trying to solve them, which is emphasized in inquiry-based learning. This can partially be explained by what the teachers said in the interviews, that students are not used to inquiry or not ready for that, and that the teachers need to give up control (which maybe takes time).

Conclusion

The teachers’ interpretation of the curriculum is crucial. In our study, we found that some elements of inquiry are present in the mathematics teachers’ intended and enacted curriculum, but we also identified some challenges. The preliminary results reported in this article can be used for further research on designing curriculum resources to support teachers' implementation of the curriculum. As well, the results emphasize the importance of combining a focus on both the intended and enacted curriculum for creating a more complete picture of the implementation of a new curriculum.

Acknowledgments

The study presented here is financed by the Norwegian Directorate of Education.

References

Alrichter, H. (2005). Curriculum implementation – limiting and facilitating factors. In P Nentwig & D. Waddington (Eds.), *Context based learning of science* (pp. 35-62). Waxmann: Munster.

Artigue, M. & Blomhøj, M. (2013). Conceptualizing inquiry-based education in mathematics. *ZDM Mathematics Education 45*, 797–810. https://doi.org/10.1007/s11858-013-0506-6

Gujarati, J. (2011). From Curriculum Guides to Classroom Enactment: Examining Early Career Elementary Teachers' Orientations Toward Standards-Based Mathematics Curriculum Implementation. *Journal of mathematics education at teachers' college, Spring-summer 2011* (2), 40-47.

Karseth, B., Kvamme, O. A. & Ottesen, E. (2020). *Fagfornyelsens læreplanverk. Politiske intensjoner, arbeidsprosesser og innhold. [The new curriculum. Political intentions, working processes and content].* Det utdanningsvitenskapelige fakultet, Universitetet i Oslo.

Ministry of Education and Research (2017). *Overordnet del – verdier og prinsipper for grunnopplæringen [Core curriculum – values and principles for primary and secondary education].* https://www.regjeringen.no/no/dokumenter/verdier-og-prinsipper-for-grunnopplaringen/id2570003/

Ministry of Education and Reseach (2018) *Retningslinjer for utforming av nasjonale og samiske læreplaner for fag i LK20 og LK20S [Guidelines for design of national and sami curiculums for subjects in LK20 and LK20S]* https://www.regjeringen.no/contentassets/3d659278ae55449f9d8373fff5de4f65/retningslinjer-for-utforming-av-nasjonale-og-samiske-lareplaner-for-fag-i-lk20-og-lk20s-fastsatt-av-kd.pdf

Ministry of Education and Research (2019). *Læreplan i matematikk 1.–10. trinn [Curriculum for Mathematics Year 1–10*]. https://www.udir.no/lk20/mat01-05?lang=eng.

Norwegian Directorate for Education (2020). *Hva er nytt i matematikk? [What is new in mathematics?].* https://www.udir.no/laring-og-trivsel/lareplanverket/fagspesifikk-stotte/nytt-i-fagene/hva-er-nytt-i-matematikk/

OECD (2019). *"Draft Change Management: Facilitating and Hindering Factors of Curriculum Implementation", OECD Future of Education and Skills 2030: Curriculum analysis*. OECD Publishing: Paris. https://www.oecd.org/education/2030-project/contact/Change_management_for_curriculum_implementation_Facilitating_and_hindering_factors_of_curriculum_implementation.pdf

Pepin, B., Gueudet, G. & Trouche, L. (2013). Investigating textbooks as crucial interfaces between culture, policy and teacher curricular practice: two contrasted case studies in France and Norway. *ZDM Mathematics education 45*(5), 685-698. https://doi.org/10.1007/s11858-013-0526-2

Priestley, M., Minty, S. & Eager, M. (2014). School-based curriculum development in Scotland: curriculum policy and enactment. *Pedagogy, culture & society 22*(2), 189-211. https://doi.org/10.1080/14681366.2013.812137

Remillard, J. (2005). Examining key concepts in research on teachers' use of mathematics curricula. *Review of Educational Research 75*(2), 211-246. https://doi.org/10.3102/00346543075002211

Remillard, J. & Heck, (2014). Conceptualizing the curriculum enactment process in mathematics education. *ZDM Mathematics Education 46*, 705–718. https://doi.org/10.1007/s11858-014-0600-4

Roehrig, G. H., Kruse, R. A. & Kern, A. (2007). Teacher and school characteristics and their influence on curriculum implementation. *Journal of Research in Science Teaching 44*(7), 883-907. https://doi.org/10.1002/tea.20180

Rogan, J. & Aldous, C. (2005). Relationships between the constructs of a theory of curriculum implementation. *Journal of Research in Science Teaching 42*(3), 313-336. https://doi.org/10.1002/tea.20054

Skovsmose, O. (2001). Landscapes of investigation. *ZDM Mathematics Education 33*(4), 123-132. https://doi.org/10.1007/BF02652747

Smith, J., & Thier, M. (2017). Challenges to common core standards implementation: views from six states. *NASSP Bulletin 101*(3), 169-187. https://doi.org/10.1177/0192636517712450

Zumwalt, K. K. (2004). Choosing to make a difference. In *A. T. Costigan & M. S. Crocco (Eds.), Learning to teach in an age of accountability* (pp. 247-257). Mahwah, NJ: Lawrence Erlbaum Associates.

LESSONS LEARNED FROM THE CO-DESIGN PROCESS OF A STUDENT CENTERED, MIDDLE GRADES MATHEMATICS CURRICULUM

Peter Karanevich
University of Colorado-Boulder
Peka7558@colorado.edu

A teacher solidarity co-design framework was used to inform the co-design of a middle school math curriculum involving teachers, mathematics education researchers and graduate students. Incorporating teacher and student voice, positioning teachers as experts in their classrooms, and using feedback cycles were all methods used to create the mathematics content and context. Results showed that there can be tension around what content to emphasize in the co-design process and that students tend to engage and converse more when the context is familiar to them which can cause lessons to take longer than originally intended. Groups pursuing a co-design process should maintain open communication between members, keep the process iterative and be open to modifications so that expertise can be highlighted, ultimately culminating in experiences that are meaningful for students.

Keywords: Curriculum, Culturally Relevant Pedagogy, Affect, Emotions, Beliefs and Attitudes

The purpose of this research was to enact and study a co-design model for creating mathematical lessons for students in which the content is something that is relatable to sixth grade math students experiencing high levels of poverty. Many curricula do not incorporate the experiences of the modern sixth grader so co-design with teachers and students was utilized to attend to students' interests in the selection of problem contexts and storylines. From the initial planning, through the first drafts, to the implementation, teachers, researchers, and math educators collaborated to create this content using ideas from students, relayed through teachers, to create a curriculum that leverages contexts of student interest to learn and practice mathematics concepts.

Through the codesign process, the researcher wanted to know how co-designing curriculums are going to help teachers. as well as how does this ultimately provoke student interest in mathematics. The researcher also wondered how involving teachers from the beginning would allow them to feel part of the team and allow them to vocalize their thoughts and opinions on the curricula, knowing that their feedback would be used to make lessons better. The guiding research question was: how do teachers discuss/reflect upon lessons based on student interest in which they have been a part of codesigning? Ultimately, the researcher anticipated that teachers would better understand the goals of the instructional materials, and when content is designed around students' interests, students will be more engaged and eager to participate in mathematics lessons. In this report, the co-design process was used for curriculum development and the data was analyzed using several different frameworks which influenced some of the lessons learned and how these conclusions were drawn.

Theoretical Frameworks

There are three key frameworks guiding the co-design process that was implemented while developing the sixth-grade math curriculum and subsequent analysis of data. The frameworks include teaching and learning as sociocultural endeavors, a teacher solidarity co-design framework, and a theory of teacher and student affect towards mathematics. These three theories

-- sociocultural learning, teacher solidarity co-design framework and affect theory in mathematics -- and their relevance to this study are outlined below.

Teaching and learning are a collaborative process. The development of curricula with teachers, students, and other math education professionals should be a collaborative process as well. Lave (1988) proposes that learners and learning can be thought of as being situated in a social context where the tendency is to reproduce similar experiences and use prior, well-known solutions as a kind of guiding force (Packer, 2001). Traditional mathematics curricula that ask students to look at a strategy for solving a certain problem, followed by procedural practice, is only setting up those students to reproduce this version of what it means to do mathematics. Problems that are situated in contexts familiar to students can help them learn new solving strategies for familiar problems as well as give them a foundation to use new strategies in unfamiliar contexts (Lave, 1988, in Greeno, 1994). A curriculum design process that leverages teacher knowledge about their students and considers direct input about student interests can help students infer that the work they are putting forth is relevant and something they can relate to.

Furthering the idea that teaching and learning are socially constructed, there is a need to acknowledge the intricacies of power that come with creating a working group -- in particular, a curriculum design team. Researchers with PhDs and many years of experience in academia can be positioned in design groups as hosts while teachers can be seen as guests, just like teachers can assume the host role with their students as guests in a classroom environment. This positionality can be challenging as hosts have the privilege of choosing when to extend hospitality towards guests to make them feel included and that their expertise is welcome (Barton & Tan 2020). The co-design framework that was employed to acknowledge and counteract this phenomenon was drawn from Phillip's (2022) teacher-solidarity co-design framework. In this framework, teachers are seen as experts in the many aspects that make up their professional settings (i.e., a blend of context and content expertise) and these points of view are used to guide lesson design. The design process acknowledges that teachers and researchers are positioned within a larger system of academia that places these individuals in different positions of power in many different ways. Validating teacher expertise and agency also aims to "counteract the deprofessionalization of teachers and teaching within the larger context of neoliberal reform in schools" (Phillip 2022 et al., pg. 60). Keeping these principles in mind during collaborative design helped in neutralizing the guest/host relationship that can be common in co-design processes.

Teacher affect toward a curriculum will influence how they teach it. Teachers who have a say in the design of a curriculum may be more inclined to follow the creators' intended design, since they were part of the design process. One of McLeod's (1992) major facets of the affective experience of mathematics is that students have beliefs about themselves and mathematics that "play an important role in the development of their affective responses to mathematical situations" (pg. 578). Extending this idea, teachers also have affective responses to mathematical situations and teachers can have emotional responses toward their teaching. The curriculum they are teaching from can be something they have positive/negative attitudes towards. By including teachers in the design process and using their opinions to help shape what curriculum looks like can create positive, pre-teaching connections with problem contexts and discourse patterns, which can make the implementation smoother and easier for the teacher; this may translate to better student engagement and understanding of the instructional materials.

Methods

In the beginning of the project, a small group of education material designers wanted to create math lessons that incorporated learning about and thinking through executive functions. From these initial steps, math education researchers and executive function scholars were included in conversations. Through the funding agency, a school district was included, and teachers/administrators joined the team. Once this team had banded together, a virtual meeting of teachers, administrators and researchers was held to discuss key goals and objectives for this new mathematics curriculum. Student voice, criticality, interactive games, and computer simulations were all things that were decided as important to incorporate into lessons.

A team made up of three sixth grade math teachers, two math education professors, and two graduate students worked together to create the mathematics part of the curriculum. The Common Core State Standards for Mathematics were the motivating content as well as input from the teachers about which standards they felt were more challenging for students or tend to be more emphasized in their state's testing. The group chose to focus on three key 6th grade math topics: statistics, equations and expressions, and ratios and rates. A 15-lesson unit was developed for each of these topics; units also included an end of unit project, and four mini assessments. During the first unit creation around equations and expressions, the larger group split into subgroups with each group tasked to develop 4-5 lessons (each lesson had a student facing page, a teacher facing page, and lesson slides). Each group was given specific standards/concepts to follow while they created their own theme in which they centered their problems around. For example, in our first iteration, the first group was assigned expressions from the Common Core State Standards for Mathematics 6.EE.A and chose to center their lessons around a grocery shopping context based on the relatability the creator's students had by having been grocery shopping alone or with parents/guardians.

Other themes like creating a story line and using graphic novel illustrations were ideas that were generated through conversations with teachers about their students' interests. Surveys were sent to students and read by teachers to collect ideas about interests and hobbies that were used to generate themes in the curriculum. The team was also able to collaborate with professional artists who created the graphic designs from suggestions made by lesson developers.

In order to track progress, a Google Sheet was created, and it helped to inform team members of how sections were coming along as well as aid in transitional lessons between teams. This first round of lesson design took about two months before the unit was "finalized". After teachers implemented the lessons, researchers performed interviews with them asking about how lessons went, what aspects students liked/disliked, timing, balance of practice and conceptual learning, layout/design of materials, etc. These interviews helped to inform how to make modifications for this unit and the next unit design.

Results

Preliminary results show that all co-designers have a desire to create something that has a positive impact on students but there is a real struggle of getting a large group of people to agree on what this looks like. The first key result is that there was a disconnect between research and teacher goals in how to balance the number of procedural/fluency problems and conceptual problems in each lesson. "It's more that procedural side lacking, and I want to see more" (post-unit interview *5/19/22,* Teacher A*)* said one teacher. Another said, "they need more work" (post-unit interview *5/18/22*, Teacher B) referencing the lack of procedural practice. Teachers face end of year state testing which impacts their evaluations, so they requested more fluency practice be embedded into the lessons so that students can get used to solving problems like those on the state tests.

Researchers initially proposed lessons focusing more on conceptual knowledge with only a few practice problems that still relied on contexts fitting the lesson's theme. The goal of conceptual understanding is to go beyond learning algorithmic procedures and learn mathematical concepts deeply (Schoenfeld 1988). For example, error analysis questions were used in a rates and ratios unit for students to identify and discuss common mistakes made while creating tables using a constant rate. These types of questions are non-procedural in that students cannot simply follow a step-by-step guide to find a solution, they must critique the argument and reasoning made while searching for a flaw. These types of questions take more time for students than a problem that asks them to simply fill out a table, given a particular ratio. Researchers find these types of conceptual problems are better for student understanding of mathematics, but researchers do not face the same pressures that mandatory testing has put on teachers so finding a balance where students can gain a deep understanding and know how to use procedures is important.

There was evidence found that some teachers leveraged the context as a jumping off point for other conversations related to students' interests. In the statistics unit, students generated their own qualitative and quantitative questions. During the implementation of this lesson, the teacher had an interaction where a question was posed about a favorite artist. This teacher took up that question to have conversations around art, which is not something usually discussed in math courses. Here's what she said about her experience,

> "And it [the question] was able to open the doors of their knowledge to different stuff and it just brought in more knowledge for them. I thought it was great because it opened doors, they just, it just, it opened their world a little bit." (Teacher A, interview *5/19/22*)

In the algebra unit, a teacher was able to use the fast-food context to broaden the discussion to one of economics and health as seen in the quote below.

> "The whole idea is to make you realize that fast-food is a cheap easy way to get in this economy, right now. But it's not healthy because we should be getting by with 2000 calories per day and your meal just added up to 1900". (Teacher B, interview *5/18/22*)

Both examples show teachers taking time to tap into students' funds of knowledge (Schenkel et al. 2021) to enhance their learning experience and make mathematics have a more personal meaning.

A consequence of students being more interested in context were issues with the timing of implementation in the classroom. Designers aimed to create 50-minute lessons with each part of the lesson given approximate time frames. Classroom teachers gave feedback that the timing of most tasks took about three times longer than originally intended by the developers due to more conversations and students engaging in the curriculum. In the second iteration of lesson design, the designers were able to take the feedback and modify lessons to still aim to keep 50-minute lessons but make the content shorter by reducing the number of practice problems embedded and shortening discussion prompts. When teachers implemented these shorter lessons, they were able to get through the lessons in the allotted time frame and said the lessons were still well balanced.

Discussion

Working as a team to accomplish a goal can be extremely rewarding, even though conflicts are bound to come up throughout the process. Finding a balance between conceptual and procedural problems came from the commitment to incorporating student and teacher voice into curriculum design. Further, as Penuel et al (2007) suggested, ensuring a cyclical process in the co-design model allows these voices to have real impact on the curriculum's redesign and future units. The more teachers can vocalize comments and concerns in all phases and design, the more likely they will enact the lessons as they were designed. Since student voice was also used to identify interesting contexts, lessons enacted with fidelity should engage students more.

We saw this increase in student engagement while teachers were doing their first rounds of implementation, which led to the lessons being much lengthier than anticipated. Encouraging students to pull from their funds of knowledge can help math content to have more meaning and be more entertaining to engage with. Students who were using numbers and logic while discussing things they enjoy are in better positions to learn how to engage in discussions about important issues that are facing our world today. Co-designing mathematics curriculum with experts in math learning as well as experts in local students (i.e. teachers) can help to create lessons that hopefully have positive long-term impacts on student learning and understanding of mathematics in their world.

Acknowledgments

I would like to thank EF+Math and the Spark math team for giving me the chance to codesign such an amazing curriculum for students to interact with. Thank you to the teachers who collaborated with us and provided their honest feedback and commitments. Shout out to my advisor, David Webb for the support through the development, data collection and analysis!

References

Barton, A. C., & Tan, E. (2020). Beyond Equity as Inclusion: A Framework of "Rightful Presence" for Guiding Justice-Oriented Studies in Teaching and Learning. Educational Researcher, 49(6), 433–440. https://doi.org/10.3102/0013189X20927363

Greeno, J. G. (1998). The Situativity of Knowing, Learning, and Research. American Psychologist, 53(1), 5–26.

McLeod, D. B. (1992). Research on Affect in Mathematics Education: A Reconceptualization. In Handbook of Research on Mathematics Teaching and Learning (pp. 575–596).

Packer, M. (2001). The Problem of Transfer, and the Sociocultural Critique of Schooling. Journal of the Learning Sciences, 10(4), 493–514. https://doi.org/10.1207/S15327809JLS1004new_4

Penuel, W. R., Roschelle, J., & Shechtman, N. (2007). DESIGNING FORMATIVE ASSESSMENT SOFTWARE WITH TEACHERS : AN ANALYSIS OF THE CO-DESIGN PROCESS. *Research and Practice in Technology Enhanced Learning*, *2*(1), 51–74.

Philip, T. M., Pham, J. H., Scott, M., & Cortez, A. (2022). Intentionally Addressing Nested Systems of Power in Schooling through Teacher Solidarity Co-Design. Cognition and Instruction, 40(1), 55–76. https://doi.org/10.1080/07370008.2021.2010208

Schenkel, K., Calabrese Barton, A., Wiersma, C., Eiden, O., Tan, E., & Calabrese Barton, S. (2021). An Engineering Funds of Knowledge Framework An approach to support teachers in leveraging students' personal knowledge, cultural expertise, and assets in engineering. *Science & Children*, *August*, 46–53. www.nsta.org/science-and-children

Schoenfeld, A. H. (1988). When Good Teaching Leads to Bad Results : The Disasters of " Well Taught " Mathematics Courses. *Educational Psychologist*, *March*. https://doi.org/10.1207/s15326985ep2302

OPPORTUNITIES FOR LEARNING: THIN SLICING CONTENT WITH VARIATION

Leah M. Frazee
Central Connecticut State University
frazee@ccsu.edu

Fabiana Cardetti
University of Connecticut
fabiana.cardetti@uconn.edu

Jillian M. Cavanna
University of Hartford
cavanna@hartford.edu

Megan Staples
University of Connecticut
megan.staples@uconn.edu

Building Thinking Classrooms (Liljedahl, 2021) provides teachers with a new method of designing and sequencing tasks called "thin slicing," which emerged from variation theory. The results of the present study indicate that an analysis of the dimensions and ranges of variation within such a task offers insights into learning opportunities available. Specifically, identifying instances where variation has not been adequately positioned against a background of sameness can highlight potentially limited opportunities for students to notice the intended mathematics. The results of this analysis can inform design decisions and modifications to the task before implementation increasing the potential of the task to support student learning.

Keywords: Instructional Activities and Practices, Curriculum, and Professional Development.

Building Thinking Classrooms (Liljedahl, 2021) is rapidly growing in relevance and impact in mathematics education across North America. Experienced teachers and preservice teachers are studying the book, sharing implementation experiences at conferences, and participating in professional development to learn new strategies for increasing student thinking in the classroom. In the summer of 2022, we selected the text as a tool for professional development for a cohort of experienced teachers. In this paper, we focus on one pedagogical strategy presented in the book called *thin slicing*. Designing thin sliced tasks involves writing a carefully sequenced series of problems utilizing small, incremental changes to support students' development of new mathematical knowledge building from their current ways of understanding. Liljedahl (2021) provides guidance for designing these task sequences grounded in variation theory (Marton et al., 2004), but much is left for teachers to work out in their classrooms. We, as mathematics teacher educators, are interested in how teachers are taking up this practice. Specifically, we are interested in the variations teachers use when designing thin sliced tasks and the possible learning opportunities afforded by this task design.

Literature Review and Framework

The National Council of Teachers of Mathematics (NCTM) recommends the use of intentionally sequenced tasks to build procedural fluency from conceptual understanding by informally drawing on students' prior knowledge, assessing students' preconceived ideas that may serve as intellectual motivation for the concepts being learned, or transitioning students from simple, concrete representations to more complex and abstract representations (Boston et al., 2017). A practice that aligns with NCTM's recommendation is the use of problem or number strings. Problem strings are carefully sequenced tasks for the purposes of facilitating students' understanding of mathematical relationships to develop certain numeracy strategies (Carpenter et al., 2003; DiBrienza & Shevell, 1998; Fosnot & Dolk, 2002; Harris, 2011) and providing rich discussion opportunities in classrooms (Bofferding & Kemmerle, 2015).

While problem strings originated at the elementary level, the practice is extending to secondary classrooms (Harris, 2011; Liljedahl, 2021; Wieman et al., 2021). Liljedahl (2021) introduces thin slicing as a method of task design similar to problem strings that uses principles of variation theory (Marton et al., 2004) to foster student thinking around curricular content. Liljedahl distinguishes the sequencing of tasks as *thin slicing*, in which there are incrementally small increases in challenge from one task to the next as students' abilities increase, from *thick slicing*, where the increase in challenge between tasks is much greater (see Liljedahl, 2021, p.151). Liljedahl suggests that teachers can design thin sliced tasks using their current curricular resources to move students from their current mathematical conceptions to deeper understandings while keeping students in the *flow* of learning, avoiding the disruptions of frustration and boredom.

In mathematics education research, task design and sequencing are often informed by learning trajectories and learning progressions which describe ways students may develop new mathematical conceptions (Battista, 2011; Confrey, 2012). A *hypothetical* learning trajectory (as opposed to an *actual* learning trajectory) is a description of the learning goal, the learning activities and a prediction of how students' thinking will develop by engaging the activities (Simon, 1995). For some topics, there are collections of tasks for which the design and sequence have been shown to effectively support students as they develop more sophisticated mathematical reasoning and understanding in relation to a specific learning goal (e.g., Battista, 2012). However, not all mathematics topics in the K-12 curriculum have been studied from the perspective of learning trajectories, and existing learning trajectories continue to be refined (Confrey, 2012). Furthermore, each learning trajectory must assume a starting point and offers one possibility among many (Rich et al., 2017).

Teachers are in the classroom each day implementing tasks and determining what works for their students. The design principles of thin sliced tasks and problem strings offer teachers a way to create tasks based on a sequence they think will support students to advance their understanding of a particular mathematical idea, defining a hypothetical learning trajectory that is then tested in the classroom. Because thin slicing is a new method of task design for many teachers, additional resources to support the design, analysis, and implementation of such tasks are not readily available.

Variation Theory

In *Building Thinking Classrooms* (BTC), readers are introduced to two principles of variation theory to guide the development of thinly sliced curricular tasks: learners see variation against the backdrop of sameness, and a required condition for learning is that only one thing is varied at a time (Liljedahl, 2021). Variation theory describes the necessary conditions for learning, with the aim of enabling the learner to engage in novel situations in powerful ways (Marton et al., 2004). Learning "implies seeing or experiencing critical aspects of an object of learning," where the *object of learning* defines the content and the objective to be learned (Kullberg et al., 2017, p. 560). The teacher's goal is to provide learning opportunities that offer students different ways to see or experience the critical features of the object. What the learner sees or notices during the learning process is impacted by past experiences and social, cultural, environmental, and mathematical dispositions and practices (Watson & Mason, 2006). If a learner does not learn the intended objective, it is because they were not able to discern the critical features of the object of learning which can only occur when learners have experienced variation against a background of sameness.

In mathematics education, Gu et al. (2004) propose two forms of variation, *conceptual variation* and *procedural variation*, to provide a deliberate and reasoned way to utilize variance and invariance in mathematics teaching. Conceptual variation is the use of examples of a concept to discern its features and non-examples of the concept to distinguish it from others. Procedural variation is dynamic and engages students in a sequence to see connections among concepts and processes and to facilitate problem solving strategies and solutions. Procedural variation includes varying a problem to provide scaffolding for higher level concepts or extensions resulting in generalizations; developing multiple methods for solving a problem; and applying the same solution method to a group of similar problems.

Variation Theory and Task Analysis. Variation theory can be used to analyze mathematical tasks or chosen examples to determine opportunities for learning (e.g., Kullberg et al., 2017; Watson & Mason, 2006). Aligning with procedural variation (Gu et al., 2004), Watson and Mason (2006) advocate for the use of variation to design exercises that foreground mathematical structure leading to generalizations. Students enact a procedure on intentionally varied problems as part of a mathematical exercise and then reflect on the results to generalize a new mathematical relationship. Before implementation, the exercises can be analyzed for the potential dimensions of variation, which are aspects of the task that may be varied. Identifying what is available for the learner to notice through variation exposes the underlying mathematical structure and learning opportunities present in a task. For a given task or exercise, the analysis of what is varied (*dimensions of variation*) and how it is varied (*range of permissible variation*) helps the teacher predict opportunities for student learning, a helpful tool for lesson planning. The predicted learning outcomes from task analysis sometimes align with what is learned by the student, but it is possible that the task designer's choices of variation do not always produce the desired student learning (Watson & Mason, 2006). While discrepancies between the intended object of learning and what is actually learned can be caused by the learning environment or the learner's prior knowledge and experience, the task itself can also impact the learner's ability to discern the critical aspects of the object of learning. If there is no variation with respect to a particular object of learning, then it is guaranteed that students will not have an opportunity to learn it (Gu et al., 2004; Kullberg et al., 2017). As teachers experiment with the principles of variation theory to design thin sliced tasks, the potential discrepancies between the intended learning objective and actual learning may result in frustration or discouragement for both teachers and students, possibly leading to time-consuming cycles of task implementation and refinement or abandonment of the task design method completely.

To support teachers in this type of design work, we examined a set of initial thin sliced designs from a group of experienced secondary math teachers. Following Watson and Mason's (2006) example of analyzing tasks before implementation, we conducted a similar analysis to better understand the features of the initial task designs by the group of teachers. The results of our analysis support the design of thin sliced tasks. The purpose of this paper is to illustrate how the tools of variation theory can be used to analyze thin sliced tasks to answer the following questions: what variations are present and what opportunities for learning are available to students in thin sliced tasks designed by teachers using the BTC framework?

Methodology

Participants & Data Collection

The teachers in the present study are part of a larger, five-year teacher development program that advances mathematics teacher leadership. All teachers have at least five years of experience teaching secondary mathematics and were nominated by their district for the program. For this report, we draw on 19 teachers' submissions from the first summer of professional development coursework. The course assignment we analyzed involved teachers working in groups of two or three to design thin sliced mathematics tasks for topics of their choice using variation theory and the BTC framework as a guide. Teachers were directed to select topics in their current curriculum for which they wished to consider alternative teaching methods. While the tasks were designed in groups, each teacher wrote their own analysis of one of the group's tasks. Both the tasks and teacher reflections were analyzed. Consistent with Liljedahl's (2021) description of how teachers can design thin sliced task sequences, the assignment asked teachers to reflect on how to ensure the goal of the task sequence was clear, how they could "slice" the mathematical content to incrementally increase the challenge for students as their abilities increase, and how to extend student learning by varying only one thing at a time.

Data Analysis

The research team analyzed a total of eight tasks and the corresponding teacher reflections. Analyses involved iterative stages of qualitative coding by the four authors of this paper (Saldaña, 2015). In the first stages of coding, the team utilized open coding with teachers as the unit of analysis to understand teachers' design decisions. The team first examined the mathematics in each task, elements of variation, and teachers' learning goals. To support reliability in coding, all four team members coded one teacher's submission together as a group and then pairs of team members double coded the remaining 18 teachers' submissions. Building on these early observations, the team shifted the unit of analysis to be individual mathematics tasks, rather than teachers. Subsequently, the team engaged in refined coding of the eight tasks looking for themes related to the objective or object of learning for each task; the type of variation; and whether the object of learning could be reached based on the variation observed. Emergent themes, such as variation for the purpose of practice or for illuminating patterns, were recorded in matrices (Saldaña, 2015). Throughout this process, the team specifically looked for disconfirming evidence (Creswell & Miller, 2000).

In the final stage of coding, the team refined codes based on the work of Watson and Mason (2006). Tasks were coded for dimensions of the task that could be varied; of those dimensions, which ones were fixed; of those dimensions, which ones were open for variation; what was the possible range of variation; and what range of variation was chosen by the task designer. The team found that by distinguishing the possible range of variation from the chosen range helped to characterize the learning opportunities available to students. Again, all tasks were double coded by pairs of researchers with the lead author coding all eight tasks. The research team met together to discuss and came to consensus on codes related to dimensions and ranges of variation.

We acknowledge that the analysis of the tasks depends on our own mathematical perspectives and teaching experience. All researchers identify as university-based mathematics teacher educators, and most have taught in K-12 schools. Two members of the research team are faculty in mathematics departments and two are members of education departments.

Findings

We observed a wide range of objectives and task structures as the teachers applied BTC recommendations to create their first thin sliced tasks. For this paper, we focus on the analysis of two tasks because they are instructive in revealing the relationship between the objective and the chosen dimensions of variation. One task was designed to help students develop new mathematical knowledge (Task 1: Exploring Negative Exponents), and one task was designed for skill practice (Task 2: The Quadratic Formula). We analyzed the open dimensions of variation and the chosen range of variation leading to the opportunities for learning in each task.

Task 1: Exploring Negative Exponents

The teachers' stated goal for the Exploring Negative Exponents task is to use patterns to help students draw "their own conclusions about the reciprocal nature of negative exponents." The teachers thinly sliced the concept of negative exponents into five "task cards" intended to be given to students in sequence. Card 1 (Figure 1) is a review of prior knowledge to establish the meaning of a base and exponent. The equations are exponential, with the open dimensions of variation being the base, exponent, and its equivalent value. The chosen range of variation for each of these dimensions is positive integers. The opportunity for learning is to recall the relationship between base, exponent, and the equivalent value they define.

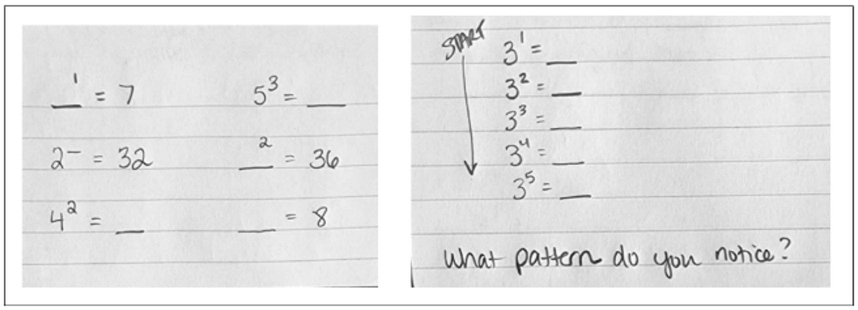


Figure 1. Exploring Negative Exponents Task Cards 1 and 2

In Card 2 (Figure 1), two dimensions of variation are closed: the value of the base and the unknown value. The base of the exponential expression is always 3, and the unknown is always the equivalent value of the exponential expression. The exponent is an open dimension of variation for which the range is constrained to positive integers from 1 to 5. It is intended that students complete the task sequentially beginning with 3^1, and students are encouraged to look for a pattern. The opportunity to learn is that the values of the equivalent expressions for each exponential increase by a factor of 3 as the exponent value increases by 1.

In Card 3 (Figure 2), three aspects are varied relative to the previous card: the base, the order of the exponents, and the inclusion of 0 as an exponent. Students are encouraged to complete the pattern sequentially beginning with 2^5. With these changes, the opportunity to learn now includes a recognition that the pattern in the equivalent values for each expression is multiplied by a factor of ½ as the exponent decreases by a value of 1. Students also have the opportunity to conjecture that 2^0 follows the same pattern, if this was not already known.

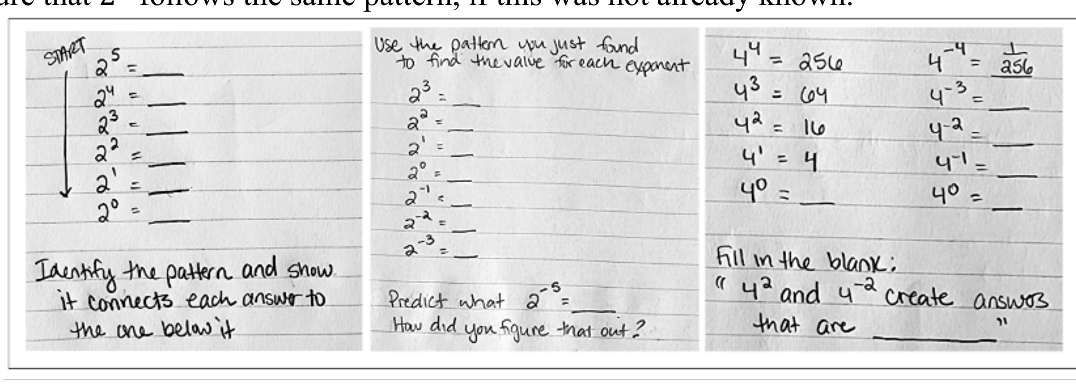


Figure 2. Exploring Negative Exponents Task Cards 3, 4, and 5

In Card 4 (Figure 2), only one aspect is varied relative to the previous card: the value of the exponents. By extending the range of variation to include negative exponents, students have the opportunity to conjecture the equivalent values for 2^{-1}, 2^{-2}, and 2^{-3}. Students are then asked to predict 2^{-5}. Since there are no explicit instructions for how to do this, some students may continue to extend the pattern, concluding that 2^{-5} will be $1/32$.

In Card 5 (Figure 2), there are four aspects that are varied from the previous: the base, the value of the exponents, some equivalent expressions are provided, and the position of the problems into two columns rather than one. The juxtaposition of the two expressions with a base of 4 with exponents of n and $-n$ opens the opportunity to notice the reciprocal relationship between 4^n and 4^{-n}. Because the students have been prompted to discover patterns, they have a way to determine the missing information if they do not yet see the reciprocal relationships.

If students use a calculator for Cards 4 and 5, the reciprocal relationship may not be available for them to notice. While this means that students could complete Card 4 without observing the intended relationship, the implementation of the task in the classroom could provide a rich discussion of the equivalence of fraction and decimal representations and the recognition that one representation is more advantageous in this instance for pattern recognition.

Task 2: The Quadratic Formula

For the Quadratic Formula task, the teachers' stated goal is to help students "feel confident substituting and then following the order of operations" to find the solution(s) to the quadratic equation. Each equation in Figure 3 is one "slice" presented to students sequentially. The open dimensions of variation for the task include the form of the equation (set equal to 0 or not); the values for coefficients *a*, *b*, and *c*; the number of unique solutions; and whether the solutions are rational or irrational. For each slice, the *a*, *b*, and *c* values are varied, and the range of variation only includes integers.

Solve for x using the quadratic formula:

1. $2x^2 + 7x + 3 = 0$
2. $x^2 + 7x + 10 = 0$
3. $x^2 - 4x + 3 = 0$
4. $-2x^2 - 4x + 3 = 0$
5. $9x^2 - 6x + 1 = 0$
6. $x^2 + 2x - 1 = 2$
7. $5x^2 - 80 = 0$

Figure 3. The Quadratic Formula Task

For Slices 1-5, the form of the equation remains consistent with 0 on one side of the equal sign. In Slice 1, the choices for *a*, *b*, and *c* result in solutions that are rational (in this case one integer and one non-integer). There is an opportunity for students to identify where *a*, *b*, and *c* are located in a quadratic equation set equal to zero with the quadratic, linear, and constant terms in descending order. There is also an opportunity to notice that values of *a*, *b*, and *c* can be positive integers, and solutions can be integer or non-integer rational numbers. In Slice 2, the only aspects varied relative to the previous slice are the values for *a* and *c*. There is an opportunity for students to recognize, conjecture, or ask the meaning of an implicit coefficient of 1 and to recognize the effect of $a = 1$ in the quadratic formula. In Slice 3, the value of *b* is a negative integer, while *a* remains 1, and *c* is a positive integer. There is an opportunity for students to see that the *b*-value can be a negative integer. There is also the opportunity for students to notice the effect of a negative *b*-value in the quadratic formula. In Slice 4, the choices of a, *b*, and *c* yield irrational solutions, giving students an opportunity to notice that irrational solutions are possible for quadratic equations. In Slice 5, the choices of a, *b*, and *c* yield one unique solution giving students an opportunity to notice that there may only be one unique solution for a quadratic equation, and the one solution can be rational (non-integer).

In Slice 6, the form of the equation is varied because it does not have 0 on one side of the equal sign. There is an opportunity for students to notice that quadratic equations are not always given in the same form. Students may have an opportunity to notice that determining the *c* value will require computation, but this observation is not certain. The introduction to the lesson (not shown in Figure 3) includes a discussion of how to identify a, *b*, and *c* when the equation is set equal to 0, but there is no indication that the form of the equation will be varied during the discussion. While students may notice that the equation is not equal to 0, there is not an explicit opportunity to notice how this impacts the value of *c*. Students are encouraged to check their solutions by substitution, which could provide the opportunity to recognize that something is incorrect if they choose $c = -1$ (rather than $c = -3$). It is also possible that students recognize that $c = -1$ is incorrect by graphing the left and right sides of the equation as two functions and examining the intersection points. However, if students graph only the left side and examine the *x*-intercepts, they will not have an opportunity to recognize that the solution is incorrect.

In Slice 7, there are two varied aspects from the previous slice: *b* is 0 (as opposed to a positive or negative integer) and the equation is set equal to 0 again. There is an opportunity for students to notice *b* can be 0. There is also the opportunity for students to notice the effect of $b = 0$ in the quadratic formula.

Discussion

When designing a task with variation theory, the open dimensions of variation and the constraints on the range of variation impact what students have the opportunity to notice. In the present study, the teachers chose constraints on the range of variation for different reasons, such as illuminating mathematical patterns or anticipating student difficulties. For the Exploration of Negative Exponents task, the teachers' goal was for students to recognize the reciprocal relationship between positive and negative exponents of 2 and generalize this relationship to a base of 4, concluding that 4^n and 4^{-n} are reciprocals. The choice of constraints on the dimensions of variation and ranges of variation was guided by the teachers' desire for the students to recognize a pattern resulting in a generalization. For instance, the teachers explained that Cards 2 and 3 were intended to help students "recognize that increasing the exponent by one creates a subsequent answer that's a scalar multiple of the base." This targeted goal of pattern recognition and generalization resulted in very few open dimensions of variation and a small range for variation of open dimensions. Consequently, the task adequately provided variation against a background of sameness. This task provides an example of how students can enact a procedure on carefully varied problems illustrating a pattern that ultimately leads to a mathematical generalization (Gu et al., 2007; Watson & Mason, 2007).

In contrast, the Solving Quadratic Equations task was designed to increase students' "confidence substituting in numbers and then following the order of operations," resulting in different reasons for the teachers' selection of constraints of variation than the exponent task. The teachers' reported design decisions indicated that they wanted students to attend to (1) coefficients of the quadratic: positive, negative, and zero values for *b*; positive and negative values for a, including a = 1; (2) solutions to a quadratic equation can be rational and irrational; (3) there can be one or two unique solutions; and (4) quadratic equations are not always set equal to 0. These ranges of variation do not account for all variation in quadratic equations. For instance, it is possible that $c = 0$, but $a \neq 0$, and a quadratic equation can have zero real solutions. We point out these additional variations not to argue that this task should include all of them, but rather to highlight that the teachers' decisions about what to constrain were not focused on illustrating the full range of variation related to quadratic equations. We conjecture that the teachers intentionally chose constraints to address students' challenges with numerical computation. For instance, the teachers included an implied $a = 1$, likely because this is something they want to reinforce with students, but they did not intentionally choose $b = 1$ or $c = 1$. Additionally, b is varied in Slices 2 and 3 from a positive integer to a negative integer, and in Slice 7, $b = 0$. We conjecture that the teachers wanted to highlight various b-values to address common errors when evaluating the expressions $-b$ and b^2 in the quadratic formula. Variation theory suggests that varying too many aspects of a problem unrelated to the targeted concept might not provide the background of sameness needed to highlight the intended concept. For example, if students have difficulty understanding the meaning of $-b$ when b is negative, they will likely get an incorrect answer in the quadratic formula but not have an opportunity to focus on the concept that led to the incorrect answer because there are many places where the computation error could have occurred. For this task, the goal of learning how to use the quadratic formula had a subgoal of accurately evaluating expressions for positive and negative integers. Since the variation of b values occurred at the same time other dimensions were varied (e.g., form of equation, values of a and c), students may not have had the opportunity to observe patterns in evaluating $-b$ and b^2 for positive and negative b-values. Another draft of this thin-sliced task might highlight the nuances of working with $-b$ and b^2 by offering a more static background by closing some dimensions of variation. This task provides an opportunity for some students to develop efficiency and accuracy with the quadratic formula, which are components of procedural fluency (NRC, 2001), but opportunities for developing fluency may be missing for students who have difficulty with the necessary numerical computations.

Conclusion

Well-designed thin sliced tasks can provide opportunities for students to learn curricular content while developing their mathematical thinking and reasoning (Liljedahl, 2021). With any task design method, there is much to learn about the nuances of design and possible learning opportunities afforded in the task as well as limitations in the design. This report illustrates how the elements of variation theory (Watson & Mason, 2006), specifically the juxtaposition of sameness and variation within and across dimensions, can be used to highlight the learning opportunities available in thin sliced tasks. The analytic approach used here to identify the dimensions of variation along with the ranges of variation seems a promising tool for teachers and mathematics teacher educators to reflect on their thin sliced task designs. Such reflection before implementation may inform task revisions resulting in increased opportunities for students to learn the intended objectives. Additionally, by examining why particular dimensions of variation were constrained and by considering the resulting opportunities to learn, teachers may identify important mathematics to discuss or call attention to when consolidating ideas (Liljedahl, 2021) at the conclusion of the learning activity.

Acknowledgments

The research described in this paper was supported by the National Science Foundation Robert Noyce Teacher Scholarship Program under Grant Number 2050659. The opinions, findings, conclusions, and recommendations, however, are those of the authors and do not necessarily reflect the views of the National Science Foundation.

References

Battista, M. T. (2011). Conceptualizations and issues related to learning progressions, learning trajectories, and levels of sophistication. *The Mathematics Enthusiast, 8*(3), 507-569.

Battista, M. T. (2012). *Cognition-based assessment and teaching*. Heinemann.

Bofferding, L. & Kemmerle, M. (2015). Elementary teacher candidates' use of number strings: creating a math-talk learning community. *Mathematics Teacher Educator, 3*(2), 99-115.

Boston, M., Dillon, F., Smith, M. S., & Miller, S. (2017). *Taking action: Implementing effective mathematics teaching practices in grades 9-12.* National Council of Teachers of Mathematics.

Carpenter, T. P., Franke, M. L, & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic & algebra in elementary school*. Heinemann.

Confrey, J. (2012). Articulating a learning sciences foundation for learning trajectories in the CCSS-M. In L. R. Van Zoest, J.-J. Lo, & J. L. Kratky (Eds.), *Proceedings of the 34th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 2-22). Kalamazoo, MI: Western Michigan University.

Creswell, J. W. & Miller, D. L. (2000). Determining validity in qualitative inquiry. *Theory Into Practice, 38*(3), 124-130.

DiBrienza, J. & Shevell, G. (1998). Number strings: Developing computational efficiency in a constructivist classroom. *The Constructivist, 13*(2), 21-25.

Fosnot, C. T., & Dolk, M. (2002). *Young mathematicians at work: Constructing fractions, decimals, and percents*. Heinemann.

Gu, L., Huang, R., & Marton, F. (2004). Teaching with variation: A Chinese way of promoting effective mathematics learning. In L. Fan, N.-Y. Wong, J. Cai, & S. Li (Eds.), *How Chinese learn mathematics. Perspectives from insiders.* World Scientific.

Harris, P. W. (2011). *Building powerful numeracy for middle and high school students*. Heinemann.

Kullberg, A., Kempe, U. R., & Marton, F. (2017). What is made possible to learn when using the variation theory of learning in teaching mathematics? *ZDM Mathematics Education, 49*, 559-569. DOI 10.1007/s11858-017-0858-4

Liljedahl, P. (2021). *Building thinking classrooms in mathematics grades K-12: 14 teaching practices for enhancing learning.* Corwin Press.

Marton, F., Tsui, A. B., Chik, P. P., Ko, P. Y., & Lo, M. L. (2004). *Classroom discourse and the space of learning*. Routledge.

National Research Council. (2001). *Adding it up: Helping children learn mathematics*. J. Kilpatrick, J. Swafford, & B. Findell (Eds.) Mathematics Learning Study Committee, Center for Education, Division of Behavioral and Social Sciences and Education. National Academy Press.

Rich, K. M., Strickland, C., Binkowski, T. A., Moran, C., & Franklin, D. (2017). K-8 learning trajectories derived from research literature: Sequence, repetition, conditionals. In *Proceedings of the 2017 ACM conference on international computing education research* (pp. 182-190).

Saldaña, J. (2015). *The coding manual for qualitative researchers* (3rd ed.). Thousand Oaks.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*(2), 114-145.

Watson, A. & Mason, J. (2006). Seeing an exercise as a single mathematical object: Using variation to structure sense-making. *Mathematical Thinking and Learning, 8*(2), 91-111.

Wieman, R., Freedman, L., Albright, P., Nolen, D., & Onda, J. (2021). From notes to strings. *Mathematics Teacher: Learning & Teaching PK-12, 114*(12), 956-963.

QUANTITATIVE INSTRUMENT REPOSITORY FOR MATHEMATICS EDUCATION RESEARCH WITH VALIDITY EVIDENCE

Erin E. Krupa
North Carolina State University
eekrupa@ncsu.edu

Jonathan David Bostic
Bowling Green State University
bosticj@tdfolge@bgsu.edu

Timothy Donald Folger
Bowling Green State University
tdfolge@bgsu.edu

Brianna Bentley
North Carolina State University
blbentle@ncsu.edu

Katie Burkett
North Carolina State University
keburket@ncsu.edu
;

The paper reports on the developments of a repository of quantitative assessments used in mathematics education contexts. This repository centralizes assessments and the associated validity evidence. The repository is public and freely available and has potential to inform future quantitative mathematics education scholarship.

Keywords: Assessment, Measurement

Finding and selecting quantitative instruments to use in mathematics and statistics education can be difficult for scholars for many reasons. First, it is not always easy to ascertain if there is a suitable instrument to measure a desired construct. Second, many published articles report results based on analyses of data from instruments they used, but they do not always publish the full instrument used. A third reason is that some manuscripts describe an assessment but do not provide details about the validity evidence related to the assessment's intended uses or score interpretations. Broadly speaking, validity describes the degree to which evidence supports an intended claim (AERA et al., 2014; Messick, 1989). A goal to fostering scholarship is building a strong knowledge base that uses assessments with strong validity evidence (Authors., 2022A; Kane, 2013). A searchable database of mathematics education assessments may help to address that goal, which is the objective of this research project. The purpose of this submission is to describe the creation of a new database of mathematics and statistics education assessments that will be broadly available to scholars. A result of addressing this purpose is to provide the mathematics and statistics education scholarly community with a tool that may help individuals locate possible assessment for use, review validity evidence about the assessment, as well as review available literature associated with the respective assessments.

Literature

Prior to 2014, there was scant discussion of validity and quantitative assessment within mathematics education scholarship (Author, 2017). In the last decade, there has been a substantial increase in scholarship exploring the degree to which validity and validity evidence is discussed in mathematics and statistics education as well as scholarship focusing on how those validity arguments are communicated (e.g., Pellegrino et al., 2016; Walkowiak et al., 2019; Wilhelm & Berebitsky, 2019). As more assessments are described in journal articles, it can be time intensive to search journals for instruments that match a desired construct as well as their validity evidence. Moreover, there are documented differences in the journals that institutions access, which is an equity issue as well as an issue of building robust research (Authors, 2022B). With a greater attention paid to the quality of information collected on quantitative assessments

in mathematics and statistics education scholarship, there is a pressing need for providing the fields of mathematics and statistics education scholars with a means to efficiently and effectively explore available measures used.

Methods

In the spring of 2020, 41 participants attended a BLINDED conference in Las Vegas. Participants of the conference included mathematics education faculty, researcher scientists, psychometricians, assessment developers, and graduate students. The goal of the conference was to create an understanding of validity within mathematics education contexts and solicit recommendations from experts about information necessary to build a repository of quantitative mathematics education instruments. Based upon the recommendations from the experts who attended the conference, a synthesis procedure was developed for identifying and categorizing validity evidence, interpretation statements, and use statements of quantitative instruments. This conference participants were divided into six synthesis groups: (1) Elementary (K-6) Tests and Instruments; (2) Secondary (7 - 12) Tests and Instruments; (3) Undergraduate and Graduate Mathematics Tests and Instruments; (4) Statistics Education (K - 20) Tests and Instruments; (5) Teacher Education Tests; and (6) Teacher Education Instruments.

Each group searched for instruments that fell within their group's parameters, identified if the instruments should be included into the repository, searched for validity evidence associated with the included instruments, and identified and categorized the existing validity evidence for each instrument. The validity evidence was categorized using the sources outlined in the *Standards for Educational and Psychological Testing* ([*Standards*] AERA et al., 2014). The *Standards* describe five validity sources: test content, response process, relations to other variables, internal structure, and consequences from testing/bias. Reliability is a related component of the *Standards* but is not one of the five sources. These sources and resulting evidence types (Figure 1) provided the foundation for our categorization. In addition, each instrument that was found by each synthesis group was tagged with key features, such as, the population being tested, the construct measured by the instrument, the types of items present in the instrument, etc. These instruments and the associated evidence are what were used to create the repository.

Product Development

From the work of the six synthesis groups, we created a free online digital repository for the categorization of instruments that also describes their associated validity evidence. To access the repository, users will create a log-in and briefly acknowledge the user agreement. Once logged in users have access to instruments, validity evidence, and training modules about validity. The website was designed to be user friendly, accessible, and evidence in the repository is aligned with the *Standards* (AERA et al., 2014). The repository has several important design features explained below: search features, researcher portal, and ease of adding new instruments.

Design Feature: Search Features

The database will have a search feature that will allow users to input any desired text and search for instruments based on that text entry. In addition, they will be able to search by any of the tagging features. We intentionally created tagging features consistent across all six synthesis groups, and ones customized to each group, so researchers can easily sort and search for instruments. Tagging features were common to all six synthesis groups and include: Synthesis Group, Population, Grade Level, Construct, Type of Instrument, Mode of Delivery, and Item Type.

Consequences of Testing	Internal Structure	Relations to Other Variables	Reliability	Response Process	Test Content
Appropriate cut score	Bayesian Network Models	Alignment with expert opinion of test user (e.g. teacher, therapist)	Alternate form	Cognitive interviews	Alignment with frameworks/standards/theory/learning trajectory
Bias as one consequence of testing	Cluster analysis	Convergent Association or Divergent Association	EAP/PV test reliability	Error related to response patterns/CTT	Construct Definition
Cost-benefit analysis	Factor Analysis- Bifactor	Correlation analysis	Generalizability theory- D-studies	Eye tracking/physiological data	Data from experts
Documentation of unintended behavior changes based on test use	Factor Analysis- Confirmatory Factor Analysis (CFA)	Discriminant validity	Generalizability theory- G-studies	fMRI (functional Magnetic Reasoning Image)	Fairness of content
Explicit intended uses and interpretations and warn against inappropriate uses	Factor Analysis- Exploratory Factor Analysis/Exploratory Structural Equation Modeling	Discrimination power	Inter-rater reliability- Kappa	Focus groups	Field Work
Impact of assessment is similar under clinical and practical implementations	Factor Analysis- Multi-trait Multi-method matrix (MTMM)	Hierarchical Linear Modeling	Inter-rater reliability- Percent agreement	Generalizability-theory (G-theory) related evidence	Literature Review
Item functioning such as DIF - unknown subgroups had to know	Factor Analysis- Not specified	Multi-trait Multi-method matrix (MTMM)	Internal consistency or alternatives- Alpha	Log data	Participant-generated content
Motivational consequences	Factor Analysis - Parallel Analysis	Statistical Testing (e.g., t-test, regression, and chi-square)	Internal consistency or alternatives- IRT or Rasch reliability	Predicted response patterns/processes based on Learning Trajectories	Revision Process
	Factor Analysis- Principal Axis Factoring (PAF)	Structural Equation Model	Internal consistency or alternatives- Omega	Rater agreement/reliability	Standard Setting
	Factor Analysis- Principal Component Analysis (PCA)	Treatment/control study	Internal consistency or alternatives- Raykov	Rater training and calibration	
	Item difficulty	Triangulation/Crystallization with qualitative data	Item remainder correlations	Sorting tasks	
	Item Scale Correlations		Kuder-Richardson formula 20	Student written work	
	Item Response Theory (IRT)		SEM measurement	Studies of respondent's speed of task completion	
	Latent Class Analysis (LCA)		Sensitivity analysis	Think alouds	
	Latent Profile Analysis (LPA)		Split-half reliability		
	Multidimensional scaling		Test - Retest		
	Rasch modeling				
	TETRAD				

Figure 1: Validity Evidence Types

Design Feature: Researcher Portal

There will be a researcher portal that will have training modules, both written and video based. These will provide users of the repository with educative content pertaining to ideas related to validity and validation and on the proper use of the repository and instruments the user may want to use for their own purposes. Once logged into the researcher portal, instruments can be saved to users' "favorites" or shared. Users will also be able to provide feedback on the results of an instrument or download the search results for a particular search.

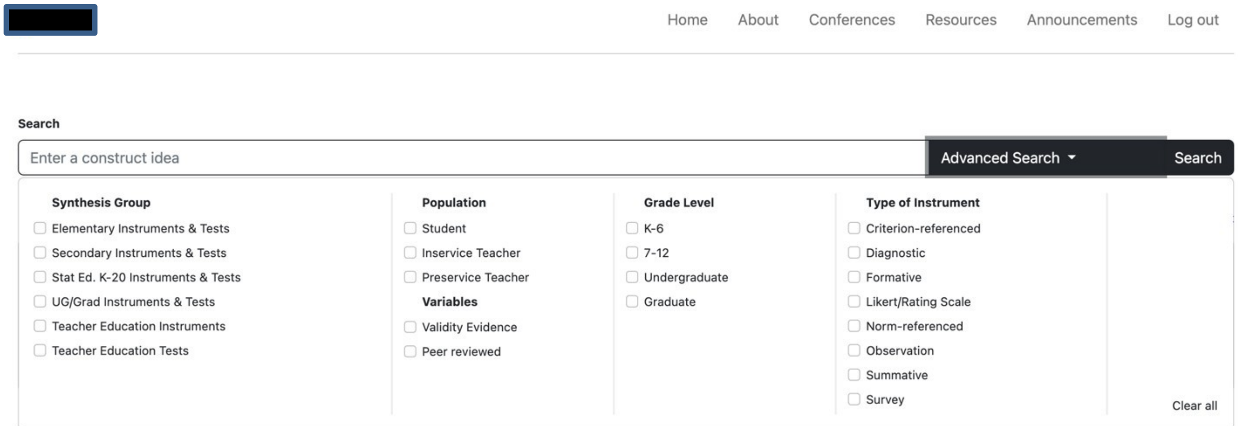


Figure 2: Search Engine and Tagging Features

Design Feature: Ease of Adding New Instrument

Initially, assessments were added by the project leaders based upon the work of the synthesis groups. However, users can now submit a request electronically to include an instrument in the repository, along with the associated validity evidence and tagging features. Once an instrument is submitted, a repository curator will vet the materials to ensure the inclusion criteria have been met. Once the criteria have been met, the instrument will be included in the repository. If it cannot be included, then the project leader will communicate what additional information is needed to include the instrument.

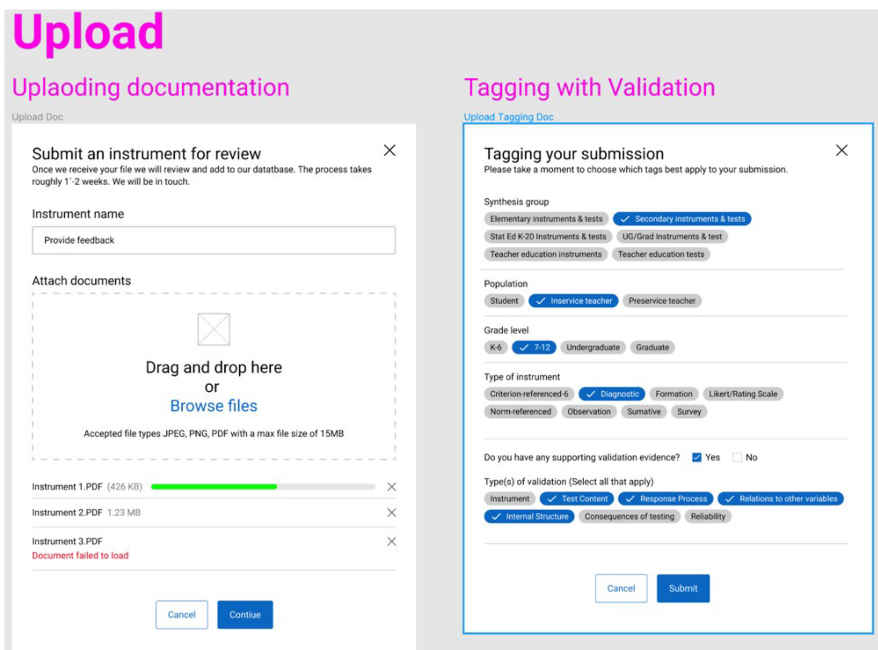


Figure 3: Adding a New Instrument Conclusions

A goal of this project was to produce a repository that could be used by scholars seeking a quantitative instrument for use in mathematics or statistics education contexts. Through developing the repository, it may be easier for scholars to locate assessments for a given construct or evaluate two viable instruments considering their validity evidence and claims.

Moreover, this repository has a propensity for others to locate an instrument and improve it, or to gather greater or more robust validity evidence. As a result of the repository, we argue it offers a start to promote equity through using assessments aligned to a desired construct and suited for an appropriate population. Engaging all learners in mathematics education and assessing whether we are engaging all learners in mathematics, should be based on research conducted from high- quality studies. Foundational to such studies are instruments with proper validity arguments.

Acknowledgments

This material is based upon work supported by the blinded foundation agency under Grant No. blinded. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the blinded foundation agency.References

American Educational Research Association, American Psychological Association, & National Council on Measurement in Education. (2014). *Standards for educational and psychological testing.* Washington, DC: American Educational Research Association.

Authors.
(2017).

Authors.
(2022A).

Authors.
(2022B)

Kane, M. T. (2013). Validating the interpretations and uses of test scores. *Journal of Educational Measurement, 50*(1), 1-73. https://doi.org/10.1111/jedm.12000

Messick, S. (1989). Meaning and values in test validation: The science and ethics of assessment. *Educational Researcher, 18*(2), 5-11. https://doi.org/10.3102/0013189X018002005

Pellegrino, J. W., DiBello, L. V., & Goldman, S. R. (2016). A framework for conceptualizing and evaluating the validity of instructionally relevant assessments. *Educational Psychologist, 51*(1), 59-81. https://doi.org/10.1080/00461520.2016.1145550

Walkowiak, T., Adams, E., & Berry III, R. (2019). In BLINDED.

Wilhelm, A. & Berebitsky, D. (2019). Validation of the mathematics teachers' sense of efficacy scale. *Investigations in Mathematics Learning, 11*(1), 29-43. https://doi.org/10.1080/19477503.2017.1375359

RETHINKING MATHEMATICS CURRICULAR COHERENCE ACROSS ELEMENTARY TEACHERS' MULTIPLE CURRICULAR MATERIALS AND MULTIPLE PROFESSIONAL OBLIGATIONS

Kristin Doherty
Michigan State University
kdoherty@msu.edu

Nan Jiang
University of Arizona
jiangn@arizona.edu

Marcy B. Wood
University of Arizona
mbwood@arizona.edu

A coherent curriculum is important for supporting students' learning of mathematics. While a single teacher textbook is created with coherence in mind, teachers are now using multiple materials to plan and teach mathematics. Additionally, teachers are held accountable to multiple professional obligations that can also affect coherence. This study reports on findings from interviews with nine elementary teachers about how they are conceptualizing mathematics curricular coherence across multiple curricular materials and multiple professional obligations.

Keywords: Curriculum, Elementary School Education

According to researchers, curricular coherence is an important component of a high-quality, equity-oriented mathematics curriculum (McNail et al., 2021; Wilkerson, 2021). Curricular coherence in mathematics is described as the connection of concepts and practices within and across grades (Wilkerson, 2021). Supporting these connections can deepen students' understanding of mathematics content (Cuoco & McCallum, 2018). While there has been significant research on curricular coherence within one curricular material (Remillard & Kim, 2020), progressions of mathematical ideas (Cai, 2014), and consistency across multiple contexts (Schmidt & Prawat, 2006), there is little understanding of (1) how teachers' professional obligations influence their decision-making about coherence and (2) how teachers are creating coherence across multiple materials. To better understand how teachers conceptualize and make decisions about coherence, our study explores the research question, *How do elementary teachers make decisions related to mathematics curricular coherence?*

Theoretical Framework

Herbst and Chazan (2011) explain that a teacher's decision-making is influenced by many *professional obligations*. These come from four sources: the *discipline* the teacher is representing, the *individual* students the teacher is serving, the *socio-cultural* context the classroom is situated in, and the *institution(s)* governing particular policies and sanctions. As teachers are making decisions about curricular coherence, one obligation they are attending to is the *disciplinary obligation*, which says that mathematics instruction must be a valid representation of the knowledge, practices, and applications of the discipline of mathematics. For curriculum designers, this obligation can be a primary focus, which allows for the creation of a coherent mathematical storyline. However, teachers also attend to additional obligations, including individual obligations, interpersonal obligations, and institutional obligations. These other obligations influence teachers' decision-making related to curricular coherence. The *individual obligation* says that students bring uniquely diverse traits and experiences that a teacher must be responsive to and incorporate into instruction. Based on Bruner's (1960) conceptualization of the interrelation of the teacher, curriculum, and students and Foster et al' s (2021) mathematics curriculum as a story framework, we conceptualize teachers as the mediator between students and curriculum who digest the curriculum and represent the content and storyline in ways relevant to their students. We consider student engagement as part of the individual obligation because teachers must consider what will be engaging for each individual. The *interpersonal obligations* says the teacher has a role in facilitating how the individual students within the classroom environment interact with one another to share time, resources, and space in socially and culturally specific ways. This obligation has connections to pedagogy that facilitates these interactions. The *institutional (schooling) obligations* says there are organizations that condition and constrain a teacher's practice; these include obligations to the department (e.g., mandates for the use of particular curricular materials), to the school (e.g., calendar, bell schedule), and to the district in which the school is situated (e.g., assessment instruments, scope and sequence mandates). The many professional obligations that teachers are held accountable to can influence their decision-making about curricular coherence.

Methods and Analysis

To better understand what factors influence decisions elementary teachers make related to mathematics curricular coherence, we first administered a survey to 524 teachers across 46 states to gain a broad understanding and then began conducting interviews for a deeper understanding. This paper reports on findings from our first set of interviews, with nine teachers from schools in two states. Five teach at a public school in the Midwest (Sunshine Elementary) and four at a public school in the Southwest (Creekside Elementary). The teaching experience of these participants ranges 1-26 years and averages 9.5 years. The participants all teach elementary, which provides an interesting grade band to study because these teachers are thinking about curricular coherence both within and across subject areas (though the focus of these interviews was on mathematics curriculum). Two 60-minute interviews were conducted with each teacher, followed by a 60-minute group interview with teachers from the same school. We asked the teachers about the curricular materials (mandated and nonmandated) that they use, including which they use as well as how and why they use them. We also asked about the types and reasons for adaptations they make to their curriculum. While these questions provided some insight into teachers' reasons and strategies for creating curricular coherence, we also asked questions explicitly about coherence. These questions asked teachers to describe the connections they are making within one lesson, between lessons in a typical week, and between units across the year as well as within one curricular material and between multiple materials. Thematic analysis of each interview transcript was conducted to code for patterns and nuances in teachers' decision-making about coherence.

Findings

All nine teachers reported using multiple curricular materials to teach mathematics. This ranged from 6 to 14 different curricular resources, including materials they designed themselves. Since a primary curricular material (i.e., teacher textbook) is created by the curriculum designer with coherence in mind, we asked teachers whether they found it challenging to create coherence across so many materials. Surprisingly, all but three reported it was not challenging. Amanda (the veteran teacher with 26 years of experience) explained, "I guess I've been doing it for a long time, so it's not harder. I can see that for a new teacher, it might be harder." Cristina (with two years of teaching experience) mentioned that she found it challenging to support students in learning concepts when different curricular materials use different vocabulary. These different responses suggest that more experienced teachers may find it easier to create coherent mathematics lessons.

Some teachers reported incoherence within their primary resource. For instance, Sarah felt that her main curriculum did not sufficiently cover the topic of elapsed time for her students, which meant she needed supplementary materials. Amanda reported something similar but with converting between fractions and mixed numbers.

> Some of the things they present for the kids to do, like with fractions, to take an improper fraction and turn it into a mixed number, they never explicitly teach the kids. They just expect they already know why or something. I don't know if that was supposed to be taught in third grade. But we get to the point where we're like, okay, you're supposed to identify mixed numbers based on these improper fractions. And they don't know how to do it. So then I have to teach that. And it's not really in the curriculum.

Here, both teachers were attending to *disciplinary* and *individual obligations* to meet the needs of their students who needed more practice with particular topics before progressing in the curriculum.

As teachers talked about coherence, their focus was often on content, with few mentions of mathematical practices. When considering their *institutional obligations*, this was not entirely surprising since they are held accountable to standardized tests that primarily assess content. The teachers at Sunshine Elementary reported that they were required to teach all of the content in their curriculum, not by the end of the school year, but by the standardized testing date in the spring. This meant a focus on breadth over depth in order to touch upon many mathematics topics over a shorter period of time. For these teachers, the institutional obligation to prepare students for the test was often in tension with the disciplinary obligation to focus on mathematical understanding.

To guide the coverage of topics and create some degree of curricular coherence, teachers reported following the sequence of units in the pacing guide provided by their school district (i.e., *institutional obligation*) as well as the sequence of topics in their mandated primary curriculum material (i.e., *disciplinary and institutional obligations*). The teachers at Sunshine Elementary reported that some grade level teams aligned these, whereas other grades (within the same school) did not. This meant some teachers were better set up than others to create coherence between some of their disciplinary and institutional obligations.

The teachers reported supplementing their primary material with other materials for various reasons. As mentioned earlier, sometimes this happened when teachers felt their primary material did not sufficiently cover a particular topic. Another common reason teachers reported supplementing their materials to provide coherence focused on the goal of student interaction and engagement (i.e., *individual* and *interpersonal obligations*). For instance, Amanda explained, “I have to also think, do I have something that's more engaging, that teaches that same content, maybe in a better way?” She was tying the individual obligation of engaging students with the disciplinary obligation of teaching content better.

Sometimes teachers combined engagement with differentiating. Marvel expressed,

> Well, I guess the reason that I would use something besides just our curriculum would be to make those connections because kids think in different ways. And the way that I'm teaching it, or the way that even [the curriculum] is teaching- wants me to teach it- might not really connect with some of the kids.

Sometimes engagement also meant connecting to students’ lives or other real world situations, which integrate individual and disciplinary obligations. Cristina shared,

> I used the apples from their breakfast bags. I gathered them, and I used them to act out the situation as a form of differentiating. And so making a connection to, you know, the situations that are being given, like real life word problems, and then using it in the classroom so that they can understand what is, what these words mean.

Sometimes engagement meant what the students would find most enjoyable or fun. When describing one supplementary material, Kasie mentioned that her students "really enjoy their videos, and I think that helps with the kids' engagement." Similarly, when evaluating which curricular materials to use, Amanda mentioned, "I can just look at those materials, and figure out which ones I think the kids would really get into." Likewise, Cristina justified her frequent use of an activity,

> So really just like games that they're familiar with, activities that they're familiar with, like I mentioned in our last interview, bingo. They love bingo. So I tried to make my lesson, make the practice more engaging for them. And making connections like a little within these materials is, this is everything that we've been practicing. So now let's use it to help us gain more understanding.

Victoria reported that finding engaging resources was a challenging connection to make, "And the reason it's challenging too is because I want to make sure that it's a video or content that keeps them interested and it's not just a person talking online or you know, I want it to still be fun and animated-interactive for them." She also added that self-paced videos promote some independence and agency in the learning process, as students can pause and rewind their video as needed.

Teachers also indicated that they wanted more support to meet individual obligations. For example, Cristina expressed needing more support in working with English language learners, "I do wish there was more guidance because not only do I wish there was more support or like more examples of how to work with students who need more differentiation, but also students who are English language learners." She mentioned that each lesson in her curriculum has a "box" with suggestions for how to support English language learners but meeting the individual needs of her own students required more nuance than the suggestions in the boxes.

Discussion

There is no doubt that a coherent curriculum is important for supporting students' learning of mathematics. As teachers thought about instruction, their decisions related to curricular coherence were influenced by many professional obligations that sometimes took the focus off of the disciplinary obligation (which is crucial for creating coherence). It seems that stronger coherence between policy makers (who mandate what should be taught, and how and when it is assessed), curricular materials (which may include content and pedagogical suggestions that can be more or less engaging or culturally relevant for particular students), and teachers' professional obligations could better set up teachers for success in creating curricular coherence. We invite the field to consider, to what extent can mathematics curricular coherence be realized in schools, given the existing policies and structures of schooling?

Acknowledgments

This research was supported through funding from the National Science Foundation (Grant Number: BLINDED). Any opinions, conclusions, or recommendations contained herein are those of the authors and do not necessarily reflect the views of NSF.

References

Bruner, J. (1960). The process of education. Harvard University Press.

Cai, J., Ding, M., & Wang, T. (2014). How do exemplary Chinese and US mathematics teachers view instructional coherence?. Educational Studies in Mathematics, 85(2), 265-280. https://doi.org/10.1007/s10649-013-9513-3

Cuoco, A., & McCallum, W. (2018). Curricular Coherence in Mathematics. In: Li, Y., Lewis, W.,

Madden, J. (eds) Mathematics Matters in Education. Advances in STEM Education. Springer, Cham. https://doi.org/10.1007/978-3-319-61434-2_13

Foster, C., Francome, T., Hewitt, D., & Shore, C. (2021). Principles for the design of a fully-resourced, coherent, research-informed school mathematics curriculum. Journal of Curriculum Studies, 53(5), 621-641.

Herbst, P., & Chazan, D. (2011). Research on practical rationality: Studying the justification of actions in mathematics teaching. The Mathematics Enthusiast, 8(3), 405–462. https://doi.org/10.54870/1551-3440.1225

McNail, T. B., McConnell, K. D., & Whitehead, D. M. (2021). Introduction to the special issue: Curricular coherence as an instructional, ethical, and equity-minded imperative. International Journal of Teaching and Learning in Higher Education, 33(1), 1-3. https://www.isetl.org/ijtlhe/pdf/IJTLHE4367.pdf

Remillard, J. T., & Kim., O. (Eds.). (2020). Elementary mathematics curriculum materials: Designs for student learning and teacher enactment. Springer. https://doi.org/10.1007/978-3-030-38588-0

Schmidt, W. H., & Prawat, R. S. (2006). Curriculum coherence and national control of education: issue or non-issue?. Journal of Curriculum Studies, 38(6), 641-658. https://doi.org/10.1080/00220270600682804

Wilkerson, T. (2021). Starting the school year with coherence and connections. President's message. National Council of Teachers of Mathematics (NCTM). https://www.nctm.org/News-and-Calendar/Messages-from-the-President/Archive/Trena-Wilkerson/Starting-the-School-Year-with-Coherence-and-Connections/

THE DEVELOPMENT OF STORYBOOKS SUPPORTING ELEMENTARY STUDENTS' MATH IDENTITY, EXECUTIVE FUNCTION, AND WORD PROBLEM SOLVING

Caroline Byrd Hornburg
Virginia Tech
chornburg@vt.edu

Amanda S. Mayes
Purdue University
asmayes@purdue.edu

Tamika L. McElveen
Purdue University
tmcelvee@purdue.edu

Sarah R. Powell
The University of Texas at Austin
srpowell@utexas.edu

Gigliana Melzi
New York University
gigliana.melzi@nyu.edu

Nydia Prishker
St. Thomas Aquinas College
nprishke@stac.edu

Ninie Asad
Virginia Tech
ninie@vt.edu

Isabel Valdivia
Virginia Tech
isabelvl@vt.edu

Ma Bernadette Andres-Salgarino
Santa Clara County Office of Education
bernadette.salgarino@gmail.com

Le M. Tran
The University of Texas at Austin
tran.le@utexas.edu

Michael D. Eiland
Purdue University
meiland@purdue.edu

Sarah H. Eason
Purdue University
season@purdue.edu

Sara A. Schmitt
University of Oregon
sschmitt@uoregon.edu

David J. Purpura
Purdue University
purpura@purdue.edu

Keywords: Instructional Activities and Practices; Elementary School Education; Equity, Inclusion, and Diversity; Problem Solving

One core component of the Our Mathematical World program, co-developed with 3rd-5th grade teachers and students, is a storybook series showcasing PULSE (Pause, Understand and Remember, Lay It Out, Solve, Evaluate), a novel metacognitive problem-solving approach that integrates executive function and problem-solving skills. Each book features Black and Latine/x youth engaged in problem solving in community-focused contexts, with math content increasing in difficulty across the series. After initial design, books were translated to Spanish by scholars representative of different Latine/x communities.

After implementation, both teacher and student feedback indicated students' engagement and enjoyment, and most students indicated that reading the books made them feel like strong math learners. Feedback from multiple implementation phases was also used to further develop and improve the books, in line with the goal to promote students' math identity, executive function, and problem-solving skills, as well as math-specific vocabulary.

Acknowledgments

The books were written by Tierra Martin, in collaboration with the Our Mathematical World team, and illustrated by Darwin Marfil. Educators and students from Hubbard Media Arts Academy, especially Danielle Letts, Sandra Kunze, and Audrey Diaz, as well as Middletown City School District, contributed to the co-design, along with Adam Smith, Grace Carroll, Matin Abdel-Qawi, Gabe Stahl, and the Learning and Development Lab. The research reported here was supported by the EF+Math Program of the Advanced Education Research and Development

Fund (AERDF) through funds provided to Purdue University. The opinions expressed are those of the authors and do not represent views of the EF+Math Program or AERDF.

PRE-SERVICE TEACHER EDUCATION

A COMPARISON OF ELEMENTARY AND SECONDARY LEVEL PRESERVICE TEACHERS' BELIEFS ABOUT LEARNING MATHEMATICS

Pingping Zhang
Winona State University
pzhang@winona.edu

Manjula Peter Joseph
Fresno Pacific University
manjula.joseph@fresno.edu

Keywords: Teacher beliefs, Elementary teacher preparation, Secondary teacher preparation

Introduction

There is considerable research to show that mathematics teachers' beliefs and their classroom practices are intricately connected (Pajares, 1992; Thompson, 1985). The need for equity in mathematics education continues to be important in teacher preparation, since "ensuring the success of each and every learner requires a deep, integrated focus on equity in every program that prepares teachers of mathematics" (AMTE, 2017, p. 1).

Theoretical Perspective

Teacher education must engage candidates in discussion about their beliefs focusing on "the nature of the student-teacher relationship, the curriculum, schooling, and society" (Ladson-Billings, 1995, p. 483). We used the work of Aguirre, Mayfield-Ingram, & Martin (2013) and focused on teacher beliefs about mathematical competencies and identities.

Methods

Twenty-two elementary PSTs and six secondary PSTs from a Midwestern public university participated in this study. This poster reports on the responses to the first question in the pre-questionnaire which was administered on the first day of the semester: *In your opinion, why is mathematics important for students to learn?*

Results

Four distinct types of reasons were revealed in all responses as shown in Table 1 below.

Table 1: Distinct types of reasons in all responses for question 1

		Elementary level		Secondary level	
1	Foundational subject	3	13.6%	1	16.7%
2	Mathematical thinking	5	22.7%	2	33.3%
3	Problem solving	6	27.3%	3	50.0%
4	Tools	13	59.1%	5	83.3%

Some of the responses covered two or three types on the list. We also found that the ranking of the four types of reasons was the same between elementary and secondary level PSTs, despite the differences between the exact percentages from both groups.

Conclusion

Based on the results provided in the previous section, we made two observations. First, secondary level PSTs were more likely to mention two or three reasons in their responses.

Secondly, despite the different experiences the two groups had with mathematics, the distinct types of reasons they provided and their rankings were identical.

References

Aguirre, J., Martin, D., & Mayfield-Ingram, K. (2013). *The impact of identity in K-8 mathematics: Rethinking equity-based practices*. National Council of Teachers of Mathematics.

Association of Mathematics Teacher Educators. (2017). *Standards for Preparing Teachers of Mathematics.* Retrieved from https://amte.net/sites/default/files/SPTM.pdf

Ladson-Billings, G. (1995). Toward a theory of culturally relevant pedagogy. *American educational research journal*, *32*(3), 465-491.

Pajares, M. F. (1992). Teachers' beliefs and educational research: Cleaning up a messy construct. *Review of educational research*, *62*(3), 307-332.

Thompson, A. G. (1985). Teachers' conceptions of mathematics and the teaching of problem solving. *Teaching and learning mathematical problem solving: Multiple research perspectives*, 281-294.

AN EMERGING FRAMEWORK FOR ELEMENTARY STUDENT TEACHER POSITIONS AND POSITIONINGS

Eric Partridge
University of Missouri
egpbk6@missouri.edu

Phi Nguyen
University of Illinois Chicago
pknguyen@uic.edu

Corey Webel
University of Missouri
webelcm@missouri.edu

Positioning theory has been used in research to understand how elementary student teachers are positioned while working with their mentor teacher. While this research has identified four general positions, there remains a lack of clarity about these positions, particularly in how they differ. This study explores three contrasting student teacher-mentor teacher pairs to explore how different student teacher positions are constructed. Through qualitative coding of interviews that followed a math lesson observation, an emerging framework is proposed along two dimensions: interactional positioning and actional positioning. Within each dimension, two sub-dimensions are proposed: discursive patterns and intellectual authority within interactional positioning and teaching experiences and planning experiences within actional positioning. This framework provides a foundation for further research on student teacher positions.

Keywords: preservice teacher education; elementary school education

Both teacher self-report (Levine, 2006) and external metrics and research (National Council for Accreditation of Teacher Education, 2010; Ronfeldt & Reininger, 2012) have shown that student teaching is an integral component of teacher preparation. Given that student teaching occurs in a mentor teacher's classroom and in close proximity to a mentor teacher, it is important to understand the nuances of student teacher-mentor teacher relationships.

Many scholars have utilized positioning theory to study student teacher-mentor teacher relationships (Bullough & Draper, 2004; Campbell & Lott, 2010; Chen & Mensah, 2018; Hart, 2020; Mosvold & Bjuland, 2016; Valencia et al., 2009), as positioning theory provides distinct definitions that differentiate roles and positions (Hart, 2020). Davies and Harré (1999) defined roles as "static, formal and ritualistic" (p. 32); the fixed, long-standing roles of student teacher and mentor teacher are entrenched components of teacher education. Conversely, positions are "seen as dynamic and fluid in nature" (Hart, 2020, p. 3) and are context specific (Chen & Mensah, 2018). As such, positioning theory allows for deeper investigation into the nuances of student teacher-mentor teacher relationships.

Research on student teacher positioning has established four general positions. Some student teachers are positioned as observers or workers in a classroom (Campbell & Lott, 2010; Chen & Mensah, 2018). Other student teachers are positioned as learners of teaching (Campbell & Lott, 2010; Chen & Mensah, 2018) or as teachers in training (Mosvold & Bjuland, 2016), where the focus is on mimicking their mentor teacher (Valencia et al., 2009). Still other student teachers are positioned as collaborators with the mentor teacher, where the student teacher offered ideas and both student teacher and mentor teacher were learning. (Campbell & Lott, 2010). Finally, some student teachers are positioned as classroom or fellow teacher, where the student teacher contributes to constructing classroom norms (Chen & Mensah, 2018).

While the names and conceptions of these four general positions have been established in the literature, there is little clarity on what differentiates these positions, particularly in how they are constructed. To explore this, our study was guided by the following research question: How are different student teacher positions constructed as elementary student teachers work with their mentor teacher around mathematics instruction? Specifically, we sought to 1) identify dimensions by which the positions differ and 2) develop an emerging framework that describes each position by dimension.

Theoretical Framework

Harré and van Langenhove (1999) defined positioning theory as the "study of local moral orders as ever-shifting patterns of mutual and contestable rights and obligations of speaking and acting" (p. 1). Positioning theory does not assume that all people involved have equal access toward performing any action (Harré, 2012), as these contestable rights and obligations inform potential boundaries of peoples' actions. For student teachers, how they are positioned provides different access to rights and obligations, and therefore, possibilities for action.

Position and positioning are two important, interconnected constructs within positioning theory. Positions have been defined as "a complex cluster of generic personal attributes, structured in various ways, which impinges on the possibilities of interpersonal, intergroup and even intrapersonal action" (Harré & van Langenhove, 1999, p. 1). Positions are dynamic and fluid as they occur within a particular context (Hart, 2020; Mosvold & Bjuland, 2016), and relatedly, are manifested through discourse (Hart, 2020). Herbel-Eisenmann et al. (2015) suggest that studying communication actions (e.g., gesture, physical position, etc.) allows for understanding discursive practices and interactions beyond just speech.

Positions are constructed through positioning, which has been defined as "the discursive process in which people use action and speech to arrange social structures through locating people in conversations" (Herbel-Eisenmann et al., 2015, p. 188). An important component of positioning is that it occurs at a moment in time, which contributes to the fluidity of positions (Wood, 2013). Positioning can occur when a person positions someone else or when a person positions themself (Kayi-Aydar & Miller, 2018).

Drawing upon this previous literature, we define position as a complex cluster of attributes that impinges (or affords) the possibilities of action, which is temporary and assigned through positioning. Conversely, we define positioning as the discursive process whereby communication actions locate oneself and others in moments in time.

Method

Data Context

The three mentor teachers in this study were members of a cohort of elementary teachers who received funding to complete an Elementary Mathematics Specialist (EMS) certification program and serve as teacher leaders in their schools for four additional years. At the time of this study, they had completed their two-year EMS program and a semester of their teacher leader service component. They had engaged in several leadership activities, including leading mathematics professional development in their schools and district, presenting at regional conferences, and working with colleagues and administrators to improve mathematics teaching in their schools. Ms. Erin was a fourth-grade teacher with 11 years of teaching experience. Ms. Molly was a third-grade teacher with 12 years of teaching experience. Ms. Julia was a kindergarten teacher with 16 years of experience. All names are pseudonyms.

The three student teachers in this study were enrolled in a program that included two mathematics methods courses during the previous school year (except for Jamie, whose program only required one methods class). These courses emphasized both mathematics, with a heavy focus on fractions and whole number operations, and pedagogy, with a focus on eliciting and responding to student thinking. In the Fall semester, student teachers were placed in elementary schools three days a week and rotated through different grade levels every two weeks. In the Spring semester, when data for this study was collected, they were placed with one teacher at the same school where they observed in the Fall. They were expected to gradually take over most of the instruction in that class, under the supervision of the mentor teacher.

Data and Participants

The data for this study come from three student teacher-EMS mentor teacher pairs: Elise and Ms. Erin; Michelle and Ms. Molly; and Jamie and Ms. Julia, respectively. All names are pseudonyms, and pseudonyms were chosen so that each student teacher-mentor teacher pair begins with the same letter. Ms. is included to indicate who is the mentor teacher. Data included seven interviews, each following an observation of a mathematics lesson. Two interviews were conducted each with the pairs of Elise and Ms. Erin, and Michelle and Ms. Molly, while three interviews were conducted with Jamie and Ms. Julia. Interviews were conducted via Zoom and lasted 11 to 48 minutes, for an average of 25 minutes. Interviews elicited participants' reflections on the math lesson, the planning of that lesson, their classroom responsibilities, as well as their goals for upcoming math teaching. All interviews were audio recorded and transcribed. The interviewer previously worked on this research project but is not one of the authors of this paper.

Data Preparation and Analysis

Analysis began by listening to the interviews and open coding for the various ways student teachers (ST) were positioned, which was informed by the literature and theoretical framework. Through discussion, an initial list of positionings was created, which was then used in a second round of listening to reduce the data. In data reduction, the authors listened separately and reconvened to agree on interview excerpts that did not include positionings of the ST. These excerpts were excluded, and a condensed transcript was prepared where excluded excerpts were summarized, and all other parts of the interview were fully transcribed.

These condensed interviews were then coded for different positionings by first employing the codes that emerged in the earlier listening rounds and adding and refining codes as needed. The first and second authors coded the first three interviews together, and then separately coded and came back to negotiate discrepancies for the remaining four interviews. After analyzing all the interviews, we then collapsed and grouped codes. First, we differentiated between positionings that emerged through ST's engagement (or lack of engagement) in the practices of teaching—what we call actional positionings—and positionings that occurred in interactions between mentor teacher (MT) and ST during the interview—what we call interactional positionings. Next, we separated the actional positionings into teaching experiences or planning experiences. Finally, we separated the interactional positionings into discursive patterns (i.e., patterns of MT and ST engaging together in the interview) or intellectual authority (i.e., to what extent the ST can make decisions or judgements). Table 1 summarizes the final organization of our data, which functions as the dimensions by which we analyze the positionings.

Table 1. Dimensions of Student Teacher Positions

Dimension		Definition	Example
Interactional Positioning	Discursive Patterns	Engagement between MT and ST in interview.	MT made space for ST: "Do you want to share what your thoughts were?"
	Intellectual Authority	To what extent the ST can make decisions or judgements.	MT as expert: "I didn't ever think to do that until MT" showed me.
Actional Positioning	Teaching Experiences	The qualitative nature of ST's teaching experiences.	ST overlooked by MT: I wish "there were a way to conference with" all students.
	Planning Experiences	The qualitative nature of ST's planning experiences.	ST with independence: "I've taken the math book home a few times" to plan.

Findings

As we explored our three cases, we saw distinct differences between the interactional and actional positionings of each student teacher. Using these interactional and actional positionings, we drew upon the previous literature on student teacher positions to identify how each student teacher was predominantly positioned: Elise as a learner of teaching, Michelle as a collaborator, and Jamie as a co-teacher. (Note that we use co-teacher rather than classroom or fellow teacher [Chen & Mensah, 2018] to highlight that ST and MT are still working together.) To facilitate presentation and reading of the findings, we first name how the student teacher was predominantly positioned and then present the corresponding positionings. However, this is the reverse of our analysis process, where we first focused on positionings, and then utilized the positionings to consider how the student teacher was predominantly positioned.

In the following sections, we explicate how different positionings constructed each student teachers' position. Organized by case, we first discuss interactional positionings, then actional positionings. As shown in Table 1, interactional positionings include discursive patterns and intellectual authority while actional positionings include teaching experiences and planning experiences. Given space limitations, we identify prominent positionings from each case that exemplify the predominant position for that student teacher.

Case 1: Elise and Ms. Erin

Elise was a student teacher in Ms. Erin's classroom. Through the interactional and actional positionings detailed below, Elise was predominantly positioned as a learner of teaching.

Interactional Positioning. Two prominent interactional positionings that represent Elise's case are Ms. Erin not making space for Elise in the interviews and both Elise and Ms. Erin positioning Ms. Erin as the expert. In this paper, we use the phrase "make space" to describe when a MT asks if the ST would like to share or offers the opportunity for the ST to share during the interview. In this sense, Ms. Erin only made space for Elise once in each interview. Moreover, when Elise did talk, her statements were brief; she only made three multi-sentence verbal statements in the first interview and six such statements in the second interview. This contrasted with Ms. Erin, who talked for extended periods in both interviews. Discursively, Ms. Erin's lack of making space for Elise contributes to her predominant position of learner of teaching as it indicates that Elise's thoughts and ideas are not as valuable as Ms. Erin's.

When considering intellectual authority, Elise positioned Ms. Erin as the expert in both interviews. For example, in the second interview, Elise said, “Ms. Erin is so good at seeing [student] work and automatically being able to tell where some holes are. And so I'd say going forward, that's a huge goal for me.” Elise is grounding her goals in Ms. Erin’s practice, which she views as the desired standard. Ms. Erin also positioned herself as the expert. When recalling Elise’s taking over of classroom instruction, Ms. Erin explained that Elise “hasn't been afraid to take over. So that's good, I feel like. And she knows that, you know, she can turn to me if she gets stuck.” Here, Ms. Erin reaffirmed herself as the expert, where she can provide guidance and support to Elise, who is still learning to teach. In both their discursive patterns in the interview and their interactions in the classroom, Elise and Ms. Erin’s positioning of Ms. Erin as the expert enforces Elise’s predominant position as a learner of teaching.

Actional Positioning. As expected, given the student teacher role, all student teachers had opportunities to teach and plan math lessons. The actional positionings (i.e., teaching experiences and planning experiences) detailed for each case focus on the qualitative nature of each student teachers’ opportunities and experiences with teaching and planning math lessons.

Regarding teaching experiences, Ms. Erin and Elise overlooked opportunities for Elise to gain more teaching experience. For example, as Ms. Erin reflected on the “kind of weak” assessment of the first lesson, she wished “there were a way to conference with [all students as] that would be better [as an assessment than a worksheet], but there’s 26 [students in the class].” Here, Ms. Erin overlooked the possibility of Elise conferencing with students, which could have strengthened the lesson’s assessment and provided Elise with more teaching experience. Separately, Elise identified that “walking around, listening, [and] assisting in any way has been really impactful” for her learning. By engaging in these basic actions, Elise lost opportunities to engage meaningfully and strategically with students during conferencing. In these examples, Ms. Erin and Elise both positioned Elise as a learner, rather than doer, of teaching by overlooking opportunities for Elise to engage in the key responsibilities of assessing student thinking and conferencing intentionally.

Elise’s lack of confidence with math was mentioned in both interviews, including that it resulted in math being the last subject she took over teaching. In the first interview, Ms. Erin said, “[Elise’s] like, I just wanna watch you for a little bit longer.” By implicitly conveying a theory of learning to teach through observing, rather than doing, Elise reinforced her position as a learner of teaching. In the second interview, Elise said, “I think the ultimate thing that changed since we [last] talked is now I'm leading every math lesson. And Ms. [Erin’s last name]’s of course there to jump in when she needs to or offer that support.” Even as Elise took over more teaching responsibilities, the hierarchy in their relationship (i.e., Elise as learner of teaching, Ms. Erin as expert) persisted, since Ms. Erin would “jump in” when needed.

Regarding planning experiences, there were times when Elise and Ms. Erin planned together and times when they did not. When the interviewer asked about the planning that went into the first lesson, Ms. Erin said she “probably didn't do a very good job of involving [Elise] in it this time, because [Ms. Erin] was planning it by [herself], (pause, then to Elise) sorry about that. We planned the Graham Fletcher part separately.” Not only is Elise missing out on planning experiences, but Ms. Erin is implicitly acknowledging they should be planning together, because these are important opportunities for Elise to learn about teaching.

Case 2: Michelle and Ms. Molly

Michelle was a student teacher in Ms. Molly’s classroom. Through the interactional and actional positionings detailed below, Michelle was predominantly positioned as a collaborator.

Interactional Positioning. Two prominent interactional positionings that represent Michelle's case are Ms. Molly's consistency in making space for Michelle in the interviews and equal status, including shared intellectual authority, between Michelle and Ms. Molly. Ms. Molly explicitly made space for Michelle six times in the first interview and four times in the second interview, though Michelle did not always take the space. Two examples of Ms. Molly making space were "did you want to kind of share what your thoughts were, and I can share what my thoughts were" and "do you wanna talk a little bit about the, how the fractions were connected?" Ms. Molly positioned Michelle as a collaborator by communicating value for her thoughts through consistently making space for her, which contrasts Elise's experience with Ms. Erin. This consistent making space for Michelle during the interviews exemplifies the discursive nature of her predominant position as a collaborator.

Michelle's predominant position as a collaborator is also exemplified through the distribution of intellectual authority in Ms. Molly's classroom, which is frequently shown through Michelle and Ms. Molly's references of equal status in their relationship. For example, in the first interview, Michelle mentioned that Ms. Molly once said, "this is your classroom just as much as it is mine," which "lets [Michelle] know that [she] can try new things." This demonstrates that Ms. Molly positioned Michelle as a collaborator who is provided with authority, intellectual and otherwise, in the classroom. In the second interview, Ms. Molly said that through their "constant communication," she thinks she and Michelle "depend on each other a lot," that they "work really well together," and that "everything is a team." Here, Ms. Molly positioned Michelle as a teammate of hers and someone she depends upon, which reinforces her predominant position as a collaborator who has equal status to Ms. Molly.

Actional Positioning. Regarding teaching experiences, Michelle, and Ms. Molly's ability to transition teaching responsibilities in the moment exemplifies Michelle's equal status in the classroom. Michelle said that "when [Ms. Molly] has to step out for one of our students…I'll just jump in and start teaching. And then sometimes, I'll be like, okay, I need to take a step back and [Ms. Molly'll] just jump in and start teaching." These smooth transitions between who is teaching in the classroom demonstrates Michelle and Ms. Molly's equal status and close collaboration, which reinforces Michelle's predominant position as a collaborator.

Regarding planning experiences, Michelle mentioned in the first interview that they "plan lessons after school together." Michelle also contrasted her experience working with Ms. Molly to her roommate's experience student teaching at another school. Michelle said her roommate's mentor teacher "goes home at the end of the day and just chooses to plan at home. And she just sends her like on her own to plan things. If she has questions, she can't really reach out to her," whereas Michelle and Ms. Molly "text all the time." By drawing on the contrasting experience of her roommate, Michelle reinforced the collaborative nature of her and Ms. Molly's relationship, which exemplifies her predominant position as a collaborator.

Case 3: Jamie and Ms. Julia

Jamie was a student teacher in Ms. Julia's classroom. Through the interactional and actional positionings detailed below, Jamie was predominantly positioned as a co-teacher.

Interactional Positioning. Three prominent interactional positionings that represent Jamie's case are Jamie interjecting in the interviews, Ms. Julia deferring to Jamie in the classroom, and Jamie making decisions about student learning. Across the three interviews, Jamie interjected to confirm Ms. Julia's statements 71 times (e.g., "mmhmm," "yeah," etc.). Similarly, Ms. Julia interjected to confirm Jamie's statements 15 times. Jamie also often added on to the conversation or answered the interviewer's question first, demonstrating that her interjections were not deferral to Ms. Julia's response. Jamie's interjections to confirm and add on to the conversation in an organic way demonstrate her equal discursive status with Ms. Julia, which contrasted with Ms. Molly's make space for Michelle and Ms. Erin's lack of making space for Elise. This exemplifies how Jamie was discursively positioned as a co-teacher.

Jamie was also positioned as a co-teacher through the intellectual authority she was both afforded by Ms. Julia and took up herself. In reflecting on the lesson, Ms. Julia said, "I kind of talked to [students], too. I was like, are you willing to share if we need you to? Like I didn't wanna be like, you are going to share." Here, Ms. Julia positioned Jamie as a co-teacher by deferring to Jamie regarding which students would share their strategies. This is particularly noteworthy because Ms. Julia thought ahead and intentionally planned her language when circulating so that Jamie would have intellectual authority over the eventual discussion.

Jamie also took up intellectual authority in the interview and in the classroom. For example, when the interviewer asked if there was anything further Jamie or Ms. Julia wished to discuss at the end of the interview, Jamie explained that in small groups, she "kind of used the kids to help create [the story problems]" by taking "turns asking them…what was your favorite dessert, and then that's what I'll base the problem around." Here, through answering the interviewer's open-ended question and establishing intellectual authority to make instructional decisions in the classroom, Jamie demonstrated her position as co-teacher.

Actional Positioning. Regarding teaching experiences, Jamie had several opportunities for more independent teaching than Elise or Michelle. For example, in the first interview, Ms. Julia mentioned that Jamie and Ms. Julia led different activities in the classroom. This trust in Jamie's instruction to be independent of Ms. Julia exemplifies Julia's predominant position as a co-teacher. Furthermore, in the first interview, Jamie and Ms. Julia mentioned that Jamie taught the class on her own for several days when Ms. Julia was absent or pulled to teach in another classroom. Both Jamie and Ms. Julia appeared comfortable with this situation; there were no comments of concern, stress, or uncertainty. This also indicates Ms. Julia's trust in Jamie as well as Jamie's confidence in herself as a teacher, particularly given this happened early in Jamie's student teaching semester. Jamie's opportunities for independent teaching and her and Ms. Julia's trust in her as a teacher reinforces Jamie's predominant position as a co-teacher.

Regarding planning experiences, Jamie also had some independence from Ms. Julia. While they "did a lot of [planning] together" (second interview) or "sat down at the beginning of the week and…plotted out what activity [they're] gonna do each day" (third interview), Jamie also said she had "taken the math [curriculum] book home a few times and kind of typed up, like what the book says, and then kind of brought it to school the next day" (second interview). In addition to planning with Ms. Julia, Jamie also had the opportunity to plan lessons on her own

Discussion

This study introduces two dimensions of positioning that construct student teacher positions: interactional positioning and actional positioning. Within each dimension, there are two sub-dimensions: discursive patterns and intellectual authority within interactional positioning, and teaching experiences and planning experiences within actional positioning. Each position is then described along these dimensions, using examples from the three cases in this study. Together, the dimensions and examples create an emerging framework that provides conceptual specificity and distinction between the student teacher positions of learner of teaching, collaborator, and co-teacher. This emerging framework, summarized in Table 2, can support further research on student teacher positions as well as teacher education structures for student teaching, particularly mentor teacher training.

Table 2. Emerging Framework of Student Teacher Positions by Dimension

		Learner of Teaching	Collaborator	Co-Teacher
Interactional Positioning	Discursive Patterns	MT does majority of the talking and makes little space for ST.	MT makes space for ST, which the ST may or may not take.	ST interjects and organically participates in discussion without MT making space.
	Intellectual Authority	MT is the expert. The goal is to watch their practice and emulate it.	MT communicates value and appreciation for ST ideas and presence.	MT defers to ST in some decision making and provides some independence.
Actional Positioning	Teaching Experiences	ST is overlooked sometimes and gets less opportunities to engage in teaching practice.	ST engages in teaching regularly and transitions easily between them and MT.	ST has some teaching experiences independent of MT.
	Planning Experiences	Sometimes MT plans without ST.	ST often plans with MT and can reach out with questions.	ST has opportunity to plan alone.

Importantly, this study does not seek to suggest a hierarchy of student teacher positions. For example, we do not view a position of co-teacher as inherently better than a position of learner of teaching. Rather, we believe there are different affordances and constraints that come with each position, and that each position may be most ideal for different student teachers or even the same student teacher at different points in their student teaching experience. Relatedly, given the fluidity of positions and the in-the-moment focus of positioning, we want to reiterate that while each student teacher was predominantly positioned as one position, there were examples in each case where the positioning in that moment would have aligned with a different position.

Given the limitation of space, we restricted the data presented in this paper to the observation-debrief interviews. Individual interviews were also conducted at the end of the semester with each student teacher and mentor teacher. One next step for this work is to analyze those interviews and consider how that data complements the data presented in this paper. Another next step is to analyze the storylines within each case and across the cases, as storylines are an important construct in positioning theory that are not always explicitly or thoroughly addressed in research (Herbel-Eisenmann et al., 2015).

Acknowledgments

This manuscript is based on work supported by the National Science Foundation under grant #1852822. Any opinions, findings, and conclusions or recommendations expressed herein are those of the authors and do not necessarily reflect the position, policy, or endorsement of the National Science Foundation.

References

Bullough Jr, R. V., & Draper, R. J. (2004). Making sense of a failed triad: Mentors, university supervisors, and positioning theory. *Journal of Teacher Education*, 55(5), 407-420.

Campbell, T., & Lott, K. (2010). Triad dynamics: Investigating social forces, roles, and storylines. *Teaching Education*, *21*(4), 349-366.

Chen, J. L., & Mensah, F. M. (2018). Teaching contexts that influence elementary preservice teachers' teacher and science teacher identity development. *Journal of Science Teacher Education*, *29*(5), 420-439.

Davies, B. & Harré, R. (1999). Positioning and personhood. In R. Harré and L. van Langenhove (Eds.), *Positioning theory*. Oxford: Blackwell Publishers.

Hart, A. (2020). Interpersonal Dynamics of the Supervisory Triad of Pre-Service Teacher Education: Lessons Learned from 15 Years of Research. *Georgia Educational Researcher*, *17*(2), 3.

Harré, R. (2012). Positioning theory: Moral dimensions of social-cultural psychology. In J. Valsiner (Ed.), *The Oxford handbook of culture and psychology* (pp. 191–206). NY: Oxford University Press.

Harré, R., & van Langenhove, L. (Eds.). (1999). *Positioning theory: Moral contexts of intentional action*. Oxford, UK: Blackwell Publishers.

Herbel-Eisenmann, B. A., Wagner, D., Johnson, K. R., Suh, H., & Figueras, H. (2015). Positioning in mathematics education: Revelations on an imported theory. *Educational Studies in Mathematics*, *89*(2), 185-204.

Kayi-Aydar, H., & Miller, E. R. (2018). Positioning in classroom discourse studies: A state-of-the-art review. *Classroom Discourse*, *9*(2), 79-94.

Levine, A. (2006). Educating school teachers. *Education Schools Project*.

Mosvold, R., & Bjuland, R. (2016). Positioning in identifying narratives of/about pre-service mathematics teachers in field practice. *Teaching and Teacher Education*, *58*, 90-98.

National Council for Accreditation of Teacher Education. (2010). *Transforming teacher education through clinical practice: A national strategy to prepare effective teachers. Report of the Blue Ribbon Panel on Clinical Preparation and Partnerships for Improved Student Learning*. ERIC Clearinghouse.

Ronfeldt, M., & Reininger, M. (2012). More or better student teaching? *Teaching and Teacher Education*, 28(8), 1091-1106.

Wood, M. B. (2013). Mathematical micro-identities: Moment-to-moment positioning and learning in a fourth-grade classroom. *Journal for Research in Mathematics Education*, *44*(5), 775-808.

Valencia, S. W., Martin, S. D., Place, N. A., & Grossman, P. (2009). Complex interactions in student teaching: Lost opportunities for learning. *Journal of Teacher Education*, *60*(3), 304-322.

ARTIFICIAL INTELLIGENCE AS TEACHING AID: POLYPHONIC LESSONS FROM THE “GREAT CALCULATOR DEBATE” IN MATHEMATICS EDUCATION

Christopher H. Dubbs
East Stroudsburg University
cdubbs@esu.edu

Keywords: Computational Thinking; Instructional Aid; Preservice Teacher Education; Technology

As a field, we are amid the latest in a series of technological that wars we have faced. In the 1960s, 70, and 80s, technological *conservatists* worried the hand-held electronic calculator would cause students to lose their ability to calculate by hand (see Usiskin, 1978 for a rebuttal of the ‘crutch’ argument). Today, the descendants of those same technological conservatist reactionaries worry that the use of artificial intelligence (AI) in schooling will develop a generation of individuals reliant upon AI tools. Rejecting this Luddite reaction, I conceive of an AI-infused mathematics teacher education as “a mathematics teacher preparation that uses AI—in its various forms—to the fullest extent possible in service of accomplishing the mathematics teacher educator’s (MTE’s) other simultaneous goals (e.g., social justice, cultural relevance, mathematical rigor, etc.)” AI, then, is not as a threat but as an instructional aid: an object to be used to enhance the learning of *mathematics* (Berger, 1973), not only for mathematics teachers (MTs) but also for MTEs.

Conceiving of AI as instructional aid, I presume a *systems thinking* approach to the teaching of mathematics. That is, I assume that it is the role of the teacher to manage the numerous resources, people, spaces, and experiences at their disposal to best meet the engagement and learning needs of all children in their classrooms. If the goal is to best manage these resources, then, it is necessary that MTs learn to best manage AI in the service of teaching mathematics.

Johnson and colleagues (1980) described affective indicators of computer literacy as those attitudes, values, and motivations we have towards computers and their use (e.g., fear or confidence, aversion or preference). They, in turn, described nine characteristics of computer literacy which I recast to consider an *AI literacy for MTs and MTEs*. AI literacy for MTs and MTEs suggests that teachers are (1) not anxious, but (2) confident about their ability to use AI programs; they value the (3) automation AI can enable, (4) the effort it can offload, (5) the information it makes available, and (6) the economic benefits of these; and they (7) enjoy, and actively seek opportunities to use AI in the teaching of mathematics. None of this is possible without (8) positive past-experiences with AI or (9) using AI in their free time.

Foucault’s historio-philisophical method of *mirrors as levers* (Fendler, 2010)—Foucault’s ironic approach to history suggests that sometimes mirrors make the best levers, that is, sometimes looking back can give us tools to change our present—enables me to recast lessons from the *Great Calculator Debate* (Barrett, 1974) to disrupt the current backlash against and aversion towards the use of AI by mathematics teachers. Consider this quote by Wheatley in which I replace *calculator* with *AI* in to make explicit the way I use history as a lever: “we encourage teachers to consider [AI] as an instructional aid and to explore ways of using them as an integral part of the teaching/learning process” (1979, p. 21). Wheately’s and others’ polyphonic voices speak historically about the calculator, contemporaneously about the present, and prophetically about the future.

References

Barrett, R. (1974). The great calculator debate. Nation's Schools & Colleges, 1(4), pp. 12-4. McGraw Hill.

Berger, E. J. (1973). Introduction. In Instructional Aids in Mathematics, pp. 1-6. NCTM.

Fendler, L. (2010). Michel Foucault. Continuum.

Johnson, D. C., Anderson, R. E., Hansen, T. P., & Klassen, D. L. (1980). Computer literacy—what is it?. The Mathematics Teacher, 73(2), pp. 91-6. https://doi.org/10.5951/MT.73.2.0091

Usiskin, Z. (1978). Are calculators a crutch? The Mathematics Teacher, 71(5), 412–413. http://www.jstor.org/stable/27961286

Wheatley, G. H. (1980). Calculators in the Classroom: A Proposal for Curricular Change. The Arithmetic Teacher, 28(4), 37–39. http://www.jstor.org/stable/41189767

BRIDGING CO-TEACHING AND METHODS COURSEWORK THROUGH SITE-BASED LEARNING COLLABORATIVES

Alyssa Sayavedra
California State University, Monterey Bay
asayavedra@csumb.edu

Nationally, there is a gap between the ambitious and equity-oriented mathematics pedagogies that pre-service teachers may learn in their university methods classes and the more traditional instruction that they typically try in their field placements (Horn & Campbell, 2015). This pilot study seeks to bridge this gap through a novel teacher learning collaborative with teacher candidates and cooperating teachers together, focused on the high-leverage practice of leading a discussion (Shaughnessy et al., 2021). Baseline findings were grim: mathematics "discussions" in teaching placements rarely included multiple student strategies or moved beyond "show and tell." However, the first learning cycle of the collaborative expanded teacher learning opportunities: one teacher candidate tried three dot talks and elicited and connected many strategies. Discussing her teaching video supported teachers' rich reflection on Agency, Ownership and Identity (Schoenfeld, 2020; 2022).

Keywords: Preservice Teacher Education, Elementary School Education, Classroom Discourse

Objectives

Nationally, there is a "gap" between the ambitious and equity-oriented practices that pre-service teachers may learn in their university methods classes and the more typical and traditional instruction that they almost always try out in their field placements (Horn & Campbell, 2015). In general, adoption of ambitious and equity-oriented teaching practices by schools and districts is a difficult problem, and is constrained by many factors including available curricula, testing, lack of sustained PD support, and competing demands on the time of administrators and teachers (Cobb et al., 2018; Schoenfeld, 2022). Powerful teacher learning can occur in learning communities, however, there is a relative scarcity of adequately facilitated teacher learning spaces that support teachers to both reflect on concrete instances of practices and make connection to broader principles and theories of ambitious and equity-oriented instruction (Horn & Little, 2010; Horn et al., 2017; Louie, 2016).

This paper is a preliminary report of findings from a pilot teacher learning collaborative, which will run from January through May, 2023. The collaborative is facilitated by the author and includes 4 TCs, their 4 cooperating teachers (CTs), and 3 other teachers at one elementary school site in Northern California. The CTs were highly recommended by the program coordinator as experienced and skilled teachers and mentors; all four teach dual language immersion in Spanish and English and have lived near the community where they teach for at least 10 years. However, the teachers have self-reported that they have less knowledge about supporting academic discourse in mathematics.

The collaborative will meet approximately monthly for a "video club" professional development (PD). The author will also observe and support the four co-teaching pairs (TCs with their CTs) approximately twice per month in the role of university clinical coach of the TCs. This preliminary report focuses primarily on analysis of baseline data, plus some analysis of the first learning cycle of the teacher collaborative in February 2023. Analysis of the baseline data is crucial to understanding the opportunities and obstacles TCs face. The first learning cycle offers insights about creating expansive opportunities for TC learning. The research questions are:

1. Prior to the start of the teacher collaborative, what opportunities do TCs have to practice and receive coaching feedback on the high-leverage teaching practice of leading a discussion in mathematics? What challenges constrain these opportunities?
2. What new opportunities and/or challenges are created for TCs in the first learning cycle of the teacher collaborative?

Theoretical Framework

The pilot intervention draws on known techniques for creating powerful teacher learning spaces, including a "video club" format to ground discussions in real teaching practice (van Es & Sherin, 2008) and the use of the Teaching for Robust Understanding framework to support principled reflection on practice (Schoenfeld, 2020; 2022). The TRU framework includes five dimensions: (1) Content, (2) Cognitive Demand, (3) Equitable Access, (4) Agency, Ownership and Identity, and (5) Formative Assessment. The intervention is innovative in that it (1) integrates support for TC and CT learning and (2) integrates the time-consuming work of coaching teachers in new pedagogies with the existing duties of clinical coach observations.

The teacher collaborative focuses on the high-leverage teaching practice of *leading a group discussion* (Ball & Forzani, 2009; Shaughnessy et al., 2015), one of two high-leverage practices that are central to our university's math methods course. Mathematical discussions, particularly those that go beyond "show and tell" to connect multiple student strategies, offer opportunities for teachers to learn about all five dimensions of TRU, and especially about Agency, Ownership and Identity. Key components of *leading a discussion* that particularly support Agency, Ownership and Identity include *recording students' ideas, orienting students to each other's ideas*, *responding to unexpected or incorrect answers, and noticing and disrupting inequitable patterns of participation.* Nevertheless, the practice is somewhat rare; one large-scale study found that even after substantial PD work in a large district, when asked to videotape their "best" mathematics lessons, only about half of the lessons included a discussion and only half of those went beyond "show and tell" (Horn, 2017).

Methods of Inquiry

Planned data sources for the teacher collaborative include a pre- and post- survey, video of PD sessions, video of at least one classroom coaching visit per TC in which that TC is trying out a new pedagogical practice, and written observation notes from all other coaching visits. At the time of this writing, the teacher learning collaborative has completed one learning cycle including two PD sessions and one video recording of a TC trying math talks which was shown in the second PD session. At least 2 additional learning cycles are planned for the remainder of Spring 2023.

In addition, written observation notes from my coaching observations prior to the start of the teacher collaborative are analyzed as a source of baseline data. I conducted 46 coaching visits with 6 TCs in Fall 2022, prior to the start of the teacher collaborative. 12 of these coaching visits (26%) occurred during mathematics instruction.

Both the 12 baseline observations and the TC video used in the first learning cycle of the teacher learning community were coded based on:

1. Class format: tutoring individual students, pair/group work and/or whole group
2. Whether the observation included a discussion, defined broadly as any whole group (5+ students with teacher) activity in which students took talk turns longer than a few words,
3. Opportunities to practice components of leading a discussion, listed in Table 1.

Results will be presented as code percentages plus qualitative excerpts from transcript and/or field notes to illustrate the nature of the learning opportunities for TCs.

Results

Baseline Data

Of the 12 baseline observations from Fall 2022, only 7 included a discussion of any kind. The remaining observations involved tutoring individual students (2 observations) or teacher demonstrations to the whole class (3 observations). Of these 7 observed discussions, the opportunities for feedback on specific aspects of *leading a discussion* are listed below.

Table 1: Number and percentage of learning opportunities in observed discussions

Launching and concluding the discussion	7 (100%)
Recording students' ideas	0 (0%)
Orienting students to each other's ideas	6 (86%)
Treating mistakes and unexpected answers as resources for learning	2 (29%)
Noticing and disrupting inequitable participation patterns	6 (86%)

In all 7 observed discussions, TCs had the opportunity to launch and conclude the discussion. However, none of the TCs recorded student ideas during the observations, and it was unusual for students to give unexpected or incorrect responses. This is because most of the discussions were either about a fairly easy question, such as "How do you know the corner of the paper is 90 degrees?" or involved one or two students coming to the front to presenting correct, standard solutions in a "show and tell" format (Stein et al., 2008). Although most discussions included some opportunities to orient students to each other's ideas and to notice and disrupt inequitable participation patterns, these opportunities were not as in-depth as they would have been for more open-ended tasks in which students considered partial or incorrect reasoning and connected multiple solution strategies.

First learning cycle

In the first PD session, the teacher learning community discussed the value of eliciting student thinking about multiple strategies, strategies for orienting students to each other's ideas, and helping them connect ideas. A TC with some previous familiarity with number talks (from my math methods course) then volunteered to lead a sequence of three dot talks and be videotaped for the "video club" PD. The second PD session introduced math talks (including number talks, dot talks, and other variations), a pedagogical practice with which all CTs were previously unfamiliar. The majority of the session was spent in a video club session analyzing the TC's dot talks. This created new, important learning affordances for learning by both the TC and all participating teachers.

First, the presenting TC had the opportunity to practice all five aspects of *leading a discussion* during her dot talks. She used three dot talks from *youcubed.org* (2019) as a basis for a 43 minute discussion with a group of six students. Students generated a total of 30 strategies for the 3 dot talks. During the first dot talk, the teacher practiced *recording students' ideas* by turning to each of the first four students and asking them to verbally construct a number sentence for their dot pattern as she wrote each number sentence on the board. After that, students came to the whiteboard to record their own ideas, including number sentences. The inclusion of so many varied strategies offered the TC rich opportunities to help *orient students to each other's ideas*, for example by asking what was different about a given strategy and two opportunities to respond to incorrect answers, for example by allowing other students to disagree and then the presenting student to revise her own answer. She *disrupted inequitable participation patterns* by supporting one lower-status student who was hesitant to present; by the last dot talk, all six students shared and were positioned as competent.

Discussion of this video during PD created important learning affordances for all participating teachers. Transcript excerpts showing teacher talk relating the dot talk video to different TRU dimensions are shown in Table 2. The session inspired future learning opportunities: two of the remaining three TCs made plans to try math talks, while the fourth made plans to support student talk in small groups.

Table 2: Teacher Talk during Professional Development Session - Learning Cycle 1

Access	I would say more access, there's access for everyone. Everyone has a different approach for it. - TC
Agency, Ownership, and Identity	I think also agency because the students are able to solve the problem different ways and explore - CT
Cognitive Demand	We connected that to cognitive demand, because the kids will push each other, "why are you saying that?" - CT

At the time of this writing, data from two additional learning cycles has just been completed, and not yet analyzed; post surveys are in process and not yet available. However, there are some early indications that the learning affordances of the first learning cycle were made available to the group as a whole: participants requested to continue focusing on math talks for the remainder of the school year; all four teacher candidates tried math talks and shared their video in PD sessions; 11/11 (100%) participants responding to a March exit ticket recommended that math talks should be required of all candidates in the credential program; the five experienced teachers who participated in the closing PD session reported that they had tried between one to six math talks each and felt that more students were becoming more comfortable explaining their thinking.

Discussion and Conclusions

Baseline data showed that prior to the PD, TCs coached by the author had very little opportunity for supported practice *leading a discussion* in mathematics in their teaching placements. TCs had opportunities to lead whole class instruction in mathematics, but typical instructional routines in placement classrooms (as in most U.S. math classrooms) offered little opportunity for TCs to elicit multiple strategies from students, record student thinking, orient students to each other's thinking, or respond to unexpected or incorrect student strategies.

The first learning cycle of the teacher learning community offered a promising intervention in that it created a new opportunity for one TC to (1) try an instructional routine - math talks - that allowed her to practice all of these elements of *leading a discussion* and (2) share a video of her new practice with the collaborative and reflect in ways that created opportunities for all 11 participating teachers to reflect on all of these elements of practice. Next steps could include: (1) Supporting all elementary candidates to rehearse math talks in the math methods course, (2) Incorporating at least one observation of a math talk into the 16 coaching observations conducted by clinical coaches with every TC (about 125 TCs/year), and (3) Seek grant funding to offer a one hour PD on math talks to all CTs and optional video clubs at school sites.

References

Ball, D., & Forzani, F. M. (2009). The Work of Teaching and the Challenge for Teacher Education. *Journal of Teacher Education*, *60*(5), 497–511. https://doi.org/10.1177/0022487109348479

Cobb, P., Jackson, K., Henrick, E., & Smith, T. M. (2018). Systems for instructional improvement: Creating coherence from the classroom to the district office. Cambridge, MA: Harvard Education Press.

Horn, I. S. (2017, July 24). *An Asset-Orientation is Everything*. Hugo Rossi Lecture, University of Utah.

Horn, I.S. & S. S. Campbell (2015) Developing pedagogical judgment in novice teachers: mediated field experience as a pedagogy for teacher education, Pedagogies: An International Journal, 10:2, 149-176, DOI: 10.1080/1554480X.2015.1021350

Horn, I. S., Garner, B., Kane, B. D., & Brasel, J. (2017). A Taxonomy of Instructional Learning Opportunities in Teachers' Workgroup Conversations. Journal of Teacher Education, 68(1), 41–54. https://doi.org/10.1177/0022487116676315

Horn, I. S., & Little, J. W. (2010). Attending to Problems of Practice: Routines and Resources for Professional Learning in Teachers' Workplace Interactions. American Educational Research Journal, 47(1), 181–217. https://doi.org/10.3102/0002831209345158

Louie, N. L. (2016). Tensions in equity- and reform-oriented learning in teachers' collaborative conversations. Teaching and Teacher Education, 53, 10–19. https://doi.org/10.1016/j.tate.2015.10.001

Shaughnessy, M., Garcia, N. M., O'Neill, M. K., Selling, S. K., Willis, A. T., Wilkes, C. E., Salazar, S. B., & Ball, D. L. (2021). Formatively assessing prospective teachers' skills in leading mathematics discussions. *Educational Studies in Mathematics*, *108*(3), 451–472. https://doi.org/10.1007/s10649-021-10070-z

Schoenfeld, A. H. (2020). Reframing teacher knowledge: A research and development agenda. ZDM, 52(2), 359–376. https://doi.org/10.1007/s11858-019-01057-5

Schoenfeld, A. H. (2022). Why Are Learning and Teaching Mathematics So Difficult? In M. Danesi (Ed.), *Handbook of Cognitive Mathematics* (pp. 1–35). Springer International Publishing. https://doi.org/10.1007/978-3-030-44982-7_10-1

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning*, *10*(4), 313–340.

van Es, E. A., & Sherin, M. G. (2008). Mathematics teachers' "learning to notice" in the context of a video club. Teaching and Teacher Education, 24(2), 244–276. https://doi.org/10.1016/j.tate.2006.11.005

youcubed (2019). *Dot Card and Number Talks. https://www.youcubed.org/wp-content/uploads/2019/08/WIM-Dot-Card-and-Number-Talks-Grades-K-12.pdf*

BROKERING MOVES TO SUPPORT ADVANCES IN A PRESERVICE ELEMENTARY TEACHER'S DISCOURSE ABOUT HIERARCHICAL GEOMETRIC RELATIONSHIPS

Ian Whitacre
Florida State University
iwhitacre@fsu.edu

Domonique Caro-Rora
Florida State University
dcarorora@fsu.edu

Azar Kamaldar
Florida State University
ak22@fsu.edu

Limited literature addresses how elementary preservice teachers (PSTs) can advance their thinking about hierarchical geometric relationships. Informed by a commognitive perspective, we investigated this phenomenon as a matter of discursive change, focusing on word use, visual mediators, and narratives. We report on a teaching experiment involving a PST whom we call Mariah. Mariah's progress seemed to be driven primarily by metalevel changes in her word use and interpretations of diagrams. These changes supported shifts in her narratives regarding hierarchical geometric relationships. Instructor moves related to these discursive changes involved brokering between multiple discourse communities. Our analysis reveals nuances of communication that we do not see highlighted in the literature on PST education.

Keywords: Communication, Geometry and Spatial Reasoning, Preservice Teacher Education

There is little literature regarding the thinking or learning of elementary preservice teachers (PSTs) concerning hierarchical geometric relationships (Browning et al., 2014). *Hierarchical geometric relationships* refer to set–subset relationships between different types of shapes based on their properties (Fujita & Jones, 2007). Some articles frame their findings in terms of deficits in PSTs' thinking (Fujita & Jones, 2007; Pickreign, 2007). Others report results of intervention studies but do not illuminate the learning process (Brunheira & da Ponte, 2019; Yi et al., 2020). The literature and our teaching experience have shown us that reasoning about and representing hierarchical geometric relationships can be challenging for PSTs. For this topic and many others, the field needs accounts of PSTs' thinking and learning that contribute insights and support theoretically grounded instructional sequences.

Clues and Unanswered Questions in the Literature

Yi et al. (2020) conducted an intervention study and reported evidence that PSTs' geometry knowledge for teaching 2-D shapes can improve significantly over a three-week period. The description of their instructional sequence mentions one activity in which PSTs "used Venn diagrams to categorize [quadrilaterals] based on their similar and different properties" (p. 5) but does not elaborate. The assessment included items on quadrilateral properties and relationships (among others) but did not address some challenging aspects (e.g., categorization of trapezoids). Thus, Yi et al. offer some evidence of improvement, but their account does not explain PSTs' difficulties with the topic of hierarchical relationships or how those difficulties can be overcome.

Brunheira and da Ponte (2019) report on a teaching experiment focused on classification of quadrilaterals and prisms. They offer examples of a PST's work that involves what seem to us two contrasting interpretations of Venn diagrams (p. 76), rather than a shift in thinking about the hierarchical relationships among the shapes. This distinction is left unexplored. The authors report progress in PSTs' reasoning but also lingering difficulties, and they suggest the need to explore "other factors" such as "language interpretation and logical reasoning" (p. 80).

Overall, the literature leaves us with unanswered questions. It does not provide insights into PSTs' geometric thinking, nor does it document viable learning processes for specific topics (Browning et al., 2014). We believe a key to understanding PSTs' thinking and supporting their learning related to hierarchical geometric relationships lies in those aforementioned "other factors" that we have not seen deeply explored in the literature to date.

Theoretical Framework

This study is primarily informed by a commognitive perspective, according to which thinking and communicating are one and the same. Through this lens, learning takes the form of discursive change. In the case of an individual PST, we can characterize aspects of their mathematical discourse at points in time, and thus document changes that took place in their discourse. These changes are tantamount to changes in the person's thinking (Sfard, 2007).

We focus here on word use, visual mediators, and narratives as properties of discourse. Mathematical discourse involves specifically mathematical vocabulary, along with everyday words: "Although shape- and number-related words may appear in nonspecialized, colloquial discourses, literate mathematical discourses as practiced in schools or in academia dictate their own more disciplined uses of these words" (Sfard, 2007, p. 571). Mathematical discourse often involves symbolic artifacts that serve important roles as *visual mediators* of mathematical communication. In our case, Venn diagrams are a symbolic artifact of particular interest. *Narratives* are statements that are "subject to *endorsement* or rejection, that is, to being labeled as *true* or *false*" (p. 572). Narratives regarding properties of quadrilaterals and relationships based on those properties are central to the topic of hierarchical geometric relationships.

We draw on related literature to characterize the role of the instructor in our study as a broker between discourse communities: "By definition, a broker is someone who can facilitate communication and fluidity of practices between different communities and who has membership status in all the different communities" (Zandieh et al., 2017, p. 97). In the case of Zandieh et al., the communities of interest were the local classroom community and the broader mathematical community. Their analysis examined ways in which an instructor helped to link student practices with elements of practice from the broader mathematical community. This connection to a "broader mathematical community" relates to Sfard's (2007) notion of a *leading discourse.* The leading discourse has special status relative to individual discourses. At the same time, the broker as interpreter is concerned with honoring students' thinking so that "interpreting between communities facilitates the students' sense of ownership of ideas" (p. 98).

In keeping with the view of learning as discursive change, we are interested in how an instructor of elementary PSTs supports that change process. The notion of brokering moves is consistent with the participationist nature of Sfard's (2007) framework, and we have found it to be a useful way of framing the instructor's role. In the case of our study, there are four types of discourse communities involved: the broader mathematical community and the state standards, both of which are leading discourses; the non-leading discourse of other PSTs and "people" who were mentioned by the instructor but not present in the sessions; the collective discourse between Mariah and the instructor; and Mariah's individual geometric discourse.

According to Sfard (2007), studies guided by the commognitive framework should be able to address the following questions:

1. Focus on the object of learning: In the case under study, what kind of change was supposed to occur as a result of learning?

2. Focus on the process: How did the students and teacher work toward this change?
3. Focus on the outcome: Has the expected change occurred? (p. 566) We address each of these points in our Method and Results sections below.

Method

We conducted an individual teaching experiment with Mariah (pseudonym). She was 18 years old and identified as a Black, Caribbean-American woman. She had been accepted into an Elementary Education program at a large research university in the Southeastern United States. The sessions took place in the summer before she began the program. The instructor (and first author) was 44 years old and identified as a White male. The teaching experiment spanned three one-hour sessions. These were conducted via Zoom and involved Google Slides and the Desmos Geometry Tool. The learning goal was for Mariah to engage in a geometric discourse based on official definitions, emphasizing set–subset relationships, and involving the use of diagrams to represent those relationships. The culminating task was to construct "one big diagram" involving several types of quadrilaterals and to explain the relationships between them.

In Session 1, Mariah was asked about her meaning for the term *Venn diagram* and her interpretations of various diagrams. She was asked to generate examples of diagrams to represent relationships, both mathematical and non-mathematical. Mathematical topics of discussion included the square–rectangle and parallelogram–rhombus relationships. In Session 2, the instructor introduced the distinction between inclusive and exclusive definitions. Mariah was invited to think about the implications of definitions. She discussed various relationships and associated diagrams involving sets and subsets. She worked to create a diagram to represent the rectangle–rhombus–square relationship. In Session 3, Mariah explored trapezoids and considered both inclusive and exclusive definitions for *trapezoid*. Mariah worked to produce "one big diagram" to represent the hierarchical relationships between quadrilaterals, parallelograms, rectangles, rhombi, squares, and trapezoids.

The instructional sequence was extracted from a geometry unit that the instructor had taught in a course for PSTs the previous semester. The summer teaching experiment with Mariah provided the opportunity to closely study the thinking and learning of an individual PST through the lens of discursive change. In this context, we asked the following research questions:

1. How did Mariah's discourse related to hierarchical geometric relationships change over the course of the teaching experiment?
2. What brokering moves did the instructor use to support Mariah in shifting her discourse?

Videos of the Zoom sessions were transcribed for analysis. The analysis in answer to Research Question 1 was informed by the commognitive framework, together with previous experience teaching the topic of hierarchical geometric relationships and initial investigations of PSTs' thinking related to this mathematical topic (Whitacre & Caro-Rora, 2022). Our analysis of Mariah's discourse focused on the following aspects: (a) her word use, especially concerning diagrams, definitions, and hierarchical relationships; (b) her interpretation and use of symbolic artifacts, especially Venn diagrams; and (c) her narratives concerning quadrilateral properties and relationships. We identified patterns in the data to produce summary characterizations of details of Mariah's discourse in each of the sessions. We made chronological comparisons within and across sessions to identify shifts in aspects of Mariah's discourse.

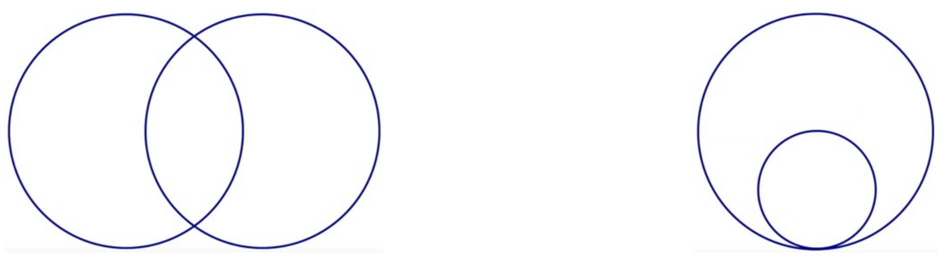
To answer Research Question 2, we analyzed the same data set with a focus on the instructor's utterances. Building on the answers to Research Question 1, we investigated the interactions that preceded and appeared to facilitate or support the observed shifts in Mariah's discourse. This analysis is informed by the notion of brokering (Rasmussen et al., 2009; Zandieh et al., 2017) because the instructor's role was marked by a focus on communication and clarification. Through our ongoing analyses, we continue to refine these characterizations of the interlocutors' interactions and the process of change that took place across the sessions.

Results

In answer to Research Question 1, we highlight changes in Mariah's discourse in terms of word use, visual mediators, and narratives across the three sessions of the teaching experiment. We present characterizations of her discourse and changes in her discourse in chronological order. In answer to Research Question 2, we highlight the instructor's brokering moves and relate them to shifts in Mariah's discourse.

Shifts in Mariah's Discourse

Session 1. Mariah initially indicated that a "Venn diagram" was for representing similarities and differences (i.e., she used the term Venn diagram to indicate a diagram that consisted of overlapping circles and was used for the purpose of comparing and contrasting). In Session 1, she began distinguishing between two types/interpretations of diagrams, which she named "compare and contrast diagrams" and "container diagrams" (Figure 1). She also introduced the language of "genre and subgenre" to describe the set and subset depicted in a "container diagram." Mariah initially talked about different arrangements of circles as determining the type of diagram: (a) overlapping circles were for showing similarities and differences, as in a "compare and contrast diagram," whereas (b) a circle within a circle showed a "genre and subgenre" relationship, as in a "container diagram."



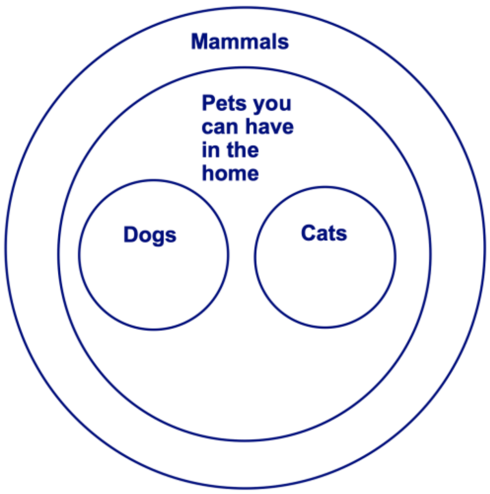
"Compare and contrast diagram" "Container diagram"

Figure 1: Mariah's initial interpretations of Venn diagrams

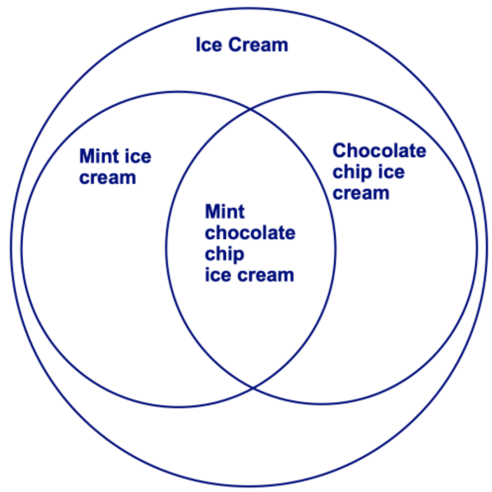
The properties that Mariah voiced, together with her answers to "Is it still?" questions (e.g., "If we move this vertex of the parallelogram construction until it becomes a rectangle, is it still a parallelogram?"), constituted her working definitions of specific quadrilaterals as of Session 1: Mariah talked about rectangles and parallelograms inclusively regarding their side lengths (e.g., she endorsed the narrative that a square is a "subgenre" of rectangle). At the same time, she talked about parallelograms exclusively regarding their angles (i.e., she endorsed the narrative that a rectangle was *not* a type of parallelogram).

Session 2. Mariah became aware of the terms *inclusive* and *exclusive* as applied to definitions of geometric shapes. She was able to distinguish between the two types of definitions and relate them to diagrams when asked to do so, but she rarely used these terms spontaneously.

Mariah recognized that for container diagrams, different arrangements were possible (beyond the circle-in-circle archetype in Figure 1), and these indicated the nature of various possible "genre and subgenre" relationships. She distinguished different arrangements of circles from types/interpretations of diagrams. Regarding container diagrams, overlap took on a different meaning in Mariah's discourse: "subgenre of both" (Figure 2b). Similarities were shown in membership in a larger set (Figure 2a), rather than in an overlapping region. She indicated that she did not think it would make sense for the "Dogs" and "Cats" ovals in Figure 2a to overlap.



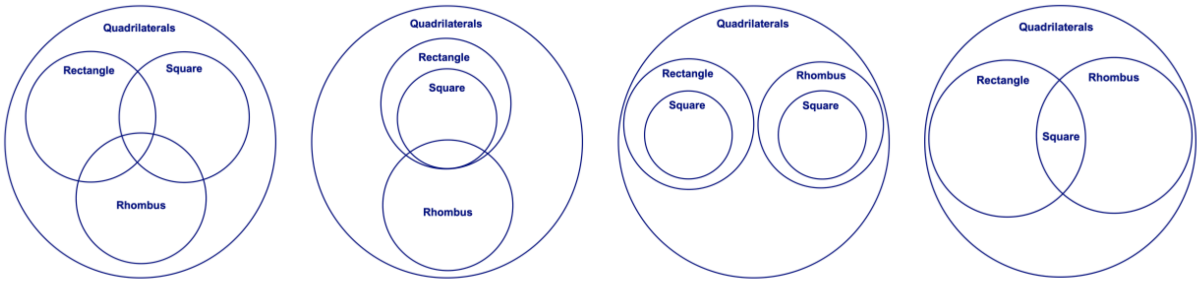
2a. Mariah's "container diagram" depicting similarities between dogs and cats as membership in common "genres"



2b. Mariah's "container diagram" using overlap to show that mint chocolate chip ice cream was a "subgenre" of both mint ice cream and chocolate chip ice cream

Figure 2: Progress in Mariah's use of "container diagrams"

When asked to create a container diagram for rectangles, rhombi, and squares, Mariah worked through four versions (Figure 3). In the first two, she spoke about rhombi as exclusive regarding angles. Given an inclusive definition of rhombus, she then described a square as a "subgenre of both" rectangle and rhombus, eventually settling on the fourth version (far right).



Mariah's first version, which she spontaneously abandoned

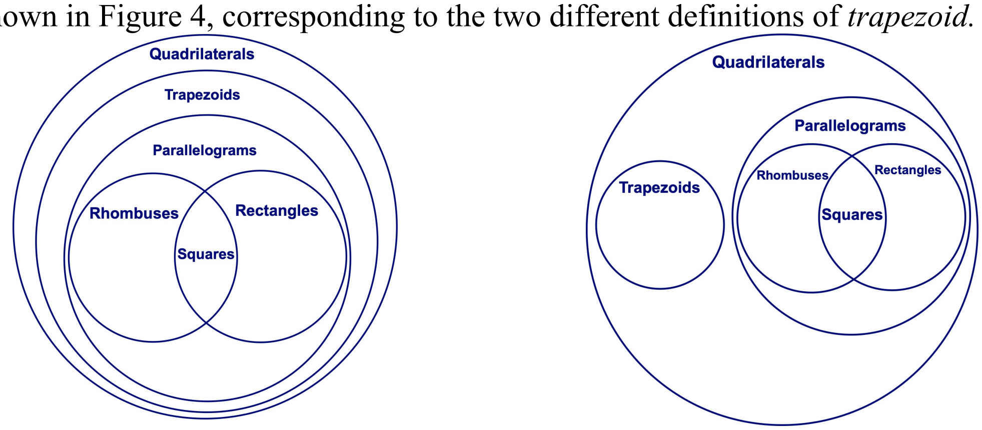
Mariah's second version, revised in response to questions about the regions

Mariah's third version, based on an inclusive definition of rhombus

Mariah's final diagram, inspired by other overlapping "container diagrams"

Figure 3: Mariah's "container" diagrams for the rectangle–rhombus–square relationship

Session 3. In this session, "inclusive" and "exclusive" became part of Mariah's individual discourse—not only terms she understood, but terms that she used repeatedly and spontaneously. Mariah also fluently used "container diagrams" to represent hierarchical relationships, including nested and overlapping relationships. She was eventually able to create diagrams to represent how several types of quadrilaterals were related, to explain each of those relationships based on definitions, and to explain how those relationships were represented in her diagrams. Mariah spoke about trapezoids exclusively. However, she was able to construct the two "container diagrams" shown in Figure 4, corresponding to the two different definitions of *trapezoid.*



Treating Trapezoids Inclusively

Treating Trapezoids Exclusively

Figure 4: Mariah's "container" diagrams relating several types of quadrilaterals Brokering Moves

The instructor did not impose narratives regarding hierarchical relationships. He alternated between acting as an interviewer inquiring about Mariah's thinking and acting as a broker informing Mariah about other discourses that would not otherwise have been available during their interactions (e.g., sharing information about standards). We identified four specific subtypes of brokering moves that appeared to support Mariah in shifting her discourse:

Sharing information about a non-leading, or peer, discourse. To motivate the need for discussion of types/interpretations of diagrams, the instructor began by sharing his observation (from courses for PSTs) that people often use and interpret diagrams and the term *Venn diagram* in different ways. This introduction set the stage for Mariah to share her interpretations and helped establish the need for these kinds of clarifications. Similarly, the instructor shared that people use "can be" and "can't be" in different ways when discussing geometric relationships; thus, it was important for people to make explicit what they meant by "can be" when discussing such relationships. Mariah clarified that she meant "can be considered" and subsequently used specific phrasing more consistently (e.g., "a square can be considered a rhombus").

Promoting and valuing individual contributions to the collective discourse. Once it was clear that Mariah herself had varying interpretations of diagrams, it became important to use terms to name types/interpretations of diagrams in the interest of clear communication. The instructor invited Mariah to suggest names, and she introduced "compare and contrast diagram," "container diagram," and "genre and subgenre." Mariah and the instructor consistently used Mariah's language from that point on, which enabled clear communication. It also signaled that Mariah's contributions were valued by making them part of the collective discourse.

Sharing technical terminology from a leading discourse. The instructor introduced terminology to name relevant distinctions that arose in Mariah's talk about quadrilaterals, especially the terms *inclusive* and *exclusive definitions*. Mariah subsequently distinguished between these two types of definitions and considered the implications of choices of definition for hierarchical relationships between quadrilaterals.

Sharing meta-rules from a leading discourse. The instructor shared information about the state's standards, especially the fact that the glossary used inclusive definitions for quadrilaterals, including for *trapezoid*. Thus, it was highly relevant for Mariah as a PST to become familiar and comfortable with those definitions. Mariah transitioned from classifying quadrilaterals based on her implicit, personal definitions to doing so based on explicit, official definitions.

Discussion

Our learning goal for the instructional unit was achieved: There were noteworthy changes in Mariah's geometric discourse, culminating in her ability to construct "one big diagram" relating all the types of quadrilaterals that were discussed, based on their explicit definitions, and to explain the relationships shown in her diagram. Given the reported difficulties that PSTs have with this topic (Fujita & Jones, 2007; Pickreign, 2007), and the challenges of communication arising from different ways of using and interpreting language and diagrams (Whitacre & Caro-Rora, 2022), this outcome is a feat. Furthermore, our findings provide insights into Mariah's learning process: (a) the major changes in Mariah's discourse were meta-level changes; (b) these changes were facilitated by the instructor's brokering moves, which primarily involved interpreting between communities; and (c) progress in Mariah's reasoning about hierarchical geometric relationships was driven primarily by changes in her word use and interpretation of diagrams, not by changes in the logic of her narratives themselves.

Meta-level Changes

We find that the major shifts identified in Mariah's discourse were meta-level discursive changes, meaning that they happened at the level of discourse about discourse (Sfard, 2007). These were primarily shifts in word use and interpretation of symbolic artifacts: (a) from "Venn diagram" meaning overlapping circles with a compare-and-contrast interpretation to "compare and contrast diagram" vs. "container diagram," (b) from associating overlap in circle diagrams with similarities under the compare-and-contrast interpretation to also being able to interpret overlap in a container diagram using the "genre and subgenre" interpretation (i.e., "a subgenre of both”), and (c) from thinking/talking about each quadrilateral in her individual manner (e.g., treating parallelograms as inclusive in terms of side lengths but exclusive in terms of angles) to thinking/talking explicitly about inclusive vs. exclusive definitions and considering the implications of definitions on hierarchical relationships—better connecting her individual discourse to the leading discourse on quadrilateral properties and relationships.

Teacher Educator as Broker

The instructor made several brokering moves to support Mariah in transforming her discourse. These moves involved interpreting between discourse communities by (a) sharing information about a non-leading, or peer, discourse; (b) promoting and valuing individual contributions to the collective discourse; (c) sharing technical terminology from a leading discourse; and (d) sharing meta-rules from a leading discourse.

We speak in terms of *a* leading discourse because there are at least two leading discourses that were relevant in our teaching experiment. As in the case of Zandieh et al. (2017), the broader mathematical community was one of these. For example, the instructor, a mathematics teacher educator, shared the fact that there is not a consensus definition of *trapezoid* in mathematics textbooks or other authoritative sources. In this case, the leading discourse was the broader mathematical community. However, another leading discourse at play in the teaching experiment was determined by the state standards. Because Mariah was a preservice teacher, the standards were highly relevant and justifiably treated as authoritative by the interlocutors. This dimension may not arise in the case of undergraduate students studying linear algebra, as in Zandieh et al.

Contributions to the Literature on Elementary PSTs and Geometry

This study offers contributions to the literature regarding PSTs’ geometric thinking and learning. First and foremost, given a thoughtfully designed instructional sequence and support in the form of the instructor making key brokering moves, Mariah was able to transform her discourse in ways indicative of substantial progress. In our view, her progress toward the learning goal did not involve shifts from incorrect ideas to correct ones; instead, the process involved making distinctions to facilitate communication. The available literature on the topic of PSTs’ geometric thinking is limited and may have thus far ignored nuances of word use and the roles of symbolic artifacts such as Venn diagrams as visual mediators in the learning process.

Responding to the call from Brunheira and da Ponte (2019) to investigate “other factors” that might explain PSTs’ difficulties with this topic, such as “language interpretation and logical reasoning” (p. 80), our findings indicate that attention to language interpretation has the potential to support PSTs’ learning in this area. Logical reasoning did not seem to be an issue for Mariah.

Mariah’s successful learning process, together with our additional teaching experiments and investigations into PSTs’ geometric thinking, indicate that the instructional sequence that we employed is potentially viable. A noteworthy feature of the instructional sequence is its explicitness about communication and the need to share interpretations, discuss terminology and facts about leading discourses, as well as to adopt local terminology (e.g., “container diagram” and “genre and subgenre”) that functions as if shared in the classroom community (Rasmussen & Stephan, 2008). Our findings also inform revisions to the instructional sequence. Most notably, it was crucial to Mariah’s progress in Session 2 to work with an explicit and official definition of *rhombus*. This and other iterations of the teaching experiment have highlighted advantages to earlier introduction of official definitions, after exploration of quadrilateral constructions and discussion of properties.

Limitations and Conclusion

Naturally, we do not regard Mariah as representative of all elementary PSTs. In our ongoing research, we are analyzing data from another iteration of a teaching experiment on this topic, involving a pair of PSTs. We have also collected data from two sections of a course for PSTs taught in Spring Semester 2023. We have observed consistent patterns in PSTs' thinking about the term *Venn diagram* and in the need to make the kinds of distinctions regarding language and interpretations that were made in the teaching experiment with Mariah.

We reiterate the point that the field needs accounts of PSTs' mathematical thinking and learning that contribute insights and support theoretically grounded instructional sequences. In the case of hierarchical geometric relationships, we have made substantial progress in that regard. We have also found Sfard's (2007) framework to be useful for the purpose.

Our findings relate to the conference theme of *Engaging All Learners* in at least three ways: First, as noted in answer to Research Question 2, the instructor valued Mariah's language and contributions, so that the teacher–student relationship was not one directional. By adopting Mariah's terms (e.g., "container diagram") and making them part of the collective discourse, he supported changes in her discourse without devaluing her ideas. Second, the meta-level focus of the instructional sequence (i.e., consisting of discussion of how people use and interpret terms and diagrams) is inclusive of multiple perspectives and does not treat them as correct vs. incorrect. Instead, it emphasizes communication and the need to "speak the same language" across multiple discourse communities, as an approach to supporting learning. Third, the instructor modeled these practices in his interactions with Mariah, who is a PST and stands to benefit from examples of how to teach mathematics in ways that honor students' ideas, including their use of language and symbolic artifacts. These kinds of instructional experiences can help prepare PSTs to teach mathematics in ways that engage all learners in their future classrooms.

Acknowledgments

We thank Mariah for her participation in the teaching experiment sessions.

References

Browning, C., Edson, A. J., Kimani, P. M., & Aslan-Tutak, F. (2014). Mathematical content knowledge for teaching elementary mathematics: A focus on geometry and measurement. The Mathematics Enthusiast, 11(2), 333–384.

Brunheira, L., & da Ponte, J. P. (2019). From the classification of quadrilaterals to the classification of prisms: An experiment with prospective teachers. Journal of Mathematical Behavior, 53, 65–80.

Fujita, T., & Jones, K. (2007). Learners' understanding of the definitions and hierarchical classification of quadrilaterals: Towards a theoretical framing. Research in Mathematics Education, 9(1–2), 3–20.

Pickreign, J. (2007). Rectangles and rhombi: How well do preservice teachers know them? Issues in the Undergraduate Mathematics Preparation of School Teachers, 1.

Rasmussen, C., & Stephan, M. (2008). A methodology for documenting collective activity. In A. E. Kelly & R. Lesh (Eds.), Design research in education (pp. 195–215). Erlbaum.

Rasmussen, C., Zandieh, M. & Wawro, M. (2009). How do you know which way the arrows go? The emergence and brokering of a classroom math practice. In Roth, W.-M. (Ed.), Mathematical representation at the interface of body and culture (pp. 171–218). Information Age Publishing.

Sfard, A. (2007). When the rules of discourse change, but nobody tells you: Making sense of mathematics learning from a commognitive standpoint. The Journal of the Learning Sciences, 16(4), 565–613.

Whitacre, I., & Caro-Rora, D. (2022). Charting a viable learning trajectory for the development of preservice elementary teachers' understanding of hierarchical geometric relationships. Lischka, A. E., Dyer, E. B., Jones,

R. S., Lovett, J., Strayer, J., & Drown, S. (2022). Proceedings of the forty-fourth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Middle Tennessee State University.

Yi, M., Flores, R., & Wang, J. (2020). Examining the influence of van Hiele theory-based instructional activities on elementary preservice teachers' geometry knowledge for teaching 2-D shapes. Teaching and Teacher Education, 91, 103038.
Zandieh, M., Wawro, M., & Rasmussen, C. (2017). An example of inquiry in linear algebra: The roles of symbolizing and brokering. PRIMUS, 27(1), 96–124.

CHANGING TEACHERS' CONCEPTION OF PROOF

Óscar Chávez
Illinois State University
ochave2@ilstu.edu

David Barker
Illinois State University
dbarker@ilstu.edu

Alicia Erwin
Illinois State University
amerwi2@ilstu.edu

Seyedehkhadijeh Azimi
Virginia State University
sazimi@vsu.edu

Proof is a fundamental aspect of mathematics. However, in the high school curriculum, it often receives uneven attention that is focused on form rather than understanding. One avenue for addressing this issue is to change and strengthen teachers' conceptions of proof. To explore this idea, we followed a group of teachers as they participated in a summer mathematics research experience. During this experience, proof was not an isolated exercise but part of the mathematical process of discovery. In this study, we analyzed pre- and post-survey data and participants' critique of proofs to uncover the influence of the mathematics research experience on their concept of proof. We present data on the criteria participants used to evaluate proofs, their conception of proof, and how the mathematics research experience changed their conception of proof.

Keywords: Reasoning and Proof, Preservice Teacher Education, Teacher Knowledge, High School Education

Introduction

Since proof is critical to mathematical activity, both educators and mathematicians agree that it should be part of the mathematics curriculum for all students. Stylianides et al. (2017) pointed out that even though proof is central to the learning of mathematics, it "has a marginal place in ordinary mathematics classrooms" (p. 237). The National Council of Teachers of Mathematics (NCTM,1989, 2000) has long recommended that students at all grade levels recognize reasoning and proof as a fundamental aspect of mathematics. Unfortunately, this has been unevenly implemented in topics outside of Geometry (Thompson et al., 2012).

Historically, proof has been part of the high school geometry curriculum in the United States. According to Harel and Sowder (1998) it was only with the *New Math* that proofs were included in secondary algebra courses. But then, "the death of the 'New Math' almost put an end to algebra proofs in school mathematics" (p. 234). Historical precedent, textbook design, and a narrow interpretation of the standards continue to influence how proof is taught and learned in schools, particularly in secondary schools. Thus, most students in American secondary classrooms experience proof first, and sometimes only, in the context of a Geometry course (Zaslavsky et al., 2012). Even then, proof tends to be taught as a separate topic, not as an integral component of how we do mathematics (Knuth, 2002b).

CadwalladerOlsker (2011) argued that the way in which mathematical proof is introduced in high school is implicitly focused on the formal act of proving at the expense of the construction of knowledge. Furthermore, teachers' experiences in college might reinforce this formalistic view of proof (Selden, 2012). Reid and Knipping (2010) summarized studies that examined teachers' understanding of proof. They noted,

> […] there seems to be not much difference when compared to students' understanding of proof. If this is the case then the level of students' understanding might be best improved by addressing teachers' understanding of proof itself, rather than exposing them to new methods of teaching about proofs and proving. (p. 71)

In this study, we examined the views of proof held by teachers who participated in a summer mathematics Research Experience for Undergraduates (REU). Although the purpose of the REU program was not to modify participants' views of proof, we conjectured that the experience of doing mathematics research would have some effect in this direction. Teacher's beliefs about proof are relevant to their practice. As Stylianides and Stylianides (2022) stated, "teachers' knowledge and beliefs about proof shape their readiness, willingness, and capacity to support students' engagement with proof" (p. 1). They further argued that our field lacks productive ways to introduce students and prospective teachers to a notion of proof that would help them see proof as relevant and important. Our research provides a potential example of engaging teachers with proof. With this in mind, we addressed the following research questions: 1) What criteria did REU participants use to evaluate proof? 2) What are the REU Participants' conceptions about the role of proof? 3) How did the REU program influence their conception of proof?

Literature

Researchers have found it useful to make distinctions between the different roles of proof in the classroom. For instance, Hanna (1990) distinguished between proofs that *prove* and proofs that *explain.* She stated that, "in the classroom the key role of proof is the promotion of mathematical understanding" (p. 5). Similarly, Schoenfeld (2009) argued that "Proofs are hardly 'mere' confirmation, verifying one's intuitions. For the mathematician, proof is a way to figure out how things work" (p. xiv). Looking at the function of proof in the work of mathematicians, de Villiers (1999) enumerated various roles of proof in mathematics: to verify that a statement is true, to explain why a statement is true, to communicate mathematical knowledge, to discover or create new mathematics, and to systematize statements into an axiomatic system.

Nevertheless, many teachers hold a narrow view of proof. Ko (2010), summarizing studies on teachers' views of proof, found that teachers,

> […] do understand that proof serves as a means of verifying the truth of a statement. Only a few teachers stated that proof serves the functions of communicating mathematics, helping students make discoveries, and systemizing results, and none of the teachers mentioned proof as a means of providing intellectual challenge. (p. 1115)

Moreover, when evaluating proofs, teachers and students seem swayed by the appearance of a proof rather than the logic of it. When teachers were asked to judge student proofs, about half the teachers rejected correct proofs written as verbal justifications (Tabach et al., 2010). In addition, understanding is not always seen as a goal of a proof. Stylianou et al. (2015) found that college students selected proofs written as deductive arguments as best, even though they believed they were not helpful in providing understanding.

In a study aimed at examining secondary mathematics teachers' conceptions of proof, Knuth (2002a, 2002b) interviewed 16 in-service mathematics teachers. Knuth investigated (a) teachers' conceptions about the role of proof, (b) what constitutes proof for teachers, and (c) what teachers found convincing. Following de Villiers (1999), Knuth classified these teachers' views of proof as a means of verifying, explaining, communicating, creating, and systematizing knowledge. In addition, Knuth found two additional roles of proof in the secondary classrooms: developing thinking and displaying thinking. Most teachers identified developing thinking skills as a primary role of proof, but none identified promoting understanding as a role. Knuth also asked teachers about the centrality that proof should have in the secondary mathematics curriculum. Although their views were diverse, most teachers thought proof was more appropriate for advanced mathematics courses. Knuth reported that teachers with a more formal interpretation of proof were more likely to limit student exposure to proof. Most teachers stated that proof should be addressed in geometry courses (Knuth, 2002b).

The teachers in Knuth's study primarily used four criteria to determine whether an argument constituted proof: valid methods, mathematically sound, sufficient detail, and knowledge dependence. In addition, features of the argument, familiarity with the argument, or the method of proof used, rather than the mathematical substance of the argument, often determined whether an argument was convincing for teachers. This is consistent with other studies. Harel and Sowder (1998) found that college students focus their attention on the format of proof, not on the content. Stylianou et al. (2015) found that undergraduate students' proof choices were strongly influenced by surface characteristics.

It is also important to point out that what teachers identify as proof and what they find convincing is not always the same. In Knuth's (2002a) study, "teachers seemed to reach a stronger level of conviction regarding the truth of a proof's conclusion by testing it with empirical evidence" (p. 410). Lesseig et al. (2019) found that preservice teachers used different criteria to evaluate arguments than when they decided what proofs they would present to students. Ko (2010) argued that teachers should be provided more opportunities to engage in proof. Moreover,

> Since mathematics teachers' conceptions of mathematical proof influence not only the experiences they provide for their students but also the expectations they hold for their students in learning proof (Knuth & Elliott, 1997), having a robust understanding of proof is important for teachers. (Ko, 2010, p. 1124).

Methodology

The work of Knuth (2002a, 2002b) served as a foundation for our study. The primary distinction was that the four secondary mathematics teachers and 11 pre-service teachers in our study were participants in a mathematics Research Experience for Undergraduates (REU). The eight-week REU program explored research topics in graph theory that allowed participants to explore research questions, generate examples, form generalizations and conjectures, and to provide careful justification. The different conceptions of proof were not directly addressed during the program, but rather, proof was experienced as an integral part of the mathematics research process. An education component was implemented each week to help participants translate their mathematics research experience to their classrooms. According to the Conference Board of the Mathematical Sciences (2012), "Teachers who have engaged in a research-like experience for a sustained period of time frequently report that it greatly affects what they teach, how they teach, what they deem important, and even the ability to make sense of standard mathematics courses" (p. 65).

The first data source used in this study was a survey instrument given at the beginning and end of the REU that included the following prompts: "What is proof? What is the role of proof in Mathematics? and What is the role of proof in the classroom?" The post-survey included the additional prompt, “Describe any changes since the beginning of the program and why your thinking has changed.” The second data source was a proof instrument given during the fourth week of the REU that asked participants to provide the strengths and weaknesses of a series of proofs. It also included the question, “What other roles does proof play in the high school classroom?” Four proofs from this instrument (paragraph, two-column, algebraic, and visual) were analyzed. The paragraph and two-column proof were correct geometric proofs of the proposition that complements of congruent angles are congruent. Both were based on the same argument but presented differently. The other two were informal arguments (one algebraic and one visual) showing that the sum of the first n consecutive positive integers equals $n(n-1)/2$.

To answer the first research question (What criteria did REU participants use to evaluate proof?) each critique was coded using the criteria outlined by Knuth (2002a) —valid methods, mathematically sound, detail, and knowledge dependent. We used the constant comparative method (Glaser & Strauss, 1967) to refine and adjust our definitions to account for statements that could not be coded using Knuth’s original criteria. This process continued until all disagreements were resolved. Once each critique was coded in terms of the criteria used, we determined whether the criterion was used to identify a strength or weakness and then further analyzed the data for patterns. A similar approach was used to answer the second research question (What are the REU Participants’ conceptions about the role of proof?). For this question, the survey prompts were initially coded using the seven roles identified by Knuth (2002a): *to verify that a statement is true, to explain why a statement is true, to communicate mathematical knowledge, to discover or create new mathematics, to systematize statements into an axiomatic system, to develop logical thinking skills, and displaying thinking.*

To answer the final research question (How did the REU program influence their conception of proof?), we identified statements in their initial and final responses where they discussed the nature of proof and compared those responses. In addition, we used open coding to generate codes for their responses to the prompt, “Describe any changes since the beginning of the program and why your thinking has changed.” We then used those codes to look for themes across the participants.

Findings

The findings of this study will be presented according to the three research questions addressed. Each section provides a perspective on the conception of proof that teachers with mathematics research experience hold and whether these experiences have the potential to change teachers’ conceptions of proof.

Criteria Used to Evaluate Proof

The teachers in Knuth’s study primarily used four criteria to determine whether an argument constituted proof: valid methods, mathematically sound, sufficient detail, and knowledge dependent. We found all these criteria in our analysis, but to different degrees. The knowledge-dependent criterion was primarily used to evaluate a proof by induction that was not included in the data for this paper, so it was removed. In addition, detail was a common theme in our participants’ critiques, but often in connection with other categories such as mathematical soundness or communication. Hence, detail was used as a subcode. For instance, if a teacher stated that a proof needed more detail in the mathematical argument, it would have been coded mathematically sound with the detail subcode. It would have also been identified as a weakness. The other criterion that we found (communication, understanding why, concreteness, and generality) were also found in the work of Knuth, but are included here because they were prominent in the critiques provided by our teachers. Table 1 provides the main criteria used by our teachers to evaluate proof and the associated definitions.

Table 1: Criteria Used to Evaluate Proof

Criteria	Definition
Methods	"The focus of the teachers who applied this criterion was primarily on the method (or perhaps the form) used in producing an argument rather than on the reasoning behind it" (Knuth, 2002a, p. 395).
Mathematically Sound	"These teachers focused explicitly on the validity of the reasoning presented in an argument" (Knuth, 2002a, p. 396)
Communication	The teacher focused on communicating mathematical ideas to the audience.
Understanding Why	The teacher's focus went beyond just proving a statement is true to focusing on why it is true or to student conceptual understanding
Concreteness	The teacher commented on the inclusion of a concrete feature, visual reference, or specific example that helped the reader.
Generality	The teacher commented on "arguments that established the truth of a statement for all relevant cases" (Knuth, 2002a, p. 399).

Table 2 provides the criteria used by the participants to evaluate the four proofs (visual, algebraic, paragraph, two-column) analyzed for this paper. The 2 in the upper left-hand corner of the table represents two participants who made a positive statement about methods when critiquing the visual proof.

Table 2: Distribution of Criteria Used

	Visual		Algebraic		Paragraph		Two-Column		Total
	Pos	Neg	Pos	Neg	Pos	Neg	Pos	Neg	
Methods	2	4	8	1	2	6	6	3	32
Mathematically Sound	1	2	2	4	1	6	4	3	23
Communication	1	1	1	7	4	5	7	2	28
Understanding Why	1		1	2	1	2	1	1	9
Concreteness	10								10
Generality		4	1		1				6

Note, the values represent the number of teachers who used each criterion for a particular proof. There were teachers who used a criterion to make both a positive and negative statement about a proof, the teacher in this case would have been included under both the positive and negative. In addition, data for the pre-service and in-service teachers were merged due to the small number of in-service teachers and our focus on other patterns within the data.

In terms of the criterion used, we found evidence that participants were often focused on the formal aspects of proving, which is consistent with the findings of CadwalladerOlsker (2011).

There were 32 instances where participants used the method of proof as a criterion, 16 of these specifically referenced format. In contrast, there were nine instances where understanding why was used as a criterion. Communication (28) and mathematical soundness (23) were also used often. Although there was a strong emphasis on format, the teachers in this study attended to students' abilities to communicate and understand through proof. This contrasts the finding of Ko (2010) that "few teachers stated that proof serves the function of communicating mathematics" (p. 1115) and Knuth (2002b) who found that none of the teachers in his study identified understanding as a role. We do not know whether this difference is a result of their experience with mathematics research, but this possibility deserves the attention of further research.

Knuth (2002b) found that features such as familiarity and method, rather than the substance of the argument, often determined whether an argument was convincing. This finding was also observed in our data. Consider the data from the two-column and paragraph proof, which were based on the same argument, but presented differently. Participants identified the paragraph proof as having the most weaknesses. Six participants made negative comments about the methods of the proof with one student stating that the proof "lacked structure and a more formal foundation." One participant stated that it left "some logical steps to the reader." The participants were divided on whether communication was a strength or weakness for the paragraph proof. Four felt that it was a strength given that the "vocabulary was very simple" and that it would not confuse a reader who was not strong in mathematics. On the other hand, five felt that it was a weakness because it did not "use mathematical language."

Contrast this with the critique of the two-column proof, which had more positive comments. Seven participants mentioned communication as a strength, with one stating that it made "every step clear to the reader." Six participants mentioned methods as a strength of the proof, with five of these referencing the formalism of the proof. "Utilizing the given information to develop a clear outline" and "Laying out each statement clearly with reasoning on the side" were among the features of the proof that were considered positive.

Method, communication, and mathematical soundness were more often considered a strength of the two-column proof and a weakness of the paragraph proof, even though both proofs contained similar information. This supports the claim that features such as familiarity and method, rather than substance, determine whether an argument is convincing. These data suggest an interplay between participants' conceptions of proof, their evaluations of proof, and their prior experiences with proof. It may not be enough to simply change teachers' conceptions of proof or the criteria they use to evaluate proof, without also addressing deeply engrained beliefs and traditions regarding proof.

The criteria used to evaluate a proof differs depending upon the type of proof. For instance, observe the data for the visual proof. Ten of the 12 participants who critiqued the visual proof commented that the concreteness was a strength, with one commenting that using specific examples "made the logic clearer." Concreteness was not used as a criterion for the other proofs. When discussing weaknesses of the proof, participants were concerned about the generality of the proof (four participants). The participants still commented on the communication, methods, and soundness of the visual proof, but their comments also included the additional criteria of concreteness and generality, which is quite different from the distribution for the other proofs. The variety observed in our data not only illustrates that teachers evaluate proofs using different criteria that may reflect different conceptions of proof, but that proofs differ in terms of the characteristics and potential roles they project.

Role of Proof

At the beginning of the research experience, 14 participants gave initial responses to the prompt, “What is the role of proof in Mathematics?” We found evidence of all five of Knuth’s (2002b) original roles of proof with all 14 participants suggesting that one of the roles was to verify that a statement is true. Five participants cited communication as a role of proof, and only two participants saw the role of proof as explaining why a statement is true.

Interestingly, in their initial responses to “What is the role of proof in the classroom?”, more participants (five) said that explaining why a statement was true was important. Knuth (2002b) noted that the teachers’ comments in his study suggested that they were talking about understanding the steps of the proof, rather than the underlying concepts. In our data, two of the five, both pre-service teachers, suggested that proofs promote understanding. The other three emphasized that explanation was helpful to “show students” why things work and are true.

All fifteen participants responded to the prompt about the role of proof in the classroom at the conclusion of the REU. We again found evidence of all five of Knuth’s (2002b) original codes and one of his additional codes related to teaching (developing logical thinking skills, Knuth, 2002a). Similar to the initial responses, the two roles of proof cited most often were to verify a statement was true (10 participants) and to explain why a statement is true (seven participants).

There were two key differences in the statements coded as proof as explanation in the pre- and post-survey data. First, more participants emphasized understanding of mathematics in the post-survey. Second, while some participants initially suggested the importance of “showing students” why, their focus on the post-survey shifted away from a teacher-focused model to discussing how proofs can help students deepen their own understanding of mathematics.

The finding of teachers citing proof as verification was consistent with Knuth’s (2002b) findings. However, the role of proof as explanation was mostly absent in responses from teachers in Knuth’s (2002b) study, but emphasized by many participants in our study, especially after the completion of the REU. In addition, we found two roles of proof cited by our participants that were not present in Knuth’s (2002a, 2002b) studies: proof to foster abstract thought and proof as a means of giving students mathematical authority. Two participants discussed the role of giving students mathematical authority, with one participant stating, “it (proof) allows more mathematical independence because you are able to determine for yourself something is true.”

Impact of the Summer Mathematics Research Program

At the beginning and end of the REU, participants were asked to respond to the prompt, “What is proof?” In their initial responses, the participants viewed proof to be a static end-product. They made statements like, “proof is a way of definitely stating some idea to be true,” “proof is the formalization of a process,” and “proof is a demonstration of a mathematical theory.” In their final responses, the participants seemed to view proof as more of a process rather than an end-product. They emphasized that proof was a way of reasoning and communicating. One participant said, “rather than memorizing properties or theorems, proofs should be a natural way to describe a process.” Another participant stated, “a proof is the reasoning we use for mathematics.”

On the final prompt, participants were asked how their thinking had changed since the beginning of the REU. Two themes arose that gave insight into these shifts in views of proof. First, several participants referred to their work on generalization during the REU; generalization was a critical aspect of the process of doing mathematics research. One participant stated that their research helped them realize that “not all proofs are about generalization, but all generalizations necessitate proof to explain why they are correct." This quote suggested a shift to viewing proof as part of the larger process of doing mathematics. Schoenfeld (2009) stated that, "for the mathematician, proof is a way to figure out how things work" (p. xiv). The participants started to view proof as a useful tool in their work of developing generalizations.

Second, several participants commented at the beginning of the REU that they preferred formal proofs due to their prior mathematical experiences. At the conclusion of the REU they saw the value of informal proofs as they can often "best represent student thinking and can be less confusing." Ko (2010) argued that instruction in proof is often influenced by the teacher's conception of proof. In this case, one might conjecture that as our participants' views of proof started to shift during the REU, their view of proof instruction also changed.

Discussion

Our findings are consistent with previous studies on teachers' and undergraduate students' conceptions of proof. Specifically, we found that many participants emphasized the form or appearance of a proof over its substance and prioritized verification among the roles of proof. Knuth (2002b) suggested that one way to help give teachers a robust understanding of proof was for them, "*as students*, to experience proof as a meaningful tool for studying and learning mathematics" (p. 403). The evidence reported in this paper suggests that the REU may have contributed to qualitative changes in teachers' perception of proof and the role of proof in the mathematics classroom. The REU seems to have helped participants understand that students can use proof for themselves rather than simply exposing students to proof through direct instruction. Indeed, participants reported seeing proof as a process used to explain and promote understanding. This new perspective on the role of proof may explain how participants viewed the role of informal proof and the value of using them with high school students. Nevertheless, participants still favored proofs that were more formal when evaluating them. This conflict is not surprising given the observations of other researchers (e.g., Tabach et al., 2010).

Our study was a small, self-selected sample of teachers that cannot be easily generalized. Nevertheless, we believe we have documented that authentic mathematical experiences for teachers can shift their understanding and conceptions of proof. Research evidence (Stylianides & Ball, 2008; Ko, 2010) suggests that these shifts in conceptions will improve the instruction that their future students' experience with proofs and proving.

References

CadwalladerOlsker, T. (2011). What do we mean by mathematical proof? Journal of Humanistic Mathematics, 1(1), 33–60. https://doi.org/10.5642/jhummath.201101.04

Conference Board of Mathematical Sciences [CBMS]. (2012). The Mathematical Education of Teachers II. Providence, RI: American Mathematical Society.

De Villiers, M. (1999). The role and function of proof with Sketchpad. Rethinking Proof with Sketchpad, 3–10.

Glaser, B., & Strauss, A. (1967). The discovery of grounded theory: Strategies for qualitative research. Sociology Press. https://doi.org/10.1097/00006199-196807000-00014

Hanna, G. (1990). Some pedagogical aspects of proof. Interchange, 21(1), 6–13. https://doi.org/10.1007/BF01809605

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. American Mathematical Society, 7, 234–283. https://doi.org/10.1090/cbmath/007/07

Knuth, E. J. (2002a, March). Teachers' conceptions of proof in the context of secondary school mathematics. Journal of Mathematics Teacher Education, 5, 61–88.

Knuth, E. J. (2002b, November 1). Secondary school mathematics teachers' conceptions of proof. Journal for Research in Mathematics Education, 33(5), 379–405. https://doi.org/10.2307/4149959

Ko, Y. Y. (2010). Mathematics teachers' conceptions of proof: Implications for educational research. International Journal of Science and Mathematics Education, 8, 1109–1129. https://doi.org/10.1007/s10763-010-9235-2

Lesseig, K., Hine, G., Na, G. S., & Boardman, K. (2019). Perceptions on proof and the teaching of proof: a comparison across preservice secondary teachers in Australia, USA and Korea. Mathematics Education Research Journal, 31, 393–418. https://doi.org/10.1007/s13394-019-00260-7

National Council of Teachers of Mathematics (1989). Curriculum and Evaluation Standards for School Mathematics. Reston, VA. NCTM

National Council of Teachers of Mathematics. (2000). Principles and standards for school mathematics. Reston, VA. NCTM

Reid, D. A., & Knipping, C. (2010). Proof in mathematics education: Research, learning and teaching. Brill. https://doi.org/10.1163/9789460912467

Schoenfeld, A. H. (2009). Series editor's foreword: The soul of mathematics. In D. A. Stylianou, M. L. Blanton, & E. J. Knuth (Eds.), Teaching and Learning Proof Across the Grades: A K-16 Perspective (xii-xvi). Routledge. https://doi.org/10.4324/9780203882009

Selden, A. (2012). Transitions and proof and proving at tertiary level. In G. Hanna & M. de Villiers (Eds.), Proof and proving in mathematics education (pp. 391–420). The Netherlands: Springer. https://doi.org/10.1007/978-94-007-2129-6

Stylianides, A. J., & Ball, D. L. (2008). Understanding and describing mathematical knowledge for teaching: knowledge about proof for engaging students in the activity of proving. Journal of Mathematics Teacher Education, 11(4), 307–332. https://doi.org/10.1007/s10857-008-9077-9

Stylianides, A. J., & Stylianides, G. J. (2022). Introducing students and prospective teachers to the notion of proof in mathematics. The Journal of Mathematical Behavior, 66, 100957. https://doi.org/10.1016/j.jmathb.2022.100957

Stylianides, G., Stylianides, A., & Weber, K. (2017). Research on the teaching and learning of proof: Taking stock and moving forward. In J. Cai (Ed.), Compendium for research in mathematics education (pp. 237–266). Reston, VA: National Council of Teachers of Mathematics.

Stylianou, D. A., Blanton, M. L., & Rotou, O. (2015). Undergraduate students' understanding of proof: Relationships between proof conceptions, beliefs, and classroom experiences with learning proof. International Journal of Research in Undergraduate Mathematics Education, 1(1), 91-134. https://doi.org/10.1007/s40753-015-0003-0

Tabach, M., Barkai, R., Tsamir, P., Tirosh, D., Dreyfus, T., & Levenson, E. (2010). Verbal justification—is it a proof? Secondary school teachers' perceptions. International Journal of Science and Mathematics Education, 8(6), 1071–1090. https://doi.org/10.1007/s10763-010-9230-7

Thompson, D. R., Senk, S. L., & Johnson, G. J. (2012). Opportunities to learn reasoning and proof in high school mathematics textbooks. Journal for Research in Mathematics Education, 43(3), 253–295. https://doi.org/10.5951/jresematheduc.43.3.0253

Zaslavsky, O., Nickerson, S. D., Stylianides, A. J., Kidron, I., & Winicki-Landman, G. (2012). The need for proof and proving: Mathematical and pedagogical perspectives. In G. Hanna & M. de Villiers (Eds.), Proof and proving in mathematics education (pp. 215–229). The Netherlands: Springer. https://doi.org/10.1007/978-94-007-2129-6

CLINICAL SIMULATIONS TO SUPPORT PROSPECTIVE TEACHERS' KNOWLEDGE OF CONTENT AND STUDENTS

Duane Graysay
Syracuse University
dtgraysa@syr.edu

Hillary Bermudez
Syracuse University
hbermude@syr.edu

We report our systematic design of nine clinical simulations that we hypothesized would, through systematic variation, support prospective teachers to develop an aspect of Mathematical Knowledge for Teaching. We report tentative findings from the case of Kristy, a prospective teacher whose experience across the simulations illustrates the potential and the limitations of our design.

Keywords: Preservice teacher education, instructional activities and practices, mathematical knowledge for teaching

Increasingly, mathematics teacher educators have been developing and using various forms of simulation to prepare prospective secondary mathematics teachers (PTs). In their report on the 2019 Simulations in Teacher Education conference in Louisville, KY, Mikeska at al. (2021) noted that research "using systematic and structured variations around specific design parameters . . . would be useful to supporting claims about how simulations work best, for whom, and under what conditions" (Implications for Teacher Education, para. 1–3). In this paper we report a case study of one PT, Kristy, and her evolving understandings of student reasoning as she engaged in clinical simulations that we designed to systematically support PTs' learning.

Design Principles

Our designs were grounded in Variation Theory (VT; Marton, 2015). The first tenet of the theory is that instruction involves intended *objects of learning*. Our intended objects were three qualitatively distinct student approaches to generalizing from sequences of spatial figures: a *numerical approach* (Rivera, 2007), a *constructive figural approach* (Rivera & Becker, 2008) in which one perceives figures as composed of disjoint components, and a *deconstructive figural approach* (Rivera & Becker, 2008) in which one perceives figures in terms of intersecting components. A second tenet is to orchestrate experiences so that critical features of the objects vary while other features are held constant. In theory, this variation is necessary for students to begin to conceptualize distinct objects of learning by contrasting features. For this work, critical features of the approaches were strict adherence to numerical data, perception of figures as composed of disjoint elements, and perception of figures as composed of intersecting elements. A third tenet is to support abstraction by orchestrating multiple experiences that represent critical features of each object while varying irrelevant features. Following these principles, we constructed a framework of variations to guide our design of nine clinical simulations by controlling variation in approach and variation in task across simulations.

For each type of approach (Figure 1) and Task (Figure 2) we created a protocol to guide an actor to simulate one approach with respect to one task. We assigned character names to each student: Nell for numerical approaches, Suzy for constructive figural approaches, and Michael for deconstructive figural approaches. Each protocol describes essential actions and statements representing critical features of each approach. For example, Suzy would describe the Patterns in the Tiling Task as five central tiles with additional tiles appended to each branch, while Michael would describe the same Patterns as perpendicular rows of tiles that overlap at the center. To ensure that each PT would experience the critical features of the simulated approach, we trained actors to simulate resistance if the PT directed them toward alternative approaches.

We grouped protocols so that Approach would vary within each group, but Task would vary only across groups. We designed three learning cycles, each incorporating one group of simulations. Based on VT we predicted that experiencing three simulations of different approaches to the same task, in close temporal proximity, would lead PTs to discern critical differences among approaches. We expected that multiple simulations of each approach across cycles would help PTs develop abstract conceptualizations of each approach. Our data collection and initial analysis were based on the following research questions:

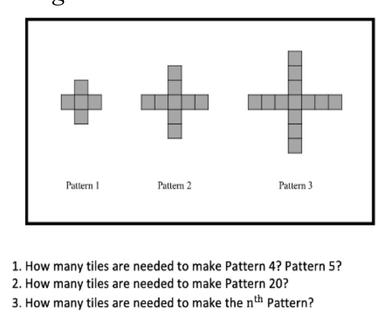
1. To what extent does a PT who experiences clinical simulations of three approaches to a single mathematical task discern critical differences among simulated approaches?
2. To what extent does a PT who experiences clinical simulations of three approaches across multiple cycles develop abstract conceptions of those approaches?

Figure 1: Dimension of Approach

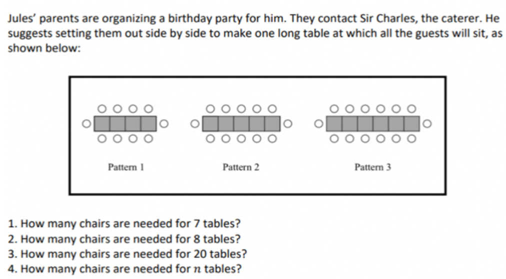
Numerical Approach (Nell)	*Constructive figural (Suzy)*	*Deconstructive figural (Michael)*
Makes a table of values and looks for patterns	Perceives patterns as constructed from disjoint components	Deconstructs patterns into intersecting components

Figure 2. Dimension of Task

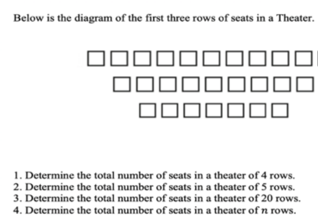
Tiling Task



Chairs Task



Theater Task



Implementation and Data Collection

We implemented our learning cycles in a methods course over a fourteen-week semester with seven prospective secondary mathematics teachers. Most relevant to this report are three PT activities from each cycle: Doing the Task and anticipating possible student approaches; participating in three live, one-on-one clinical simulations; and debriefing the simulation experience to reflect on the experience of the simulations.

Analysis and Findings

We hypothesized that data about differences that PTs discerned among simulated approaches would most likely emerge in post-simulation debriefing or in planning for the Chairs simulations and Theater simulations. We anticipated that data about PTs' conceptualizations of each approach would be most likely found in their planning conversations. Therefore, we chose to focus our initial analysis on transcripts of small-group and whole-group conversations during pre-simulation planning and post-simulation debriefing across the Cycles. We are at an early stage of analysis, so we chose to focus initial analyses on conversations involving Kristy, a masters student participant. We chose Kristy because we noted, during the planning activity for Theater Task simulations, that she spontaneously expressed the approaches she anticipated each simulated student would use. We decided that tracking Kristy's progression from her planning for Tiling through her planning for Theater simulations would be a fruitful start.

Findings and Discussion

From our analysis, we found that Kristy discerned critical differences between numerical and figural approaches starting from the Tiling simulations. Repeated experience across the Tiling simulations and Chairs simulations led her to associate specific approaches with the characters Nell and Suzy and to accurately predict the approaches that those characters would apply to the Theater Task. Kristy recognized that Michael simulated a qualitatively different approach than the other two. However, her conceptualization of Michael's approach was not yet robust enough for her to predict Michael's response to the Theater Task.

Prior to the Tiling simulations Kristy and her peer, Denise, compared how they had each solved the Tiling Task. Kristy perceived each Pattern constructively as five central tiles, with four tiles added to make Pattern 2 and two sets of four tiles to make Pattern 3. This led her to 5+4(n-1) as a constructive generalization. However, she expressed confusion about the reasoning behind one of Denise's expressions, 5n-(n-1). Denise described the approach in purely numerical terms:

> I looked at this first one, right? So I'm like, five times one is five. Okay, good. Right, now we have five times two. Because the pattern two, but then [the total is] 10 minus one. And then I noticed that this [total for pattern 3] was now [fifteen] minus two. So then . . . [the next values] are gonna be [twenty] minus three, then [25] minus four, and so on.

Denise and Kristy anticipated that students would be much more likely to use the images than to reason from numerical data, as Denise had done.

After experiencing the Tiling simulations, Kristy provided evidence that she discerned the three approaches as distinctly different. However, she had not anticipated Michael's approach:

> I did not even consider [Michael's] way. . . I had to stop and think, "I know what he's trying to say, but I don't even know how to get to a generalization from this." . . . In a real-life scenario, we might be more comfortable [saying], "Let me work this out really quick," but I felt more pressure – "I should know this answer." . . . A student might have a way of solving a problem that you didn't even consider, but it is equally as valid. You have to learn how to be flexible with that.

At this stage, Kristy saw Michael's approach as different, but was not certain that she understood.

Kristy planned for the Chairs Task with a peer, Toni. Kristy again described the Patterns constructively: two rows of chairs across the top and bottom of each figure, with a pair of chairs added to the ends of each figure: "For four tables, it's four, four, [and] two. For five tables it's five, five, [and] two. So I think for n tables, it's just two n plus two." As the PTs anticipated ways that students might approach the task, Kristy anticipated two different possibilities: A numerical approach or a constructive figural approach.

> [Someone] is gonna say four [tables] and 10 [chairs]. And they're gonna say five and 12 and six and 14, they're gonna say [the number of chairs is] always double [the number of tables] plus two. My guess is someone [else] is not going to look at the pattern in the picture. They'll just look at number of tables, number of chairs and say, "It's always something plus two." . . . [A student] might [also] see the ends as two groups of three, and then the middle. I could see someone doing that.

Note that at this stage, Kristy's descriptions differentiate between approaches that "look at the pattern in the picture" and those that attend strictly to numerical data. Her conceptions of these approaches do not seem to be attached to any one character, and do not include a deconstructive approach. This latter fact is important in light of Kristy's anticipation of student approaches in the subsequent planning for the Theater Task.

Kristy approached the Theater Task with a constructive figural approach while her planning partner, Diana, used a numerical approach. The first author overheard Kristy anticipating that in the simulation, Nell would use an approach similar to Diana's, Suzy would approach the task in a way similar to Kristy, and Michael would do something different that Kristy could not yet predict. The first author paraphrased Kristy's observation for another group:

> *Instructor:* Michael so far, for [Kristy], has caught her off guard each time. Is that correct?
> *Kristy*: It's something I never have expected.

We note that, at this stage, Kristy associated two qualitatively distinct approaches to generalizing with specific simulated students: Nell would attend to numerical data, and Suzy would attend to figures in ways that Kristy would recognize. She discerned that Michael's approach was distinctive from those of Nell or Suzy, but was not yet able to anticipate how Michael might approach the task.

Conclusions

Mikeska et al. (2021) noted an ongoing need to examine how design of simulations impact PTs' learning. We designed a set of simulations designed to systematically vary within and across subsets of simulations to support PTs' conceptualizations of distinct student approaches to linear and quadratic generalizing tasks. To a limited extent, Kristy's case suggests that VT can be a productive theory of action for designing simulations. Kristy's experiences of the simulations led to increasing awareness of distinct approaches to figural generalizing tasks and increasing capacity to describe critical features of those approaches. That knowledge and capacity are important components of Mathematical Knowledge for Teaching. However, Kristy's experiences may not have supported her in developing an understanding robust enough to anticipate approaches involving deconstructive perceptions of figures. This suggests a need for ongoing inquiry into how to design and orchestrate simulations to strengthen learning opportunities.

References

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? Journal of Teacher Education, 59(5), 389–407. https://doi.org/10.1177/0022487108324554

Marton, F. (2015). Necessary Conditions of Learning. Routledge.

Mikeska, J. N., Howell, H., Dieker, L., & Hynes, M. (2021). Understanding the role of simulations in k-12 mathematics and science teacher education: Outcomes from a teacher education simulation conference. Contemporary Issues in Technology and Teacher Education, 21(3)

Rivera, F.. (2007). Visualizing as a mathematical way of knowing: Understanding figural generalization. The Mathematics Teacher, 101(1), 69–75.

Rivera, F. D., & Becker, J. R. (2008). Middle school children's cognitive perceptions of constructive and deconstructive generalizations involving linear figural patterns. ZDM Mathematics Education, 40, 85–82. https://doi.org/10.1007/s11858-007-0062-z

CULTIVATING POSITIVE MATHEMATICS IDENTITIES IN PRESERVICE TEACHERS

Glenn Waddell, Jr.
University of Nevada, Reno
gwaddell@unr.edu

Cheryll Crowe Johnson
Asbury University
cheryll.johnson@asbury.edu

Dawn M. Woods
Oakland University
dawnwoods@oakland.edu

Jennifer Bay-Williams
University of Louisville
j.baywilliams@louisville.edu

Aligned with the AMTE Standards for Preparing Teachers of Mathematics, Bay-Williams et. al. constructed an Identity Survey and research team to composed of several institutions ranging from R1 to private teaching universities. The team implemented the common Identity Survey and research design in 2021 and has initial results to share. Participants will learn about the design and explore the initial quantitative and qualitative data outcomes.

Keywords: Teacher Beliefs, Teacher Educators, Undergraduate Education

Introduction

In 2020 the Association of Mathematics Teacher Educators (AMTE) published the Standards for Preparing Teachers of Mathematics, along with supplementary materials which included an instrument for developing an understanding for how preservice (PST) and inservice (IST) teachers develop their identity as 'teachers of mathematics' to replace their identity of 'learner of mathematics' (Association of Mathematics Teacher Educators, 2017; Bay-Williams et al., 2021). This instrument has been given at several institutions across the United States with elementary and secondary preservice teachers over three years, and the team has quantitatively and qualitatively analyzed the data.

The instrument has several sections, including qualitative and quantitative questions to uncover how PSTs make the essential shift from 'learners of' to 'teachers of' mathematics (Allen & Schnell, 2016; Anderson, 2007; Dunleavy et al., 2021; Leatham & Hill, 2010). Further, it is clear that principles of equity are essential in teaching and developing the identity of successful teachers of mathematics, but it is not clear how to build the required political and equitable knowledge (Berry III, 2019; Gutiérrez, 2013a, 2013b). Mathematics teacher organizations have published materials to assist and support mathematics teacher educators in this work (Association of Mathematics Teacher Educators, 2017; National Council of Teachers of Mathematics, 2018, 2020b, 2020a), however there is a lack of research focused on the PST that this instrument hopes to fill (Bay-Williams et al., 2021). The instrument in its final form borrows activities from Leatham and Hill's (2010) work on middle school mathematics teachers as well as Silver, Strong, & Perini's (2000) work on multiple intelligences in developing an understanding of how individuals build their identities.

Discussion

The development of identities in PSTs is an essential topic that needs more research, more data, and more study. This initial study has shown some interesting differences between elementary and secondary attitudes towards developing their own identity. Elementary PSTs have a wide range of emotional attitudes towards their own mathematical thinking, and secondary PSTs have a much more narrow and positive view. While it may not be surprising to mathematics teacher educators that elementary PSTs find themselves struggling in learning mathematics, a large number of them also had low anxiety, high confidence, and high enjoyment of mathematics. These emotional outcomes can be used to build as strong foundation of mathematics identity in these future teachers.

In the secondary PSTs, they had a very high opinion of their mathematics skills, confidence, and enjoyment of mathematics, despite the fact they also had a great deal of anxiety. And even though they have such high opinions, they did not write about mathematics as being an essential component of their identity. The disconnect between their opinion of their skills and what they considered important about themselves creates an opportunity for supporting them in creating a stronger mathematics identity.

References

Allen, K., & Schnell, K. (2016). Developing mathematics identity. Mathematics Teaching in the Middle School, 21(7), 398–405. https://doi.org/10.5951/mathteacmiddscho.21.7.0398

Anderson, R. (2007). Being a mathematics learner: Four faces of identity. Mathematics Educator, 17(1), 7–14.

Association of Mathematics Teacher Educators. (2017). Standards for preparing teachers of mathematics. 199.

Bay-Williams, J., Johnson, C. C., & Morris, S. (2021). Mathematics identity. https://amte.net/content/mathematics-identity

Berry III, R. Q. (2019, May). Examining equitable teaching using the mathematics teaching framework. NCTM President's Messages. https://www.nctm.org/News-and-Calendar/Messages-from-the-President/Archive/Robert-Q_-Berry-III/Examining-Equitable-Teaching-Using-the-Mathematics-Teaching-Framework/

Dunleavy, T. K., Marzocchi, A. S., & Gholson, M. L. (2021). Teacher candidates' silhouettes: Supporting mathematics teacher identity development in secondary mathematics methods courses. Investigations in Mathematics Learning, 13(1), 43–58.

Gutiérrez, R. (2013a). Why (urban) mathematics teachers need political knowledge. Journal of Urban Mathematics Education, 6(2), 7–19.

Gutiérrez, R. (2013b). The sociopolitical turn in mathematics education. Journal for Research in Mathematics Education, 44(1), 37–68.

Leatham, K. R., & Hill, D. S. (2010). Exploring our complex math identities. Mathematics Teaching in the Middle School, 16(4), 224–231.

National Council of Teachers of Mathematics (Ed.). (2018). Catalyzing change in high school mathematics: Initiating critical conversations. The National Council of Teachers of Mathematics, Inc.

National Council of Teachers of Mathematics (Ed.). (2020a). Catalyzing change in early childhood and elementary mathematics: Initiating critical conversations (National Council of Teachers of Mathematics. 1906 Association Drive, Reston, VA 20191. Tel: 800-235-7566; Tel: 703-620-9840; Fax: 703-476-2570; e-mail: NCTM@nctm.org; Web site: https://www.nctm.org/). National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics (Ed.). (2020b). Catalyzing change in middle school mathematics: Initiating critical conversations (National Council of Teachers of Mathematics. 1906 Association Drive, Reston, VA 20191. Tel: 800-235-7566; Tel: 703-620-9840; Fax: 703-476-2570; e-mail: NCTM@nctm.org; Web site: https://www.nctm.org/). National Council of Teachers of Mathematics.

Silver, H. F., Strong, R. W., & Perini, M. J. (2000). So each may learn: Integrating learning styles and multiple intelligences (Association for Supervision and Curriculum Development, 1703 North Beauregard Street, Alexandria, VA 22311-1714; Tel: 703-575-5400 or 800-933-2723 (toll-free); e-mail: member@ascd.org; Web site: http://www.ascd.org (members, $17.95; nonmembers, $21.95).).

DESIGNING AN EDUCATION ABROAD PROGRAM FOR PRESERVICE MATHEMATICS TEACHERS

Blair Izard
State University of New York (SUNY) Empire State
blair.izard@sunyempire.edu

Keywords: Preservice Teacher Education, Culturally Relevant Pedagogy

The National Council of Teachers of Mathematics (2014) has articulated that to create, support, and sustain a culture of access and equity within mathematics education, teachers must be "responsive to students' backgrounds, experiences, cultural perspectives, traditions, and knowledge" (2014, p. 1). Further, the Association of Mathematics Teacher Educators (2020) emphasized that we must develop mathematics teachers who "implement practices that draw on students' mathematical, cultural, and linguistic resources/strengths" (2020, p. 2). Thus, teacher education programs are considering: How do we prepare mathematics teachers for this work?

I share one approach for developing preservice teachers' ability to recognize, appreciate, and incorporate other cultures into the classroom: education abroad. Research from education abroad indicates that when students are immersed in a different culture, they develop a sense of cultural awareness and the capacity to work across cultures (Marx & Moss, 2011). Therefore, education abroad has the potential to support preservice mathematics teachers in developing the mindset and skills to draw on their future students' backgrounds and cultures.

The Developmental Model of Intercultural Sensitivity (DMIS; Bennett, 1986) provides a framework for this work. It establishes a continuum of ways to respond to cultural difference ranging from monocultural to intercultural. Monocultural is "the experience of one's own culture as 'central to reality'" (Bennet, 2004, p. 62); while intercultural is "the experience of one's own beliefs and behaviors as just one organization of reality among many viable possibilities" (Bennett, 2004, p. 62). There are two categories related to monocultural (Denial and Polarization), two categories within intercultural (Acceptance and Adaptation), and a category (Minimization) which is considered a transition between monocultural and intercultural. The Intercultural Development Inventory (IDI; Hammer & Bennett, 1998) is a validated instrument that places respondents along this continuum.

This poster shares details of a semester-long education abroad program in the United Kingdom that was designed specifically for preservice mathematics teachers. Data were collected from 13 preservice teachers who participated in this program over the course of 3 years. Data sources include the IDI, interviews, and journals. The average IDI score for these thirteen preservice teachers at the start of the program was 82.33, reflecting a category of polarization. By the end of the program, the average IDI score was 97.05, reflecting a category of minimization. This difference in pre-post means is statistically significant ($t=2.179$, $df=12$, $p<.001$), meaning that preservice teachers demonstrated higher levels of intercultural sensitivity following participation in this program.

While the IDI data and analysis of student interviews/journals suggest that education abroad has potential for developing mathematics teachers who are prepared to work across cultures, this poster focuses on sharing details of this program. I describe the semester schedule, course requirements, student internships, stakeholders, key features of the program, and other important logistics so that others are provided with a full picture of this program as a means of considering the possibilities of implementing this type of work (or components of this work) elsewhere.

References

Association of Mathematics Teacher Educators. (2020). AMTE Statement on Systemic Racism.

Bennett, M. J. (1986). Towards ethnorelativism: A developmental model of intercultural sensitivity. In R. M. Paige (Ed.), Cross-cultural orientation: New conceptualizations and applications (pp. 27–70). New York: University Press of America.

Bennett, M. J. (2004). Becoming interculturally competent. In J. Wurzel (Ed.), Towards multiculturalism: A reader in multicultural education (2nd ed., pp. 62-77). Newton, MA: Intercultural Resource.

Hammer, M. R., & Bennett, M. J. (1998). The intercultural development inventory (IDI) manual. Intercultural Communication Institute.

Marx, H., & Moss, D. M. (2011a). Please mind the culture gap: Intercultural development during a teacher education study abroad program. Journal of teacher education, 62(1), 35-47.

National Council of Teachers of Mathematics. (2014b). Access and Equity in Mathematics Education: A position of the National Council of Teachers of Mathematics. Retrieved from https://www.nctm.org/Standards-and-Positions/Position-Statements/Access-and-Equity-in-Mathematics-Education/.

DEVELOPING LEARNERS' ALGEBRAIC MANIPULATION ABILITY: A MATHEMATICS TEACHER EDUCATOR REFLECTS ON PRE-SERVICE TEACHERS' INITIAL THOUGHTS

Barbara M. Kinach
Arizona State University
barbara.kinach@asu.edu

Keywords: Algebra and Algebraic Thinking, Mathematical Knowledge for Teaching, Pre-service Teacher Education

Introduction, Perspectives & Context

> THE STUDY PROMPT: In this activity, we are exploring first thoughts on teaching students to simplify expressions. How would you teach students to simplify these expressions: -(x-y), 2-(x+3), 2-(x-3)? Write a detailed step-by-step explanation of what you would say and write symbolically.

This exploratory study examines whether the hypothetical learning trajectory (Simon, 1995) and cognitive strategy (Kinach, 2002) that the author uses in her middle grades/early secondary mathematics methods course (n = 18) to develop pre-service teachers' mathematical knowledge for teaching (MKT) integer subtraction and structural analysis of integer subtraction expressions to determine the multiple meanings of the "-" symbol (i.e, subtract, negative, opposite) could be expanded to serve as the foundation for a new HLT on MKT for simplifying algebraic expressions. The prompt above was used for this purpose.

Participants, Data Collection, Analysis

The 18 methods students were invited to share their first thoughts on ways to explain simplifying the prompt expressions to students "from scratch" by preparing their explanations on google slides. Everyone reviewed the slides prior to class, and then analyzed them through whole-class discussion. Analysis of the data entailed downloading the slides, counting instances of accurate interpretation of the "-" symbol, and listing the alternative strategies used to explain simplifying the expressions.

Results

Anticipating PSTs would apply the previously learned definition of integer subtraction (p-q=p+-q) and the structural analysis of subtraction expressions to determine the meaning of the "–" symbol within algebraic expressions, the mathematics teacher educator was surprised by the varied ways PSTs interpreted the "–" symbol. While illustrating flexible thinking, some explanations exhibit the same student difficulties identified in early algebra research (Booth, 1984; MacGregor & Stacey, 1997; Tirosh, Even & Robinson, 1998; Kieran, 2007). Others call for additional instruction on syntax to accurately interpret the "–" symbol while others call for instruction on transformation properties (e.g., opposite of a sum = sum of the opposites) and the field axioms (e.g., identity element, distributive property of multiplication over addition, commutative property). To flesh out the MKT task sequence for simplifying algebraic expressions the poster will list examples of student difficulties identified in early algebra research, detail PSTs strategies for each of the prompt expressions, and propose a revised task sequence and reading list for the MKT learning trajectory on simplifying algebraic expressions.

References

Ball, D. L., Hill, H.C, & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? American Educator, 29(1), pp. 14-17, 20-22, 43-46. http://hdl.handle.net/2027.42/65072

Booth, L. R. (1984). Algebra: Childrens' strategies and errors. Windsor: NFER-Nelson.

De Araujo, Z., Dougherty, B. J., & Zenigami, F. (2018). Putting essential understanding of expressions and equations into practice in grades 6 – 8. Reston, VA: National Council of Teachers of Mathematics, Inc.

Kieran, C. (2007). Learning and teaching algebra at the middle school through college levels: Building meaning for symbols and their manipulation. In F. K. Lester (Ed.), Second handbook of research on mathematics teaching and learning (Vol. 2, pp. 707–762). Greenwich, CT: Information Age Publishing.

Kinach, B. M. (2002). A cognitive strategy for developing prospective teachers' pedagogical content knowledge in the secondary mathematics methods course: Toward a model of effective practice. Teaching and Teacher Education 18(1), 51-71. https://doi.org/10.1016/S0742-051X(01)00050-6

Lloyd, G., Herbel_Eisenmann, B., & Starr, J. R. (2011). Developing essential understanding of expressions, equations & functions for teaching mathematics in grades 6 – 8. Reston, VA: National Council of Teachers of Mathematics.

MacGregor, M., & Stacey, K. (1997). Students' understanding of algebraic notation:11–15. Educational Studies in Mathematics, 33(1), 1–19.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. Journal for Research in Mathematics Education, 26(2), 114-145. https://doi.org/10.5951/jresematheduc.26.2.0114

Tirosh, D., Even, R., & Robinson, N. (1998). Simplifying algebraic expressions: Teacher awareness and teaching approaches. Educational Studies in Mathematics, 35(1), 51-64.

ELEMENTARY PRESERVICE TEACHERS SUGGESTED INSTRUCTIONAL STRATEGIES FOR EQUIPARTITIONING

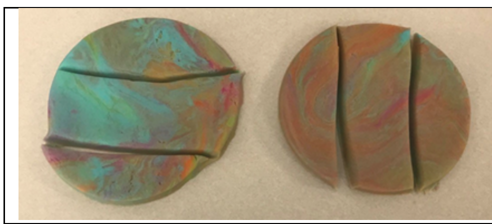
Maryam Zolfaghari
Kent State University
mzolfagh@kent.edu

Hanan Alyami
Purdue University in Fort Wayne
alyamih@pfw.edu

Keywords: Early Fraction Knowledge, Geometry, Elementary Preservice Teachers.

This poster explores elementary preservice teachers' [PSTs] suggestions to help children with partitioning circles into odd number of parts. Learners often know how to partition a continuous model from a very early age (Empson, 1995), through various partitioning schemes (Hackenberg et al., 2016). Learners use various models to demonstrate partitioning scheme such as length, rectangle, and circular fraction models (Tunç-Pekkan, 2015). While these models are beneficial in scaffolding equal partitioning knowledge, circle models are more challenging than other models for students to partition (Boyce & Moss, 2022), especially equipartitioning a circle into odd numbers (Zolfaghari, 2023). This suggests that circle models can add an extra layer of difficulty for equal partitioning into third and fifth. The research question we explored is: What strategies do PSTs suggest to support a second-grader's partitioning of a circle into thirds?

To answer the research question, 23 PSTs responded to the following task (Figure 1), and their written responses were categorized using thematic analysis (Saldaña, 2013).



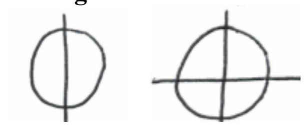
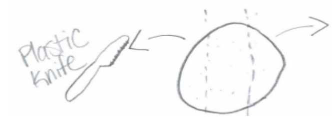
A 2nd grade student (who has demonstrated how to accurately partition a rectangle into thirds) partitioned a circle in the way demonstrated on the left. If you were the teacher of this student, how would you facilitate instructions to support them to accurately partition the circle into thirds? Be specific in your explanation. Clarify how you will use any drawings or manipulatives to facilitate the instructions.

Figure 1. Task Given to PSTs of a Second-Grader's Partitioning a Circle into Thirds

Findings and Discussion

PSTs' writing had three basic themes (see Table 1). First, PSTs' suggestion to distinguish rectangle and circle partitioning relates to studies demonstrating conceptual connections among models (Rau & Matthews, 2017). Children perform better with partitioning rectangles into thirds and fifths than partitioning circles, indicating the need to map relevant concepts they learned via rectangle models to transition and scaffold understanding of equipartitioning circles. The PSTs described partitioning the circle into "easier parts like ½ and ¼" to find the center of the circle relates to ideas of human's tendency to favor symmetry (Lipka et al., 2019). Cutting circles into a half and fourth might help children find the center. PSTs also suggested real life modeling of cutting round food items such as pizza to share among three friends.

Table 1: PSTs' Instructional Strategies Suggested for Equipartitioning a Circle into Thirds

Rectangle and Circle Distinction	Finding the Circle’s Center	Real-world Modeling
		

References

Boyce, S., & Moss, D. (2022). The role of representation in supporting prospective teachers’ understandings of fractions as measures. International Journal for Mathematics Teaching and Learning, 23(1), 64-93.

Empson, S. B. (1995). Using sharing situations to help children learn fractions. Teaching Children Mathematics, 2(2), 110-115.https://doi.org/10.5951/TCM.2.2.0110

Gould, P. (2013). Australia's next top fraction model. Australian Primary Mathematics Classroom, 18(3), 5-12. https://search.informit.org/doi/abs/10.3316/INFORMIT.642802798480259

Hackenberg, A. J., Norton, A., & Wright, R. J. (2016). Developing fractions knowledge. Sage.

Lipka, J., Adams, B., Wong, M., Koester, D., & Francois, K. (2019). Symmetry and measuring: Ways to teach the foundations of mathematics inspired by Yupiaq Elders. Journal for Humanistic Mathematics, 9(1), 107-157. https://doi.org/10.5642/jhummath.201901.07

Rau, M. A., & Matthews, P. G. (2017). How to make ‘more’better? Principles for effective use of multiple representations to enhance students’ learning about fractions. ZDM, 49, 531-544. https://doi.org/10.1007/s11858-017-0846-8

Saldaña, J. (2013). The coding manual for qualitative researchers. Sage.

Tunç-Pekkan, Z. (2015). An analysis of elementary school children’s fractional knowledge depicted with circle, rectangle, and number line representations. Educational Studies in Mathematics, 89(3), 419-441. https://doi.org/10.1007/s10649-015-9606-2

Zolfaghari, M. (2023). Elementary Grade Students’ Demonstrated Fragmenting with Visual Static Models (Doctoral dissertation, Kent State University). http://orcid.org/0000-0003-3285-3326

ELICITING CYCLES AS PRE-SERVICE TEACHERS LEAD WHOLE-CLASS DISCUSSIONS: AN INVESTIGATION OF MOMENTS OF TENSION

Kristen N. Bieda
Michigan State University
kbieda@msu.edu

Brady Tyburski
Michigan State University
tybursk2@msu.edu

Rileigh Luczak
Michigan State University
luczakri@msu.edu

Michelle Cirillo
University of Delaware
mcirillo@udel.edu

Eliciting student thinking is a core practice of ambitious mathematics teaching, yet it is one practice that is difficult for novice teachers to do consistently and proficiently. To better understand how to support pre-service teachers (PSTs) to enact this practice, our study sought to investigate how secondary PSTs engage in eliciting when they experience instructional dilemmas during whole-class mathematics discussion. We analyzed video-recorded data of 18 PSTs' instruction in a university-based early field experience setting in a College Algebra course. The findings shed light upon aspects of eliciting student thinking that PSTs need support with as they encounter common dilemmas of leading whole-class discussions.

Keywords: Preservice Teacher Education, Instructional Activities and Practices, Classroom Discourse

Eliciting student thinking is a necessary component of ambitious mathematics teaching practice (NCTM, 2014), yet it is one practice that PSTs may have had little experience with even as modeled by their own K-12 teachers. The practice sits at the intersection of PSTs' mathematical knowledge for teaching and PSTs' relational work with students (Shaughnessy & Boerst, 2018). Understanding what is difficult for PSTs to do when eliciting student thinking, particularly early in their teacher preparation program, can inform the design of teacher preparation curriculum to improve the preparation of PSTs to enact this ambitious practice skillfully as beginning teachers. In this study, using data collected from PSTs' teaching in an early field experience, we focused specifically on instances of eliciting preceding and following instances where breaks could be perceived in the instructional flow during whole-class discussion phases of the lesson. This focus was in service of the goal to understand the relationship between PSTs' nascent practice with eliciting student thinking and their fluency with leading whole-class discussions. Specifically, this study was guided by the question: *What are the characteristics of secondary mathematics PSTs' practice of eliciting student thinking immediately prior to, and after resolving, breaks in instructional flow during facilitation of whole-class discussions in an early field experience?*

Conceptual Framework

The Eliciting Cycle

Building from existing work that articulates eliciting using a cyclic metaphor (Shaughnessy & Boerst, 2018; TeachingWorks, 2020), we conceptualize an *eliciting cycle* as an episode of instruction that involves back-and-forth moments of the teacher providing additional questioning, revoicing, and providing clarification (Continued Eliciting), and student response

(Response). Cycles are initiated by an elicitation (often a question) which launches the eliciting cycle (Initial Elicitation) as well as the teacher moves that close the episode before moving on to a new topic (Cycle End). In actual classroom practice, it is not uncommon for the Initial Elicitation or Continuing Eliciting practice to receive no response from students, so this potential outcome (No Response) is also considered in the model presented in Figure 1.

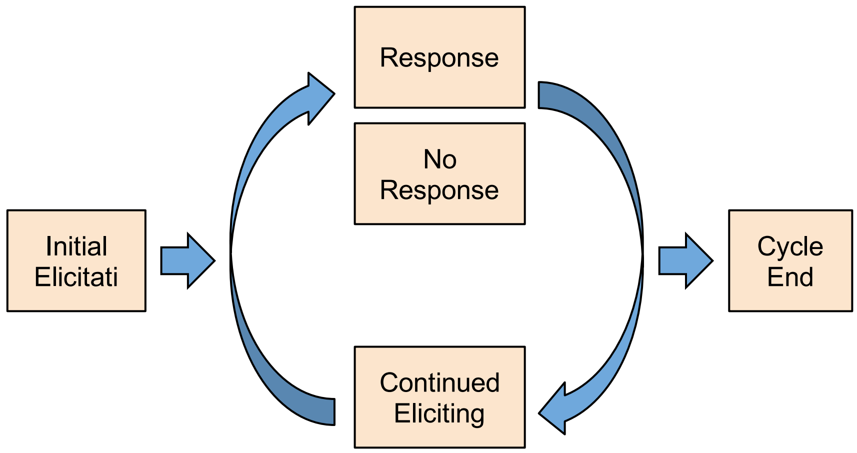


Figure 1: Model of Eliciting Cycle and its Constituent Phases

Instructional Dilemmas During Eliciting Cycles

Existing work has attended to teachers' experiences with challenging moments during instruction—where teachers face uncertainties about how to proceed or must make choices that leave them unable to address important obligations of teaching (Lampert, 1985; Kennedy, 2016)—and how teachers reason about and address these challenging moments during instruction. We contend that attending to these moments is crucial in that teachers' responses to these moments may result in outcomes such as reduced cognitive demand of instruction (Stein, Grover & Henningsen, 1996) and support for students to engage in productive struggle (Warshauer, 2015). Research has shown that PSTs find several aspects of leading whole-class discussions challenging, including practices specific to eliciting such as lack of response to elicitations (Ghousseini, 2015). Our study seeks to understand how PSTs approach eliciting that occurs before, during, and after they have resolved breaks in the instructional flow—what we call *instructional interruptions*—that could be indicative of PSTs' facing instructional dilemmas while leading whole-class discussions. The results can offer insights about how to design activities and learning experiences to specifically address PSTs' challenges when engaging in eliciting cycles during whole-class discussions.

Methods

Context and Data Collection

In our collaborative, multi-year NSF-funded project across three universities, we implemented an early field experience where PSTs planned, enacted, and reflected on lessons they taught in an entry-level university mathematics course. PSTs received real-time coaching and feedback from a mathematics teacher educator (MTE), who also served as the instructor of the teaching methods course that PSTs were concurrently taking. These lessons were recorded using Swivl technology (http://www.swivl.com).

Data was collected in Fall 2018 at one site where the PSTs taught in a College Algebra course. Each lesson was co-taught by a pair of PSTs, while the other 16 PSTs observed and provided instructional support while placed individually to work with small groups.

Data Analysis

A timestamped transcript was generated for each eliciting cycle in the three recordings in which the speaker was identified to the greatest extent possible given the limitations of the video recording (inaudible, speaker off-screen, etc.). Because each PST that was teaching wore an audio tracking device, it was always clear which PST was speaking.

First, we coded the phases of the eliciting cycle (Initial Elicitation, Continued Eliciting, Response, No Response, and Cycle End) by identifying an initial elicitation. Then, we marked the Initial Elicitation phase to include any teacher moves or context that was provided leading up to this initial elicitation. To identify the Cycle End, we considered teacher moves that likely signaled to a student that discussion of the elicitation that began that cycle had ended, such as when PSTs explicitly summarized the discussion or shifted to the next problem. Across the three lessons, there were 40 eliciting cycles with a median length of 1.5 minutes.

Next, we reviewed each lesson to identify moments during the recordings in which there was a break in the instructional flow, potentially indicating that PSTs were experiencing an instructional dilemma. The criteria for identifying these moments were: (a) there was a pause in the teacher action where PSTs had to make a choice or (b) an MTE intervened during the PSTs' lesson. We refer to such moments as *instructional interruptions* to acknowledge that the situation may not have been perceived as problematic by the PST(s) involved. Each instructional interruption was composed of three parts: Start, Tension, and Stop. Tension was identified as the (part of the) phase of an eliciting cycle that most directly caused the interruption (e.g., student provided an incorrect response). Backing up 30-60 seconds, Start was identified as the (part of the) phase of an eliciting cycle that directly prompted the "tension" (e.g., PST asked a question). Stop was identified as the (part of the) phase of an eliciting cycle in which the PST(s) reflected a closure in their response to the tension (e.g., providing the correct answer after receiving an incorrect response from a student). For this brief report, we chose to analyze only the cases PST pairs handled themselves (i.e., instructional interruptions that did not involve MTE intervention and coaching). Interruptions that involved difficulty with classroom resources or classroom management but did not otherwise appear to impact the teacher's eliciting practice were also omitted from the data set. In total, this data set of eliciting cycles across three lessons featured 25 instructional interruptions that met our criteria for inclusion.

Results and Discussion

Across the episodes of eliciting that occurred during instructional interruptions, we found PSTs used a variety of strategies to manage the moments of tension in each episode. First, for episodes of interruptions that featured a repeated lack of student response (n=5), PSTs tended to make their elicitations more specific after one or more phases of students not responding. In one example, a PST's initial elicitation requested a process ("If we have a graph that is exponential and want to create a model or equation from that, how do we do that?), but after receiving no response from students, their subsequent elicitations changed to request a single answer ("What equation do you typically use for exponential functions?"). Another common approach when met with a repeated lack of student response was for PSTs to repeat relevant information or provide specific hints. For instance, one PST was waiting for additional responses after a student introduced an incorrect initial response. After waiting a few moments, she asked a follow-up elicitation that also served as a hint ("What was the objective from Tuesday?"). When she received no responses, she followed with a more specific hint (It's a different kind of function"), to which one student offered a partially correct answer. As in the example, this pattern of making the elicitation more specific, reiterating relevant information, and then providing hints, did typically prompt students to provide at least a partial response to the initial elicitation. However, the openness of the initial elicitation was often reduced (e.g., from requesting an explanation or the process a student used to requesting a single answer), resulting in eventual student responses that tended to be single answers as opposed to sharing their mathematical processes or underlying thinking.

A second finding was that, across a variety of types of instructional interruptions, PSTs often sought clarification about a student's response by engaging in continuing eliciting to ask a simpler question until they appeared to understand the student's thinking. Sometimes, this strategy was used to encourage students to expand on a response that had initially caused a PST to pause for a few moments before continuing their eliciting practice. Other times, it was used by the PST to encourage the student to share more. In one instance, a PST used this strategy to take up the only short and incomplete student response they received after several No Response phases. In one example, a student had shared their process for finding a common denominator of two rational expressions. The PST took time to pause after receiving this response, but then asked a series of follow-up questions to clarify ambiguities that were present in the students' words:

> PST: So you're going to multiply each other's denominator? Are you just going to multiply the denominator, or…?
> Student: The entire fraction
> PST: So you're going to say the top and bottom?
> Student: Yeah

Although some of these cases may have been instances of funneling (Wood, 1998), other instances—as in the previous example—were explicit attempts to fully elicit, unpack, and understand aspects of a student's process and thinking before proceeding with the mathematical goals of the lesson and eliciting cycle. Less frequently, PSTs asked open-ended and ambiguous follow-up elicitations that appeared to serve a similar purpose, including "Could you say more" and "Could you elaborate?". It was also rare for PSTs to involve more than just the student who gave the initial answer in this clarification, such as asking someone else to explain what a particular student may be thinking.

Conclusion

Our findings offer a glimpse into the baseline practices of secondary PSTs who have knowledge about the practice of eliciting student thinking but are making their initial attempts at enacting the practice in their own teaching. These findings shed light not only on the tensions they encounter (lack of student response; making sense of a student's contribution) but also on the moves they make to address and manage these tensions. While the findings show that PSTs are tenacious in their attempts to elicit student thinking, we contend that it would be beneficial for PSTs to have opportunities to plan for ways to respond to common dilemmas of leading whole-class discussions. This could reduce the occurrence of instructional interruptions but, more importantly, could also support PSTs in responding to these situations in ways that do not reduce the openness of the mathematical inquiry and, ultimately, the cognitive demand (Stein, Grover & Henningsen, 1996).

References

Ghousseini, H. (2015). Core practices and problems of practice in learning to lead classroom discussions. *The Elementary School Journal*, *115*(3), 334-357.

National Council for Teachers of Mathematics. (2014). *Principles to action: Ensuring mathematical success for all.* Reston, VA: NCTM.

Shaughnessy, M., & Boerst, T. A. (2018). Uncovering the skills that preservice teachers bring to teacher education: The practice of eliciting a student's thinking. *Journal of Teacher Education, 69*(1), 40–55. https://doi.org/10.1177/0022487117702574

Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal*, *33*(2), 455-488.

TeachingWorks. (2018). *High leverage practices*. Retrieved from http://www.teaching works.org/work-of-teaching/high-leverage-practices

Warshauer, H. K. (2015). Productive struggle in middle school mathematics classrooms. *Journal of Mathematics Teacher Education*, *18*, 375-400.

Wood, T. (1998). Alternative patterns of communication in mathematics classes: Funneling or focusing? In H. Steinbring, M. Bartolini-Bussi & A. Sierpinski (Eds.), *Language and Communication in the Mathematics Classroom* (pp. 167–78). Reston, VA: NCTM.

EMPHASIZING PROSPECTIVE SECONDARY TEACHERS' ACTIONS AND IDEAS TO INTRODUCE STUDENT-CENTERED PEDAGOGY IN TEACHING REHEARSALS

Alicia Erwin
Illinois State University
amerwi2@ilstu.edu

Matthew Winsor
Illinois State University
mwinsor@ilstu.edu

This study investigated the effect of teaching rehearsals on 22 preservice secondary mathematics teachers (PSMTs) who were in their first mathematics pedagogy course. The objective of the teaching rehearsal was to introduce PSMTs to student-centered teaching. PSMTs worked in groups to complete a learning cycle consisting of analyzing a mathematics task, preparing to teach the task, implementing the task with their peers (the teaching rehearsal), and reflecting on their experience. Although PSMTs struggled to teach using student-centered practices, they gained more insight into student-centered practices via self-reflection and class discussion that was centered on PSMT's experiences during the rehearsal. Through the learning cycle, PSMTs also started to develop a vision for creating a classroom culture that values student thinking.

Keywords: High School Education; Instructional Activities and Practices; Preservice Teacher Education; Teacher Educators

Preservice secondary mathematics teachers (PSMTs) bring previous mathematical experiences to their university courses. Many PSMTs experienced a traditional teacher-centered math classroom environment, where direct instruction was prevalent (Prescott & Cavanaugh, 2006). In contrast, the national K-12 mathematics recommendations call for a student-centered approach to teaching mathematics that emphasizes understanding in addition to procedural skill (NCTM, 2014). Moreover, students should be engaged in making conjectures and explaining their reasoning in mathematic courses (National Council of Teachers of Mathematics [NCTM], 2000; National Governors Association Center for Best Practices & Council of Chief State School Officers [NGACBP & CCSSO], 2010).

Recommendations for mathematics teacher preparation programs note that K-12 preservice teachers (PSTs) need to experience student-centered teaching as a learner to be able to enact those practices with their own students (Association of Mathematics Teacher Educators [AMTE], 2017; NCTM, 2014). Furthermore, PSTs need to practice implementing student-centered pedagogy (AMTE, 2017). However, PSTs have few opportunities to enact student-centered pedagogy due to the structure of the practicum experience (Wilson et al., 2002; Winsor et al., 2018) or to the role of the cooperating teacher (Borko & Mayfield, 1995; Solomon et al., 2017). Recently, teacher educators have used teaching rehearsals to increase PST's opportunities to implement student-centered pedagogy (e.g., Arbaugh et al., 2019; Lampert et al., 2013). Prior to this study, our mathematics department did not use teaching rehearsals with PSMTs. We hypothesized that using teaching rehearsals with PSMTs taking an introductory pedagogy class would introduce PSMTs to student-centered practices in a structured environment. The purpose of this study was to investigate how teaching rehearsals focused on student-centered pedagogy might help PSMTs gain an initial understanding of how to implement student-centered pedagogy in future teaching.

Relevant Literature

Pedagogies of Practice

Teaching rehearsals are often situated within a pedagogies of practice framework, which posits that core teaching practices should be at the center of teacher education programs (see Grossman et al., 2009). Without opportunities to practice student-centered pedagogy, teachers are often unsuccessful at applying student-centered pedagogy in their classroom (Grossman & Dean, 2019). Additionally, because the art of teaching is complex, Grossman and Dean stated that, “Attending to practice links the focus on skill with the focus on knowledge” (p. 158), resulting in teachers viewing their teaching practices as mastered and usable knowledge. Core practices that serve as the focus for pedagogies of practice vary (Grossman & McDonald, 2008). Some core teaching practices include launching an activity, managing student engagement, responding to students, and assessing student understanding (Lampert et al., 2013).

*Principles to Action*s (NCTM, 2014) also emphasized the importance of specific teaching practices that help students develop a conceptual understanding of mathematics. The eight Mathematics Teaching Practices were created to provide teachers with " a core set of high-leverage practices and essential teaching skills necessary to promote deep learning of mathematics” (NCTM, 2014, p.3). Some practices include “Implement tasks that promote reasoning and problem solving”, “Pose purposeful questions”, and “Elicit and use evidence of student thinking” (NCTM, 2014, p. 8). In both university and classroom settings, the identification of core practices helps break down the complexity of teaching. Moreover, it helps teacher educators focus their efforts on impactful teaching practices.

Approximations of Practice: Teaching Rehearsals

One way for PSTs to learn a core practice is through an approximation of practice. Ghousseini and Herbst (2016) defined approximations of practice as “activities in which novice teachers engage in experiences akin to real practice that reproduce some of the complexity of teaching” (p. 83). Teaching rehearsals are considered an approximation of practice and are structured as part of a learning cycle that requires the PSTs to interact with the practice, teach using the practice, and then reflect on their experience (Kazemi et al., 2016). Janssen et al. (2015) noted that teaching rehearsals can vary according to their level of focus, level of authenticity, and level of scaffolding.

Focus. The level of focus in a teaching rehearsal can vary from focusing on a single component of a practice to focusing on a full practice (Janssen et al., 2015). Many reported teaching rehearsals have focused on one specific teaching practice during the learning cycle (e.g., Arbaugh et al. 2019; Lampert et al., 2013). At the secondary level, Campbell and Elliott (2015) focused on having PSMTs build a definition of a mathematics topic from an investigation. Arbaugh et al. (2019) focused on specific kinds of questions PSMTs could use to elicit student thinking while engaged in problem solving.

One challenge of focusing on a particular practice is that the PSMTs may miss how teaching practices, such as NCTM's (2014) Mathematics Teaching Practices, are interdependent. Smith and Stein (2018) noted that teaching practices do not occur in isolation. For example, one of NCTM’s (2014) practices is that teachers facilitate meaningful mathematical discourse. Teachers who effectively facilitate mathematical discussions do so in part because they use three other practices: they have clear goals for the lesson, they ask purposeful questions, and they elicit and use student thinking (Smith & Stein, 2018). Additionally, these afore mentioned teaching practices depend on implementing a high-level task (Smith & Stein, 2018). Tasks classified as high-level are more likely to help students reason mathematically, have multiple entry points, have varied solution strategies, and promote connections between mathematical ideas (NCTM, 2014; Smith & Stein, 2018).

Authenticity. The level of authenticity of teaching rehearsals depends on where preservice teachers implement their practice lessons. Previous studies (e.g., Campbell & Elliott, 2015; Lampert et al., 2013) valued a high level of authenticity while implementing practice lessons. In those studies, PSTs engaged in teaching rehearsals in a controlled environment before also implementing the lesson in front of actual students. Campbell and Elliott (2015) found that implementing a practice lesson in an authentic setting allowed PSMTs to be responsive to a particular classroom environment, which they noted is an important part of teaching.

However, other studies did not have PSMTs teach in an actual classroom (e.g., Arbaugh et al., 2019; Dotger et al, 2014). Dotger et al. (2014) purposefully used clinical simulations to give PSMTs a shared experience. They found that the use of the simulation, along with the collective reflection, illuminated the PSMTs "mathematical knowledge, instructional abilities, and practices in need of refinement" (p. 599). Arbaugh et al. (2019) used mathematics education graduate students as practice secondary students so that the PSMT would have a "low-risk approximation of practice" (p.24). The authors found evidence of PSMTs' growth in several dimensions of teaching, such as increased knowledge of students and connecting knowledge of students with specific teaching practices.

Scaffolding. Finally, the level of scaffolding can vary with the level of feedback and support the teacher educator gives during the teaching rehearsal (Janssen et al., 2015). Many teacher educators pause preservice teachers' teaching during their rehearsal to give feedback (Arbaugh et al., 2019; Averill et al., 2016; Lampert et al., 2013). Lampert et al. (2013) used scaffolding during rehearsals to draw the PSTs' attention to certain practices, pausing the PSTs during the rehearsal to suggest a next teaching move, or to give an evaluation of a certain teaching move employed by the PST.

In contrast, Peercy and Troyan (2020) have suggested that a lower amount of scaffolding during rehearsal may be beneficial. They found that directive statements from the mathematics teacher educators led to the teacher educators being positioned as the expert and the contributions of PSMTs to be less emphasized. Conversely, when discussions of the teaching enactment emphasized PSMTs' contributions, the PSMTs "had more opportunities to engage and consider questions about the work of teaching" (p. 10). In short, Peercy and Troyan (2020) suggested that mathematics teacher educators must carefully consider and respond to PSMT thinking during approximations of practice in a similar manner to how they ask PSMTs to respond to student thinking.

Research Question

Previous studies involving teaching rehearsals were designed to include a narrower focus (Arbaugh et al., 2019; Campbell & Elliott, 2015; Lampert et al., 2013), an authentic setting (Campbell & Elliott, 2015; Lampert et al., 2013), and a high level of scaffolding (Arbaugh et al., 2019; Averill et al., 2016; Lampert et al., 2013). However, we hypothesized that a broader focus might give PSMTs in their first pedagogy course a better idea of what student-centered teaching entails. Additionally, we hypothesized that a controlled setting—having PSMTs teach their peers—might create a low-risk teaching environment for the PSMTs. Our research question was: How do PSMTs react to participating in a teaching rehearsal focused on implementing a high-level task using student-centered pedagogy?

Methods

Participant and Course Description

This study took place at a midwestern university with 22 PSMTs who were in their first secondary mathematics pedagogy course. The overarching goal of the course was to introduce PSMTs to the field of mathematics education via examination of *Principles and Standards for School Mathematics* (NCTM, 2000). The second author served as the instructor of the course. His pedagogy consisted of having PSMTs experience standards-based teaching, read about and discuss different pedagogical moves, and implement student-centered pedagogy. The first author attended the course both as an observer and researcher, working closely with the second author to design the teaching rehearsal and learning cycle.

Research Design

We designed the learning cycle to have PSMTs interact with the content of the lesson, plan to teach the lesson, teach the lesson (the teaching rehearsal), and then reflect on their experience (Kazemi et al., 2016). However, to elevate the voices and ideas of the PSMTs (Peercy & Troyan, 2020), we included an additional part of our learning cycle, based on observations of PSMTs' teaching and their initial reflections. The final part of our cycle included a whole-class discussion, asking PSMTs to read and respond to an article on eliciting student thinking (Reinhart, 2000), and providing PSMTs feedback on their rehearsal. To close, PSMTs reflected on the final segment of the teaching cycle.

To begin the learning cycle, PSMTs worked in groups of four to analyze a preselected high-level task. Prior to the learning cycle, the instructor chose six articles from *The Mathematics Teacher* that included a high-level task, instruction for implementing that task, and focused on using student-centered practices (NCTM, 2000, 2014). PSMTs analyzed the task by completing the tasks, developing learning goals based on the task, and describing how the task differed from a more traditional lesson on the mathematical concept.

During the planning phase of the cycle, the PSMTs were to plan their lesson using the launch-explore-discuss model (Smith & Stein, 2018). PSMTs continued to work in groups of four, planning the launch of the task, questions they would pose to students, and writing a description of how they would facilitate a whole-class discussion.

PSMT groups taught their lessons to fellow PSMTs, who were instructed to act as students. PSMTs started each lesson by stating what grade level the lesson was intended for, which allowed their classmates to act appropriately. We did not give feedback during the PSMTs' implementation of the task to give more weight to the PSMTs' own thoughts and actions (Peercy & Troyan, 2020).

After implementing the high-level task, PSMTs completed a written reflection on what they learned from planning and implementing their task. PSMTs also reflected on their experience as "students" and how they might use that information as a teacher. We used observations of PSMTs' teaching and their initial reflections to inform the design of the final part of the cycle. For the final part of the learning cycle, we started by having a whole-class discussion that was focused on their actions as teachers and ideas they referenced in their initial reflections. We also had the PSMTs read an article that focused on specific teaching practices that teachers can use as they plan questions that elicit student thinking (Reinhart, 2000). Then we gave each group written feedback on their lesson implementation related to those practices. Finally, we asked PSMTs to write a final reflection related to feedback received and the article's content.

Data Collection

During the learning cycle, the first author observed all class sessions and took field notes. From each group of PSMTs, we collected their task analyses, lesson plans, and initial reflections as teachers. From each individual PSMT, we collected their initial reflections as student participants and their final reflections. We also videotaped each group's teaching rehearsal and the whole-class discussion. For the purposes of this paper, we focused on the lesson videos and the three reflections completed by the PSMTs.

Data Analysis

We analyzed data at two points in the study. Our initial analysis included generate themes from the teaching rehearsal videos and the initial reflections. We used the generated themes to inform the structure of the whole-class discussion and the final reflection of the learning cycle. Because we did not have preconceived notions of what kind of themes we were looking for, we used open coding (Merriam & Tisdell, 2016) to analyze the data and then combined the codes into themes. For the second analysis, we analyzed their final reflections and coded according to the themes we found in the initial analysis. We then compared how the PSMTs discussed the generated themes in their initial reflections to how they discussed them in their final reflections.

Findings

Two broad themes emerged from our initial data analysis. The first theme of eliciting and using student thinking (NCTM, 2014) was based on the disconnect between our observations and their initial reflections as teachers during the teaching rehearsal. The second theme of student engagement and learning was based on their reflections as student participants. We used these themes to structure our whole-class discussion. One part of our discussion focused on recognizing how different types of questions promoted or hindered student thinking. The second part of our whole-class discussion focused on the level of student engagement in response to using a high-level task or the implementation of the task by the teachers.

In the following section, we organized our findings based on the themes we generated. We include a brief description of key findings at each stage of the learning cycle. Our intent in presenting the findings in this manner is to illustrate differences between PSMTs actions as teachers and their responses throughout their reflections.

Eliciting and Using Student Thinking

PSMT actions as teachers. In the initial analysis, we found that PSMTs struggled to ask open-ended questions during their lesson to elicit student thinking. At the conclusion of each teaching rehearsal, the PSMTs tended to ask closed-ended questions, most requiring one-word responses, to their students (fellow PSMTs). When the PSMTs did pose an open-ended question, they often offered little wait time before they rephrased the question, making it easier for the students to answer. For example, in Group 5, the PSMT leading the discussion started by asking students how they would know where to shade when graphing a linear inequality. After a brief pause where no one answered, the PSMT followed up with: "Which region would you shade?" Then continuing to provide little wait time, the PSMT asked, "Would you shade above the line or below the line?" which relegated the question to a single-word answer.

PSMTs seemed to overlook the importance of using student thinking as they led whole-class discussions. PSMTs circulated around the class during the student work time but did not document any student work or comments that could be used during the whole-class discussion. Five of the six groups started their whole-class discussion by asking for volunteers without purposefully highlighting student work they had seen as they walked around.

Initial reflections. In their initial reflections as teachers, PSMTs were not able to critique themselves about the ways they elicited (or failed to elicit) student thinking during their lessons. When asked about the strengths and weaknesses of their lessons, PSMTs focused on external factors like timing or how well the groups worked together, making no mention of how their teaching actions might have influenced student learning.

Several PSMTs did note, however, how anticipating student thinking before the lesson would have benefitted their teaching. Group 1 said, “We learned that preparation is key…If we prepare for multiple scenarios, it will increase our chances of being able to help students complete high-level tasks.” Group 4 said that “When preparing the task, you almost want to know how your students will answer,” and Group 5 added that “It would also be good to think ahead about some possible problems that may arise and think about how to solve them."

Final reflections. As part of the final reflection, PSMTs were asked to reflect on what changes they would make to their lesson, based on instructor feedback and ideas found in the article. In the final reflection, PSMTs were more specific about the ways they struggled to elicit student thinking while teaching, when compared to their initial reflections. Two PSMTs observed how their whole-class discussion became more of a lecture. One PSMT noted, “A lot of the class discussion was just me talking and a lot of that was because we were running out of time, but also I didn't know how to get the students to actually respond.” Several PSMTs noted that the article helped them realize lecturing or asking closed-ended questions requires students to do little thinking. One PSMT even said the article “helped me realize how little many students do think.”

Several other PSMTs commented on the need for planning more intentional questions to pose during the lesson because they realized their questions did not require much student thinking. One PSMT realized they were “asking a lot of questions that required one-word answers instead of whole responses. We weren't asking the right questions since we were clearly asking students to regurgitate something from memory and not to really answer and consider the question.” The idea of preparing questions beforehand seemed new to most of the PSMTs, with one of them commenting, “I always just assumed questions were something that came up when students were confused and weren’t planned ahead to invoke specific kinds of thinking. I really like this idea, though!”

Student Engagement and Learning

PSMT actions as teachers. In the initial analysis, PSMTs tended to revert to traditional teaching methods during their teaching rehearsals, rather than student-centered teaching practices. Despite engaging in discussions focused on launching a high-level task, several groups altered the task, lowering the cognitive demand of the task, making the task easier for the students to solve. Many groups seemed uncomfortable with student struggle. For example, Group 5 began their lesson with a review of mathematical concepts instead of a launch question. Group 2 posed a launch question but quickly transitioned to showing examples because they sensed confusion among the students.

Several groups also reverted to lecturing at some point during their lesson. PSMTs in Groups 4 and 5 both attempted posing questions to elicit student thinking but ended up offering their own lengthy explanation instead. During a task focused on determining the distance formula by deriving it from the Pythagorean Theorem, Group 6’s whole class discussion focused on presenting the connection versus allowing students to share their own discoveries.

Initial reflections. We were surprised by PSMTs’ reflections about their experiences as “practice students”. Even though PSMTs struggled to identify and critique student-centered practices in their own teaching, they were able to identify benefits of experiencing student-centered practices as students. One PSMT said, "It [the task] made me think through the process of finding possibilities on my own and not just being given information and memorizing it," and another said she learned to "ask what they [students] see rather than telling them what they're supposed to see." Other PSMTs noted the benefits of not telling students the answers right away, with many discussing how figuring out something on their own or in a group led to better understanding of the topic. PSMTs also expressed the importance of students having a conceptual understanding, rather than just a procedural understanding. PSMTs noted how the tasks in which they engaged fostered connections which led to that deeper understanding. One PSMT commented that "As a teacher, I want to make sure to integrate the "how" of math more often into my lessons than my own high school teachers did."

Final reflections. Several PSMTs, in their comments on the article, referenced giving students time and space to think and struggle during a task. One PSMT said the article helped her learn that "students are capable of a lot and if teachers just give them all the answers, then students will never learn how to do it on their own." A few PSMTs commented that in high school they were not given enough time to think through problems, which resulted in a lack of understanding and waiting for teachers to provide the students with a method and solution. PSMTs noted that as teachers they want to provide students with sufficient time to think though problems to promote learning and understanding.

Additionally, many PSMTs discussed the desire to establish a classroom culture focused on students' conceptual understanding. PSMTs mentioned wanting a classroom where students are comfortable sharing ideas and asking questions. One PSMT noted that, in his future classroom, he wanted to "replace lectures with sets of questions. Just like the author says, my initial instinct is to teach a concept in the form of a lecture where I can just tell the students what they should know. However, this does not encourage critical thinking."

Discussion

Teaching a lesson that had the structure of launch-explore-discuss allowed PSMTs to experience aspects of student-centered instruction. During their lessons, PSMTs struggled implementing student-centered practices, specifically, eliciting and using student thinking. This was not surprising because it was likely these PSMTs' first time implementing a high-level task. Moreover, we did not focus on specific practices of the lesson beforehand, which may have led to PSMTs' struggles.

On their initial reflections, PSMTs commented favorably about experiencing student-centered practices as a learner during the teaching rehearsal but struggled to give specific critiques of their own teaching. Prescott and Cavanaugh (2006) note that preservice teachers view their past school experiences "through the lens of the student rather than the teacher" (p. 429). PSMTs did not have enough teacher knowledge/experiences to effectively critique themselves.

In their final reflections, PSMTs were able to identify more specific teaching practices they could employ during teaching rehearsals. Consequently, PSMTs began developing a vision for their future classrooms that went beyond specific practices and instead focused on a classroom culture that values and uses student thinking.

The structure of the learning cycle helped facilitate PSMTs' understanding of student-centered instruction. The broad focus of the teaching rehearsal allowed the PSMTs to experience of the complexity of student-centered teaching. Moreover, PSMTs started to understand the kind of classroom culture that promotes student thinking. Although other studies using rehearsals have successfully targeted a single teaching practice (e.g., Arbaugh et al., 2019; Campbell & Elliott, 2015; Dotger et al., 2015), this study demonstrates that a broad focus can also be beneficial.

The lack of scaffolding during the teaching rehearsal, in addition to the broad focus, allowed us to build on practices and ideas that were lacking in PSMTs teaching and initial reflections as we designed the end of the learning cycle. This served to "create room for TL [teacher learner] voices in the pedagogies of rehearsal" (Peercy & Troyan, 2020). The whole-class discussion, our feedback, and reading the practitioner article helped PSMTs focus on specific teaching strategies. PSMTs were then able to critique in their own teaching and reflect on how to use student-centered pedagogy. The actual teaching experience gave PSMTs a reference point during discussions focused on teaching at the conclusion of the learning cycle because it was no longer hypothetical for them. In the final reflection, PSMTs' comments about future practices were tied to their experience as teachers during the rehearsal, to our whole-class discussion, and to the feedback.

Additionally, although Campbell and Elliott (2015) noted that "practice-focused designs are better tied to and derived from the activity of teaching in schools" (p. 151), the lack of authenticity did not seem to have negative consequences in this study. Rather, PSMTs were able to experience teaching in a low-risk setting with no pressure to "get it right" when they taught the lesson again in front of students. The low-risk setting gave PSMTs more time and space to reflect on their teaching. The final reflection at the end of the learning cycle seemed to prompt new understanding of ways to change their teaching moving forward. This finding is similar to studies that show the value of PSMTs reflecting on their own beliefs and being given space to examine and challenge those beliefs (Conner et al., 2011; White et al., 2005).

Conclusion

PSMTs had many favorable reactions to participating in a teaching rehearsal focused on high-level task implementation. They had a chance to experience teaching for the first time in a student-centered manner. PSMTs were able to use their teaching experience to reflect on the kinds of teaching practices and classroom cultures that elicit student thinking. One limitation of the study was that PSMTs did not complete a second teaching rehearsal. In their final reflections, PSMTs were able to describe specific practices they would like to use in their future teaching. Another teaching rehearsal would have provided PSMTs the chance to use those practices.

This study furthers the field of teaching rehearsals. The teaching rehearsal in this study differed from rehearsals in previous studies in its level of focus, scaffolding, and authenticity. An implication of this study is that a broad level of focus in a teaching rehearsal could help PSMTs gain a sense of the complexity of student-centered instruction. Additionally, a low level of scaffolding during the rehearsal allows for highlighting PSMTs' own ideas and actions. Finally, the lack of authenticity did not detract from the experience of PSMTs, which suggests that other universities may find it helpful for PSMTs to have similar experiences early in their university programs.

References

Arbaugh, F., Graysay, D., Konuk, N., & Freeburn, B. (2019). The three-minute-rehearsal cycle of enactment and investigation: Preservice secondary mathematics teachers learning to elicit and use evidence of student thinking. Mathematics Teacher Educator, 8(1), 22–48. https://doi.org/10.5951/mathteaceduc.8.1.0022

Association of Mathematics Teacher Educators. (2017). Standards for Preparing Teachers of Mathematics. https://amte.net/standards

Borko, H., & Mayfield, V. (1995). The roles of the cooperating teacher and university supervisor in learning to teach. Teaching and teacher education, 11(5), 501–518. https://doi.org/10.1016/0742-051X(95)00008-8

Campbell, M. P., & Elliott, R. (2015). Designing approximations of practice and conceptualizing responsive and practice-focused secondary mathematics teacher education. Mathematics Teacher Education and Development, 17(2), 146–164.

Conner, A., Edenfield, K. W., Gleason, B. W., & Ersoz, F. A. (2011). Impact of a content and methods course sequence on prospective secondary mathematics teachers' beliefs. Journal of Mathematics Teacher Education, 14(6), 483–504. https://doi.org/10.1007/s10857-011-9186-8

Dotger, B., Masingila, J., Bearkland, M., & Dotger, S. (2015). Exploring iconic interpretation and mathematics teacher development through clinical simulations. Journal of Mathematics Teacher Education, 18(6), 577–601. https://doi.org/10.1007/s10857-014-9290-7

Ghousseini, H., & Herbst, P. (2016). Pedagogies of practice and opportunities to learn about classroom mathematics discussions. Journal of Mathematics Teacher Education, 19(1), 79–103. https://doi.org/10.1007/s10857-014-9296-1

Grossman, P., Hammerness, K., & McDonald, M. (2009). Redefining teaching, re-imagining teacher education. Teachers and Teaching: Theory and Practice, 15(2), 273–289. https://doi.org/10.1080/13540600902875340

Grossman, P., & Dean, C. G. P. (2019). Negotiating a common language and shared understanding about core practices: The case of discussion. Teaching and Teacher Education, 80, 157–166. https://doi.org/10.1016/j.tate.2019.01.009

Grossman, P., & McDonald, M. (2008). Back to the future: Directions for research in teaching and teacher education. American Educational Research Journal, 45(1), 184–205. https://doi.org/10.3102/0002831207312906

Janssen, F., Grossman, P., & Westbroek, H. (2015). Facilitating decomposition and recomposition in practice-based teacher education: The power of modularity. Teaching and Teacher Education, 51, 137–146. https://doi.org/10.1016/j.tate.2015.06.009

Kazemi, E., Ghousseini, H., Cunard, A., & Turrou, A. C. (2016). Getting inside rehearsals: Insights from teacher educators to support work on complex practice. Journal of Teacher Education, 67(1), 18–31. https://doi.org/10.1177/0022487115615191

Lampert, M., Franke, M. L., Kazemi, E., Ghousseini, H., Turrou, A. C., Beasley, H., Cunard, A., & Crowe, K. (2013). Keeping it complex: Using rehearsals to support novice teacher learning of ambitious teaching. Journal of Teacher Education, 64(3), 226–243. https://doi.org/10.1177/0022487112473837

Merriam, S. B., & Tisdell, E. J. (2016). Qualitative research: A guide to design and implementation (4th ed.). Jossey-Bass.

National Council of Teachers of Mathematics. (2000). Principles and standards for school mathematics.

National Council of Teachers of Mathematics. (2014). Principles to actions: Ensuring mathematical success for all.

National Governor's Association Center for Best Practices & Council of Chief State School Officers. (2010). Common Core Standards for Mathematics. http://www.corestandards.org/Math/Practice/

Peercy, M. M., & Troyan, F. J. (2020). "Am I doing it wrong?" Critically examining mediation in lesson rehearsal. Teaching and Teacher Education, 93, 1–12. https://doi.org/10.1016/j.tate.2020.103082

Prescott, A. E., & Cavanagh, M. (2006). An investigation of pre-service secondary mathematics teachers' beliefs as they begin their teacher training. In Conference of the Mathematics Education research Group of Australasia. MERGA.

Reinhart, S. C. (2000). Never say anything a kid can say!. Mathematics Teaching in the Middle School, 5(8), 478–483. https://doi.org/10.5951/MTMS.5.8.0478

Smith, M. S., & Stein, M. K. (2018). 5 practices for orchestrating productive mathematics discussions. National Council of Teachers of Mathematics.

Solomon, Y., Eriksen, E., Smestad, B., Rodal, C., & Bjerke, A. H. (2017). Prospective teachers navigating intersecting communities of practice: Early school placement. Journal of Mathematics Teacher Education, 20(2), 141–158. https://doi.org/10.1007/s10857-015-9327-6

White, A. L., Way, J., Perry, B., & Southwell, B. (2005). Mathematical attitudes, beliefs and achievement in primary pre-service mathematics teacher education. Mathematics Teacher Education and Development, 7, 33¬–52.

Wilson, S. M., Floden, R. E., & Ferrini-Mundy, J. (2002). Teacher preparation research: An insider's view from the outside. Journal of teacher education, 53(3), 190–204. https://doi.org/10.1177/0022487102053003002

Winsor, M. S., Kirwan, J. V., & Ssebaggala, L. (2018). What do we know about secondary mathematics teacher preparation in the United States? The Mathematics Educator, 27(2), 73–106.

ENGAGING AND EMPOWERING PROSPECTIVE TEACHERS THROUGH THEIR ABILITY TO COORDINATE UNITS

Rich Busi
James Madison University
busirp@jmu.edu

Alexis Stevens
James Madison University
stevenal@jmu.edu

Jennifer Gibson
James Madison University
gibso2js@jmu.edu

LouAnn Lovin
James Madison University
lovinla@jmu.edu

This study will investigate a teaching intervention designed to improve prospective teachers' abilities to coordinate levels of units within their understanding of fractions. Unit coordination is a well-defined construct within the literature in terms of how students understand fractions and progress through understanding them. However, much is to be learned about the specific ways in which we can identify and impact this understanding within undergraduate mathematics content courses for future teachers. This study will consider an intervention that allows prospective teachers to explore fractional situations whose solutions require the coordination of three levels of units. We hypothesize participants might better understand and internalize this coordination through explorations and activities in classroom settings. Specifics for data collection/analysis and next steps are shared.

Keywords: Teacher Knowledge; Mathematical Knowledge for Teaching; Rational Numbers

Proficiency with rational numbers, including fractions, has and continues to be a serious challenge for many teachers and students. It is all too common that learners feel overwhelmed and underprepared when it comes to understanding fractions and fraction operations (e.g., Ball, 1990; Bentley & Bosse, 2018; Borko et al., 1992; Izsák et al., 2010; Menon, 2009; Olanoff et al., 2014; Rathouz, 2010; Rizvi & Lawson, 2007; Schneider & Siegler, 2010; Stafylidou & Vosniadou, 2004; Tirosh, 2000). More specifically, in a previous study of preservice PreK-8 teachers' (PSTs') fraction knowledge (Busi et al., 2015; Stevens et al., 2018), we found considerable evidence that many PSTs struggled with the reasoning needed for fluency with fractions. Subsequently, we investigated ways to improve PSTs' fraction content knowledge through changes in our pedagogy (Stevens et al., 2018). It became clear through this investigation that one of the major challenges for PSTs is the coordination of units in fractional situations.

It is well established (e.g., Norton & Wilkins, 2012; Steffe & Olive, 2010; Wilkins & Norton, 2011) that understanding fractions relies on the levels of units a learner can coordinate simultaneously when faced with a fractional context. Specifically, to reach the higher levels of reasoning with fractions, learners must be able to coordinate three levels of units simultaneously (3UC) – meaning they can anticipate the outcome of this coordination before they do it. Having this anticipation is known as interiorizing the ability to coordinate units. Alternatively, if someone is unable to anticipate the outcome of the coordination, they may either not have acquired this coordination or may be coordinating the units in action as they solve the problem.

Through our most recent work (Stevens et al., 2021), we have established and refined an observation protocol for identifying 3UC with PSTs. This process involves examining them solving fractional problems either live or through recorded video. By witnessing PSTs' work and thinking as it happens, we are better able to identify PSTs' ability or inability to coordinate three levels of units. However, there remains questions in regard to best practices for supporting PSTs who are identified as lacking 3UC. In this study, we will investigate a teaching intervention to determine its effects on building and improving 3UC for PSTs.

Theoretical Framework

An existing developmental trajectory of fraction schemes and operations serves as our framework. This trajectory was first validated for upper elementary and middle school students (Norton & Wilkins, 2012; 2013; Wilkins & Norton, 2011) and later validated for PSTs (Busi et al., 2015; Stevens et al., 2018). These schemes and operations can be grouped into three bands of developmental knowledge of fractions with each subsequent band relying on an increasing number of levels of units the learner can coordinate: fractions as solely part-whole concepts (only requires the coordination of one level of unit); fractions as measures (requires the coordination of two levels of units); and fractions as fractional numbers (requires the coordination of three levels of units) (Hackenberg et al., 2016). Our previous work discovered a majority of PSTs were not proficient in being able to reason about fractions as fractional numbers, which corroborates existing research (e.g., Chinnappan, 2000; Olanoff et al. 2016; Son & Crespo, 2009; Son & Lee, 2016). The catalyst for moving into this more sophisticated way of thinking about fractions is to be able to coordinate three levels of units (Steffe & Olive, 2010). We are currently focused on how we might better support the development of 3UC in PSTs; therefore, this study focuses specifically on the 3UC portion of this developmental trajectory of fractions. (For more information about this hierarchy, please see Norton & Wilkins (2009), Wilkins & Norton (2011), Norton & Wilkins (2012), Busi et al., (2015), Steffe (2002), Steffe & Olive (2010).)

Methods

Participants and Instrument

Participants in this study are 20-30 undergraduate students enrolled in the first of two required mathematics content courses for PSTs at a southeastern university. The first course in this sequence focuses on number concepts and operations, with significant time dedicated to developing fraction understanding. Students in two sections of this course are participating in the study.

Because the motivation for the study is to investigate teaching strategies that may improve PSTs' ability to coordinate units, a written 3UC assessment will be used, as was done in previous studies (Stevens et al., 2018; 2021). This written assessment will be completed by participants as they are video recorded so the researchers can view the work and thinking as it happens rather than only having access to final written work. Participants will be asked to share aloud their reasoning as they complete their work. This assessment consists of eight questions specifically designed to assess 3UC ability of PSTs (Stevens et al., 2021). An example of a question is provided in Figure 1 below.

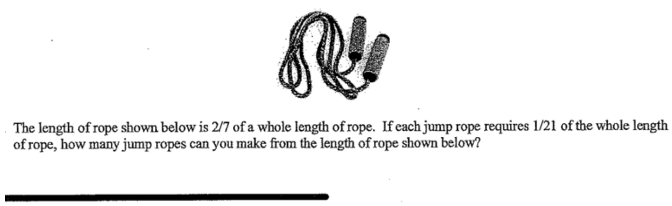


Figure 1: Example of 3UC Assessment Question

Intervention and Data Collection

The intervention for this study is a fraction unit taught in the participants' mathematics content course for prospective teachers. The intervention unit will take approximately two weeks and will cover meanings of fractions, comparing fractions, fractional equivalence, and relating fractions to other forms of rational numbers. Data collection will take place before and after the fraction intervention unit in a pretest/posttest format.

This unit was chosen as the intervention specifically for its focus on unit coordination. The problems and explorations to be used center on modeling fractions in ways that require PSTs to acknowledge and manipulate multiple levels of units in a single context. The intervention models include pattern blocks, number lines, circular area models, noncircular area models, and set models. For example, PSTs could be asked to reason, with any of the above-mentioned models and without reverting to a procedure, how many tenths are in a given length, area, or set that represents three-fifths of a whole. In order to support this argument, PSTs must relate the size of fifths and the size of tenths to the whole as well as to each other.

Participants will complete the eight-item assessment designed to determine whether or not they were able to coordinate three levels of units simultaneously. This will be done as a class assignment where each participant will record themselves solving the problem. For each item, the PSTs will be asked to provide both a solution and a demonstration of their reasoning by talking through their thinking as they solve. Sharing one's reasoning is a well-established class norm for these students.

The recordings afford the researchers the ability to observe work and reasoning. This is an essential portion of the study because it enables the researchers to watch the participants' approaches in action, rather than solely evaluating written evidence of their strategies after they submitted the assessment. Upon submission of these video recordings, the researchers will review each video and label each participant as either having or lacking 3UC.

The assessment begins with two items with no accompanying representations or context. For example, PSTs will be posed the following question: "Envision 2/3 of a whole. Now consider 1/12 of the same whole. How many 1/12s are in the 2/3 you originally envisioned?" For these items, the PSTs will be asked to solve and share their thinking freely. The remaining six items are written within a specific context (e.g., an amount of pizza or the length of a jump rope) and provide a specific representation (e.g., a portion of a circle or a line). For these items, the PSTs will be asked to use the provided representation to diagram their reasoning. The fractions used in both sections are varied in structure; the denominators either allow for halving strategies (e.g., relating 3/4 and 1/8) or require strategies other than halving (e.g., relating 3/5 and 1/15). The representations are also varied in form; the referent whole is either a circle, a portion of a circle, a rectangle, or a line. Previous explorations of developing students' understanding of fractions proport that success with varied denominators and representations is required for true fluency (e.g., Siegler et al., 2010).

To help further investigate the intervention and its impacts, interview data will be collected from a subset of participants. Specifically, we will be interviewing two participants who showed change from pretest to posttest as well as two participants who did not. These interviews will be conducted in an open interview format and will be framed around participants' work on the eight-item assessment. The open format approach is designed to give participants autonomy by having initial organic conversations with the researchers and then the ability to prepare their thoughts before the actual interviews (Robinson et al. 2021).

Data Analysis

PSTs who have interiorized the ability to coordinate three levels of units have an immediate, productive plan to solve a 3UC fraction task and can anticipate the results irrespective of context, denominator choice, or representation; they do not rely on their written work to discover a productive strategy in action (Hackenberg et al., 2016). This is the core criterion used when viewing and evaluating the participants' work.

The evaluation will take place based on the participant's ability to process the different unit sizes and their relative nature quickly and meaningfully to each other. For example, if a problem involves thirds, ninths, and the whole unit, our data analysis focuses on how immediately and deliberately the participant moves to see the number of ninths per whole, the number of thirds per whole, and finally the number of ninths per third. Because there is an inherent level of subjectivity to this analysis (Stevens et al., 2021), each participant's video will be evaluated by all four researchers in this study independently. The researchers will then discuss their evaluations and come to a consensus based on the evidence provided by the comparisons. This analysis design helps to ensure reliability of rating each participant's interiorization of 3UC.

The researchers hypothesize that interacting with the fractional intervention unit may impact the PSTs' ability to coordinate units. To fully examine this hypothesis, data will be analyzed with a mixed methods design. From a quantitative standpoint, pretest and posttest data will capture each participant's ability with 3UC – ultimately each participant will be labeled as either "yes" or "no" for both pretest and posttest. Those data will be analyzed for differences using the McNemar test for a dichotomous dependent variable as we have done with previous 3UC data (Busi et al., 2015). From a qualitative standpoint, the open interview data will be analyzed using a thematic approach (Kiger & Varpio, 2020).

Discussion

Through this study, we hope to begin to better understand how teaching fractional concepts within the context of mathematics content courses can impact PSTs' ability to coordinate units. Research has shown that the ability to coordinate three levels of units is critical in understanding fractions in meaningful ways (Norton & Wilkins, 2012; Steffe & Olive, 2010; Wilkins & Norton, 2011). Our initial hypothesis is that teaching fractional units centered on the conceptual bases of fractional meaning, comparing fractions, fractional equivalence, and relating fractions to other forms of rational numbers can have a positive impact on PSTs' ability to coordinate units.

References

Ball. D. L. (1990). The mathematical understandings that prospective teachers bring to teacher education. *The Elementary School Journal*, *90*, 449-466. https://www.jstor.org/stable/1001941

Bently, B. & Bosse, M. J. (2018). College students' understandings of fractions. *International Electronic Journal of Mathematics Education, 13*(3), 233-247. https://doi.org/10.12973/iejme/3881

Borko, H., Eisenhart, M., Brown, C. A., Underhill, R. G., Jones, D., & Agard, P. C. (1992). Learning to teach hard mathematics: Do novice teachers give up too easily? *Journal for Research in Mathematics Education*, *23*(3), 194-222. https://doi.org/10.2307/749118

Busi, R., Lovin, L. A., Norton, A., Siegfried, J. M., Stevens, A. L., & Wilkins, J. L. M. (2015). An investigation of PreK–8 preservice teachers' construction of fraction schemes and operations. *Proceedings of the 37rd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 616–623). Michigan State University: East Lansing, MI.

Chinnappan, M. (2000). Preservice teachers' understanding and representation of fractions in a JavaBars environment. *Mathematics Education Research Journal, 12*(3), 234–253. https://doi.org/10.1007/BF03217087

Hackenberg, A. J., Norton, A., & Wright, R. J. (2016). *Developing fractions knowledge*. SAGE Publications.

Izsák, A., Orrill, C. H., Cohen, A. S., & Brown, R. E. (2010). Measuring middle grades teachers' understanding of rational numbers with the Mixture Rasch model. *The Elementary School Journal*, *110*(3), 279-300. https://doi.org/10.1086/648979

Kiger, M. E. & Varpio, L. (2020). Thematic analysis of qualitative data: AMEE Guide No. 131. *Medical teacher*, *42*(8), 846-854. DOI: 10.1080/0142159X.2020.1755030

Menon, R. (2009). Preservice teachers' subject matter knowledge of mathematics. *International Journal for Mathematics Teaching and Learning*, *12*, 89-109.

Norton, A. & Wilkins, J. L. M. (2009). A quantitative analysis of children's splitting operations and fraction schemes. *Journal of Mathematical Behavior*, *28*(2/3), 150-161. https://doi.org/10.1016/j.jmathb.2009.06.002

Norton, A. & Wilkins, J. L. M. (2012). The splitting group. *Journal for Research in Mathematics Education*, *43*(5), 557-583, https://doi.org/10.5951/jresematheduc.43.5.0557

Norton, A. & Wilkins, J. L. M. (2013). Supporting students' constructions of the splitting operation. *Cognition and Instruction*, *31*(1), 2-28. http://www.jstor.org/stable/23525189

Olanoff, D., Lo, J., & Tobias, J. (2014). Mathematical content knowledge for teaching elementary mathematics: A focus on fractions. *The Mathematics Enthusiast, 11*(2), 267-310. DOI:10.54870/1551-3440.1304

Olanoff, D., Feldman, Z., Welder, R. M., Tobias, J. M., Thanheiser, E., & Hillen, A. F. (2016). Greater number of larger pieces: A strategy to promote prospective teachers' fraction number sense development. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tuscan, AZ: The University of Arizona, 799-807.

Rathouz, M. (2010). Ambiguity in units and their referents: Two cases in rational number operations. *For the Learning of Mathematics*, *30*(1), 43–51. http://www.jstor.org/stable/20749438

Rizvi, N.F., & Lawson, M.J. (2007). Prospective teachers' knowledge: Concept of division. *International Education Journal, 8*(2), 377-392.

Robinson, J., Barron, A., & Pottinger, L. (2021). Open interviews. In A. Barron, A. L. Brown, U. Ehgarter, S. M. Hall, L. Pottinger, & J. Ritson (Eds.), *Methods for change: Impactful social science methodologies for 21st century problems*. Manchester: Aspect and The University of Manchester.

Schneider, M. & Siegler, R. S. (2010). Representations of the magnitudes of fractions. *Journal of Experimental Psychology: Human Perception and Performance, 36*(5), 1227–1238. https://doi.org/10.1037/a0018170

Siegler, R., Carpenter, T., Fennell, F., Geary, D., Lewis, J., Okamoto, Y., Thompson, L., & Wray, J. (2010). *Developing effective fractions instruction for kindergarten through 8th grade: A practice guide* (NCEE #2010-4039). Washington, DC: National Center for Education Evaluation and Regional Assistance, Institute of Education Sciences, U.S. Department of Education. Retrieved from whatworks.ed.gov/ publications/practiceguides.

Son, J. & Crespo, S. (2009). Prospective teachers' reasoning about students' non-traditional strategies when diving fractions. *Journal of Mathematics Teacher Education, 12*(4), 236–261. https://doi.org/10.1007/s10857-009-9112-5

Son, J. & Lee, J. (2016). Prospective teachers' understanding of fraction multiplication, representational knowledge, and computational skills. *Mathematics Teacher Education and Development, 18*(2), 5-28.

Stafylidou, S. & Vosniadou, S. (2004). The development of students' understanding of the numerical value of fractions. *Learning and Instruction, 14*(5), 503–518. https://doi.org/10.1016/j.learninstruc.2004.06.015

Steffe, L. P. (2002). A new hypothesis concerning children's fractional knowledge. *Journal of Mathematical Behavior*, *20*(3), 267-307. https://doi.org/10.1016/S0732-3123(02)00075-5

Steffe, L. P. & Olive, J. (Eds.). (2010). *Children's fractional knowledge*. Springer.

Stevens, A., Busi, R., Gibson, J., & Lovin, L. A. (2021). Rethinking how we assess 3UC in prospective elementary teachers. *Proceedings of the 43rd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (832-841). Philadelphia, PA.

Stevens, A., Wilkins, J., Lovin, L. A., Siegfried, J., Norton, A., & Busi, R. (2018). Promoting sophisticated fraction constructs through instructional changes in a mathematics course for preK-8 prospective teachers. *Journal of Mathematics Teacher Education, 21*(3), 207-235.
Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for Research in Mathematics Education*, *31*(1), 5-25. https://doi.org/10.2307/749817
Wilkins, J. L. M. & Norton, A. (2011). The splitting loope. Journal for Research in Mathematics Education, 42(4), 386-416. https://doi.org/10.5951/jresematheduc.42.4.0386

ENGAGING PRESERVICE TEACHERS' LEARNING THROUGH INTEGRATED STEM INQUIRY IN A MATHEMATICS CONTENT COURSE

Su Liang
University of Texas, San Antonio
su.liang@utsa.edu

Sasha Wang
Boise State University
SashaWang@boisestate.edu

This paper reports on a sub-study of a National Science Foundation funded project to investigate preservice elementary teachers' (PSETs) mathematics learning through an integrated STEM inquiry in a mathematics content course. In particular, PSETs participated in a 3D printing STEM module to explore and visualize the symmetry of a variety of crystal lattice structure solids. The results show that the PSETs perceived the STEM module to be engaging. Their learning increased in STEM teaching and learning.

Keywords: Teacher Education, Integrated STEM, Technology

Science, Technology, Engineering and Mathematics (STEM) education aims to ensure STEM literacy for all learners (Bybee, 2013) and it is critical to start STEM education in elementary school if not earlier. The recent movement in STEM education advocates an integrated approach that uses inquiry-based learning to engage students in solving complex problems that reflect real-world situations. This approach enables teachers and students to see how concepts and skills learned in school are connected and applied in the context of real-world issues (Buckner & Boyd, 2015; Bybee, 2013). As such, it is likely to enhance motivation for learning and improve student interest, achievement, and persistence (National Research Council, 2014). Elementary teachers need professional training to teach STEM in their classrooms in their professional training. Experiencing STEM inquiries in college courses enable pre-service elementary teachers (PSETs) to understand how STEM disciplines are connected and thus gain relevant knowledge and skills for future teaching. As PSETs take mathematics content courses in their teacher preparation program, integrating STEM inquiries into these courses affords excellent opportunities for PSETs to develop deeper understanding of how mathematics concepts are applicable in the real world, and broaden their perspectives on the roles of mathematics in solving STEM problems. In this paper, we address the following research question: To what extent does the integrated STEM module engage PSETs' learning?

Integrated STEM Approach

In this project the STEM module is considered "Integrated STEM Inquiries" because they are designed to enhance PSETs' mathematics learning in the context of science, technology, and engineering. Prior research has shown the advantage of engaging students in an integrated STEM inquiry which requires the application of multiple STEM concepts and skills, as opposed to focusing on instruction in discrete subject areas. Teachers benefit from integrated STEM inquiry projects that complement and even reinforce their existing curriculum.

Inquiry-based Learning is a unique approach to the design of engaging activities that enable students to investigate authentic topics and enact learning processes that transcend traditional methods of instruction. Hands-on elements such as designing artifacts or conducting experiments are embedded within inquiry-based learning, allowing PSETs to work in teams or small groups, PSETs and instructors to work side-by-side, and the creation of new opportunities for relevant,

productive classroom norms. In the inquiry-based learning environment, PSETs experiment with concepts in mathematics and the application of mathematics in other STEM disciplines.
Scaffolded Knowledge Integration (SKI) guides the design of the IME STEM inquiry module. It is developed by Marcia Linn and colleagues and is employed to encourage PSETs to develop an integrated understanding of a complex domain and emphasize collaboration, problem solving skills, and critical thinking. The main tenets of SKI are: (a) make learning relevant to students, connected to prior knowledge, and related to their physical surroundings; (b)make students' thinking visible, asking them to describe how they recognize new ideas, and to reorganize and connect new prior ideas; (c) provide social supports for pair and group collaborations; and (d) provide opportunities for reflection and revisiting what students have learned. By applying SKI, PSETs learn the desired content, reflect, and collaborate, and instructors have ample opportunities to engage, support and enhance PSETs' mathematics learning.
3D Crystal Structure of Solids Visualization and Printing provides PSETs a context for geometry in science and engineering. We developed five Graphical User Interface (GUI) applications using more than 2,000 lines of codes in Python and JavaScripts languages for PSETs to explore primitive cubic (cP) structure, face-centered cubic (fcc) structure and ideal perovskite structure. These structures can be observed in metal polonium (cP), sodium chloride or rock salt (fcc), silver chloride (fcc), Calcium Titanate (perovskite) and Strontium Titanate (perovskite). They are selected for visualization and printing not only because they preserve a great deal of symmetry and have solid structures that align with the course content, but also because they are relevant to daily lives (e.g. salt) or state-of-art research in energy. Solar cells with perovskite structures have shown remarkable progress in improving energy conversion efficiency, and thus are thought to be a frontier energy solution with great importance in current material science research. Figure 1 illustrates an alignment of integrated STEM approach (Wang et al, 2020).

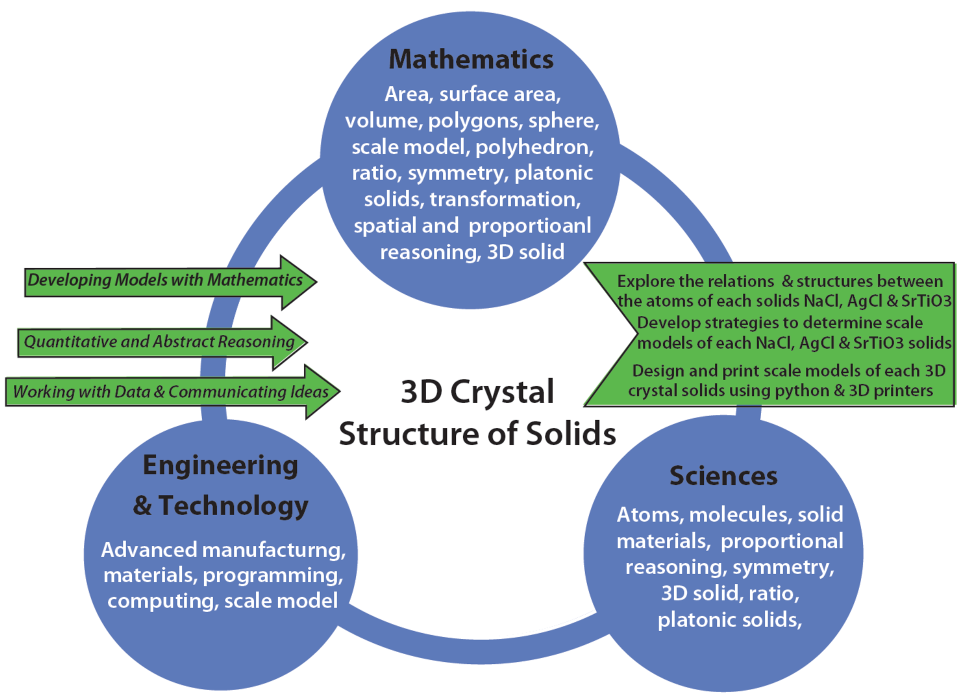


Figure 1: Crosscutting concepts in the 3D Crystal Structure of Solids Module

Reflective Practices According to Dewey (1933), it is not the experiencing itself but reflective thinking on that experience that generates learning. Critical reflection is a high level of the reflective thinking process to look for the meaning of experience. This process involves assessing, inquiring, analyzing, synthesizing, and making connections between related factors (Dewey, 1938; Brookfield, 1995; Ash & Clayton, 2009). Kolb (1984) developed a holistic model of the experiential learning cycle taking the original intellectual experiential work from educational theorists such as Piaget and Dewey. Kolb's model demonstrates a learning cycle starting with a concrete experience, following reflective observation, abstract conceptualization, active experimentation, and finally back to the starting point of concrete experience. We adapted Kolb's learning cycle to guide and document PSETs' learning experiences through their critical reflections.

Methods

Thirty-nine PSETs enrolled in a mathematic content course in a large university in the US participated in the implemented integrated STEM module, 3D Crystal Structure of Solids: Visualization and 3D Printing. First, all PSETs attended four 75-minute class meetings, which provided them a context for understanding how geometry (e.g., regular polygons, platonic solids, symmetry, etc.) is applied in science and engineering. After they are exposed to the STEM module, they also participated in the community-based experiential learning (CBEL) as a course component. The CBEL activity is designed to provide PSETs with early teaching experiences as they facilitate the STEM module in afterschool programs. The CBEL course experience closes the gap of PSETs' lack of understanding of the knowledge and skills needed for future teaching, and thus is pedagogical and links PSETs' academic learning (mathematics), personal development (e.g., attitude toward math and STEM education, civic engagement, etc.), and their future workplace skills (teacher). For this paper, we will use PSETs' critical written reflections as data sources to gain a better understanding of their learning through integrated STEM modules.

Results

Guided by the research question, PSETs' perceived learning were assessed through their critical reflections in two categories: their engagement in STEM learning and in STEM teaching. **PSETs' Engagement in STEM Learning** almost all PSETs reported that they learned about using 3D printers in their reflections. Some PSETs' reflection quotes are as follows:

- "…I have learned how to use a 3D printer, such as how to turn it on, heat it up, and connect the software on the computer to the printer. I also understand how to use the (3D modeling) software and how to control the size and placement of the solid in addition to learning how to split an object in half."
- "By doing the STEM module/ 3d structure I was able to get a better understanding of what 3d printing is and how it works. By using crystalline structures, I was able to see how the printer would have to go layer by layer."
- "Then for the 3D lattice structure assignment, we needed to rotate the shape and check what type of symmetry it had. I learned how the STEM module/3D crystal structure solids can compare and find their radius and volumes."

- "I learned about the radius and volume of the spheres of the crystalline structure and how they affect the way the structure looks. If the radius is bigger, the lines between the spheres are smaller."

PSETs' Engagement in STEM Teaching

By participating the CBEL activity facilitating the STEM module with a group of elementary students at the local community centers, many PSETs found the activity engaging. Some PSETs' reflections are as follows:

- "Our CBEL experience allowed me to appreciate teaching more than before. Having my group of students be excited to learn and create/explore STEM through 3D printing and crystal lattice structures was heartwarming"
- "I learned that students get excited when they see the engagement you have with them. You can make it fun and exciting while it still connects to the STEM/3D structure."
- "they were excited and how they understood what they had learned when I was showing them the different types of symmetry and how many lines of symmetry did the shape have. Using this in my future teaching classroom will be beneficial way to the students and for me to see that they understood the lesson."
- "What I can do with this topic that I learned in class I can show the students an example of a 3D shape that I had created and ask them questions about what type of symmetry it is. Then they will be doing a hands-on activity that the students can build, and they can write on paper what type of symmetry it has."

Conclusion

The results of this study support the practice of incorporating the integrated STEM module in a college mathematics content course for PSETs. The integration of 3D printing technology and integrated STEM injury in the module supports hands-on learning and visualization for geometry and measurement concepts, making complicated concepts more accessible and understandable for PSETs. While 3D printing technology was new to most of the participants, they were able to learn the technology and tried to problem solve when issues occurred. Seeing how mathematics concepts are connected to and used in chemistry, engineering, and technology helps PSETs contextualize their understanding of these concepts in a deeper, connected, and applicable manner. Considering this is an ongoing project, PSETs' critical reflections also voiced their needs in a more coherent and transparent curriculum design for the mathematics content course.

Acknowledgments

We wish to thank the National Science Foundation (NSF) for the support of this study under Grant Number 2111549. Any opinions, findings, and conclusions or recommendations expressed in these materials are those of the authors and do not necessarily reflect the views of the NSF.

References

Ash, S. L. & Clayton, P. H. (2009). Generating, deepening, and documenting Learning: The power of critical reflection in applied learning. Journal of Applied Learning in Higher Education, 1, 25-48.

Brookfield, S. (1995). Becoming a critically reflective teacher. San Francisco: Jossey-Bass.

Brown, J. S., Collins, A., & Duguid, P. (1989). Situated cognition and the culture of learning. Educational researcher, 18(1), 32-42. https://doiorg.ezproxy.lib.purdue.edu/10.3102/0013189X018001032

Buckner, T., & Boyd, B. (2015). STEM Leadership: How Do I Create a STEM Culture in My. School?(ASCD Arias). ASCD.
Bybee, R. (2013). The case for STEM education: Challenges and opportunities. Arlington, VA: National. Science Teachers Association.
Dewey, J. (1933). How we think: A restatement of the relation of reflective thinking to the education process. Boston, MA: D. C. Heath.
Dewey, J. (1938). Experience and education. New York: Macmillan.
Frykholm, J., & Glasson, G. (2005). Connecting science and mathematics instruction: pedagogical context knowledge for teachers. School Science and Mathematics, 105(3), 127–141.
Han, J., Kelley, T. R., Mentzer, N., & Knowles, J. G. (2021). Community of Practice in Integrated STEM. Education: A Systematic Literature Review. Journal of STEM Teacher Education: Vol. 56: Iss. 2 , Article 5.
Kelley, T. R., and Knowles, J. G. (2016). A conceptual framework for integrated. STEMeducation. International Journal of STEM Education, 3(1), 11.
https://doi-org.ezproxy.lib.purdue.edu/10.1186/s40594-016-0046-z
Lajoie, S. P., Guerrera, C., Munsie, S. D., & Lavigne, N. C. (2001). Constructing knowledge in the context of BioWorld. Instructional Science, 29(2), 155-186. https://doiorg.ezproxy.lib.purdue.edu/10.1023/A:1003996000775
Leung, A. (2020). Boundary crossing pedagogy in STEM education. International Journal of STEMEducation, 7, 1-11. https://doi.org/10.1186/s40594-020-00212-9
Linn, M.C., & Hsi, C.D. (2000). Integrated STEM education: A review of the literature. Review of Educational Research, 70(2), 129-180
Loof, H. D., Pauw, J. B., & Petegem, P. V. (2021). Engaging Students with Integrated STEM. Education: a Happy Marriage or a Failed Engagement? International Journal of Science and Mathematics Education, 20(2).DOI: 10.1007/s10763-021-10159-0
Luft, J.A., Bailey, K.A., & O'Neill, K.M. (2019). Integrated STEM education: A review of the research. Science Education, 103(6), 1407-1435.
McKnight, D. (2002). Field experience handbook: A guide for the teacher intern and mentor teacher. College Park: University of Maryland.
National Research Council. (2014). STEM integration in K-12 education: status, perspectives, and agenda for research. Washington, D.C: National Academy Press.
Ryan, K., & Cooper, J. M. (2006). Those who can, teach. Boston: Houghton Mifflin.
Sarican, G. & Akgunduz, D. (2018). The impact of integrated STEM education on academic achievement, reflective thinking skills towards problem solving and permanence in learning in science education. 13(1), 94-107.
Stohlmann, M., Moore, T., & Roehrig, G. (2012). Considerations for teaching integrated STEM education. Journal of Pre-College Engineering Education Research., 2(1), 28–34. doi:10.5703/1288284314653.
Putnam, R.T., & Borko, H. (2000). What do new views of knowledge and thinking have to say about research on teacher learning? Educational Researcher. 29(1), 4-15.
Wang, S., Ching, Yh., Yang, D., Swanson, S., Baek, Y., Chittoori, B. (2020). Developing US Elementary. Students' STEM Practices and Concepts in an Afterschool Integrated STEM Project. In: Anderson, J., Li, Y. (eds) Integrated Approaches to STEM Education. Advances in STEM Education. Springer, Cham. https://doi.org/10.1007/978-3-030-52229-2_12

EXPLORING PRESERVICE TEACHERS' EMBODIED NOTICING OF STUDENTS' FRACTION DIVISION

Karl W. Kosko
Kent State University
kkosko1@kent.edu

Temitope Egbedeyi
Kent State University
tegbedey@kent.edu

Enrico Gandolfi
Kent State University
egandol1@kent.edu

There is emerging evidence that professional noticing is embodied. Yet, there is still a need to better under embodied noticing at a fundamental level, especially from the preservice teachers. This study used traditional and holographic video, along with eye-tracking technology, to examine how preservice teachers' physical act of looking interacts with their professional noticing. The findings revealed that many participants focused on less sophisticated forms of mathematical noticing of students' reasoning. Additionally, results from eye-tracking data suggest that the more participants described students' conceptual reasoning, the more likely they were to focus on how recorded students used their hands to engage in the mathematics.

Keywords: Teacher Noticing; Technology; Preservice Teacher Education.

Professional noticing involves attending to key pedagogical events and interpreting those events based on one's knowledge and experience(s) (Jacobs et al., 2010; van Es & Sherin, 2021). At the same time, noticing is an "embodied way of accessing, exploring, and engaging with the world of classroom events" (Scheiner, 2021, p. 88). Thus, what teachers perceive and attend to both informs and is informed by the environment they notice within, and their physiological resources (eyes, body positioning, etc.). The notion of noticing as tied to the senses is not new, with analyses and conceptual descriptions focusing both on what teachers see and hear in a classroom context (van Es & Sherin, 2002). Because of its ability to convey both realistic auditory and visual information, traditional video is the most common medium for studying noticing (Santagata et al., 2021). Yet recently, teachers' professional noticing has been examined with a new format of media that conveys more physiological information than traditional video. *Extended reality* (XR) is an umbrella term for media that blends the digital and physical worlds and has been used to examine noticing through virtual reality via 360 video (Buchbinder et al., 2021; Kosko et al., 2021) and digitized humans (Huang et al., 2021). The medium allows for novel ways for decomposing practice by not only articulating what might be attended to in a scene, but where one may look and/or listen within one's recorded environment (Weston & Amador, 2021). Rather, such media have higher degrees of *perceptual capacity*, or "a medium's capacity for aspects of the scenario to be perceivable" (Kosko et al., 2021, p. 286). This higher degree of perceptual capacity has allowed for study of how written descriptions and embodied attending (where one looks) are associated (Kosko et al., 2021; Walshe & Driver, 2019), as well as facilitating teachers' growth in noticing by discussing their physical actions associated with attending (Weston & Amador, 2021). In addition to XR-based media, scholars have also used other technology, such as eye-tracking (Huang et al., 2021) and field of view (Kosko et al., 2021) to examine physiological factors related to noticing.

This paper builds upon the emerging literature on embodied noticing by focusing on the use of holographic representations of practice (Kosko, 2022) and . Past scholarship on XR and noticing has focused predominately on 360 video, which allows for to look in any direction in an omnidirectionally recorded scenario; thus, breaking away from a fixed location such as those recorded by traditional video (Buchbinder et al., 2021; Walshe & Driver, 2019). By contrast, holographic representations convey the depth and volume of recorded subjects (Kosko, 2022). Thus, a teacher viewing a holographic student can lean in to look at them writing on their desk and also move to different sides of that student. We conjecture that this added affordance better approximates the sense of being with an actual flesh-and-blood student and may allow for a better understanding of noticing as a construct. Although there is mounting evidence for noticing as an embodied act, there is a need to better understand embodied noticing at a fundamental level. Thus, the purpose of the present study is to examine how PSTs' physical act of looking interacts with their professional noticing. To do so, we examined data from PSTs' eye-tracking and written noticing when viewing both holographic and standard video of students solving fraction division tasks.

Background Literature & Theoretical Perspectives

Professional Noticing of Children's Mathematics

Professional teacher noticing involves selecting aspects from perceived experience to attend to and interpret those aspects for the purpose of shaping the contexts and interactions around them (van Es & Sherin, 2021). As people begin their teacher education and are asked to attend to children's mathematical thinking, they initially attend to generic aspects such as student participation or their teacher's classroom management. Teachers' interpretations of these events are similarly limited to either generic descriptions of students or a focus on students' answers to the math problem (Stockero et al., 2017; van Es et al., 2017). Over time, what teachers attend to and how they interpret become more specific and student-centered. van Es et al. (2017) suggest that teachers transition to focusing on students' procedural reasoning before learning to attend to and interpret their conceptual reasoning at hand. Jacobs et al. (2010) observed a similar trend. Examining how teachers attended to children solving 43 × 6, Jacobs et al. (2010) observed some teachers described how children wrote specific numerals and added them in a certain order whereas other teachers attended specifically to children's use of partial products (i.e., 40 × 6 and 3 × 6). This led to the claim that "the skill of attending to children's strategies" (p. 193) to be of primary importance in developing teachers' knowledge for teaching mathematics.

The notion of more effective professional noticing being student-centered has found support in research examining the embodied nature of noticing. For example, Kosko et al. (2022) recorded where PSTs looked when viewing a 360 video of a lesson on the Commutative Property of Multiplication. They found that when PSTs looked more directly at students, as opposed to students being located at the edge of PSTs' field of view, they were more likely to describe students' reasoning about the Commutative Property. PSTs who looked more directly at the teacher in the 360 video were more likely to describe aspects of classroom management such as use of groups to facilitate learning. Others have observed similar trends regarding where PSTs look in viewing a 360 video and the quality of their mathematical noticing (Buchbinder et al., 2021; Weston & Amador, 2021). In addition to XR-related scholarship, use of eye-tracking technology provides additional support for the embodied nature of noticing. Results from such work suggests that more experienced, knowledgeable teachers have shorter eye-gaze durations when looking at students in a classroom, but that they look at students more often than PSTs (Dessus et al., 2016; Cortina et al., 2015). Indeed, PSTs have less focused gaze behavior to the point that though they have longer durations of focus on students, the total amount of time looking at any one student is less than more experienced teachers (Stahnke & Blömeke, 2021). Rather, experienced teachers spend more time looking at students overall, as experienced teachers attend to multiple students for short durations per gaze but for a larger number of gazes across a lesson.

Professional noticing is both a psychological and embodied act (Scheiner, 2021). To better understand how physiological factors correspond with teachers' attending and interpreting of children's mathematics, there is a need to use technology that corresponds with the use of such physiological resources. In the next section, we review aspects of immersive representations of practice and how they facilitate study of embodied noticing.

Immersive Representations of Practice

There are various "different ways that practice is represented in professional education" (Grossman et al., 2009, 2058), with traditional video being the most common in teacher education (Austin et al., in press; Christ et al., 2017). However, XR-based experiences like virtual reality and 360 video are becoming more prevalent (Austin et al., 2022; in press) and have been found to support PSTs' engagement and understanding of pedagogy (Buchbinder et al., 2021; Ferdig et al., 2020; Gandolfi et al., 2021; Walshe & Driver, 2019). XR-based representations are more immersive than traditional media because they have higher degrees of perceptual capacity and, thus, "convey [more] aspects perceivable through human experience" (Austin et al., in press, p. 2). Figure 1 juxtaposes a diagram of a user-teacher viewing a standard video on a screen versus a 360 video. For a standard video, teachers must view what is on the screen and provided to them by the videographer. For 360 video, teachers may choose to look elsewhere at a different group of students or at some other aspect in the classroom.

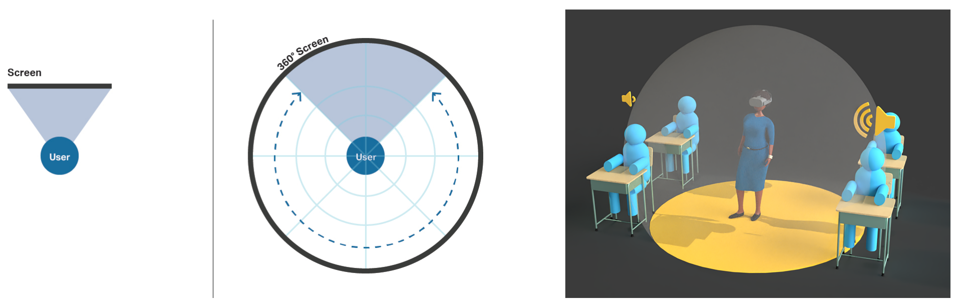


Figure 1: Traditional video versus 360 video with illustration of a teacher in VR.

XR facilitates embodied cognition, which is based on the "reactivation and reuse of processes and representations involved in perception and action" (Fincher-Kiefer, 2019, p. 10). For example, Buchbinder et al. (2021) had students record their own teaching with 360 video and then rewatch them. Rather, preservice teachers (PSTs) reflected while teaching as well as while viewing the 360 video of their own teaching. By being able to look at multiple students and themselves in the 360 video, PSTs were able to attend to "aspects of teaching pertaining to both their own teaching and to their student learning" (p. 301). The ability to look, spatially, in any direction allowed for a more nuanced connection to their lived experience of teaching the lesson than standard video would have alone (Walshe & Driver, 2019; Weston & Amador, 2021). Similar connections to embodied cognition have been made when PSTs view 360 video of others' teaching. For example, Kosko et al. (2022) observed that PSTs who looked more directly at students, and not the teacher, during the recorded class discussion were more likely to describe the mathematics concepts students were engaging. Rather, by centering students in their field of view, PSTs were more likely to describe the mathematics they were learning at a conceptual level.

Although scholarship on 360 video is promising, XR is a broad field with an ongoing turnover of technology-mediated solutions. A recent innovation that requires attention is holograms. Holographic representations of practice record "3D images onto a space" (Yoo et al., 2022, p. 2) and convey "a sense of depth and volume such that the viewer can move around and closer/further from the recorded hologram" (Kosko, 2022, 1). There is preliminary evidence that viewing holograms of students can facilitate more detailed noticing than standard video alone (Kosko et al., 2022). One potential rationale for Kosko's (2022) findings is that the volumetric aspects of the holographic recordings signal the human eye to focus on events perceived to be proximally closer. Gibson (1966) conjectured that the human eye is drawn to aspects proximally closer due partly to the distance that light travels between further and closer events. Figure 2 provides a rough illustration of how holographic representations of students may signal the eye to look more closely at the student's desk, and, by consequence, they work they are doing with the mathematics at hand. The same recording in the format of traditional video would allow light to travel from the same distance anywhere on the screen, with only the perception of movement (not proximal distance) to signal attention to the teacher's eyes. In the current paper, we used eye-tracking glasses to examine for this facet.

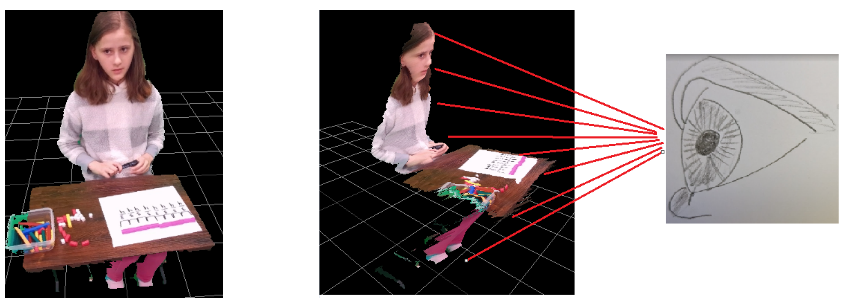


Figure 2: Perceptual distance of elements in a holographic student recorded.

More sophisticated teacher noticing involves attending to students' mathematics at a more conceptual level (Jacobs et al., 2010; van Es et al., 2017). There is evidence that such noticing corresponds with physiological data associated with looking more specifically at students (Kosko et al., 2022). In this paper, we sought to further explore this relationship by using eye-tracking technology to study patterns in where PSTs focus when observing students' mathematics, but also by incorporating holographic video to examine for the effect of spatial information. Thus, the purpose of the present study is to examine how PSTs' physical act of looking interacts with their professional noticing. To do so, we examined data from PSTs' eye-tracking when viewing both holographic and standard video of students solving fraction division tasks.

Method

Participants & Data

Participants included a convenience sample of 13 PSTs from a Midwestern U.S. university and majoring either in primary (n=11), middle grades mathematics (n=1) or secondary mathematics education (n=1). Data were collected in Fall 2022 from participants enrolled in a required educational technology course that preceded their mathematics methods coursework.

Participants predominantly identified as female (12 females; 1 male) and White (9 White; 3 Black; 1 as Asian, White, & Black)—the sole male participant identified as Black. As noted, participants were enrolled in an educational technology course. Students enrolled in this class receive course credit for participating in research studies, and each participant in this study received such credit.

Participants engaged in one-on-one sessions in which they wore eye-tracking glasses and viewed a set of recordings of two students. The eye-tracking glasses were Pupil Labs Core and include two 120Hz cameras recording pupil movement (one camera per eye) and one 30Hz camera recording in the direction the participant's head is pointed. Raw data from the glasses are transformed into viewable video with eye-gaze superimposed via the Pupil Labs software (Ehinger et al., 2019). PSTs' eye-tracking video while viewing recordings of students were a primary source of data.

Video viewed by participants included recordings of two students, Ben and Katherine, who solved tow fraction division problems using a length-based manipulative (Cuisenaire rods). The total video was 8 minutes 14 seconds long and included a scene of Ben working independently to solve the fraction division tasks of $6 \div \frac{3}{4}$ and $4 \div \frac{3}{4}$, followed by Katherine solving these independently, and then a scene with Ben and Katherine discussing how they solved $6 \div \frac{3}{4}$. For example, in solving $6 \div \frac{3}{4}$, Ben laid out 6 purple rods and 24 white cubes underneath (4 white cubes per 1 purple rod) before partitioning into four equal groups and counting all the 1s in the first three groups to find an answer of 18. Katherine also laid out 6 purple rods, but created groupings of 3 white cubes, identified each group as $\frac{3}{4}$, and counted the number of groups to find an answer of 8 (see Figure 3).

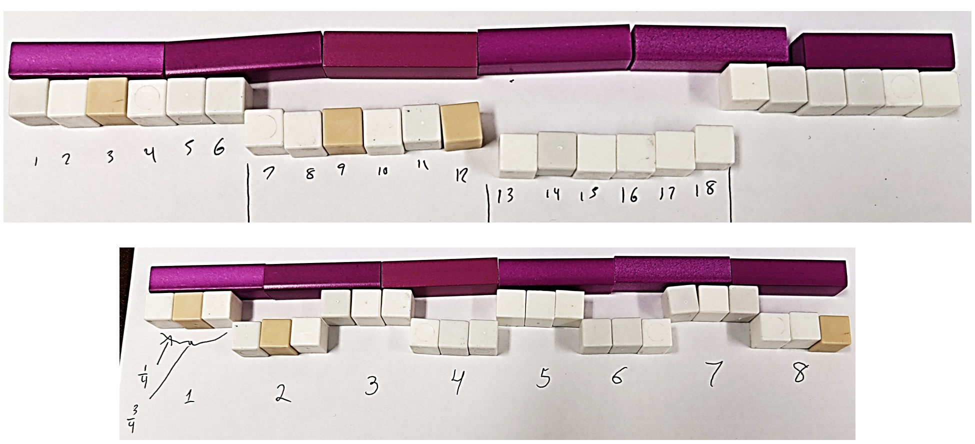


Figure 3: Illustration of Ben (top) & Katherine's (bottom) work for $6 \div \frac{3}{4}$.

The video included child actors playing the role of students using specific mathematical strategies. Both their actions and dialogue were scripted, which Herbst (2017) distinguishes as a useful feature in specific representations of practice. We used DepthKit software and Azure Kinect depth-sensing cameras to record a holographic video of the student actors. The recording can be viewed on a holographic display such as a LookingGlass Portrait (used in the present study) or as a standard video on a flatscreen device (i.e., tablet, computer screen). Figure 4 illustrates how viewing the holograms on the LookingGlass allowed for participants to move their heads to the sides to see different angles of students, and that students' use of manipulatives appeared to be physically closer than students' faces/bodies (see also Figure 2). After viewing either a standard or holographic version of the video, participants then assessed each students' fraction reasoning and were solicited to provide at least two examples of evidence from the video to justify their assessment.

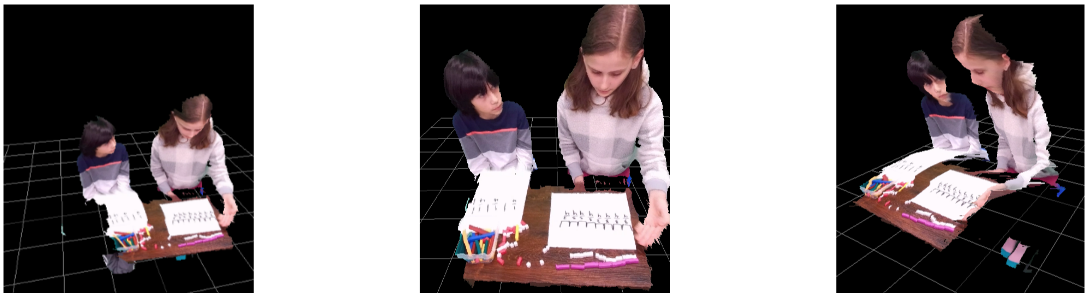


Figure 4: Scene from the holographic video from three different viewing perspectives.

Analysis & Results

Analysis in this paper focused on 13 PSTs' written noticing regarding what they attended to and interpreted of students' fraction thinking. For each participant, we also qualitatively examined videos of their eye-tracking data to examine for patterns in how and where PSTs looked when observing the recorded students. Finally, we examined for how themes from observed from written noticing and video viewing corresponded, and whether the medium of viewing affected this interaction (standard vs holographic video).

PSTs' written noticings. We used Systemic Functional Linguistics (SFL) to examine PSTs' written noticing. SFL analyzes of how grammar conveys meaning. In this study, we focused on the grammatical resource of reference chains. *Reference chains* are constructed grammatically through the repeated use of referents throughout a text. As a referent is included in text, the participant can connect it to prior references, and transform, expand, or clarify the conveyed meaning of what they reference (Eggins, 2004). In the present study, we focused on how participants referenced fractions and students' thinking. Following Kosko's (2022) approach to examining PSTs' written noticing, we used reference chains to examine the sophistication reasoning that mathematics teachers assessed and used these hierarchical themes as a priori categories in our own analysis.

Table 1 provides brief excerpts from five participants demonstrating one of four levels of sophistication. At the lowest level (Level 0), PSTs referenced a student's ability to find the answer and use procedures. For example, the participant in Table 1 references the number of "groups" in the fraction division problem, but only once and used it to support the reference chain for finding a correct answer via "solving the problem." At Level 1, PSTs' reference chains conveyed a focus on assessing children's ability to create and use fractional parts (i.e., partitioning). For example, the PST in Table 1 used the transitive processes (in bold) to "split" and "divide", but also "counted" the cubes in each group. At no point in this excerpt or the whole of this participant's writing did they reference coordination of parts to a whole but focused predominately on the child's partitioning and counting such partitions. At Level 2, PSTs referenced children's efforts to coordinate parts to the whole. The excerpt in Table 1 illustrates this with the participant's references to "groups of 3 to represent ¾" and noting that one white block represented ¼ of a purple. Such referents are significant in that they explicitly convey particular entities as relationally bound to other objects. Level 3 noticings were characterized by references to children's coordination of non-unit fractions. In Table 1, the PST at Level 3 had referenced Katherine's grouping of three white blocks into groups labeled as ¾ each. In the excerpt below, the PST focused on how Katherine counted the ¾ groups within six wholes. Thus, this participant went beyond mere reference to part-whole coordination and referenced the student's operation with non-unit fractions. In this case, it was counting how many ¾ fit in 6.

Table 1: Excerpts of PSTs' Written Noticing by Classification

Level 0 *Ben = 46.2%* *Katherine = 30.8%*	[Katherine] *was able to* **solve** the problem correctly while **explaining** the method she used to solve it. She also **understood** how many "groups" **were needed** to **solve** the problem.
Level 1 *Ben = 38.5%* *Katherine = 30.8%*	Ben **counted** each cube and **split** them into groups correctly but **did not divide** by the fraction.
Level 2 *Ben = 0.0%* *Katherine = 15.4%*	[Katherine] **knew** // she *had to* **break** the blocks up into groups of 3 to **represent** ¾ // since one small block **represented** ¼.
Level 3 *Ben = 15.4%* *Katherine = 23.1%*	Katherine **separated** the white blocks into multiple groups of ¾ and then **counted** how many groups she had to **get** the answer.

The first two authors qualitatively analyzed participants' written noticing independently before comparing findings. We used the weighted Kappa statistic and found sufficient agreement for participants' written noticing of Ben's (κ = .434) and Katherine's (κ =.590) mathematics. As indicated in Table 1, the distribution for which PSTs attended to fraction reasoning differed by the student they assessed. The bulk of participants focused on either students' answers or their partitioning for both Ben (84.6%) and Katherine (61.5%), with only two participants assessing both Ben and Katherine's coordination of non-unit fractions (Level 3).

Eye-tracking video. Next, we sought to understand where participants focused their attention. To do this, we used the Pupil Core eye-tracking glasses and recorded pupil fixation using two eye-directed cameras and one camera pointed to where the head was directed. Each participant's gaze was calibrated, with an accuracy within 0.6 degrees and we recorded video with eye-gaze dots for each participant's viewing of the standard and holographic recordings. Next, the first and third author qualitatively examined participants' eye-gaze patterns for emergent themes. Early in analysis, it was clear that PSTs viewing the holograms focused more prevalently on student work area (see Figure 5). This followed our initial conjecture that the area appearing physically closer would draw more attention by participants eye-gaze behavior. However, after an iterative process of examining for emergent themes, we noticed that some participants viewing both video and holograms followed students' hands more closely. It was whether and how participants' eye-gaze followed students' hands that mattered. Consider the center and right-hand images in Figure 5 where one participant is looking at the desk area, but not specifically following the student's hand movements (middle). By contrast, the image on the right is from a participant whose gaze went between what the student wrote (labeling 3 white rods as ¾) and the rods the labels applied. Overall, 7 participants did not follow students' hands and 6 did. Interestingly, participants whose eye-gaze followed Ben and Katherine's hands while they solved the fraction division problems tended to be the same participants who referenced either part-whole reasoning or coordination of non-unit fractions.

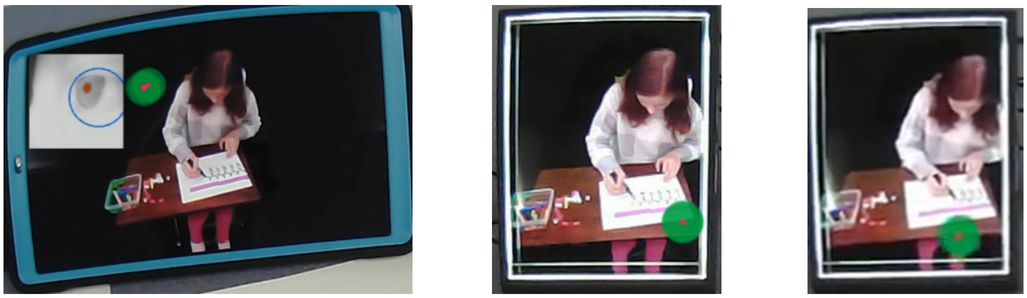


Figure 5: Two PSTs' eye-gaze not attending to student's hands while viewing standard video (left) and holographic video (middle), versus following student's hands (right).

Discussion

More sophisticated professional noticing involves attending to students' conceptual reasoning of mathematics, while less sophisticated noticing is either focused on procedures, answers, or generic aspects of classroom management (van Es & Sherin, 2021). The level of sophistication in teachers' noticing corresponds with physiological data with teachers focusing on more generic aspects of teaching looking at the classroom teacher instead of students (Kosko et al., 2022; Weston & Amador, 2021). Likewise, teachers who focus more on students tend to talk more in depth about the mathematics and students' conceptual understanding of it (Buchbinder et al., 2021; Kosko et al., 2022). Findings from the present study expand upon such scholarship by going beyond whether a teacher looks at a student and examining how they look at the student. Although many PSTs looked at Ben and Katherine's written work, only those PSTs who followed the students' hands as they coordinated the Cuisenaire rods referenced students' part-whole reasoning and coordination of the fractions.

Kosko (2022) observed that PSTs who watched holographic videos of Ben and Katherine were statistically more likely to describe their reasoning with more detail than those who watched these students with traditional video. Yet, findings here are less clear. PSTs' viewing holograms did focus more on students' desk area, but this did not necessarily translate to more sophisticated noticing either via writing or eye-tracking behavior. One potential reason for this is the sample in each study. Kosko (2022) studied PSTs late in their coursework with more experience in classroom field placements, and the sample here included PSTs with little field experience. Such a difference is worth further study. Sherin et al. (2008) argued that new technologies applied to the study and facilitation of professional noticing require study with larger samples and in varying contexts. Future work here will include a larger sample, with a focus on examining variation in its effect due to PSTs' prior field experiences.

Findings from this study are preliminary but provide additional empirical evidence for the embodied nature of professional noticing. Following Jacobs et al. (2010) and Kosko et al. (2022), findings here strongly support the notion of noticing as embodied and more effective noticing as student-centered. One clear practical implication is for teacher educators to decompose practice in video (standard, holographic, or 360) by attending to how students are coordinating manipulatives with their hands and/or how and when they write about their mathematics. Regardless, this paper builds upon prior scholarship suggesting noticing is embodied (Kosko et al., 2021; Scheiner, 2021).

References

Austin, C. K., Kosko, K. W., & Hiesler, J. L. (in press). Mathematics and science teacher educators' use of representations of practice: A mixed methods study. School Science and Mathematics Journal. http://doi.org/10.1111/ssm.12563

Buchbinder, O., Brisard, S., Butler, R., & McCrone, S. (2021). Preservice secondary mathematics teachers' reflective noticing from 360-degree video recordings of their own teaching. Journal of Technology and Teacher Education, 29(3), 279–308. https://www.learntechlib.org/primary/p/219593/paper_219593.pdf

Christ, T., Arya, P., & Chiu, M. M. (2017). Video use in teacher education: an international survey of practices. Teaching and Teacher Education, 63, 22–35. https://doi.org/10.1016/j.tate.2016.12.005

Cortina, K. S., Miller, K. F., McKenzie, R., & Epstein, A. (2015). Where low and high inference data converge: Validation of CLASS assessment of mathematics instruction using mobile eye tracking with expert and novice teachers. International Journal of Science and Mathematics Education, 13(2), 389-403. https://doi.org/10.1007/s10763-014-9610-5

Dessus, P., Cosnefroy, O., & Luengo, V. (2016). Keep your eyes on 'em all!: A mobile eye-tracking analysis of teachers' sensitivity to students. In European Conference on Technology Enhanced Learning (pp. 72-84). New York: Springer. https://doi.org/10.1007/978-3-319-45153-4_6

Eggins, S. (2004). An introduction to systemic functional linguistics (2nd ed.). London: Continuum International Publishing Group.

Ehinger, B. V., Gross, K., Ibs, I., & Konig, P. (2019). A new comprehensive eye-tracking test battery concurrently evaluating the Pupil Labs glasses and the EyeLink 1000. PeerJ, 7, e7086. https://doi.org/10.7717/peerj.7086

Ferdig, R. E., & Kosko, K. W. (2020) Implementing 360 video to increase immersion, perceptual capacity, and teacher noticing. TechTrends, 64, 849-859. https://link.springer.com/content/pdf/10.1007/s11528-020-00522-3.pdf

Ferdig, R. E., Kosko, K. W., & Gandolfi, E. (2020). Effect and influence of ambisonic audio in viewing 360 video. Journal of Virtual Worlds Research, 13(2-3), 1-14. https://jvwr.net/wp-content/uploads/2020-Assembled-The-Use-of-Ambisonic-Audio-to-Improve-Presence.pdf

Fincher-Kiefer, R. (2019). How the body shapes knowledge: Empirical support for embodied cognition. Washington, D.C: American Psychological Association. https://doi. org/10.1037/0000136-000

Gandolfi, E., Austin, C., Heisler, J., & Zolfaghari, M. (2021). Immersive presence for future educators: Deconstructing the concept of presence in extended reality environments for preservice teachers. Journal of Technology and Teacher Education, 29(3), 339-367.

Gibson, J. J. (1966). The senses considered as perceptual systems. George Allen & Unwin LTD.

Grossman, P., Compton, C., Igra, D., Ronfeldt, M., Shahan, E., & Williamson, P. W. (2009), Teaching practice: A cross-professional perspective. Teachers College Record, 111(9), 2055–2100. https://doi.org/10.1177%2F016146810911100905

Herbst, P. (2017). On dialogue and stories as representations of practice: An introduction. In R. Zazkis & P. Herbst (Eds.), Scripting approaches in mathematics education: Mathematical dialogues in research and practice (pp. 1-20). Springer. https://doi.org/10.1007/978-3-319-62692-5_1

Huang, Y., Richter, E., Kleickmann, T., Wiepke, A., & Richter, D. (2021b). Classroom complexity affects student teachers' behavior in a VR classroom. Computers & Education, 163. https://doi.org/10.1016/j.compedu.2020.104100

Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional Noticing of Children's Mathematical Thinking, Journal for Research in Mathematics Education, 41 (2), 169-202. https://doi.org/10.5951/jresematheduc.41.2.0169

Kosko, K. W. (2022). Preservice teachers' professional noticing when viewing standard and holographic recordings of children's mathematics. International Electronic Journal of Mathematics Education, 17(4), em0706. https://doi.org/10.29333/iejme/12310

Kosko, K. W., Ferdig, R. E., & Zolfaghari, M. (2021). Preservice teachers' professional noticing when viewing standard and 360 video. Journal of Teacher Education, 72(3), 284-297. https://doi.org/10.1177/0022487120939544
Kosko, K. W., Heisler, J., & Gandolfi, E. (2022). Using 360-degree video to explore teachers' professional noticing. Computers and Education, 180, 1-13. https://doi.org/10.1016/j.compedu.2022.104443
Santagata, R., König, J., Scheiner, T., Nguyen, H., Adleff, A. K., Yang, X., & Kaiser, G. (2021). Mathematics teacher learning to notice: A systematic review of studies of video-based programs. ZDM–Mathematics Education, 53 119-134. https://doi.org/10.1007/s11858-020-01216-z
Scheiner, T. (2021). Towards a more comprehensive model of teacher noticing. ZDM–Mathematics Education, 53(1), 85-94. https://doi.org/10.1007/s11858-020-01202-5
Sherin, M. G., Russ, R. S., Sherin, B. L., & Colestock, A. (2008). Professional vision in action: An exploratory study. Issues in Teacher Education, 17(2), 27-46.
Stahnke, R., & Blömeke, S. (2021). Novice and expert teachers' noticing of classroom management in whole-group and partner work activities: Evidence from teachers' gaze and identification of events. Learning and Instruction, 74, 101464. https://doi.org/10.1016/j.learninstruc.2021.101464
Stockero, S. L. Rupnow, R. L., & Pascoe, A. E. (2017). Learning to notice important student mathematical thinking in complex classroom interactions. Teaching and Teacher Education, 63, 384-395. http://dx.doi.org/10.1016/j.tate.2017.01.006
van Es., E. A., Cashen, M., Barnhart, T., & Auger, A. (2017). Learning to notice mathematics instruction: Using video to develop preservice teachers' vision of ambitious pedagogy. Cognition and Instruction, 35(3), 165-187. https://doi.org/10.1080/07370008.2017.1317125
van Es, E., & Sherin, M. (2002). Learning to notice: Scaffolding new teachers' interpretations of classroom interactions. Journal of Information Technology for Teacher Education, 10(4), 571-596. https://www.learntechlib.org/p/9171/
van Es, E. A., & Sherin, M. G. (2021). Expanding on prior conceptualizations of teacher noticing. ZDM, 53, 17-27. https://doi.org/10.1007/s11858-020-01211-4
Walshe, N., & Driver, P. (2019). Developing reflective trainee teacher practice with 360-degree video. Teaching and Teacher Education, 78, 97-105. https://doi.org/10.1016/j.tate.2018.11.009
Weston, T. L., & Amador, J. M. (2021). Investigating student teachers' noticing using 360 video of their own teaching. Journal of Technology and Teacher Education, 29(3), 309–338. https://www.learntechlib.org/primary/p/219535/paper_219535.pdf
Yoo, H., Jang, J., Oh, H., & Park, I. (2022). The potentials and trends of holography in education: A scoping review. Computers & Education, 186, 1-16. https://doi.org/10.1016/j.compedu.2022.104533

EXPLORING PROSPECTIVE TEACHERS' INTERPRETATIONS OF MATHEMATICS AND EQUITY-BASED TEACHING PRACTICES

Kathleen M. Nitta
Gonzaga University
nitta@gonzaga.edu

Keywords: Preservice teacher education, mathematics teaching practices, equity-based teaching practices, ambitious teaching

Ambitious mathematics teaching is a vision of teaching that views learners as sense-makers who bring valuable assets to a space of collaborative inquiry and where learners are given opportunities to engage in authentic, challenging tasks that support the co-construction of deep conceptual understanding (Huinker & Bill, 2017). *Principles to Action: Ensuring Mathematical Success for All* outlines research-based mathematics teaching practices intended to guide the development ambitious teaching (NCTM, 2014). In concert with these practices, the equity-based practices advocated by Aguirre et al. (2013) constitute a set of core practices that align the vision of ambitious teaching to the critical work of nurturing positive mathematical identities and developing agency in PK-12 learners (Aguirre et al, 2013; Huinker & Bill, 2017).

Preparing prospective teachers (PTs) to enact instructional practice cultivating equitable learning opportunities and rooted in principles of ambitious teaching is a challenging, multifaceted issue in teacher education. In particular, PTs need to develop understanding of the undergirding ideas of the mathematics teaching practices and equity-based teaching practices. Further, PTs need to be able to point out instructional practice that represents the enactment of those ideas. The practices are commonly developed in university coursework and the application of practice is observed in clinical experience settings. The *Standards for Preparing Teachers of Mathematics* (AMTE, 2017) highlight the importance of explicitly linking methods coursework to clinical experiences and assert that opportunities to integrate research-informed theory and pedagogical practices in PK-12 classrooms are critical to PTs' professional development.

This project is an exploratory inquiry aspiring to better understand how PTs in an elementary methods course and associated clinical experience construct meaning of the mathematics and equity-base practices. I seek to uncover connections PTs' are making between pedagogical practices discussed in methods coursework to instructional practices observed and enacted in the clinical experience. Concurrently, the project aims to reveal how PTs' express their commitment to a vision of ambitious teaching. The PTs in this inquiry were enrolled in a methods course and a corresponding clinical experience in an elementary classroom. In an end-of-course reflection assignment PTs were prompted to describe connections they made between the mathematics and equity-based practices and their experiences of teaching and learning in the clinical placement. The assignment also asked PTs to describe how their emerging identity as an ambitious teacher of mathematics was influenced by their engagement in the course and their interactions with learners in the elementary classroom. The reflection assignment served as a primary data source and was coded using the descriptive elements of the mathematics teaching practices and equity-based mathematics teaching practices. Preliminary data analysis indicates that PTs made some connection between foundational ideas of mathematics and equity-based teaching practices and instructional practices observed and enacted in the clinical experience. In my poster session I will share emerging themes from the data that point to questions for future research studies. I will also discuss PTs' interpretations of the principles of ambitious teaching.

References

Aguirre, J. M., Mayfield-Ingram, K., & Martin, D. B. (2013). The Impact of Identity in K-8 Mathematics: Rethinking Equity-based Practices. National Council of Teachers of Mathematics.

Association of Mathematics Teacher Educators [AMTE]. (2017). Standards for Preparing Teachers of Mathematics. Available online at amte.net/standards.

Huinker, D., & Bill, V., (2017). Taking action: Implementing effective teaching practices in K-grade 5 (M. Smith, Ed.). National Council of Teachers of Mathematics, Incorporated.

National Council of Teachers of Mathematics. (2014). Principles to action: Ensuring mathematical success for all. Author.

FASCINATED, SURPRISED, CURIOUS: THE ENGAGEMENT OF ELEMENTARY PRESERVICE TEACHERS IN OPEN MATHEMATICS TASKS

Emily A. Mainzer
Pennsylvania State University
ead202@psu.edu

Whole-class discussion of open mathematics tasks is an instructional practice K-12 mathematics educators report has the potential to engage all learners. Because this practice has not been extensively and systematically researched, this study aims to describe and analyze the engagement and experience of learners in open mathematics tasks. Drawing on a holistic conceptualization of engagement with behavioral, cognitive, affective, and aesthetic dimensions, the study specifically analyzes the engagement of three elementary preservice teachers as they participated in tasks and accompanying discussions in their elementary mathematics methods course. The three preservice teachers were selected because of their varied mathematical identities. While the engagement of the three focus preservice teachers varied, results suggest the openness of the tasks was an important factor in making their engagement possible.

Keywords: Instructional Activities and Practices; Affect, Emotion, Beliefs, and Attitudes; Preservice Teacher Education

In the effort to engage all learners in mathematics, a vital source of knowledge is the experiences of mathematics educators in the field. This knowledge includes instructional practices for which mathematics educators report anecdotal benefits for engagement across learners. While a practice may not yet have been formally and extensively researched, the fact that practitioners 'on the ground' are noticing a practice making an impact is reason to further explore the practice's potential. One such practice that has gained popularity in recent years is the use of open mathematics tasks for whole class discussion. These tasks—including "Which One Doesn't Belong?" (WODB, Danielson, 2016), "Notice and Wonder" (N&W, Fetter, 2021; Ray-Riek, 2013) and "How Many?" (HM, Danielson, 2018)—stand out in that they not only have multiple entry points, possible strategies, and solutions, but have ways learners can approach, interact with, and solve the tasks that an educator may not even anticipate. Practitioners and practitioner resources report that working with and discussing these tasks have positive impacts for the engagement and mathematical identities of a wide range of learners (Danielson, 2016; 2018; Illustrative Mathematics, 2021; Newell & Orton, 2019; Ray-Riek; Rumack & Huinker, 2019).

However, the field of mathematics education has yet to systematically examine these impacts and to identify characteristics of the tasks that may contribute to the impacts. In this regard, open mathematics tasks are a "black hole" of research, meaning they are "an instructional practice for which there is a scarcity of blind-peer-reviewed research evidence supporting its efficacy, yet has attained critical gravity in the teaching field" (Matney et al., 2020, p. 247). My study aims to cast light on the practice by reporting and analyzing a specific population's engagement with and experience of the tasks: preservice elementary teachers (PSTs). PSTs are a population of interest because they were recently K-12 students and because elementary PSTs often require intervention to address the narrow and sometimes negative mathematical experiences of their past (Ball, 1990). Two research questions guided my study: "How do PSTs engage with and experience open mathematics tasks?” and “What aspects of PSTs' engagement and experience can be attributed to the tasks' openness?”

Theoretical Framework

In addressing the two research questions, it is necessary to consider theories that inform how tasks play a role in influencing learners’ engagement and those that identify innate characteristics with which these specific tasks may make a connection. Three theories inform this work: sociocultural positioning theory (Hand & Gresalfi, 2015; Langer-Osuna & Esmonde, 2017), a theory of aesthetic experience (Sinclair, 2001, 2004; Wong, 2007) that stems from the work of Dewey (1934), and self-determination theory (SDT, Ryan & Deci, 2017, 2020). Sociocultural positioning theory provides a framework for considering the tasks’ role, while aesthetic experience and SDT offer ways to think about what it means to be human.

Sociocultural positioning theory highlights the role context plays in shaping events in a classroom. This context includes the way members of the classroom position one another, which, for the teacher, includes the selection and implementation of tasks. In essence, positioning theory asserts that “what someone does in a particular activity is always done in relation to what one has opportunities to do” (Hand & Gresalfi, 2015, p. 191). Tasks are one way the environment structures a learner’s opportunities.

While positioning theory provides a foundation from which to consider the role of tasks, aesthetic experience and SDT identify innate aspects of being human with which tasks may connect. Recent theoretical literature on aesthetic engagement (e.g., Sinclair, 2001, 2004; Wong, 2007) builds on ideas from Dewey (1934), who asserted that aesthetic engagement is necessary in order to live a fulfilling life. Sinclair (2001) specifically addressed educational implications of aesthetics, explaining that “aesthetically rich learning environments…legitimize students’ expressions of innate sensibilities and subjective impressions—they ‘work with’ such perceptions rather than exclude or deny them” (p. 26).

SDT’s robust empirical research base is a helpful complement to aesthetic theory’s more philosophical foundation. SDT asserts that all humans have basic psychological needs in addition to physical needs: autonomy, competence, and relatedness. Autonomy—or the extent to which a person feels regulation over their own experiences and behavior—is particularly relevant for this study. People who act autonomously engage in behaviors “whole-heartedly” (Ryan & Deci, 2017, p. 10).

Conceptual Framework

As conceptualizations of engagement vary widely, it is necessary to specify the definition of engagement I am using for this study. I draw upon the conceptualization of engagement put forth by Middleton et al. (2017):

> The in-the-moment relationship between someone and her immediate environment, including the tasks, internal states, and others with whom she interacts. Engagement manifests itself in activity, including both observable behavior and mental activity involving attention, effort, cognition, and emotion. (p. 667)

Particularly important to highlight in this definition is the consideration of engagement in the moment, the acknowledgement of internal and external factors, and the understanding of engagement as having behavioral, cognitive, and emotional components. As elaborated below, I also include aesthetic experience as a component of engagement, meaning engagement is not only a ‘taking up of’ but a being “caught up in” (Wong, 2007, p. 209).

It is also important to define what it means for a mathematics task to be open. Literature (e.g., Leikin, 2018; Mitchell & Carbone, 2011; Silver, 1995) has proposed ways to classify openness in mathematics tasks. These frameworks typically address two dimensions: number of possible strategies or approaches and number of possible solutions, with some reference to entry points and problem posing. Drawing on each of these frameworks as well as elaboration made necessary by recently emerging tasks, I propose that tasks can be open along three dimensions: multiple entry points, multiple strategies, and multiple solutions. There is also a range of extent to which tasks are open along each dimension. Figure 1 depicts this relative openness along dimensions, including the approximate openness of the four tasks used in this study (WODB, HM, N&W, and a word problem [WP] of high cognitive demand).

Entry Points		Strategies		Solutions	
	One entry point		One strategy	**WP**	One solution
WP **HM** **WODB**	Multiple but limited entry points	**WP**	Multiple but limited strategies		Multiple but limited solutions
N&W	Unbounded entry points	**HM** **WODB** **N&W**	Unbounded strategies	**WODB** **HM** **N&W**	Unbounded solutions

Figure 1: Dimensions of Openness in Mathematics Tasks

Methodology

When examining an existing practice that has anecdotally reported benefits, it makes sense to draw on multiple relevant methodologies rather than being narrowly confined to one. This pragmatic approach (e.g., Coyle, 2010; Frost & Nolas, 2011; Frost et al., 2010; Morgan, 2007) allows for strategic choice of specific methods that best capture the phenomenon in question. This study draws on three methodologies: case study, ethnography, and grounded theory. Case study (Merriam, 1998) is most strongly represented in this report, as case study involves thorough consideration of a clearly defined phenomenon such as individual PSTs’ engagement with open mathematics tasks. Ethnography (Macgilchrist & Van Hout, 2011) is relevant because of its emphasis on the participant’s perspective, which is necessary to emphasize when creating a holistic representation of engagement. Grounded theory (Glaser & Strauss, 1967) allows for new ideas to emerge rather than relying solely on previously defined constructs, which is particularly useful when considering a practice that is itself relatively new.

Context

This study was conducted at a large Mid-Atlantic public university in a semester-long undergraduate elementary mathematics methods course for PSTs. I worked with three sections of the course, all taught by the same instructor. This report focuses on the cases of three PSTs from one section of the course. There were 12 PSTs enrolled in the course. The PSTs were majority White and female; one PST was male.

Methods

The instructor of the course and I collaborated to choose four tasks for the PSTs to engage with as part of their elementary mathematics methods course. Three of the tasks (WODB, HM, N&W) did not have pre-determined answers and were of the most interest for this study. The fourth task was a word problem (WP) of high cognitive demand that allowed for multiple strategies but was implemented for the purpose of comparison as it only had one solution. Figure 1 above provides an approximate sense of the openness of each task in terms of entry points, strategies, and solutions.

The images and/or prompts for each task are included in Figure 2. The WODB task presented PSTs with four addition expressions and asked PSTs to choose which one did not belong and to explain why. Any of the four options was justifiable, and there were many possibilities as to why each one might not belong. In the HM task, PSTs were asked to consider an image of a large box of chalk. PSTs were prompted to choose what to count in the image and to be prepared to share what they counted (including the unit) and their counting strategy. For the N&W task, PSTs were shown three bulk packages of different brands of toilet paper and were asked to share what they noticed about the image and what they wondered. For the WP, PSTs needed to figure out the number of boxes of two types of cupcakes given the total number of cupcakes, boxes, and parameters as to how many cupcakes of each type could fit in a box.

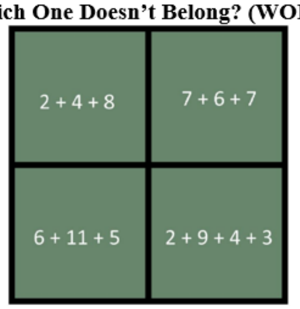
Which One Doesn't Belong? (WODB)	How Many? (HM)
	
Notice and Wonder (N&W)	**Word Problem (WP)**
	A baker made chocolate and vanilla cupcakes. He packaged the vanilla ones in boxes of 4 and the chocolate ones in boxes of 6. He made 38 cupcakes and used 8 boxes. How many boxes of vanilla and how many boxes of chocolate did he make?

Figure 2: Open Tasks Used in Study
WODB from Heon (2013); HM image from Color Splash (n.d.); N&W images from Charmin (n.d.), Quilted Northern (n.d.), Angel Soft (n.d.); WP task from Hallman-Thrasher and Spangler (2020). N&W images representative, may not match exact task.

The instructor facilitated the tasks during four separate class sessions. I attended each of these sessions to video-record and observe. For each task, the instructor presented the task and accompanying prompt. The PSTs were given 3-5 minutes to work individually. Then, the instructor facilitated a discussion of the task. The length of the discussions ranged from approximately 10-24 minutes. Following each task, participants responded to a 5-item questionnaire that asked them about their strategies and emotions during the task and discussion, the appeal of the task, and how the task compared to how they typically thought about and experienced mathematics. I also solicited volunteers to participate in a semi-structured interview at the conclusion of the semester, during which I asked participants about their overall questionnaire responses as well as their specific impressions of the tasks. During the interviews, I used video-stimulated recall to ask participants about their experience of specific parts of the task sessions. In addition to the task session recordings, questionnaire responses, and interview transcripts, I also collected identity webs PSTs completed at the beginning of the semester. For the identity web assignment, PSTs described their identity as a mathematics learner by sharing and reflecting on memories of their experiences with mathematics prior to the course.

Because practitioners have reported that these tasks engage a wide range of learners, I began my analysis with PSTs' identity webs to get a sense of their relationship with mathematics. For the analysis for this report, I aimed to select three PSTs who varied in terms of the valence of their past experiences with mathematics. I also considered variation in the number of contributions PSTs made to discussions (since this is also information that would be immediately available to practitioners). If possible, I prioritized PSTs who had participated in an interview, since the most information was available in terms of their engagement.

After selecting three PSTs, I examined their questionnaire responses—and, if available, interview transcripts—for evidence of cognitive and emotional engagement as well as aesthetic experience. I coded deductively (Maxwell, 2013) according to established constructs as well as inductively (Glaser & Strauss, 1967), using constant comparison to identify themes. Generally speaking, items coded as cognitive engagement involved how students were thinking and affective engagement involved how students were feeling (Fredricks et al., 2004; Middleton et al., 2017). Aesthetic experience included items related to curiosity, creativity, and exploration (Sinclair 2001, 2004; Wong, 2007), as well as the opportunity to express oneself (Ryan & Deci, 2017). I also made note of any reports of behavioral engagement (i.e., a PST describing what they were doing). Then, I returned to the videos and transcripts of the task sessions to code for behavioral engagement, paying attention to PSTs' actions throughout the task sessions, not only the numbers of times they contributed to the discussion or to the content of their contributions. I made note of any evidence of cognitive, affective, and aesthetic engagement as well.

Because I did not collect PSTs' individual written work or notes, I did not include these items in my analysis. These artifacts may have offered additional insight; however, as demonstrated in the results, I was able to glean considerable information about PSTs' engagement from their own reports and from my observations.

Identifying themes across each individual PST's engagement and experience from the collected data, I crafted profiles of the three PSTs, provided in the results below. Finally, I examined all of the coded items for instances that could be clearly linked to the openness of the tasks.

Results

Out of 12 participating PSTs, 10 were present at every task session. From those 10, I identified three who ranged from negative to positive valence in terms of their past experiences with mathematics and also ranged in the number of contributions they made during class discussions. The three PSTs were Andrew, Faith, and Katie (pseudonyms). Andrew presented as male, and Faith and Katie presented as female. Andrew did not choose to participate in a follow-up interview, while Faith and Katie did. The profiles of their engagement are presented below.

Andrew: Mixed Past, High Verbal Participation, Engagement Task-Dependent

Andrew's identity web reported a mix of past experiences with mathematics. He described his elementary and middle school experiences as negative—that he was "always one class behind" and one year was "awful"—but that high school math was "awesome" and in college he encountered his first math class that "made sense."

Andrew contributed verbally in three of the four task discussions and had the largest number of contributions overall (15) in comparison to his peers. In the WODB, his contributions included building on others' strategies as well as offering his own ideas both simple and complex. When not speaking, he continually looked towards the board or occasionally towards his device. During the WP discussion, he leaned back in his chair as he looked towards the board or towards who was speaking. He offered insight into what the variables would represent in solving the task with a system of equations. He did not speak during the HM discussion, and was often on his device, an activity which at times may not have been related to the task. However, he spoke frequently during the N&W discussion, offering several observations and wondering about the implications of the numbers in the image.

After all four tasks, Andrew reported that his strategy involved starting with something "easy" or obvious. Otherwise, his engagement seemed to take on a different quality in the WP and HM tasks as opposed to the WODB and N&W tasks. In his questionnaire responses following the WP and HM tasks, he mentioned evaluating others' strategies as too complicated, and, for the HM task, losing interest. He also essentially reported no emotions for these tasks. For the WODB and the N&W tasks, he reported continuing past his initial responses to consider possibilities that were less obvious and to listen to others' strategies in order to build or add on. It was also during these tasks that he reported emotions directly related to the task, feeling "challenged" by the WODB and "involved" with and enjoying the N&W task. His reports ventured into the aesthetic, explaining that the WODB was appealing because of "the ability to have your own answer" and for the N&W, that he "honestly could've kept going for awhile." During the N&W discussion, he shared that he had looked up more about the information in the task because he was "really curious," and exclaimed, "I'm fascinated by this."

Faith: "Overall" Positive Past, Low Verbal Participation, Engagement Informed by Peers' Ideas

On Faith's identity web, she stated explicitly that she "overall enjoy[s] math," using adjectives like "amazing" and "great" in describing past experiences and even recalling that she "loved" tests (including timed ones). Any negative aspects she mentioned had to do with specific teachers or struggling with emphasizing grades.

Faith contributed verbally to two out of the four task discussions and contributed to each of those two discussions once. In the HM task, she shared her strategy for counting and multiplying the rows and columns of chalk. In the N&W, she asked about the definition of a term and asserted the importance of an additional consideration. Despite relatively low verbal contributions, however, her behavior suggested she was paying attention, as she usually looked toward the board or at the person speaking, reacted by nodding or laughing, and occasionally spoke with others at her table about the task. Faith shared that she usually participates more actively but did not feel the need to speak up once someone shared the same idea that she had been thinking or when the conversation had already moved in another direction. In other words, her behavioral engagement was impacted by her peers' sharing of ideas.

Her cognitive and emotional engagement were also influenced by her peers; after the WODB, HM, and N&W tasks, she mentioned being surprised and interested by others' strategies. Her detailed recall of the strategies suggested she was cognitively as well as emotionally engaged with them. After the N&W discussion she explained she was uninterested initially but became interested once she heard how others viewed the task differently.

Faith felt most comfortable with the WP, stating she tends to think more "logically." In this regard, she set her math identity in contrast to the openness of the other three tasks. However, she acknowledged that this perception of math "as one-dimensional right or wrong" was "how it always has been presented" to her. She contrasted this description with social studies, which she described as "full of ideas and engaging conversation or different opinions." She was open to shifts in her perceptions and identity, however, as she mentioned the WODB causing her to question her "previous beliefs about math" and, in general, shared that being able to determine her own answer was "very weird and eye opening for people… like me who think logically with one set answer to like expand ideas and be like… there is no right answer."

Katie: Somewhat Negative Past, Moderately High Participation, Freed to Engage by the Absence of "Right" and "Wrong" Answers

On Katie's identity web, she made the general statement that she "always had to try harder" at math, and that it "doesn't come easily." She did not remember much about elementary school math but mentioned struggling and studying hard in middle and high school. In her interview, she explained she must be an "English and people person," as a family member had a "math and science brain" and never seemed to have to work at it like she did. She feels more of a connection with language-oriented subjects because they were more creative and open-ended.

Katie's engagement was fairly consistent across all four tasks. She raised her hand several times during all four discussions. She contributed verbally to each discussion, with a total of seven contributions. When she was not actively participating, she usually looked towards the board or at the task on her own computer screen. She sometimes wrote in her notebook, and occasionally spoke with others at her table about the task. In her interview, Katie stated that this high level of participation was not typical for her in math-related classes in general but was probably typical of her participation in this class.

Katie's cognitive engagement was evident in her verbal contributions and on her questionnaire. Her contributions included pointing out patterns in the WODB, asking about how to use variables to correctly represent the objects in the WP, sharing counting strategies and pointing out differences in numbers of colors in the HM, and sharing information she noticed in the N&W. She described her strategies in detail on her questionnaire responses and mentioned listening to others' ideas both out of interest and to compare them with her own.

Katie reported similar emotions across all four tasks, feeling motivated, content, and curious, as well as interested and excited to hear what others had to say. She initially felt confused about the WP but felt excited after "figuring it out." A key for Katie seemed to be the opportunity to share without expectation of being right or wrong and the ability to be assured of her own answer because she was the one who determined it. After the WODB, she stated, "I have always experienced math as right or wrong. This is much better and more encouraging." Even though she assessed the WP to be "a pretty average word problem," she did feel "freedom" in being able to solve the problem in her own way. She mentioned freedom to "do what [she] wanted" with the HM and that she could just "speak [her] mind" in the N&W. She specifically identified the desire to share something unique in the N&W as different from her past experiences, as the expectation to do math in a prescribed way prevented creativity.

Discussion

While each of the PSTs' engagement varied and for different reasons, the openness of the tasks played a role in how each of them engaged with and experienced the tasks.

The role of the tasks' openness can most plainly be seen in the PSTs' own references. All three PSTs mentioned having a different experience because of the ability to choose their own answer or being freed from the pressure of finding one right answer. Andrew identified this aspect as the most appealing part of the WODB task, adding that "it would have improved how [he] saw math if [he] had seen something like this" in his previous experiences. Katie explained that the N&W task was "appealing because it didn't stress [her] out or include anything [she] couldn't do." Sociocultural positioning asserts that the teacher's acts—including choice of tasks—can make certain identities available to students and not others (Langer-Osuna & Esmonde, 2017). It seems the openness of these tasks afforded the opportunity for PSTs to identify as mathematically capable. The tasks also seemed to allow PSTs to be the "arbiter" of their own mathematical experience rather than not being able to have their own "personally felt emotion guiding the selecting and assembling of the materials presented" (Dewey, 1934, p. 68).

All three PSTs mentioned the wide range of possibilities of answers, which both gave them multiple opportunities to engage with the task and made them interested to hear and engage with ideas that others would share. When I asked Faith how typical it was for her to be surprised by what others had to share in a math-related class, she stated typically she was not. Math for Faith had always been that "you have a right answer, and you have all these wrong answers," communicating that usually in math, there was no room for anything to surprise her.

The way the tasks positioned PSTs is also evident in the opportunities that PSTs had that they would not have otherwise had if the task were not open. Andrew, Katie, and Faith all talked about looking for unique or challenging ways to interact with the tasks. While this desire could be a goal as part of a math task that allows for multiple strategies (a freedom Katie felt with the WP), the multitude of possibilities in being able to determine one's own answer was a significant shift for the PSTs. In the discussion of the HM task, Katie was inspired to find something different to count after hearing another PST's idea of counting chalk that looked used versus chalk that looked unused. For Faith, the opportunity to share with such wide parameters was so unusual that she worried that the idea she shared in the HM task was somehow "cheating." In the N&W task, Andrew described his fascination with the image and took the initiative to look up more information, an action that likely would not have taken place had the task been aiming for one correct answer that could be solved with the information given. It seems that Andrew's sensibilities were awakened (Sinclair, 2001) through the process of engaging with these open tasks.

In essence, in response to the first research question regarding engagement and experience with open mathematics tasks, PSTs demonstrated engagement in all dimensions (cognitive, behavioral, affective, aesthetic) to degrees that varied by PST and to some extent by task. The WODB and N&W tasks seemed to be the most engaging across PSTs and dimensions. With regards to the second research question, PSTs' engagement with and experience of the tasks was linked to the tasks' openness. PSTs' own reports support this claim, as well as the fact that aspects of the experience to which PSTs responded would not have been possible if the tasks were not open. It is also striking that the WODB and N&W tasks—which were the most open in terms of entry points and strategies and open to a high degree in terms of number of solutions—saw considerable engagement from three PSTs who differed from one another in terms of their past experiences with mathematics.

Implications

Of course, the openness of the tasks in this study was only one of a myriad of intertwined factors shaping PSTs' engagement—aspects like the PSTs' characteristics, other PSTs, the instructor's prompts, and the content of the tasks were also influential—but the openness was, nevertheless, a factor. The possibility that the openness of these tasks prompted PSTs to fully engage in a way that contrasts with their previous experiences suggests the benefits of further research on these tasks. More generally, these results suggest that the openness of tasks in mathematics is a characteristic to be more consistently considered in task design. Finally, the fact that these results are consistent with mathematics educators' reports about learners' engagement with these tasks highlights the importance of listening to and following up on what practitioners are noticing in the field.

References

Angel Soft. (n.d.) Angel soft double roll bath tissue [Product information page]. Kroger. https://www.kroger.com/p/angel-soft-double-roll-bath-tissue/0003040079211

Ball, D. L. (1990). Breaking with experience in learning to teach mathematics: The role of a preservice methods course. For the Learning of Mathematics, 10(2), 10–16. https://www.jstor.org/stable/40247978

Charmin. (n.d.). Charmin ultra-soft toilet paper rolls [Product information page]. The Home Depot. https://www.homedepot.com/p/Charmin-Ultra-Soft-Toilet-Paper-Rolls-275-Sheets-Per-Roll-12-Mega-Plus-Rolls-003077201600/319112522

Color Splash (n.d.). Color splash jumbo sidewalk chalk assorted colors [Product information page]. Target. https://www.target.com/p/color-splash-jumbo-sidewalk-chalk-assorted-colors-126-box-sc877/-/A-85401451#lnk

Coyle, A. (2010). Qualitative research and anomalous experience: A call for interpretative pluralism. Qualitative Research in Psychology, 7(1), 79–83. https://doi.org/10.1080/14780880903304600

Danielson, C. (2016). A shapes book: Which one doesn't belong? (teacher's guide). Stenhouse.

Danielson, C. (2018). How many: A counting book (teacher's guide). Stenhouse.

Dewey, J. (1934). Art as experience. Minton, Balch, & Company.

Fetter, A. (2021, August 10). Making the shift towards sense making [PowerPoint slides]. https://files.mathematicalthinking.org/annie/FetterGSDMC2021Slides.pdf

Fredricks, J. A., Blumenfeld, P. C., & Paris, A. (2004). School engagement: Potential of the concept: State of the evidence. Review of Educational Research, 74(1), 59–119. https://doi.org/10.3102/00346543074001059

Frost, N. A., & Nolas, S. (2011). Exploring and expanding on pluralism in qualitative research in psychology. Qualitative Research in Psychology, 8(2), 115–119. https://doi.org/10.1080/14780887.2011.572728

Frost, N., Nolas, S. M., Brooks-Gordon, B., Esin, C., Holt, A., Mehdizadeh, L. & Shinebourne, P. (2010). Pluralism in qualitative research: the impact of different researchers and qualitative approaches on the analysis of qualitative data. Qualitative Research, 10(4), 441–460. https://doi.org/10.1177/1468794110366802

Glaser, B. S. & Strauss, A. M. (1967). Discovery of grounded theory: Strategies for qualitative research. Routledge.

Hallman-Thrasher, A., & Spangler, D. A. (2020). Purposeful questioning with high cognitive–demand tasks. Mathematics Teacher: Learning and Teaching PK-12, 113(6), 446-459. https://doi.org/10.5951/MTLT.2019https://doi.org/10.5951/MTLT.2019.0297.0297

Hand, V., & Gresalfi, M. G. (2015). The joint accomplishment of identity. Educational Psychologist, 50(3), 190-203. https://doi.org/10.1080/00461520.2015.1075401

Heon, H. (2013). Number 48 [Image]. Which One Doesn't Belong? https://wodb.ca/numbers.html

Humphreys, C., & Parker R. (2015). Making number talks matter: Developing mathematical practices and deepening understanding, grades 3–10. Stenhouse.

Illustrative Mathematics. (2021, September 15). Experience IM K-5 math: A focus on student thinking [webinar recording]. https://illustrativemathematics.org/experience-im-k-5-math-a-focus-on-student-thinking/

Langer-Osuna, J.M., & Esmonde, I. (2017). Identity in research on mathematics education. In J.Cai (Ed.), Compendium for research in mathematics education. National Council of Teachers of Mathematics.

Leikin, R. (2018). Openness and constraints associated with creativity-directed activities in mathematics for all students. In N. Amado, S. Carreira, & K. Jones (Eds.), Broadening the scope of research on mathematical problem solving (pp. 387-397). Springer.
Macgilchrist, F., & Van Hout, T. (2011). Ethnographic discourse analysis and social science. Forum Qualitative Sozialforschung / Forum: Qualitative Social Research, 12(1). http://dx.doi.org/10.17169/fqs-12.1.1600
Matney, G., Lustgarten, A., & Nicholson, T. (2020). Black holes of research on instructional practice: The case of number talks. Investigations in Mathematics Learning, 12(4), 246-260. https://doi.org/10.1080/19477503.2020.1804273
Maxwell, J. A. (2013). Qualitative research design: An interactive approach (3rd ed). SAGE.
Merriam, S. B. (1998). Qualitative research and case study applications in education. Jossey-Bass Inc.
Middleton, J. A., Jansen, A., & Goldin, G. (2017). The complexities of mathematical engagement: Motivation, affect, and social interactions. In J. Cai (Ed.), Compendium for research in mathematics education. National Council of Teachers of Mathematics.
Mitchell, I. & Carbone, A. (2011). A typology of task characteristics and their effects on student engagement. International Journal of Educational Research, 50(5-6). https://doi.org/10.1016/j.ijer.2011.05.001
Morgan, D. L. (2007). Paradigms lost and pragmatism regained. Journal of Mixed Methods Research, 1(1), 48–76. https://doi.org/10.1177/2345678906292462
Newell, C., & Orton, C. (2018). Classroom routines: An invitation to discourse. Teaching Children Mathematics, 25(2), 94-102. https://doi.org/10.5951/teacchilmath.25.2.0094
Ray-Riek, M. (2013). Powerful problem solving: Activities for sense making with the mathematical practices. Heinemann.
Rumack, A. M., & Huinker, D. (2019). Capturing mathematical curiosity with notice and wonder. Mathematics Teaching in the Middle School, 24(7), 394-399. https://doi.org/10.5951/mathteacmiddscho.24.7.0394
Ryan, R. M., & Deci, E. L. (2017). Self-determination theory. The Guilford Press.
Ryan, R. M., & Deci, E. L. (2020). Intrinsic and extrinsic motivation from a self-determination theory perspective: Definitions, theory, practices, and future directions. Contemporary Educational Psychology, 61. https://doi.org/10.1016/j.cedpsych.2020.101860
Silver, E. A. (1995). The nature and use of open problems in mathematics education: Mathematical and pedagogical perspectives. ZDM-International Journal of Mathematics Education, 27(2), 67–72.
Sinclair, N. (2001). The aesthetic is relevant. For the Learning of Mathematics, 21(1), 25–33. https://www.jstor.org/stable/40248343
Sinclair, N. (2004). The roles of the aesthetic in mathematical inquiry. Mathematical Thinking and Learning, 6(3), 261–284. https://doi.org/10.1207/s15327833mtl0603_1
Quilted Northern (n.d.). Quilted northern ultra plush toilet paper [Product information page]. Walmart. https://www.walmart.com/ip/Quilted-Northern-Ultra-Plush-Toilet-Paper-6-Mega-Rolls/605956796
Wong, D. (2007). Beyond control and rationality: Dewey, aesthetics, motivation, and educative experiences. Teachers College Record, 109(1), 192–220. https://www.tcrecord.org/Content.asp?ContentId=12740

FIRST IMPRESSIONS MATTER: AN ANALYSIS OF PROSPECTIVE TEACHERS' NOTICING OF CURRICULUM MATERIALS

Scott S. Block	Kelsey Quaisley	Lorraine M. Males
University of Nebraska - Lincoln	University of Nebraska - Lincoln	University of Nebraska - Lincoln
scott.block@huskers.unl.edu	kquigley@huskers.unl.edu	lmales2@unl.edu

Given the ubiquity of curriculum materials and complexity of their usage, it is imperative that teacher education programs prepare prospective teachers (PSTs) to use curriculum materials. In this paper, we focus on what PSTs notice when they are interacting with curriculum materials, and how their initial impressions of curriculum materials influence their later understandings of curriculum materials. We found that PSTs' 20-second impressions may be indicative of their longer impressions of curriculum materials, which can include their preferences, values, beliefs, and approaches to using curriculum materials. We suggest that teacher educators expose PSTs to a variety of curriculum materials to better support PSTs in planning and enacting lessons.

Keywords: curriculum; preservice teacher education; teacher noticing; instructional activities and practices

Research indicates that teacher education programs need to do more to prepare teachers to learn to use curriculum materials in adaptive and flexible ways (Drake et al., 2014). Given that 88% of mathematics teachers reported having a designated commercially published textbook (Banilower et al., 2018) and that research on teacher-intended curriculum with prospective secondary teachers (PSTs) emphasizes the complexity of using materials (e.g., Lloyd & Behm, 2005; Van Zoest & Bohl, 2002), it is imperative that teacher educators support PSTs in learning to interact with curriculum materials. Furthermore, AMTE (2017) stresses the importance of preparing beginning teachers of mathematics to plan and use curriculum materials (Indicators C.2.2 and C.1.4) and for programs to provide opportunities to learn to teach mathematics (Standard P.3).

To leverage the potential of curriculum materials, Drake et al. (2014) advocate for PSTs to have more experiences reading, interpreting, analyzing, and making decisions about how best to use a variety of materials. This requires teacher educators to design experiences that draw on PSTs' knowledge and dispositions towards curriculum and extends this to productively engage them in interacting with materials. Our study is part of a larger research project intended to understand how PSTs learn to use curriculum materials and to develop methods for productively engaging PSTs in this learning. In this paper we focus on what PSTs notice when they are interacting with curriculum materials, and how their initial impressions of curriculum materials influence their later understandings of curriculum materials.

Theoretical Framing

We frame our study using two complementary theoretical perspectives, the perspective that there is a participatory relationship between teachers and curriculum materials (Remillard, 2005) and that teachers generate documents through documentation genesis (Gueudet & Trouche, 2009). To describe the ways in which teachers participate with instructional materials and develop documents through the processes of instrumentation and instrumentalization (Pepin, Gueudet, & Trouche, 2013) we use the Curricular Noticing Framework (Authors, year), which is informed by the work in the professional noticing of children's mathematical thinking (Jacobs, Lamb, & Philipp, 2010). Curricular noticing is comprised of three interrelated skills: Curricular Attending, Curricular Interpreting, and Curricular Responding, which "enable teachers to recognize, make sense of, and strategically employ opportunities available within their curriculum materials" (Authors, Year, p. x). Figure 1 depicts the relationships between these frameworks.

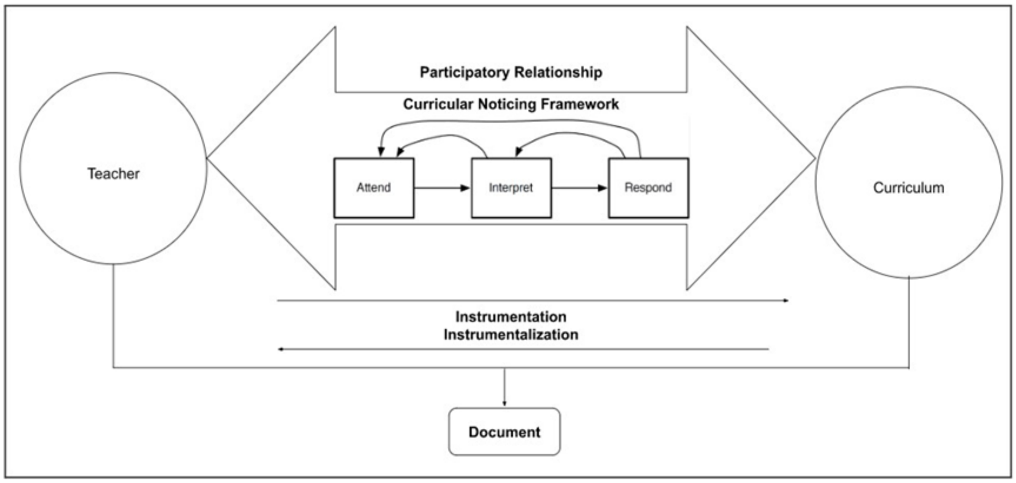


Figure 1: Relationships between Theoretical and Analytical Frameworks

Purpose of the Study

Our study focuses on PSTs' noticing of different sets of curriculum materials and aims to inform the work of teacher educators in developing ways to support PSTs to use curriculum materials with intentionality. We address the following questions:

1. What do prospective teachers notice when given the first page of a lesson from two different sets of curriculum materials for a short amount of time (i.e., what are their first impressions)?
2. How might prospective teachers' first impressions be related to their later understanding of the full lessons from both sets of curriculum materials?

Methods

Participants & Data Collection

The participants of the study were six PSTs enrolled in a 6-12 mathematics teacher certification program, either not yet enrolled in or enrolled in the first of two mathematics teaching methods courses. Note that all names are pseudonyms, and, in the results, we use the pronoun they to refer to each PST.

Each PST participated in a semi-structured interview which lasted approximately 45 minutes. We video-recorded the interviews using both a regular video camera facing the PST and eye-tracking glasses that the PST wore. See Figure 2 for images from the recordings. Note that the red circle indicates the PSTs gaze (I.e., what they were looking at).

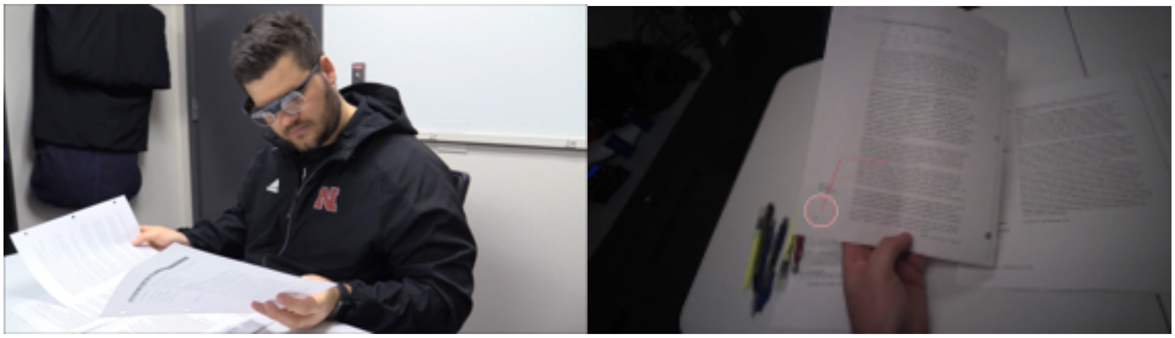


Figure 2: Images from the Video Recording from the Face Camera and Glasses

To address our research questions, we conducted two parts of the interview where we gave PSTs materials and asked them to talk aloud. For the first part, we gave them 20 seconds to describe what they noticed when we gave them the first page of a lesson from CPM Algebra Core Connections (Dietiker et al., 2014) [CPM] and then another 20 seconds when we gave them the first page of a lesson from Meaningful Math Algebra 1 (Fendel et al., 2014) [MM]. After each 20-second exposure, we asked each PST what they noticed. During the second part of the interview, we gave them an unlimited amount of time to look at the full lessons and compare. We then asked questions about what they noticed and their comparisons, including what they imagined might be supportive of both students and teachers.

Data Analysis

We imported the video produced by the eye tracking glasses into Tobii Labs analysis software. The eye-tracking glasses allowed us to see what PSTs were looking at during the semi-structured interviews and then utilize software to visualize what and how long they looked at various items (i.e., heat maps) and the order in which they looked at them (i.e., gaze plots). We also utilized the eye tracking video to open-code the verbal data--"identifying any segment that might be useful" (Merriam & Tisdell, 2016, p. 204), writing summaries of what PSTs noticed during both sections of the interview, and then consolidating our summaries into themes.

Results

Following analysis, we noticed connections across PSTs' 20-second exposures, their untimed comparisons, and the portion in which we asked them follow-up questions. Notably, PSTs had ideas in the initial 20-second portion of the interview that remained consistent or grew stronger in subsequent portions of the interview. Table 1 summarizes our findings.

Table 1: Summary of Findings

PST	Part I: 20 seconds		Consistencies Found between Part 1 & Part 2
	CPM	MM	
Eddy	Instructional; Student tasks and activities; Organized; Clear objective; Student engagement	Highlighted key concepts and vocabulary; Supportive of teachers; Learning goals; Instruction manual	Instruction manual; Student activities
Cam	Lesson goals	Organized; Lesson goals	Lesson goals
Jim	Organized; Wordy	Organized; Easy to read	Wordy; Easy to read

Andy	PST summarized lesson; Wordy; Difficult to discern what's going on; No visuals	PST summarized lesson; Lesson goal; Key words	Wordy; Visual representations; Key words
Mike	Clear lesson goals; Summary of the lesson; Lesson length	Clear; Not distracting; Concise; Easy to read	Clarity; Concise structure
Ken	Objective; Lesson length; Standards; Suggested activities	Lesson length; Materials needed; Lesson description	Lesson components & Structure

For each PST, we found consistencies between their first impressions (i.e., the 20-second exposure) and their later noticings. Some PSTs attended to formatting in the curriculum materials, either commenting on their own preferences or how different curriculum materials were formatted. Aspects of formatting that PSTs mentioned involved features such as how pages were organized, the number of words that they noticed, and how readable they felt the materials were. Specifically, PSTs noted features such as vocabulary words, the visuals that were present, and lesson goals/objectives/intent. In addition, we found that some PSTs were consistent in their noticings with respect to how they wanted to use the curriculum materials and the manners in which the lessons were organized and/or structured.

Overall, we found that each PST had at least one idea that they voiced during the first portion of the interview and preserved throughout the rest of the interview. Their idea was often something that they elaborated on in the later portion of the interview without prompting, as well as something that they emphasized as important to them when we asked them questions. In the next sections we describe the findings in more detail for each PST.

Eddy

Eddy's focus was on looking for activities students will do and how they will do them. They mentioned looking for "explicit instruction" and guidance related to leading discussions with students to support student learning. During the 20-second portion of the interview, Eddy's heat map indicated significant time spent on the CPM section about the suggested lesson activity, aligning with their desire to know what the teacher might be doing and how to develop the experience for students. Additionally, Eddy stated that MM, "read more like an instruction manual" during the 20-second portion of the interview. While comparing materials, Eddy stated:

> Um, this (CPM) lesson plan seems to not, uh, maybe I just haven't read it yet, but doesn't give like explicit instruction on like how the discussion is happening, whether it's like as a class or individually or like with a partner. And this one (MM) is a little more explicit and like having students work in groups and then later come together as a class. But I mean, it doesn't say that you can't do that, but this one seems very, uh, more, uh, straightforward in terms of just following instructions.

Cam

Cam noticed the goal/objective of each lesson in the initial 20-second portion of the interview, while reading more thoroughly, and after reading. After reading, Cam described how CPM and MM approached representing relationships, where they both did this, and how they had slight differences in their respective approaches. Cam mentioned CPM having more suggested activities for teachers, while MM seemed more formalized for establishing procedures and getting students to interpret what was happening in each visualization. During the 20-second exposure, Cam's gaze plot for CPM indicated looking at the lesson objective, moving to other portions of the page, and then re-visiting the lesson objective. This suggests Cam may have engaged in sense-making while they were reading and potentially re-visited what they felt was important to understand the lesson. During questioning at the end of the interview, Cam talked about MM and how it aligned with what they felt was the intent of the lesson, saying:

> I think I mentioned this in an earlier question, but the intent it (MM) says it's setting the groundwork for students to learn techniques of graphing and making connections and graphs. So, it's not about regular rigorously graphing functions yet. It's about being able to look at a graph and get a general idea of what's going on and what the graph represents maybe without, you know, specific data points necessarily.

Jim

A prominent finding throughout Jim's interview was their focus on how "wordy" curriculum materials appeared. During the initial 20-second portion of the interview, Jim identified CPM as "wordy". In the latter portion of the interview, Jim noted that CPM "being so wordy can cause some confusion." Jim continued to emphasize wordiness after reading, describing how easy it was to read the materials and even comparing CPM's ease of reading to that of MM. Eye-tracking picked up on this theme of wordiness as well, as heat maps indicated flickering attention to the initial first page in CPM and more sustained reading in the first page of MM (see Figure 3).

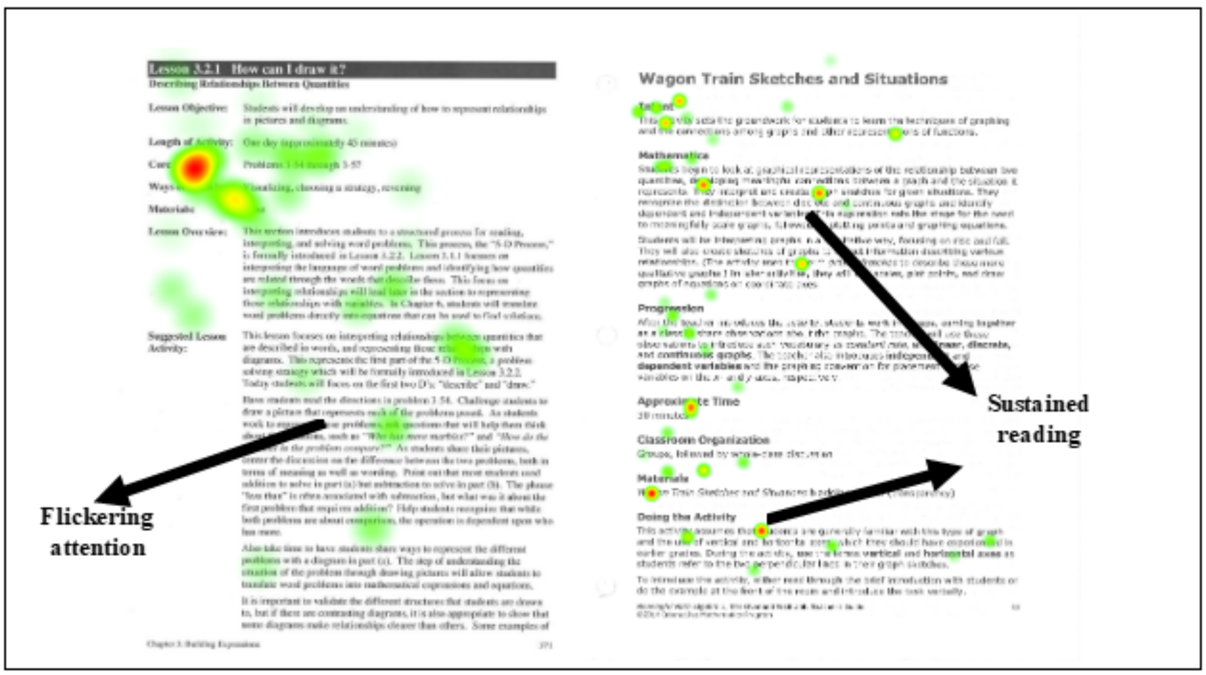


Figure 3: Heat Map of Jim's 20-second Exposure to CPM (left) and MM (right)

Andy

Wordiness was also attended to by Andy, as they stated after seeing CPM initially during the 20-second portion that it "was very wordy and it was hard to discern what was actually going on. It's kind of, yea, it wasn't easy to, there wasn't any pictures or anything. Okay. So, it was hard to figure out what's going on." Andy's challenges related to making sense of the lesson in CPM contrast with their sense making of MM. While comparing CPM and MM, Andy stated that MM is, “a lot better than...” CPM because MM “...has questions for me and minimal text to talk about it,” whereas CPM is “too wordy.” Andy’s heat maps contrasted with Jim’s, even though they both had similar conclusions related to CPM being too wordy. Andy’s heat maps suggested a focused attention on one portion of the CPM text, possibly engagement in sense-making. We did not see with this MM, but instead saw smaller heat centers across the page, likely indicating sustained reading (see Figure 4). Large clusters of focused attention may be related to Andy’s impressions of wordiness and challenges to make sense of the curriculum material.

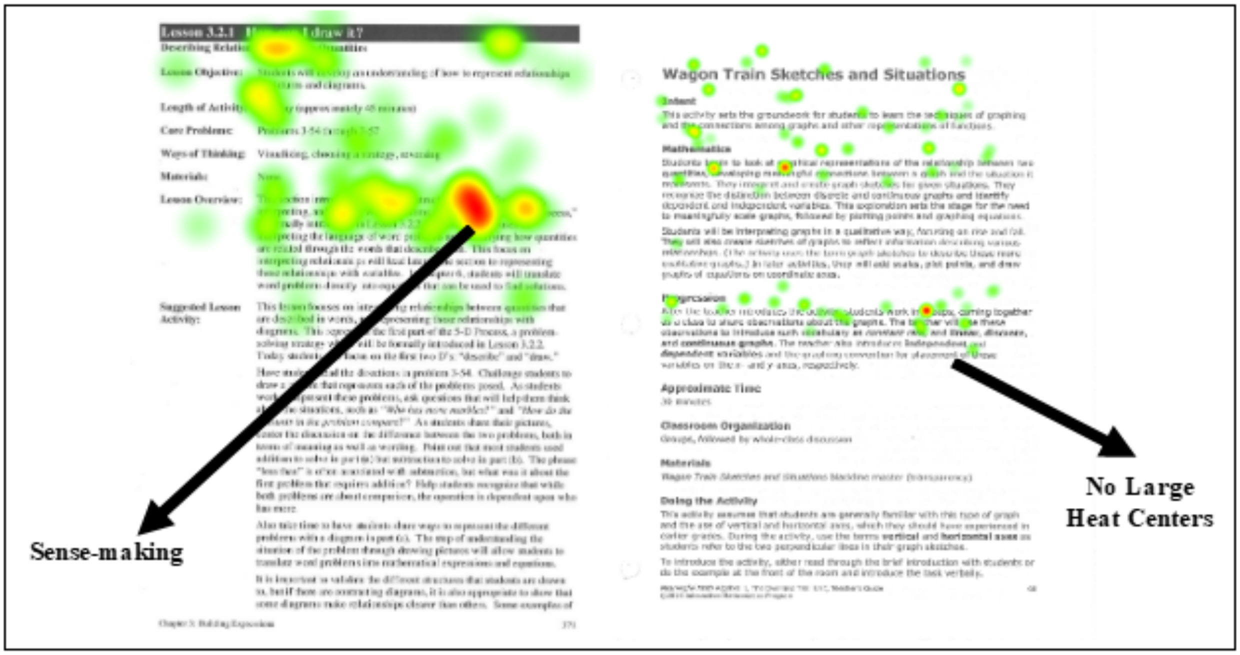


Figure 4: Heat Map of Andy’s 20-second Exposure to CPM (left) and MM (right)

In addition to wordiness, Andy also attended to key words. Andy noticed key words in MM during the initial 20-second portion of the interview, while comparing the materials, and later in the interview. Andy stated how helpful they found key words and that they find them to be something they need most in a lesson. Andy went on to state:

> I think (what) I need the most is just to be able to construct a lesson plan (that) is okay. The key words, the key terms, the key, uh, like discrete, continuous, dependent, independent, uh, words like that...The wagon train problem, um, or the questions are good examples and the key terms are good to keep in mind.

Mike

During Mike’s interview he noticed structure. In the 20-second portion of the interview, Mike emphasized a preference for clarity related to the lesson objective and concise summaries. Later on in the interview, Mike continued to emphasize structure and indicated that MM’s conciseness was more useful when preparing to teach a lesson, as compared to CPM. Mike’s gaze plots provided further evidence of this, indicating a linear progression through the CPM page, but returning to core problems and the lesson objective after already noticing the suggested activities section on the page. Mike’s gaze plot for MM, though, was completely linear with no visual revisiting (see Figure 5). Mike's gaze plots may be a sign that conciseness, clarity, and ease of reading can be indicated through less visual revisiting.

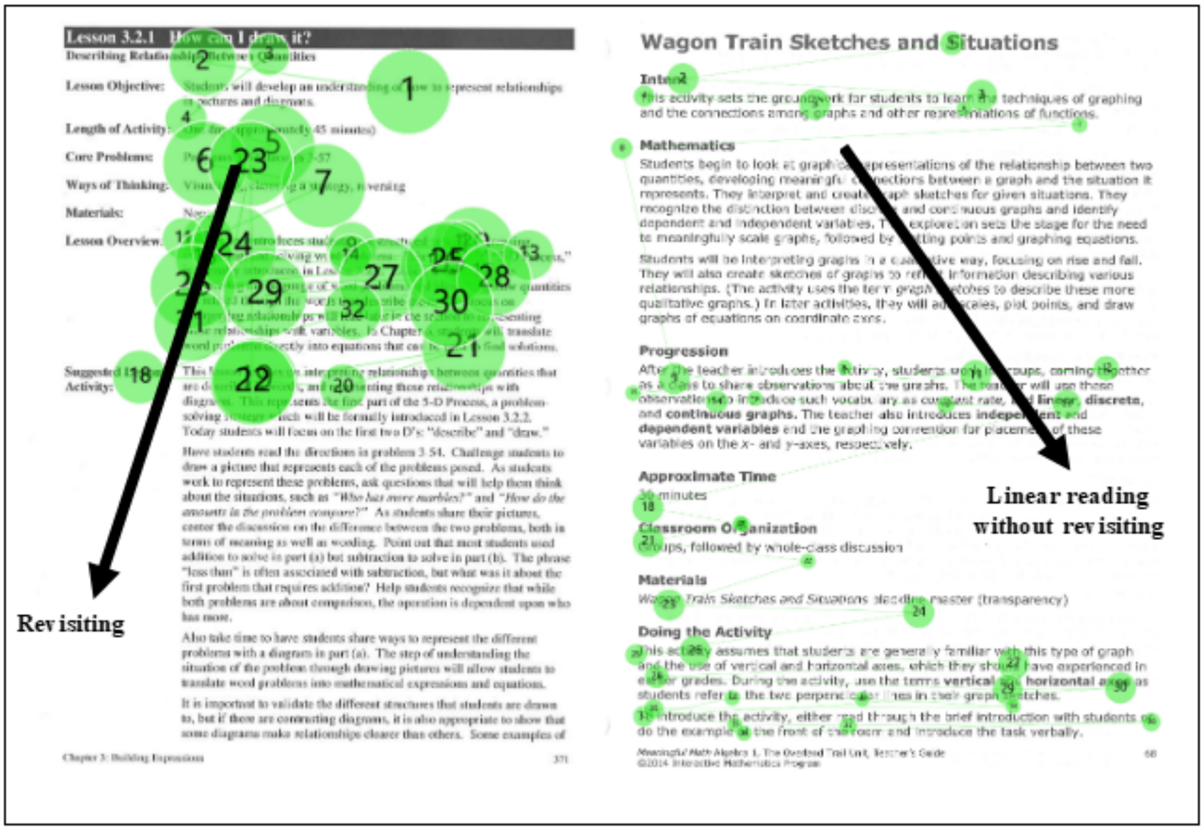


Figure 5: Gaze Plot of Mike's 20-second Exposure to CPM (left) and MM (right)

Ken

Ken noticed structure as well but had a different sense than Mike as far as what it meant. Ken emphasized an ability to identify various lesson components, such as lesson length, lesson objective, required materials, and what the teacher and students would be doing as indicated both verbally and with the eye-tracking glasses. In the latter portion of the interview, Ken was asked about what seemed important while comparing the lessons. Ken responded:

> Uh, the structure. So, like what would the teacher be doing was like the big thing, and like it was easy to find out what the teacher's doing in this one (CPM), cause, like, writes all of it down. But this one (MM), I'm still kind of trying to piece together what the teacher's exactly supposed to do.

Ken's sense of structure in identifying lesson components indicated a preference towards CPM, stating that MM's structure was confusing and that they would, "have to read it like word-for-word...(to) see exactly what the teacher is supposed to do."

Discussion

PSTs search for specific things when looking at curriculum materials related to their own preferences. When interpreting curriculum materials, they approach the materials in various manners, looking for key words or concepts, as well as how they might facilitate or enact a lesson, treating the curriculum materials like an instruction manual. What PSTs look for, or attend to, when they look at curriculum materials is noticeable through eye-tracking technology, as indicated specifically by gaze plots and heat maps. Through interpretations of these tools, PSTs preferences and approaches are discoverable, as shown in features such as sustained focus on one area, revisiting other areas, rapid gazes, and linear progression through the materials. Overall, PSTs' 20-second impressions may be indicative of longer impressions of curriculum materials, which can include their preferences, values, beliefs, and approaches to using curriculum materials.

In the work of preparing PSTs to engage students and support learning, it is imperative that PSTs familiarize themselves with various curriculum materials. Awareness of different curriculum materials and familiarity with them can help PSTs understand various approaches to the teaching and learning of mathematics, as well as how they might plan for and enact lessons. In other words, preparing PSTs to plan with curriculum materials is essential to the future work of engaging students in their classrooms.

This work of exposing PSTs to different curriculum materials is important because PSTs make initial judgements which can be developed and sustained over time. Teacher educators have opportunities to discuss what PSTs notice in a safe, low-stakes environment, such as a methods course. Taking the time to do this work is important because PSTs have beliefs, values, and orientations to materials, which may or may not be aligned with engaging and supporting each and every mathematics learner. Helping PSTs work with curriculum materials might assist them in thinking through their personal judgements before making decisions related to curriculum materials, as well as help them plan lessons efficiently and effectively.

Acknowledgements

We want to thank each of the PSTs for participating in the study. This material is based upon work supported by the CPM Educational Program (#ANONYMIZED) and National Science Foundation (#ANONYMIZED). Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

References

Association of Mathematics Teacher Educators (AMTE). (2017). Standards for preparing teachers of mathematics. Available online at http://amte.net/standards.

Authors. (Year).

Banilower, E. R., Smith, P. S., Malzahn, K. A., Plumley, C. L., Gordon, E. M., & Hayes, M. L. (2018). Report of the 2018 NSSME+. Retrieved from Chapel Hill, NC: Horizon Research, Inc.

Dietiker, L., Kysh, J., Sallee, T., & Hoey, B. (2014). Algebra core connections. Sacramento, CA: CPM Educational Program.

Drake, C., Land, T. J., & Tyminski, A. M. (2014). Using educative curriculum materials to support the development of prospective teachers' knowledge. Educational Researcher, 43, 154-162. doi:10.3102/0013189x14528039

Fendel, D., Resek, D., Alper, L., & Fraser, S. (2014). Meaningful Math Algebra 1. Greenwich, CT: Activate Learning.

Gueudet, G., & Trouche, L. (2009). Towards new documentation systems for mathematics teachers? Educational Studies in Mathematics, 71, 199–218. doi:10.1007/s10649-008-9159-8

Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. Journal for Research in Mathematics Education, 41(2), 169-202. doi:10.5951/jresematheduc.41.2.0169

Lloyd, G. M., & Behm, S. L. (2005). Preservice elementary teachers' analysis of mathematics instructional materials. Action in Teacher Education, 26(4), 48-62. doi:10.1080/01626620.2005.10463342

Merriam, S. B., & Tisdell, E. J. (2016). Qualitative research: A guide to design and implementation (4th ed.). San Francisco, CA: Jossey-Bass.
Pepin B, Gueudet G, Trouche L. (2013) Re-sourcing teachers' work and interactions: a collective perspective on resources, their use and transformations. ZDM 45, 929–944. doi:10.1007/s11858-013-0534-2
Remillard, J. T. (2005). Examining key concepts in research on teachers' use of mathematics curricula. Review of Educational Research, 75, 211-246.
Van Zoest, L. R., & Bohl, J. V. (2002). The role of reform curricular materials in an internship: The case of Alice and Gregory. Journal of Mathematics Teacher Education, 5, 265-288.

HOW INSTRUCTORS OF UNDERGRADUATE MATHEMATICS COURSES MANAGE TENSIONS RELATED TO TEACHING COURSES FOR TEACHERS?

Amanda Brown
University of Michigan
amilewsk@umich.edu

Patricio Herbst
University of Michigan
pgherbst@umich.edu

Mike Ion
University of Michigan
mikeion@umich.edu

For centuries, there has been a debate about the role of undergraduate education in society. Some have argued that universities should focus on practical skills and knowledge to prepare students for the workforce, while others have supported the idea that universities should prioritize providing a broad understanding of disciplinary knowledge and practices. In this paper, we leverage data collected from 32 interviews to explore how instructors of the undergraduate geometry course for teachers (GeT) talk about the various tensions they experience in their work. Three distinct ways of talking about tensions emerged from the data: the tension as a dilemma that needs to be managed, the tension as a place to take sides, the tension as an opportunity to reframe aspects of the work. In closing we draw connections between these patterns in the data and the two perspectives about the role of undergraduate mathematics courses in preparing PTs for the work of teaching.

Keywords: Undergraduate Education, Geometry and Spatial Reasoning, Preservice Teacher Education, Teacher Educators

This research has been founded on a reasonable conjecture that more knowledgeable teachers would be better prepared to lead instruction of higher mathematical quality and that the latter outcome should result in better mathematics learning from students. Efforts to flesh out that conjecture have produced substantial progress in refining conceptualizations of the phenomena involved, particularly teacher knowledge and its indicators. While at one time, teacher knowledge of mathematics was thought of as only disciplinary knowledge and indicated with degrees achieved or courses taken, research using those operationalizations of the construct have shown conflicting results (Begle, 1979; Monk, 1994). Also, a mathematics degree or the accrual of mathematics credits have not been reliable predictors of success in teaching (Hill, 2012; McDiarmid & Wilson, 1991). This has led scholars to promote a conceptualization of the mathematical knowledge needed for teaching based on an analysis of the recurrent work of teaching (Ball, Lubienski, & Mewborn, 2001). Relatedly, there has also been groups of scholars reconsidering the ways that undergraduate mathematics courses might be adapted to better meet the needs of prospective teachers (PTs) by equipping them with such knowledge (Adler et al., 2014; Appova et al., 2014; Speer & Wagner, 2009; Wasserman et al., 2022). While these two bodies of scholarship have done much to advance the field's conceptualizations of the knowledge needed for teaching and the role that undergraduate mathematics courses might potentially play in developing that knowledge, there has yet to be an adequate accounting of the perspectives of instructors who teach these undergraduate mathematics courses regarding these conceptualizations (Lai, 2019).

In this report, we build on previous work in which we drew on interviews conducted with 32 GeT instructors (Herbst et al., 2023) to report on how instructors perceive their position and the work they do in the GeT course in relation to institutional stakeholders. In those previous reports, we characterized the tensions that we detected beneath instructors' descriptions of the GeT courses they teach. The 5 tensions that emerged in those reports can be understood as the dilemmas that can result from an instructor's consideration of the following questions:

1. The **content tension**: What is the content students need to learn in the GeT course?
2. The **experience tension**: What experiences can support students' learning in the GeT course?
3. The **students tension**: Who are the students that populate the GeT course and what do they need and want from the course?
4. The **instructor tension**: Who am I and how have my experiences prepared me to teach the GeT course?
5. The **institutions tension:** Which institutions might benefit from, and support GeT improvement?

In the section entitled *Prior Results*, we elaborate on these tensions by illustrating ways that the answer to these questions can present instructors with a dilemma (Berlak & Berlak, 1981/2011; Lampert, 1985; Elbow, 1983)—forcing the GeT instructors to choose between two different courses of action, both of which are problematic for different reasons. These two distinct courses of action help illustrate the poles of the tension (or horns of the dilemma). More than simply identifying and defining those tensions according to their poles, our prior work accounts for these tensions as emerging from two distinct ways of organizing the work specific to teaching undergraduate mathematics courses for teachers: undergraduate mathematics education versus secondary mathematics teacher preparation.

In those prior reports, our focus was primarily on identifying the tensions, in terms of their poles, for the group of instructors. In this report, we leverage the same data corpus to explore the variability in the ways that different instructors experience these tensions—as a ***dilemma that needs to be managed***, as a place to ***take sides*** by identifying with one pole of a particular tension, or as an opportunity to ***reframe aspects*** of their work by satisficing the demands from both poles of the tension. The study was guided by the following questions: What are the different ways that instructors relate to the tensions in their work that can be detected in the ways that instructors talk about those tensions? How can these differences be accounted for in terms of the ways of relating to the ongoing debates regarding the role of undergraduate mathematics courses in preparing PTs for the work of teaching?

Prior Results

In our previous work (Herbst et al., 2023), we have detailed five tensions that emerged from our interviews with instructors. Here, we provide a very brief description of those tensions to provide the necessary background for engaging with the results of this paper. The ***content tension*** describes the kinds of challenges an instructor might face when it comes to making decisions about the content addressed within a GeT course. On the one hand, instructors must consider the type of knowledge that future high school geometry teachers need to acquire to be effective in their roles. On the other hand, these courses also serve as an opportunity to expose mathematics majors to centuries of geometric research, including advanced ideas and evolving approaches to posing and addressing geometric questions. This creates a tension in course design as instructors must navigate the expectations of both groups of students and find a way to provide meaningful and relevant instruction to all.

The ***experiences tension*** relates to the types of practices students enrolled in the course are expected to be apprenticing into and the ways that their experiences in the course accommodate that apprenticeship. As an undergraduate mathematics course, students enrolled in the GeT course could reasonably be expected to be engaged substantively in mathematical practice valued in the disciplines: thinking about mathematics, doing mathematical activities (such as conjecturing, proving) and learning to communicate about mathematics according to disciplinary conventions (such as the process of publishing or writing in LaTeX). However, since these courses are also service courses for PTs, it is reasonable to expect that students will have the opportunity to apprentice into the work of teaching (such as learning how to address the entire class, pose questions to students, and respond to student contributions and inquiries). Similar to the content tension, our data suggests that attending to both kinds of practices may be challenging for instructors of the GeT course.

The ***students tension*** arises from the instructors' need to consider the diverse group of individuals enrolled in the course and how the instruction can cater to their individual needs. While the course is often purportedly offered to satisfy the needs of PTs, it also includes students pursuing different majors, such as pure mathematics, physics, or engineering. PTs have unique needs and expectations for the course. For instance, it has been suggested that they require explicit discussions on how the mathematical ideas and practices learned in the course are connected to the high school geometry course (Kilpatrick, 2019). Furthermore, it has been recommended that PTs would benefit from clear discussions on the pedagogical practices used to support mathematics teaching and learning (Wasserman et al., 2022). However, other students have conflicting expectations for the course, such as the hope that it would provide opportunities to learn new geometric content useful beyond the requirements of any specific profession.

The ***instructor tension*** highlights the challenges that instructors face in preparing to teach a course that requires expertise in multiple domains. In the case of the GeT course, instructors need to have a solid understanding of both mathematics education and mathematics research, which are two distinct but related areas of expertise. On the one hand, formal education and practical experience in mathematics education can help instructors develop the pedagogical skills and knowledge needed to effectively teach high school geometry. This might include experience teaching the subject, writing textbooks, supervising student teachers, or conducting research on effective teaching practices. On the other hand, the GeT course also requires instructors to have a deep understanding of the theoretical aspects of mathematics, particularly as they relate to proof and mathematical reasoning. This might involve experience conducting research in mathematical fields or engaging in other activities that require advanced mathematical skills. However, few instructors are experts in both of these areas of knowledge. This can create challenges in their work, particularly when they are required to make decisions or provide guidance that draw on both kinds of expertise.

The ***institutions tension*** stems from the complex set of demands placed on instructors by various institutions that support and oversee the course. Instructors are expected to navigate and reconcile the sometimes-conflicting expectations of both the mathematics department and teacher education programs. On the one hand, instructors rely on the mathematics department for guidance on the content and structure of the course. They must ensure that the course meets the department's standards and expectations for an upper-level undergraduate mathematics course.

On the other hand, instructors must also take into account the requirements and expectations of teacher education programs. In many institutions, the GeT course was originally designed to fulfill a programmatic requirement related to preparing PTs for the unique challenges of teaching high school geometry, and as such, must meet the standards and expectations of these programs. Instructors must ensure that the course provides opportunities for pre-service teachers to develop the knowledge and skills necessary to effectively teach the subject, as well as to meet any program-specific requirements. While some GeT instructors recognize the importance of balancing the demands of both institutions, in practice, it can be challenging to reconcile the sometimes-divergent expectations and requirements.

To be clear, the nature of all five of these tensions are such that they are not easily resolved, even if instructors recognize them and avail themselves of resources that might help them manage those tensions. Furthermore, our data reveals that not all instructors experience these tensions in the same way, as evidenced by the ways they speak about the tensions. In this paper, we hope to elaborate on these differences and help account for them as drawing crucially from longstanding debates about the societal role of universities, in general, and university mathematics departments, in particular.

Literature Review/Theoretical Framework

For centuries, there has been a debate about the role of universities in society. Some have argued that universities should focus on practical skills and knowledge to prepare students for the workforce (Eliot, 1869; Cappeli, 2015), while others have argued that universities should prioritize providing a broad understanding of disciplinary knowledge and practices (Newman, 1891; Roth, 2014). The debate is relevant to mathematics departments, with some arguing that undergraduate mathematics courses should provide practical skills for future teachers (Hill et al., 2007; Leikin et al., 2018; Wasserman et al., 2018), while others prioritize educating individuals about the broader knowledge and practices drawn from the discipline of mathematics (Yacek & Kimball, 2017, see also Kimball, 2015; Silverman & Thompson, 2008). These ongoing debates have meaningful connections to the tensions that we will elaborate on in the closing remarks.

Methods

As part of a multi-year project aimed at developing an inter-institutional network for instructors of Geometry for Teachers (GeT) courses, data was collected through online video-conferencing interviews with GeT instructors. The purpose of the interviews, conducted over the first two years of the project, was to gain a better understanding of the problem space individual instructors may identify as worthwhile for the community to address. Audio and video records of the interactions were captured and transcribed for analysis. The interviews followed a semi-structured protocol with three sections. The first section, which is the exclusive focus of this analysis, consisted of 16 questions aimed at understanding the nature of the course, such as the profile of students taking the course and how faculty in the mathematics department came to teach the geometry for teachers' course. In addition, instructors were asked about the role they saw the course playing in improving capacity for high school geometry teaching, including how students' mathematical experiences in secondary schools could be influenced by the course.

We conducted interviews with a total of 32 instructors (21 men and 11 women) from 30 universities across the United States. All of the participating institutions were public, but they varied in size and focus, with some being primarily undergraduate-focused and others being doctoral-granting. All of the interviewees had recently taught a geometry course for prospective secondary teachers. Of the 32 interviewees, 30 were faculty members at various ranks, and the remaining two were graduate students in mathematics or mathematics education programs located in mathematics departments.

Analysis

Initially, the transcript data from the interviews was initially analyzed for the tensions alone. Because our goal was to identify these tensions for the group of instructors, rather than describe the variability in how individuals experience these tensions, we used the instructors' testimonies (regardless of form) to enrich the description of each of the poles in tension. That said, in that round of coding, we did notice that the data was not uniform, there were different ways that the tension was emerging in the data. In our second round of coding, we went back to the data and coded the instructors' speech about the tensions according to these distinctions, which we illustrate in the following section.

Results

In this section, we describe the three different ways that we observed GeT instructors relate to the five tensions that could be detected in the ways that instructors talk about those tensions. To be clear these distinctions showed up across the data drawn from all five tensions, but due to space constraints, we elect to illustrate them across only two of the five tensions: the *content tension* and the *instructor tension*.

In some cases, instructors talked about the tension as a ***dilemma that needs to be managed***—making it clear in their speech that they were explicitly aware of the tension in their work. In these cases, instructors included more explicit expressions regarding the challenges they faced in managing their own or others' expectations about how to handle the two, often conflicting, poles of the tension. For example, in the course of describing the GeT course, one instructor talked about the *content tension* in the following way:

> One of our algebraists, we don't have any geometers like no one who has geometry as their area, so one of the algebraists started borrowing a set of notes from someone down at [blinded university] and kind of made a book out of it. The book is slow in the sense that they tried to make half the semester about trying to prove how points are arranged on a line. Then eventually halfway through the semester we have two lines it's—it's really, really basic and very, very axiomatic and very formal. And it's probably really not the optimal out of arrangement for math for teachers. So, there are benefits to taking that approach but I think there are a lot of fall backs

In this quote we see an instructor acknowledging the tension by noting the ways that a GeT instructor might find themselves in a situation in which they are assigned to teach a course that has been previously designed in ways that are less than optimal for PTs. That said, while identifying this kind of organization (with a focus on a rigorous set of axioms, such as those formulated by Hilbert, 1899) as suboptimal for PTs, the instructor was also able to admit that this organization has its benefits, perhaps the ways that such an organization highlights disciplinary concepts such as the importance of consistency and completeness in axiomatic systems.

This way of talking about the tension also surfaced in the ways that some instructors talked about the *instructor tension.* For example, when responding to the question "In what ways is it important for mathematicians to be involved in teaching and improving this course?", one instructor talked about the *instructor tension* in the following way:

> So, I do think it's important for mathematicians to teach this course. I think it's important for them to teach the course with the guidance from math educators and from the knowledge in the field. You know, it's not like I think mathematicians should take this and just say we're

> going to do this our way, you know, come what may. But um, so I think it's important that it is done with the guidance of math educators, but ultimately it's a surprisingly mathematically sophisticated course. I think that it's mathematically sophisticated enough that mathematicians should be teaching it. Like it's, you know, even if you're not going to teach non-Euclidean geometry, the appreciation for the connections to and the differences to non-Euclidean geometry and just all these kind of very subtle mathematical things that come up in the context of the course I think do require a pretty high level of mathematical education.

Here we see an instructor acknowledging the tension by recognizing the importance of the knowledge and experiences of both the mathematician and mathematics educator as playing a crucial role in shaping the course. And while this individual ends up noting the importance for mathematicians in teaching the course, they recognize the unique knowledge that mathematics educators bring to the table with regards to the design of the course.

In other cases, instructors talked about the tensions as a place in which they had ***taken a side***—identifying with one pole of a particular tension. In these cases, instructors failed to represent the tension as something they experienced as tensionful, and instead used the opportunity to provide descriptions and sometimes justifications related to their personal alignment or misalignment with one or the other poles in tension. For example, when asked about the importance for the field that the GeT course be taught to PTs, one instructor who identifies as a mathematician talked about the *content tension* by saying:

> Um, I would hope that um, it is not just the material that they would teach in a high school, uh, but that it does include more. So, I guess in my case, the non-Euclidean geometry I'm doing, they would not do that in high school, but you need to have a little bit more about the idea of what else is out there other than just let's prove side, side, side criteria and for triangles or whatever it is that they're going to be doing in high school.

Different from the prior quotes, this instructor provides less evidence for an awareness of the *content tension*. Instead, the instructor takes sides by sharing details about the decisions they have made to align themselves with the side of the tension that argues for the need to focus on geometry from a more advanced perspective, rather than the knowledge needed by teachers.

Similarly, when asked about the organization of the course another instructor, who had been a high school teacher, talked about the *content tension* by saying:

> So, when I was in undergrad, I took a college geometry course. And we did nothing that looked like high school geometry in that course. So, I did not feel like it prepared me for being a high school geometry teacher that I became. We need to make a decision about what are the components that will be important or valuable for future teachers.

Like the prior quote, this way of talking about the *content tension* provides little assurances that the instructor recognizes the tension. Instead, the instructor takes sides by aligning themselves with the other pole of the tension, namely the need to focus the course on supplying the knowledge needed to teach geometry.

We also observed instructors prone to talk about the *instructor tension* in ways that revealed a propensity to *take sides*. For example, when asked why it's important for mathematicians to be involved with the GeT course, one instructor talked about the *instructor tension* by saying:

> Well, you know mathematicians know what proof is. They just have a broader perspective of – and I mean who else would [teach the course] if it wasn't us. I guess it would be math

> education specialists. Math education specialists just don't have the same perspective that we do. Even when they're teaching calculus, they sometimes don't see how it's put together

When comparing this way of talking about the instructor tension with those from the previous subsection, we can see an individual who seems to lack the sensitivity necessary to recognize and acknowledge competing perspectives about the kind of knowledge and expertise needed to design and teach the GeT course—taking the stance that those trained as mathematicians are really the only appropriate choice for staffing the course.

Lest we begin to think that mathematicians might be alone in this lack of sensitivity, here we offer another example, from an instructor who identifies as a mathematics educator. When asked questions about how the course is staffed and the expectations for the course are made, this instructor brought up the *instructor tension* by saying:

> The one issue that I have with it at a school like mine is that outside of me, the other people don't have a direct connection to K12 education. They are math faculty members who think they know what happens in schools, but they do not. They'll say, schools do this and this but I taught high school geometry for six years and I would beg to differ.

Like the previous quote, the instructor here seems equally unaware or unconcerned with the competing perspectives about the requisite expertise needed by instructors of the GeT course. Here the individual seems to have decided that having a direct connection to K12 education and knowing what happens in schools is the only or most important kind of knowledge to highlight.

Finally, we also observed instructors handling the tension by ***reframing aspects of their work*** in ways that satisficed one or both of the poles of the tension—electing to settle for a suboptimal solution to lessen the sense that there is a salient tension to wrestle with. In these cases, instructors had ways of talking about one or more of the poles of the tension in ways that were quite different from the ways that others had talked about the same tension.

Related to the *content tension*, some instructors found ways to reframe, and therefore minimize, the challenges related to the *content tension* by redefining one of the two bodies of knowledge. For example, to justify the choice to focus on more advanced mathematical topics, one instructor said,

> So I want them to get depth and breadth of topics which are related to geometry that they will be teaching at the secondary level but also beyond that; in the sense that we are touching on non-euclidean geometry, we are explaining the big ideas behind the axioms. This course is supposed to introduce them to axiomatic structure ... I don't feel that it's okay for a teacher to graduate without even seeing what non-euclidean geometry is. I think it's just a part of their general education.

That is, rather than frame non-euclidean geometry as part of the more advanced perspectives of geometry (as so many instructors do), this instructor identifies this topic as belonging to the the "general education" needed for all secondary mathematics teachers—suggesting non-euclidean geometry to simply be part of the canon of knowledge needed by all undergraduates.

Related to the *instructors' tension*, we also observed instructors engaged in this kind of reframing. For example, when asked about the needed characteristics of individuals that teach the GeT course, one instructor said, "*There's only two of us [in the mathematics department] that like really give a sh*t about geometry.*" Here, we see an instructor taking the attention off of the typical poles of the argument related to the *instructors' tension* by naming a purportedly more pressing concern about staffing the course, namely instructors are generally unwilling to teach it. This way of talking about the *instructors' tension* draws the listeners attention away from issues related to the educational and practical experiences of an instructor and reduces the conversation about faculty qualifications to one about willingness, or perhaps motivation to teach the course.

Discussion/Conclusion

In this paper, we have described the different ways that instructors talk about the various kinds of tensions that exist within their work. We see these three ways of talking about the tension as potentially having some meaningful differences not only in the ways that a given individual might experience a tension, but also in the ways that an individual participates in the broader arguments about the role of the university. To begin, the way in which an instructor positions themselves relative to the longstanding debates about the role of the university in society could play a substantial role in the way they make sense of the *content tension* and the *instructor tension.* If the priority for universities and mathematics departments is on preparing students for the workforce, then it follows quite naturally that the purpose of the GeT course is to prepare PTs for the work of teaching by supplying PTs with an instructor who is uniquely qualified to support them in gaining the knowledge needed for teaching; and for this, no one is more uniquely qualified than a mathematics educator well versed in the work of a geometry teacher. If on the other hand, the priority for universities and mathematics departments is to prepare well rounded students with a broadly defined education that can be used flexibly in a variety of contexts, then the purpose of the GeT course is to avoid the trap of focusing too narrowly (on a single profession) and instead focus on supporting students in gaining the canon of disciplinary knowledge that has been built up across many centuries; and for this, no one is more qualified than a mathematician well versed in such knowledge. These two examples illustrate how an individual with those views might find themselves prone to relating to the *content* and *instructor tensions* by *taking sides.* Of course, the source of the variation here does not rest solely with the individual, as instructors work in different kinds of institutions (e.g., liberal arts, technical schools, land grant institutions) which have historical ties to these larger arguments that have led them to organize their programs in ways that might make these competing perspectives more salient. With that as a background, we make sense of the instances in which an instructor elects to deal with the tension by *reframing aspects of their work* as somehow the opposite of *taking sides*. That is, by reframing the poles of the tension, we see ways that the instructor can avoid the need to take sides with colleagues by settling for a less than optimal solution. We think such a technique could be a useful skill set for an instructor needing to navigate and (perhaps) avoid difficult or contentious conversations with colleagues seeking to come to agreement about challenging aspects of the work. Finally, we see those places in which an instructor talks about the tension as a *dilemma to be managed* as neither one of *taking sides* or *reframing*, but as a decision to hold the poles of the tension, in tension. In his work *Embracing Contraries*, Elbow (1983) argues for the importance of undergraduate instructors coming to see tensions in the work as inherent dilemmas stemming from the very nature of teaching. In this way, Elbow himself conceives of tensions as objects deserving not only our careful attention, but deserving an embrace that holds together the integrity of the work of teaching in all of its complexities. While we are still mulling these difference over, in terms of what they might mean for instructors' participation in larger discourses, our intuition leads us to believe that when it comes to the work of an instructor, the propensity to perceive of the tensions as *dilemmas to be* *managed* as the more productive than the propensity to treat the tensions as a problem to be solved (by *taking sides* or *reframing)*.

Acknowledgements

This work has been done with support of NSF grant DUE-1725837. All opinions are those of the authors and do not necessarily represent the views of the National Science Foundation.

References

Adler, J., Hossain, S., Stevenson, M., Clarke, J., Archer, R., & Grantham, B. (2014). Mathematics for teaching and deep subject knowledge: voices of Mathematics Enhancement Course students in England. Journal of Mathematics Teacher Education, 17(2), 129–148. https://doi.org/10.1007/s10857-013-9259-y

Appova, A. & Taylor, C.E. (2019). Expert mathematics teacher educators' purposes and practices for providing prospective teachers with opportunities to develop pedagogical content knowledge in content courses. Journal of Mathematics Teacher Education 22(2), 179–204 (2019). https://doi-org.proxy.lib.umich.edu/10.1007/s10857-017-9385-z

Aveni, A. (2014). Class not dismissed: Reflections on undergraduate education and teaching the liberal arts. University Press of Colorado.

Cappelli, P. H. (2015). Skill Gaps, Skill Shortages, and Skill Mismatches: Evidence and Arguments for the United States. ILR Review, 68(2), 251–290. https://doi.org/10.1177/0019793914564961

Clinebell, S. K., & Clinebell, J. M. (2008). The tension in business education between academic rigor and real-world relevance: The role of executive professors. Academy of Management Learning & Education, 7(1), 99-107.

Conference Board of the Mathematical Sciences. (2001). The mathematical education of teachers I. American Mathematical Society/Mathematical Association of America.

Conference Board of the Mathematical Sciences. (2012). The mathematical education of teachers II. American Mathematical Society/Mathematical Association of America.

Elbow, P. (1983). Embracing contraries in the teaching process. College English, 45(4), 327-339.

Herbst, P., Brown, A.M., Ion, M. et al. (2023). Teaching Geometry for Secondary Teachers: What are the Tensions Instructors Need to Manage?. Int. J. Res. Undergrad. Math. Ed. (2023). https://doi.org/10.1007/s40753-023-00216-0

Hill, H. C., Blunk, M. L., Charalambous, C. Y., Lewis, J. M., Phelps, G. C., Sleep. L. & Ball, D.L. (2008) Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. Cognition and Instruction, 25(4), 371-430.

Jackson, B. S., Napier, D., Newman, B., Odom, S., Ressler, J., Ridgeway, S., ... & Spector, N. (2008). Nursing faculty qualifications and roles. National Council of State Boards of Nursing (NCSBN) Faculty Qualifications Committee.

Lai, Y. (2019). Accounting for mathematicians' priorities in mathematics courses for secondary teachers. The Journal of Mathematical Behavior, 53, 164-178. https://doi.org/10.1016/j.jmathb.2018.08.001

Leikin, R., Zazkis, R. & Meller, M. (2018). Research mathematicians as teacher educators: focusing on mathematics for secondary mathematics teachers. J Math Teacher Educ 21, 451–473. https://doi.org/10.1007/s10857-017-9388-9

Lenz, L. (2015). Active learning in a math for liberal arts classroom. Primus, 25(3), 279-296.

Levine, A., & Van Pelt, S. J. (2021). The great upheaval: Higher education's past, present, and uncertain future. JHU Press.

Marcus, R., Fukawa-Connelly, T., Conklin, M., & Fey, J. T. (2007). New thinking about college mathematics: Implications for high school teaching. The Mathematics Teacher, 101(5), 354-358.

Mounce, P. H., Mauldin, D. S., & Braun, R. L. (2004). The importance of relevant practical experience among accounting faculty: An empirical analysis of students' perceptions. Issues in Accounting Education, 19(4), 399-411.

Newman, J. H. (1891). The idea of a university defined and illustrated, 1852. Newman Reader.

Roberts, J., Propsom, P., & Tobin, W. (2018). Developing Shared Vision: A Case Study Documenting a STEM General Education Change Process at a Small Liberal Arts School. Journal of college science teaching, 47(6).

Roth, M. S. (2014). Beyond the university: Why liberal education matters. Yale University Press.

Speer, N. M., & Wagner, J. F. (2009). Knowledge needed by a teacher to provide analytic scaffolding during undergraduate mathematics classroom discussions. Journal for Research in Mathematics Education, 40(5), 530-562. https://doi.org/10.5951/jresematheduc.40.5.0530

Smith, J. (2013). Amateur hour? Experience and faculty qualifications in US intelligence courses. Journal of Strategic Security, 6(3), 25-39.

Wasserman, N.H., Fukawa-Connelly, T., Weber, K., Pablo Mejia-Ramos, J., Abbott, S. (2022). Understanding analysis and its connections to secondary mathematics teaching. Springer

Wasserman, N., Weber, K., Villanueva, M., & Mejia-Ramos, J. P. (2018). Mathematics teachers' views about the limited utility of real analysis: A transport model hypothesis. Journal of Mathematical Behavior, 50(1), 74–89.

IMPACT OF TAKE-HOME MANIPULATIVE KITS IN AN ELEMENTARY MATHEMATICS CONTENT COURSE

Barbara Ann Swartz
West Chester University
bswartz@wcupa.edu

Katherine DeRosa
West Chester University
kd923281@wcupa.edu

Keywords: Mathematical Knowledge for Teaching; Preservice Teacher Education

Theoretical Framework and Research Question

The Conference Board of the Mathematical Sciences (2010) recommends, "before beginning to teach, an elementary teacher should study in depth, and *from a teacher's perspective*, the vast majority of K-5 mathematics, its connections to prekindergarten mathematics, and its connections to grades 6-8 mathematics" (p. 23, emphasis added) in addition to courses in mathematical pedagogy. Mathematical tools (e.g., manipulatives) are helpful in developing understanding of the mathematical concepts (National Research Council, 2009). Thus, teacher educators need to explicitly engage future teachers in using mathematical tools to represent abstract mathematical ideas to help learners make sense of how and why those procedures work the way they do (Morris & Hiebert, 2017). We investigated the impact of using such tools in ways that align with NCTM's (2014) vision for teaching mathematics in an elementary mathematics content course on teacher candidate learning and dispositions. The research question guiding this exploratory study was: In what ways and in what topics, did implementing NCTM's (2014) effective mathematics teaching practices impact teacher candidate (TC) learning and/or dispositions in an undergraduate elementary mathematics content course?

Methodology and Findings

The TCs were enrolled in a 100-level elementary mathematics content course for teachers at a mid-sized public university in the mid-Atlantic United States. To document their learning and changes in mathematical dispositions, seventy-nine TCs responded to a survey at the start and end of the course (to compare pre/post responses). This survey asked TCs to rate their understanding of each of the course's learning goals on a Likert Scale from 1-4 with 1 meaning, "I don't know what this (statement) means" and 4 meaning "I fully understand what this means and can explain completely and accurately with confidence." We then used thematic qualitative analysis methods to summarize emergent themes for the open-ended questions following.

Summary of Results and Implications

TCs reported growth in understanding the mathematical content explored from *a teacher's perspective* on the post-survey – most noteworthy was their deepened understanding of fractions. The themes that emerged from responses of the open-ended questions suggest that TCs' dispositions grew as a result of implementing NCTM's (2014) effective teaching practices. Many TCs came into the class with very negative opinions of mathematics but left the course believing that mathematics was something they could do and understand because of the manipulatives and rich mathematical discourse. While the limitations of self-reported data are well documented, these responses help to provide insights to the valuable learning experiences teacher education courses can provide for future teachers and provide insights to how we can help future teachers develop positive mathematical identities so that beginning teachers can create their math classes to be an enjoyable and worthwhile experience for every student!

References

American Mathematical Society (AMS) with the Mathematical Association of America (MAA). (2012). Conference board of the mathematical sciences issues in mathematics education (vol 17): The mathematical education of teachers II. https://www.cbmsweb.org/archive/MET2/met2.pdf

Morris, A. K., & Hiebert, J. (2017). Effects of teacher preparation courses: Do graduates use what they learned to plan mathematics lessons? American Educational Research Journal, 54(3), 524–567. DOI: 10.3102/0002831217695217

National Council of Teachers of Mathematics (2014). Principles to actions: Ensuring mathematical success for all. Reston, VA: NCTM.

National Research Council. (2009). Mathematics Learning in Early Childhood: Paths toward Excellence and Equity. In C. T. Cross, T. A. Woods, & H. Schweingruber, (eds.), Committee on Early Childhood Mathematics, Center for Education, Division of Behavioral and Social Sciences and Education. Washington, D.C.: National Academies Press.

IMPLEMENTING CO-PLANNING TO INTEGRATE ACADEMIC AND PRACTITIONER KNOWLEDGE

Emily Deal
Illinois State University
eedeal@ilstu.edu

David Barker
Illinois State University
dbarker@ilstu.edu

Óscar Chávez
Illinois State University
ochave2@ilstu.edu

Keywords: Preservice Teacher Education, Teacher Educators

There is often a disconnect for preservice teachers (PST) between the knowledge gained from university-based and PK-12 field-based experiences (Zeichner, 2010). The purpose of this study is to combat this disconnect by working to understand how PSTs, university instructors (UI), and mentor teachers (MT) draw upon and integrate academic knowledge (AK) and practitioner knowledge (PK) while co-planning secondary mathematics instruction in a third space. Co-planning in a third space allows for the interplay of multiple knowledge sources in inclusive and potentially transformational ways (Zeichner, 2010). For this study, we have defined AK as knowledge learned through interactions during university-based coursework, PK as anything learned through classroom-based experiences (McCulloch et al., 2020; Zeichner, 2010), and integrated knowledge (IK) as the mental collaboration of both AK and PK.

This case study was conducted with participants from a secondary mathematics education methods course at a large midwestern university during the Fall of 2022. Participants included two PSTs, two UIs, and one MT. Both of the UIs and the MT served as co-principal investigators for this study. Data were collected in four phases: initial background questionnaires, initial planning questionnaires, the co-planning session, and post co-planning questionnaires. Self-reported data from questionnaires and transcripts from the co-planning session were coded according to role (i.e. PST, MT, or UI) and knowledge type (i.e. AK, PK, and IK). Data from the co-planning session were also coded by topic (i.e. pedagogy, students, mathematical content).

Consistent with the literature, there were more self-reported instances of AK and PK by PSTs than instances of IK (Zeichner, 2010). A closer analysis of these knowledge types revealed several interesting findings. First, it was observed during the co-planning session that the interplay of both AK and PK played a role in the continual development and refinement of PSTs' instructional decisions. This process of development and refinement may be a means of better understanding how AK and PK become integrated over time.

Second, it was difficult in several instances to delimit if the knowledge referenced was learned in a university setting, a PK-12 setting, or some combination of both. It is reasonable to consider that participants may find it challenging to remember if knowledge they possess first originated in a methods course or from a classroom experience. Furthermore, it is reasonable to acknowledge that although one may have been exposed to an idea in one context, the development of that idea may be attributed to another context.

Third, the current definitions of each knowledge type are either not clear enough, are a misrepresentation of the body of knowledge they seek to describe or may need further elaboration. For example, PK may be better defined as the refinement or adjustment of AK. It should also be considered that the interplay between knowledge types, or IK, may be best understood as a continuum rather than as a discrete category. For example, some study participants merely combined AK and PK while others seemed to tightly weave AK and PK.

We conclude that the notion of identifying and integrating AK and PK is not so clean cut and provides fruitful opportunities for further discussion.

References

McCulloch, A. W., Bailey, N. G., Fye, K., & Scott, G. (2020). Creating a third space for learning to design technology-based mathematics tasks. Mathematics Teacher Educator, 9(1), 7–22. https://doi.org/10.5951/MTE.2020.0011

Zeichner, K. (2010). Rethinking the connections between campus courses and field experiences in college- and university-based teacher education. Journal of Teacher Education, 61(1-2), 89–99. https://doi.org/10.1177/0022487109347671

INTEGRATING SOCIAL EMOTIONAL LEARNING (SEL) INTO MATHEMATICS TEACHER PREPARATION: A TRAUMA-INFORMED APPROACH

Jenna R. O'Dell
Bemidji State University
Jenna.ODell@bemidjistate.edu

Rebecca Hoffman
Bemidji State University
Rebecca.Hoffman@bemidjistate.edu

Todd Frauenholtz
Bemidji State University
Todd.Frauenholtz@bemidjistate.edu

Stacy Bender
Peacemakers Resources
stacy@peacemakerresources.org

In this study, we investigated preservice mathematics teachers' growth and application of their emotional intelligence skills after participating in monthly social emotional learning (SEL) workshops for an academic year. SEL workshops were led by a local nonprofit specializing in universal prevention tools for systemic changes that benefit individuals and communities. We interviewed three preservice teachers who participated in all the workshops about their experiences with SEL skills in their personal lives, practicum experiences, and how they will use it in their future classrooms. We found the most common SEL competency areas they discussed were responsible decision making and social awareness. The undergraduate preservice mathematics teachers were clearly able to identify instances in both their personal and professional lives where they observed SEL skills being used.

Keywords: Affect, Emotions, Beliefs, and Attitudes; Instructional Activities and Practices; Communication; Preservice Teacher Education

In the United States there is a national shortage of mathematics teachers, and it is even more difficult to hire and keep highly qualified mathematics teachers in rural areas (Monk, 2007). This was especially true in our area, so we applied for, and received, a grant from the National Science Foundation to give full support to sixteen preservice teachers for the last two years of school if they double major in mathematics and mathematics education. In addition, they must participate in two learning community events each month during the school year. One of these learning community events would take place with our local nonprofit, Peacemakers Resources. Our relationship with this nonprofit represents a collaboration between our university and a partner in our local community to build trauma-informed approaches in schools. During the monthly meetings, the staff and students would learn about Social Emotional Learning (SEL) and how to use SEL skills in their classrooms.

Researchers (Felitti, 2002; Shonkoff, Boyce, & McEwen, 2009) have documented adverse child experiences (ACEs) and trauma impact how students' brains develop and function. Subsequent researchers have found teaching social emotional learning to students can reverse the effects of chronic stress (Thompson, 2014). Additionally, Ottmar et al. (2015) stated teachers trained in SEL are more capable of perusing ambitious mathematics teaching and Durlak et al. (2011) found students showed an 11 percentile-point gain on standardized achievement tests. Because of this research, we incorporated the SEL training into our teacher preparation project. We created this partnership to help prepare our preservice teachers to engage all students and design learning environments that take students' emotional and learning needs into account.

Research Question

The following question guided our research study: how have preservice teachers demonstrated growth and application of their emotional intelligence skills after participating in an SEL learning community for one year?

Theoretical Framework

Learning happens through social practices and people construct their understanding of ideas through active participation in learning communities. Communities of practice (CoP) (Wenger, 1998) was our theoretical framework for this study. When people are learning, they are "active participants in the practices of social communities and constructing identities in relation to these communities" (Wenger, 1998, p. 4). Our preservice teachers engaged in a required SEL learning community for one year as part of their teacher preparation project. They had shared experiences centered around their mathematics coursework, SEL workshops, and learning community activities. The community of practice framework guided the organization and structure of our teacher preparation project with eight preservice teachers.

Methods

As part of a five-year long grant project on preparing preservice teachers to teach mathematics in high need rural areas, we investigated three of the preservice teachers' growth and application of their SEL training. These students completed their junior year at a public university in the Midwest. The students double majored in mathematics and secondary mathematics education. All three students participated in the community of practice activities, including the SEL workshops with the local nonprofit. The first workshop was a full day training to introduce them to the impacts of trauma on brain development, ACEs, historical trauma, and strategies to build resilience (SEL). They then participated in monthly two-hour workshops on integrating SEL skills for personal and professional practice. At the end of the academic year and the workshops, we interviewed three preservice teachers using semi-structured interviews (Glesne, 2011). The interviews were each completed with one preservice teacher, two of the researchers who participated in the SEL trainings, and the leader of the SEL training.

Each interview lasted approximately twenty minutes. We allowed for discussion around each question and follow up. We asked the preservice teachers four questions:

1. How are you using what you learned in our SEL sessions in your personal life?
2. How are you seeing SEL in your practicum experiences?
3. What is your vision for incorporating SEL into your future classroom?
4. Do you have any suggestions for the SEL training next year?

The interviews were videotaped and transcribed. We recruited a faculty member from the Social Work department at our university with expertise in SEL to join our research team, then together we created an analytic framework (see Table 1) based on work from the Collaborative for Academic, Social, and Emotional Learning (CASEL, 2022). We used CASEL's five broad but interrelated competency areas of SEL. We then used CASEL's examples of each of the five categories as subcodes (common examples).

After creating the analytic framework, we examined all the transcripts as a team of three (two researchers who participated in the SEL training and social work faculty member) and used the framework to document anytime a student discussed SEL. We first documented it as one of the five competency areas followed by a common example. We did this process together, getting one hundred percent agreement before moving on. We coded two of the interviews a second time to check for consistency. We then tallied each time we documented a competency area during the interview.

Table 1: Social Emotional Learning Competencies

Competency	Definition	Common Examples
Self-Awareness	Accurately recognizing one's emotions and thoughts, and their influence on behavior	Self-efficacy; growth mindset; sense of purpose
Self-Management	Regulating throughs, emotions, and behaviors in different situations and being able to set and achieve personal and academic goals	Courage to take initiative; setting personal and collective goals; identifying and using stress-management techniques
Social awareness	Taking the perspective of and empathizing with others from diverse backgrounds	Empathy and compassion; showing concerns for other's feelings; taking others' perspectives
Relationship skills	Establishing and maintaining constructive relationships with diverse individuals and groups	Developing positive relationships; communicating effectively; collaborative problem-solving
Responsible decision-making	Making ethical and respectful choices about personal behavior and social interactions	Demonstrating curiosity and open-mindedness; making reasoned judgment after analyzing information; evaluating impacts

Results and Discussion

During analysis, we coded each time the preservice teachers shared things related to the five SEL competency areas and the 40 corresponding examples (CASEL, 2022). Through analysis, we found social awareness and responsible decision-making were the most common competency areas discussed by the preservice teachers (see Table 2).

Table 2: Competency Frequency by Preservice Teacher

	Self-Awareness	Self-Management	Social Awareness	Relationship Skills	Responsible Decision-Making
Katie	4	6	9	8	7
Lola	2	5	3	3	7
Noah	3	4	10	0	9
Total	9	15	22	11	23

Self-Awareness

We documented a total of nine comments fitting the definition of self-awareness making it the least frequently observed. The most common expressions of self-awareness included self-efficacy, growth mindset, and sense of purpose. Noah demonstrated a growth mindset in reflecting on how he is seeing SEL in his practicum experiences. After discussing successful and unsuccessful teachers he has observed during this time, Noah reflected on the positive teachers he has observed:

> Noah: I am kind of observing practices I want to use as a future teacher. And because I've had kind of the experiences, learning about the SEL stuff, I know what to look for and know what works and why it works and stuff like that.

This is an example of Noah showing a growth mindset in relation to his self-awareness.

Katie also demonstrated several examples of self-awareness. When speaking about how the SEL trainings have influenced her, she demonstrated a growth mindset, a sense of purpose, and self-efficacy in reflecting on her experiences:

> Katie: Another thing I felt like when I got into mathematics, I was more introverted, and (these) experiences definitely made me more extroverted and like I feel better to express opinions and like answer questions in class. And it just made me a better student as well and person.
>
> Author: Why do you think that is? I am just curious.
>
> Katie: Well, I think in a way like the first couple of SEL sessions that we had just kind of like forced me out of my bubble. Like, here is all these new people, talk to them. So seeing that like, oh, that went well. Like, I actually made connections really well. Like it just gave me the confidence I needed.

This understanding of self-awareness is important because recognizing one's strengths and limitations leads to professional identify formation as a new mathematics educator. It is important for them to have a strong sense of self-efficacy and a growth mindset.

Self-Management

Self-management builds upon self-awareness. This was the third most common SEL competency area the preservice teachers discussed during their interviews. The most common expressions of self-management were identifying and using stress-management strategies, showing the courage to take initiative, self-management skills, and setting personal and collective goals. We documented self-management 15 times.

Lola had five instances where she showed reflection on self-management. An example of this in relation to stress management was when she spoke about needing to take breaks, breathe, and doing hot yoga since starting the SEL trainings:

> Lola: I definitely take a break and breathe sometimes. So, I definitely use those mindful moments. Also, just like expressing my emotions more rather than just bottling it up all the time. So, that is how I use it in my personal life. I also just do like breathing exercise more to calm down and stuff so. There was a lot of deep breathing this morning.
>
> Author: Is this a new thing or have you always done it?
>
> Lola: No, I haven't done that before, and I also started taking hot yoga as I feel there is a lot of SEL in that.

Katie displayed six instances of self-management. Her discussions around self-management focused on her showing the courage to take initiative and setting personal and collective goals for her future classroom. Katie spoke of wanting to be attentive to her students' needs. After speaking about a teacher, she observed who was very attentive to what students might need she said:

> Katie: I want to incorporate that, but I guess that's more on me. That's more of like a personality trait than it is something that I incorporate into my classroom.

Katie was also very reflective on how she viewed the future SEL workshops the next year. She spoke of the importance of getting to know the new eight preservice teachers who would be joining the group.

Noah also spoke of ideas related to self-management four different times during his interview. While sharing how he is using what he learned in his personal life, Noah reflected how it is important:

> Noah: Knowing what makes me happy and how can I add more of that to life.

Noah identified aspects of a trauma-informed classroom that prioritizes students' wellbeing and self-regulation:

> Noah: I'm going to just create a classroom environment where students feel like they are set up for success. They know I want their success; they know their peers want them to succeed and stuff like that. And this taking time throughout the day, even though it feels like there's a lot to get done in a class to get the students into the mindset of learning before getting into the learning.

The preservice teachers reflected on the how they were using or would use self-management skills in their personal lives or future classrooms. Demonstrating personal and collective agency is essential to a diversity-informed mathematics education environment where all students can thrive.

Social Awareness

Documented 22 times, social awareness was the second most common theme of the SEL competencies during the interviews. Common occurrences of social awareness were taking others' perspectives, showing concern for the feelings of others, and showing empathy and compassion for others.

When Noah was asked about how he is using SEL skills after the trainings in his personal life he demonstrated taking others' perspectives:

> Noah: I think overall it is just kind of being more mindful of kind of like the place where other people are coming from. I mean, with my job, I work with a bunch of students from different backgrounds and things like that. And it is kind of just taking a step back and thinking about, okay, what are they going through? What are their motivations behind their actions and stuff like that, and kind of seeing them as people rather than the problems or the successes they have and stuff like that.

When Noah was asked about using SEL in his future classroom he stated:

> Noah: Just seeing students as students and as people who have backgrounds.

Noah’s last statement represents a critical element of trauma informed pedagogy. Social awareness reflects educators’ ability to ensure their students are felt and heard in the classroom.

Relationship Skills

We found students mentioned ideas related to relationship skills a total of 11 times making it the fourth most observed SEL competency. Noah did not have any instances where he mentioned ideas related to relationship skills. The times we documented relationship skills for both Lola and Katie were focused around communicating effectively, developing positive relationships, and practicing collaborative problem-solving.

Lola spoke of a teacher she observed several times and how they had an active learning classroom environment that she would like to implement when she begins teaching. This is an example of collaboratively solving a mathematics problem together at the whiteboard, she said:

> Lola: Like the kids are at a whiteboard and they are communicating with one another. And one student has a marker, so there is only one person writing. So, it is more communication based.

Katie also had eight instances where she spoke related to relationship skills. Many of the ideas were how the training had influenced her in her personal life and as a student. She spoke of resolving conflicts constructively. First, she reflected on how the SEL workshops had influenced her ability to manage her relationships with her roommates. She said:

> Katie: Now instead of like yelling at them to do the dishes I can understand. You didn’t do the dishes like maybe we should have a conversation about this.

Katie also spoke of how the learning community activities have helped her develop positive relationships:

> Katie: Well, I think in a way like the first couple of sessions that we had just kind of like forced me out of my bubble. Like, here’s all these new people talk to them. So, seeing that like, oh, that went well. Like I actually made connections really well. Like it just gave me the confidence I needed.

We found this to be a powerful statement of the positive aspect of developing relationship skills. Central to developing a healthy classroom community, relationship management skills reflect the ability to constructively engage with diverse groups.

Responsible Decision-Making

Documented 23 times, responsible decision-making was the most common competency when the preservice teachers were discussing their noticing and learning of SEL skills. Common occurrences of this were identifying solutions for personal and social problems, reflecting on one’s role to promote community well-being through reflecting on SEL skills, demonstrating curiosity and open-mindedness to shifting perspective and evaluating the idea of interpersonal impacts.

Lola spoke about ideas she wanted to do in her future classroom in relation to how she would help build a community of students. She stated:

> Lola: I personally would like to help students make their own agreements to like what their core values are. What they want to see in their classroom.

When Katie was reflecting on what she has learned through the SEL trainings, she explained how her mindset has shifted from how she would start the school year with her future students.

> Katie: At first, I feel like I was a little scared to take an entire week not devoted to math. I feel like a lot of opinions that I've heard are like you got to start with the math. You don't have enough time in the year to get through it all. But, then like what we've been talking about with like mindful moments and then when the teacher came in to talk about how she uses that week, like, I am not scared of it anymore. Like, the math can wait a week.

Katie demonstrated curiosity and open-mindedness to shifting her perspective of how the first few days of school should look and evaluated the idea of interpersonal impacts. A few minutes later Katie reflected on the trainings:

> Katie: The only thing left was that I feel like this is giving me a good perspective of what teaching is and I've seen more that it's not just about teaching content. It's not just about math. It's about connections and meaningful relationships.

Noah also reflected on how the SEL sessions might influence his future classroom. Noah reflected:

> Noah: I feel like my own classroom feels so far away even though it really isn't. So, I should be thinking about these things, but I'm not thinking about these things all the time and I'm kind of the person where I'm like, I don't know what I don't know until I need to know that. I'm like, I should have thought about that before. So, it's definitely been nice to kind of have this experience to think ahead, kind of reflection on what I want my classroom to look like and kind of emulate some of the positive experiences I've observed and had in classrooms.

The most common theme we found in relation to the preservice teachers' reflections of SEL skills was responsible decision-making. When reflecting on things they learned throughout the academic year and through their field experiences they found many examples of SEL. They made reasoned judgements, and demonstrated curiosity and open-mindedness based on what they saw and learned, ultimately embracing their role in promoting personal, family, and community well-being.

Conclusions

Researchers (e.g., Durlak et al., 2011; Ottmar et al., 2015) have documented the importance of training preservice teachers in SEL. Our preservice mathematics teachers participated in monthly SEL workshops. After one year, the preservice teachers demonstrated growth in their emotional intelligence skills in relation to the five broad SEL categories (CASEL, 2022). The most common competency areas in which they spoke of growth were social awareness and responsible decision making. Social awareness is critical because it will allow a teacher to be cognizant of their students and understand and empathize with students from diverse backgrounds. Responsible decision making is important because it supports teachers in making caring and constructive choices that benefit the collective well-being and promote student learning for all students.

Our teacher preparation project created a learning community where students were active participants. Through these experiences the students demonstrated growth in different ways. Katie spoke of how the participation made her a better student and gave her the confidence to be more outgoing and build more relationships as a mathematics student. Noah shared that the experiences have allowed him to see students as people rather than problems. Lola shared she was better able to manage stress after the trainings. While all three of the interviewed students learned and grew in different ways, we found the learning community participation and SEL workshops to be extremely beneficial for the students and us as the authors.

This training helped preservice teachers design classroom environments that take students and learning into account, including students with a history of trauma exposure. The dearth of literature regarding the integration of SEL content in preservice teacher education calls for the articulation of methods that both successfully build SEL skills and pedagogical approaches. In this case, a CoP approach was used that paired personal integration of SEL practices with professional identity formation. Preservice mathematics educators indicated emerging personal SEL skills that integrated well with their pedagogical formation. With this study, our preservice teachers also documented a growth in skills central to resiliency that will support them in persisting in rural mathematics education positions.

Acknowledgments

Work on this project was supported by the National Science Foundation under Grant No. 1949915. The opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the funding agencies. We would also like to acknowledge the nonprofit, Peacemakers Resources, for the support and learning opportunities of this project.

References

CASEL (2022). Interactive CASEL Wheel. https://casel.org/fundamentals-of-sel/what-is-the-casel-framework/#interactive-casel-wheel accessed Feb 13, 2023.

Durlak, J.A., Weisberg, R.P., Dymnicki, A.B., Taylor, R.D., & Schellinger, K. (2011). The impact of enhancing students' social and emotional learning: A meta-analysis of school-based universal interventions. Child Development. 82(1), 405–432. doi:10.1111/j.1467-8624.2010.01564x

Felitti, V. (2002). The relation between adverse childhood experiences and adult health: Turning gold into lead. The Permanente Journal, 6(1). 44–47. doi:10.7812/tpp/02.994

Glesne, C. (2011). Becoming qualitative researchers: An introduction. Boston, MA: Pearson.

Monk, D. H. (2007). Recruiting and retaining high-quality teachers in rural areas. The Future of Children, 17(1), 155–174. doi:10.1353/foc.2007.0009

Ottmar, E. R., Rimm-Kaufman, S. E., Larsen, R. A., & Berry, R. Q. (2015). Mathematical knowledge for teaching, standards-based mathematics teaching practices, and student achievement in the context of the responsive classroom approach. American Educational Research Journal, 52(4), 787–821. doi: 10.3102/0002831215579484

Shonkoff J.P., Boyce W.T., McEwen B.S (2009). Neuroscience, molecular biology, and the childhood roots of health disparities: Building a new framework for health promotion and disease prevention. JAMA 301(21), 2252–2259. doi:10.1001/jama.2009.754

Thompson, R. (2014). Stress and Child Development. The Future of Children, 24(1), 41–59. doi: 10.1353/foc.2015.0004

Wenger, E. (1998). Communities of practice: Learning, meaning, and identity. Cambridge, MA: Cambridge University Press. doi:10.1017/CBO9780511803932

INVESTIGATING PRESERVICE TEACHERS' DESIRE TO EMPHASIZE FUN IN THE MATHEMATICS CLASSROOM

Jordan Allred
University of Central Arkansas
jallred5@cub.uca.edu

Lisa Skultety
University of Central Arkansas
lskultety@uca.edu

Keywords: Elementary School Education, Preservice Teacher Education, Teacher Beliefs, Affect, Emotion, Beliefs, and Attitudes.

Elementary preservice teachers (EPSTs) can sometimes trace feelings of mathematics anxiety or mathematics teaching anxiety to their own negative experiences as elementary students (Brown et al, 2011; Hobden & Mitchell, 2011). Teachers who performed poorly in mathematics class often come to dread teaching mathematics themselves (Brown et al, 2011), and EPSTs, having often focused on subject areas outside of mathematics, are particularly prone to this dread (McCulloch et al., 2013; Swars 2009). Based on these experiences, many EPSTs may hope to help their future students avoid negative experiences in the mathematics classrooms, which can risk translating into setting up students to avoid difficult or stressful situations in mathematics entirely. In this case, mathematics classes risk becoming less about teaching students the mathematical content and more about trying to ensure that students do not grow to dislike mathematics in the way they do. Gellert (2000) found that EPSTs desire to teach elementary mathematics in ways children would find fun, like focusing on games, competitions, and similar activities. While it is certainly beneficial for students to enjoy mathematics, some of these methods risk communicating that teachers must externally introduce fun into the classroom through activities that distract students from the mathematics. The implication is that mathematics is not innately fun or engaging for students. To thrive in mathematics, students must become comfortable with and willing to handle difficult problems (Bishop, 2012). Encouraging students to accept struggling with mathematics is then a necessary aspect of mathematics education; however, some EPSTs may not believe this themselves.

In this study, we present our findings to the following questions: *What assumptions do elementary preservice teachers make about their students' feelings towards mathematics?* and *In what ways do these assumptions impact their future classroom plans?* Thirty-seven EPSTs completed mathematics autobiographies in which they discussed their perceptions of students' mathematical experiences and their plans to manage these experiences in their own future classrooms. Using an emergent coding process (Creswell, 2017), we categorized these responses according to participants' overall perception of students' feelings, their reasoning for their perceptions, and the ways they planned to teach mathematics in their future classrooms. Participants most commonly perceived students' opinions of mathematics to be negative (46%), with 59% of those respondents citing that students felt negatively about mathematics due to bad teachers and 65% believing the negative perception is due to the intrinsic difficulty of mathematics. The majority of EPSTs planned to emphasize "fun" (54%), hands-on activities (54%) and classroom collaboration in their classrooms (43%), with a desire to foster student confidence (22%); EPSTs form these plans under the assumption that all students must have had the same mathematics experiences that they did. This suggests the participants perceive learning mathematics as an inherently harmful or traumatic experience that must be carefully mitigated within their classrooms.

References

Bishop, J. P. (2012). “She's always been the smart one. I've always been the dumb one”: Identities in the mathematics classroom. Journal for Research in Mathematics Education, 43(1), 34-74.

Brown, A. B., Westenskow, A., & Moyer-Packenham, P. S. (2011). Elementary Pre-Service Teachers: Can They Experience Mathematics Teaching Anxiety without Having Mathematics Anxiety?. Issues in the Undergraduate Mathematics Preparation of School Teachers, 5.

Creswell, John W. (2017). Qualitative Inquiry and Research Design: Choosing among Five Approaches, 3rd ed. Thousand Oaks: Sage.

Gellert, U. (2000). Mathematics instruction in safe space: Prospective elementary teachers' views of mathematics education. Journal of mathematics teacher education, 3(3), 251-270.

Hobden, S., & Mitchell, C. (2011). Maths and me: Using mathematics autobiographies to gain insight into the breakdown of mathematics learning. Education as Change, 15(1), 33-46.

McCulloch, A. W., Marshall, P. L., DeCuir-Gunby, J. T., & Caldwell, T. S. (2013). Math autobiographies: A window into teachers' identities as mathematics learners. School Science and Mathematics, 113(8), 380-389.

Swars, S. L., Smith, S. Z., Smith, M. E., & Hart, L. C. (2009). A longitudinal study of effects of a developmental teacher preparation program on elementary prospective teachers’ mathematics beliefs. Journal of Mathematics Teacher Education, 12, 47-66.

INVESTIGATING PRE-SERVICE TEACHERS' PROBLEM-SOLVING ABILITY AND THEIR CURRICULAR NOTICING ABILITY THROUGH PEDAGOGICAL SEQUENCING ACTIVITIES

Mi Yeon Lee
Arizona State University
mlee115@asu.edu

Ji-Eun Lee
Oakland University
lee2345@oakland.edu

This study investigated how 155 pre-service teachers solved three pattern generalization problems in a two-part written test and sequenced them for teaching purposes to demonstrate their curricular noticing. Participants' solutions were analyzed using inductive content analysis, which showed that only 8.4% of PSTs produced correct answers to all three problems, and 14.2% proposed the same sequence as the researchers. As criteria for their sequencing, 52.9% used the problem's difficulty, variously associated with visuality, complexity, and feasibility, and 11% used their own understanding of how to solve the problem. It was also suggested that some participants' incorrect problem-solving influenced their curricular noticing. Implications for helping pre-service teachers improve their curricular noticing ability are discussed.

Keywords: pre-service teacher education, problem solving, teacher noticing

Introduction

The purpose of this study was to examine elementary pre-service teachers' (PSTs) problem solving abilities and their curricular noticing through problem-solving and problem-sequencing activities. With increased interest in curricular and pedagogical innovation in mathematics education, curricular noticing (Dietiker et al., 2018; Males et al., 2015), including proper sequencing of problems to scaffold students' understanding, has emerged as critical as noticing students' mathematical thinking (Dietiker et al., 2018; Males et al., 2015; Remillard, 2005). Thus, Males et al. (2015) extended the construct of noticing to teachers' treatment of curriculum materials and coined the term *curricular noticing*, defined as "the process through which teachers make sense of the complexity of content and pedagogical opportunities in written or digital curricular materials (p. 88)."

Teachers' ability to effectively use curriculum materials is a key component of teaching expertise and quality of instruction (Dietiker et al., 2018; Males et al., 2015; Remillard, 2005). For example, Amador et al. (2017) found that non-routine fractional tasks may support PSTs' interpretations of core mathematical properties of tasks they encounter in curricular materials. Also, Males et al. (2015) examined how PSTs engaged with curricular materials by asking them to create a lesson plan after providing Grade 6 teacher materials on the division of fractions from four curricular programs.

Although several studies have investigated PSTs' curricular noticing in fractions, research has rarely been conducted on pattern generalization, which has been emphasized as important for advancing learners' algebraic reasoning (Kaput, 1999; Radford, 2008). Also, as noted, Males et al. (2015) investigated PSTs' curricular noticing by having them create lesson plans with curricular materials from four sources provided by the researchers. However, PSTs may not be ready for noticing with such a range of curriculum materials. Rather, in the early stages of teacher preparation, they may need to develop a lens for curricular noticing in a controlled situation with specific tasks or activities.

In this study, the act of curricular noticing was channeled through sequencing three particular problems rather than through the provision of multiple curriculum materials. The following research questions guided this study: (1) How do the PSTs solve the three pattern generalization problems? (2) What reasoning do the PSTs think is required to solve each of the three problems? and (3) How do the PSTs sequence the three problems and rationalize their sequencing?

Theoretical perspectives: Curricular Noticing

Teacher noticing is one of the key teaching practices that PSTs should begin to develop in teacher education programs. Jacobs et al. (2010) described this construct as involving the three interrelated processes of attending to students' mathematical thinking, interpreting its meaning and importance, and making instructional decisions to further support their mathematical reasoning. In this construct, the first two components are required to attain the third component successfully, which is the core of noticing. Jacobs et al. termed this combination of interrelated processes as *professional noticing,* in that teacher's noticing has the specialized purpose of making appropriate pedagogical decisions. Dietiker et al. (2018) later expanded the professional noticing framework to include curricular noticing, consisting of three aspects, attending, interpreting, and responding. Attending involves "looking at, reading, and recognizing aspects of curricular materials," interpreting refers to "making sense of that to which the teacher attended," and responding requires "making curricular decisions based on the interpretation (e.g., generating a lesson plan, a visualization, or an enactment)" (p. 89).

As a framework in this study, we followed Dietiker et al.'s (2018) conceptualization of curricular noticing. However, because PSTs have few opportunities to engage in teaching in a classroom, we focused on one activity from each component of the curricular noticing framework to design a task using prompts corresponding to the three interrelated curricular noticing skills. We limited to asking PSTs to solve three problems to examine how they *perceived* the problems (Attending), to report what they thought was required to solve them to probe how they *comprehended* the problems (Interpreting), and to sequence the three problems with rationales to investigate how they would use them in *lesson planning* (Responding).

Methods

Data in this study comprised the responses to a written task. The 155 PSTs enrolled in several sections of a mathematics methods course at a large Southwestern university. The participants were enrolled in an elementary teacher education program leading to certification to teach from pre-K to eighth grade in the state where the data were collected. The PSTs had completed three mathematics content courses about number and operations, geometric reasoning, and algebraic reasoning before taking this methods course. The written task used in the study consisted of two parts (see Fig. 1). In the first part, the PSTs were asked to solve three pattern generalization problems developed by Stump et al. (2012) and to explain the reasoning/knowledge required to solve the problems. In the second part, they were asked to determine the proper sequence of the three problems to support student learning and to explain the rationale for their sequencing.

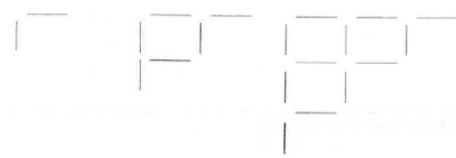
Part 1: Problem Solving
The following three problems are designed to help students learn to generalize pattern relationships. For each problem, (1) find an expression that you could use to find the *n*th term, and then (2) provide the kind of reasoning/knowledge required to solve each problem.

A. Hidden rectangle

Investigate the relationship between the number of rectangles that can be found in a row of squares and the length of the row by filling in the table. Remember that a square is a special kind of rectangle. For example, in the second figure, count each square and the rectangle formed from two squares, so there are three rectangles in the second figure.

B. Herringbone patterns

Shown below are the first three patterns in a sequence of herringbone designs. Investigate the relationship between the width of the pattern and the total number of line segments in the design by filling in the table.



C. Number of squares

Investigate the number of squares in each of the rectangles (including the hidden squares) by filling in the table. Find if there is a connection between the width of the rectangle and the total number of squares. For example, in the second figure, there are four 1x1 squares and one 2x2 square for a total of five squares.

Part 2: Sequencing the Three Problems

Revisit the three problems above and sequence the three problems in the best order for teaching upper elementary grade students (6-8th grade). Then provide your rationale for sequencing the three problems in that order.

Figure 1: Main Task of This Study

We applied an inductive content analysis approach (Grbich, 2007), which involved five processes: (a) reading each PST's response and creating codes based on the raw data, (b) identifying the correctness and themes of the responses, (c) creating categories and subcategories based on features of the PSTs' responses and solutions, (d) coding the categories and subcategories, and (e) interpreting the data quantitatively and qualitatively.

Findings

PSTs' Solutions

To probe how the PSTs solved the three pattern generalization problems, we examined whether they correctly identified the relationships and generalized expressions for the relationships they presented in their solutions. Table 1 shows the valid number sequence each problem contains, the generalized expressions for the number sequences identified, and the frequency of PSTs' correct/incorrect responses.

Table 1. Correctness of PSTs' Responses

Problem	number sequence	Generalized expression	Frequency (n = 155)		
			Correct	Incorrect	No attempt
Hidden rectangles	1, 3, 6, 10, 15…	$\frac{n^2+n}{2}$	13 (8.4 %)	129 (83.2 %)	13 (8.4 %)
Herringbone patterns	2, 6, 12, 20, 30…	n^2+n	88 (56.8 %)	52 (33.5 %)	15 (9.7 %)
Number of squares	2, 5, 8, 11, 24…	$3n-1$	51 (32.9 %)	97 (62.6 %)	7 (4.5 %)

As shown in the above table, there was a wide range of differences in correct and incorrect responses across the three problems. Only a small number of PSTs (8.4%) produced correct answers to all three problems. In particular, the incorrect responses were predominant in the hidden rectangles problem.

PSTs' Thoughts about Required Skills and Understanding

When asked about the necessary reasoning to solve each problem based on what they did to solve it, the PSTs proposed various ideas, from which three main themes emerged as common across all three problems: (a) knowing formulas, (b) explaining and identifying relationships, and (c) using problem-solving strategies. Also, generic statements and restating the generalized expression without additional explanations appeared across all three problems. For the hidden rectangles problem and the number of squares problem, geometry knowledge was additionally mentioned as a prerequisite for problem-solving. Table 2 shows PSTs' ideas about the required skills and understanding for solving the three problems.

Table 2. PSTs' Ideas about Required Skills and Understanding

	Examples	Hidden rectangles	Herring-bone patterns	Number of squares
Knowing formulas	• Use the formula • Remember the formula • Know the arithmetic sequence formula • Know the quadratic function formula	19 (12.3%)	21 (13.5%)	23 (14.8%)
Explaining /identifying relationships	• You must recognize that the number of rectangles is not increasing by a constant difference or ratio. • Each new rectangle adds 2 small squares and 1 hidden square, increasing by 3 each time.	41 (26.5%)	22 (14.2%)	11 (7.1%)
Problem-solving strategies	• Use a table • Trial and error: No distinct difference, so I made my own formula through trial and error. • Draw it out	11 (7.1%)	17 (11.0%)	13 (8.4%)
Generic statement without specifics	• Ability to find the pattern • Logical thinking • Algebra • Find relationships	27 (17.4%)	53 (34.2%)	24 (15.5%)

Simply Restating the generalized expression	• For (n + 1): Each place's answer is always that place x that place + 1 • For 3n + 1: Multiply 3 and add 1	20 (12.9%)	27 (17.4%)	45 (29.0%)
Geometry knowledge	• Need to know the definition of a square • Know that a square is also a rectangle • Need to see the rectangles inside rectangles	30 (19.4%)	0 (0 %)	24 (15.5%)
Other	• It is hard. • Unsure how to solve. • I don't know	0 (0 %)	4 (2.6%)	3 (1.9%)
No response		7 (4.5%)	11 (7.1%)	12 (7.7%)

PSTs' Suggested Problem Sequences and Rationales

PSTs were asked to sequence the three problems to support students' learning. Considering the cognitive demand and complexity, we agreed that the following is the proper sequence (C-B-A): C-Number of squares problem, B-Herringbone patterns problem, and A-Hidden rectangles problem. Table 3 shows the sequences PSTs proposed.

Table 3. Proposed Sequence by PSTs

	Problem A	Problem B	Problem C
First order	59 (38.1%) ***A***-B-C: 18 (11.6%) ***A***-C-B: 41 (26.5%)	9 (5.5%) ***B***-C-A: 4 (2.3%) ***B***-A-C: 5 (3.2%)	71 (45.8%) ***C***-A-B: 49 (31.6%) ***C***-B-A: 22 (14.2%)
Second order	54 (34.8%) B-***A***-C: 5 (3.2%) C-***A***-B: 49 (31.6%)	40 (25.8%) A-***B***-C: 18 (11.6%) C-***B***-A: 22 (14.2%)	45 (28.8%) A-***C***-B: 41 (26.5%) B-***C***-A: 4 (2.3%)
Third order	26 (16.5%) B-C-***A***: 4 (2.3%) C-B-***A***: 22 (14.2%)	90 (58.1%) A-C-***B***: 41 (26.5%) C-A-***B***: 49 (31.6%)	23 (14.8%) A-B-***C***: 18 (11.6%) B-A-***C***: 5 (3.2%)
No response	16 (10.3%)		

Twenty-two (14.2%) of the PSTs proposed the same sequence we proposed (C-B-A), including the 13 PSTs who solved all three problems correctly, whereas nine who produced the sequence of C-B-A did not solve all three problems correctly. Overall, Problem C was most frequently placed first in the recommended order, Problem A in second place, and Problem B in third.

The PSTs' justifications for their proposed sequencing of the problems are summarized in Table 4:

Table 4. PSTs' Justifications for Their Sequence

Category	Examples	Frequency
Difficulty	• Easier to harder (without explanation) • Easy: Start with the one that has a clear common difference • Easy: Start with the one that formula can be used • First, Problem C because it is easy to see the pattern, then	82 (52.9%)

	Problem B because it is easy to visualize, then Problem A because recursive reasoning can be confusing	
Scaffolding	• Problem A is the easiest, and Problem C is similar to Problem A • Because the first two build at each other and help them understand the third one.	20 (12.9%)
PSTs' own understanding/skills	• Starting with Problem C makes sense to me. I was able to understand this better than the others. • Problem B was complex for me, so I would not use it with students. • I will suggest the order in which I could complete them. • Problem B was more difficult for me, so I feel that the kids would also have difficulty understanding.	17 (11.0%)
Knowledge of geometry	• It requires knowing "squares are rectangles." • It is hard to find all of the hidden squares.	10 (6.5%)
Prior knowledge	• Use of known formula • Solving Problem B requires knowledge of exponents	6 (3.9%)
Importance	• Problem C is the most important	1 (0.6%)
General	• Critical thinking • Need to understand algebra	6 (3.9%)
No response		13 (8.4%)

More than 50% of the PSTs used the problem's difficulty as a criterion for the proposed sequence. However, we noted that the PSTs referred to the difficulty in many different ways, such as visuality, complexity, feasibility for directly applying formulas, and connections with multiple mathematical topics. We also noted that about 11% of the PSTs sequenced the problems based on their own understanding and skills, often mentioning what was easy or difficult for them. Although this category was also related to how the PSTs defined difficulty, we reported this rationale separately because these PSTs offered purely subjective interpretations based on their own abilities. Some PSTs also explained that Problems A and C were related because of their connection with geometry and juxtaposed them in the sequence. It is worth noting that the PSTs' incorrect problem-solving might have influenced their proposed sequence. For example, many PSTs produced incorrect answers by counting only 1x1 squares, resulting in straightforward arithmetic sequences, which they felt were easy.

Discussion and Implications

Gaps to Fill in PSTs' Understanding and Execution of the Problem

We noted that only a small number of PSTs (8.4%) found correct answers in all three problems. It is worth noting that these PSTs did not rely on the mechanical application of existing arithmetic or geometric formulas. Instead, they first looked for the problem's structure and generated patterns.

Some of our observations are worth further discussion when reviewing the incorrect responses. First, we found that most errors in the two problems (hidden rectangles problem and number of squares problem) occurred due to misunderstanding the problems. Specifically, when PSTs did not count all rectangles or squares (e.g., only counting 1x1 unit squares), they ended up with straightforward arithmetic sequences, such as the sequence with a common different 1 or 2. We intentionally used tasks that could create more than basic arithmetic or geometric sequences where the common difference or common ratio was not apparent using geometric contexts. However, unlike our expectations, our tasks could not fully facilitate many PSTs' in-depth investigative work. We conjecture that many PSTs were familiar with working on typical number patterns neatly presented for directly applying formulas. Their prior experience of what and how they learned pattern generalization does matter. If PSTs' prior experiences in pattern generalization were mainly focused on a rule-driven approach, it would be necessary to provide opportunities for PSTs to investigate atypical number patterns that may integrate other domains of mathematics, such as geometry.

Second, many PSTs who could identify the correct number patterns still failed to present generalizable expressions. This result was particularly prevalent in the herringbone pattern problem (e.g., 75% of the incorrect cases). A few apparent strategies included guess and check, as the generalized expression only worked for some, not for all terms. Also, regardless of the correctly identified number patterns, some PSTs presented the general form of an arithmetic sequence, considering the difference between the first and the second terms as the common difference for the entire patterns, showing the application of formulas in a rote manner. In Common Core State Standards for Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010), students are expected to explore patterns using various modes of representations, such as physical materials, drawings, tables, words, and symbols, to make sense of the regularities and to see the changes quantitively and abstractly. If this kind of exploration was not a substantial part of PSTs' prior experiences, they need to have such experiences in the teacher education program.

Instructional Sequence: What Matters

When asked to sequence the three pattern generalization problems, we noted that a small number of PSTs (22 PSTs, 14.2%) suggested the instructional sequence of the problems that was the same as the researchers' sequence. Apparently, the PSTs who misunderstood the problems and yielded simple arithmetic sequences for the hidden rectangles problem and the number of squares problem were unable to sequence the problems properly. Of these 22 PSTs, 13 PSTs were those who solved all three problems correctly. This result implies that teachers' content knowledge is necessary, although we cannot say it is the only sufficient factor for the appropriate instructional decisions.

In examining PSTs' justifications for their proposed sequence, "difficulty" was the most predominant reason. However, we noted that there was no consensus on what it meant by "difficulty." While some PSTs simply said "from easier to harder" without explanations, other factors, such as visuality, complexity, the feasibility of direct application of arithmetic sequence formula, and the connection with other topics, were also addressed. It is also noted that about 11% of PSTs relied on their own problem-solving ability and comfort/confidence level in sequencing the problems, assuming that students' problem-solving ability and comfort/confidence level would be similar to theirs. Also, this tendency might hinder PSTs from eliciting and interpreting students' thinking and offering appropriate interventions as needed, especially when students' approaches are different from PSTs'. Thus, teacher educators need to provide more opportunities for PSTs to examine and analyze strategies different from theirs and decipher the reasoning behind those approaches. It is not our intention to suggest that there is a pre-defined instructional sequence, and that PSTs need to master the pre-determined sequence. Instead, we suggest PSTs need to be aware of various factors that should be considered in the decision of instructional sequence and to build the ability to discern what factors should be foregrounded and backgrounded.

This study contributes to the current literature on curricular noticing and the knowledge base of teacher education. In particular, this study has implications for teacher educators working on designing mathematics education courses for PSTs and researchers interested in a better understanding teachers' knowledge and pedagogical strategies. That is, the findings of this study suggest that elementary mathematics teacher education programs need to include more curricular noticing activities in the form of sequencing tasks and other activities that enable PSTs to experience curricular noticing in the early stages of their teaching careers. This study also suggests the importance of further investigations of PSTs' curricular noticing ability as demonstrated in other mathematics teaching contexts.

References

Amador, J., Males, L., Earnest, D., &Dietiker, L. (2017). Curricular noticing: Theory on and practice of teachers' curricular use. E. O. Schack et al., (eds.), Teacher Noticing: Bridging and broadening perspectives, contexts, and frameworks, research in mathematics education (pp. 427-443). Cham, Switzerland: Springer

Dietiker, L., Males, L., Amador, J., & Earnest, D. (2018). Curricular noticing: A framework to describe teachers' interactions with curriculum materials. Journal for Research in Mathematics Education, 49(5), 521-532.

Grbich, C. (2007). Qualitative data analysis: An introduction. Thousand Oaks, CA: Sage.

Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. Journal for Research in Mathematics Education, 41(2), 169-202.

Kaput, J. (1999). Teaching and learning a new algebra. In E. Fennema & T. Romberg (Eds.), Mathematics classrooms that promote understanding (pp. 133-155). Erlbaum.

Males, L., Earnest, D., Dietiker, L., & Amador, J. (2015). Examining K-12 prospective teachers' curricular noticing. In T.G Bartell, K. N. Bieda, R. T. Putnam, K. Bradified, & H. Dominguez (Eds.), Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (pp. 88-95). East Lansing, Mi: Michigan State University.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). Common Core State Standards for Mathematics. Washington, DC: Authors.

Radford, L. (2008). Iconicity and contraction: A semiotic investigation of forms of algebraic generalizations of patterns in different contexts. ZDM: The International Journal on Mathematics Education, 40(1), 83-96.

Remillard, J. T. (2005). Examining key concepts in research on teachers' use of mathematics curricula. Review of Educational Research, 75, 211-246.

Stump S., Roebuck., K., Bishop, J. (2012). Algebra for elementary and middle school teachers: An inquiry approach (2nd ed). Pearson Learning Solutions.

"I UNDERSTAND IT EVEN MORE!" PROMOTING PRESERVICE TEACHERS' RELATIONAL UNDERSTANDING OF FRACTIONS

Jinqing Liu
University of California Irvine
jinqinl@uci.edu

Yuling Zhuang
Emporia State University
yzhuang@emporia.edu

Preservice teachers (PSTs) are expected to possess a relational understanding (i.e., knowing how to do and why) of mathematics for ambitious instruction. This study aimed to shed some light on the possibilities of supporting PSTs' development of relational understanding of fractions through engaging them in writing collective argumentation. Drawing data from a larger project; we explored the development of a PST's understanding of fractions through the engagement of collective argumentation. The results indicated that the PST's relational understanding of fractions developed from both structural and content perspectives. Some educational implications for teacher education are discussed.

Keywords: Mathematical Knowledge for Teaching, Preservice Teacher Education, Number Concepts and Operations, Elementary School Education

Developing mathematical understanding is the core of mathematics education. Skemp (1976) conceptualized that one's mathematical understanding can be considered as a continuum moving from *instrumental* (processing procedures or mathematical facts without understanding how they are connected) to *relationa*l (knowing how and why mathematical facts and procedures are related). In the field of mathematics education, there is agreement that we should engage All students in meaningful mathematics by supporting students in developing a relational understanding of mathematics (Hiebert, 1997; Ma, 2020; NCTM, 2014; Van de Walle et al., 2020). Given teachers' mathematical knowledge for teaching (Ball et al., 2008) plays an influential role in affecting students' opportunities of engaging in developing a relational understanding of mathematics (Hill et al., 2005), it is essential to prepare preservice teachers (PSTs) with a relational understanding of mathematics (AMTE, 2017). This study serves this broad goal: promoting PSTs' relational understanding of mathematics, with a focus on fractions.

The data of this study draw from a larger project which was designed to promote PSTs' fraction proficiency via engaging them in *writing collective argumentations online* (WCAO), an approach asking PSTs to write their justification for how and why a certain mathematical strategy (e.g., the common denominator strategy for ordering fractions) works (Liu, 2021). The larger project focused on the topic of fractions and the population of PSTs because of the following three major reasons. First, fractions play a fundamental role in mathematical learning, which can predict students' overall mathematics achievement in upper elementary, middle school, and high schools (Bailey et al., 2012; Brown & Quinn, 2007; Hansen et al., 2017; Siegler et al., 2012; Torbeyns et al., 2015). Second, students are often struggled with fractions and lack an understanding of fractional procedures (Booth & Newton, 2012; Carpenter, 1981; Cramer et al., 2002; Gómez & Dartnell, 2019; Liu & Jacobson, 2022; NMAP, 2008; Siegler & Pyke, 2013; Stafylidou & Vosniadou, 2004). In other words, although fractions are especially important, students' understanding of fractions is limited, which makes improving students' fraction learning opportunities urgent. Scholars argued teachers' fraction knowledge might play a role in this situation, which is related to the third reason that teachers, including pre-and in-service teachers, possess inadequate fraction knowledge for teaching fractions for understanding (Olanoff et al., 2014).

Situating in the broad context of the large project, the current study aims to shed light on understanding the possibility of improving PSTs' relational understanding of fractions and exploring an effective analytical approach to capture the progress toward a more relational understanding. Specifically, we zoomed in on one case, a PST named Hope, in which she used the common denominator strategy to solve a fraction ordering task to answer the following research question: *What relational understanding has the PST developed after engaging in WCAO?*

Theoretical Background

A Continuum of Mathematical Understanding: From Instrumental to Relational

This study employed a constructivist perspective that individuals construct mathematical understanding and meanings by connecting what they know and the new information they encounter. Skemp (1976) proposed that one's mathematical understanding can be thought of as a continuum that progresses from instrumental to relational (see Figure 1). Instrumental understanding refers to a situation in which an individual's knowledge is primarily made up of isolated facts, meaningless procedures, and rules learned by rote (see the left circle in Figure 1). On the other hand, a relational understanding enables an individual to comprehend how and why mathematical concepts and procedures are logically related (see the right circle in Figure 1). Van de Walle et al. (2020) argued that "understanding is a measure of the quality and quantity of connections a new idea has with existing ideas. The greater the number of connections to a network of ideas, the better the understanding" (p. 20). Thus, a more relational understanding can be understood as a process of making more qualified connections between mathematical ideas to one's network.

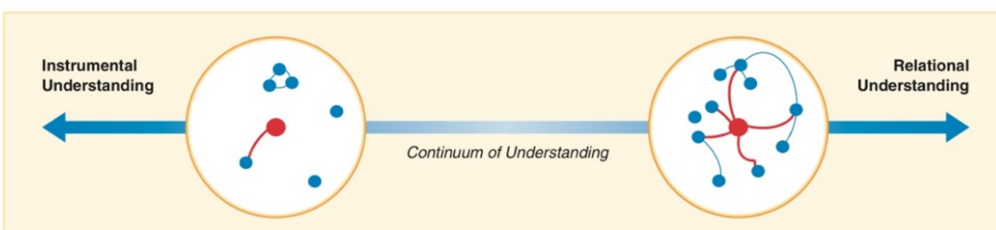


Figure 1: An Illustrative Model of Understanding Development (Van de Walle, 2020, p.20)

Toulmin's (1958/2003) Model for Collective Argumentation

As shown in Figure 2, according to Toulmin (1958/2003), an argument involves some combination of claims (statements whose validity is being established), data (support provided for the claims), warrants (statements that connect data with claims), rebuttals (statements describing circumstances under which the warrants would not be valid), qualifiers (statements describing the certainty with which a claim is made), and backings (usually unstated, dealing with the field in which the argument occurs).

In order to explore how a person's relational understanding of a certain mathematical object shifts, it is crucial to analyze both the quality and/or quantity of new connections an individual made between mathematical ideas. In this study, we adapted Toulmin's (1958/2003) model as an analytical framework because this model has been widely used in mathematics education literature to analyze how collective argumentation developed in a social setting concerning the content and structure of arguments (e.g., Krummheuer, 1995; Zhuang & Conner, 2022). In addition, Toulmin's (1958/2003) model provides us with a tool to identify how mathematical ideas are connected by examining how argument components (e.g., claims, warrants) are constructed.

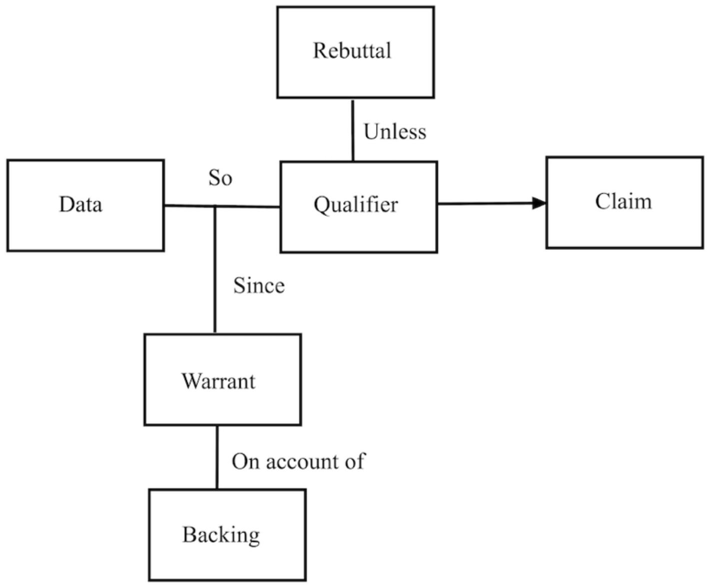


Figure 2: Structure of a Generic Argument (adapted from Toulmin, 1958/2003)

To frame one's understanding, this study followed the idea from Conner (2008) to use colors to denote different types of argument components: given data (green square), claim or data claim (blue square), explicit warrant (yellow circle), implicit (brown circle). This allows us to explore the structure of an argument and analyze the content of a specific argument component. In this way, from the lens of the structure of an argument, we assess the *quantity* of the connections according to the structure of an argument; and from the lens of the content, we assess the *quality* of the connections. These two lenses together help us to determine to what extent an individual develops a more relational understanding. For the purpose of this specific study, we focused on three main core components of argument (i.e., claim, data, warrant). If a component functions as both data in one argument and as a claim in a sub-argument, we label it as a data/claim. Sometimes, parts of an argument may not be explicitly stated but can be inferred were labeled as implicit.

Methodology

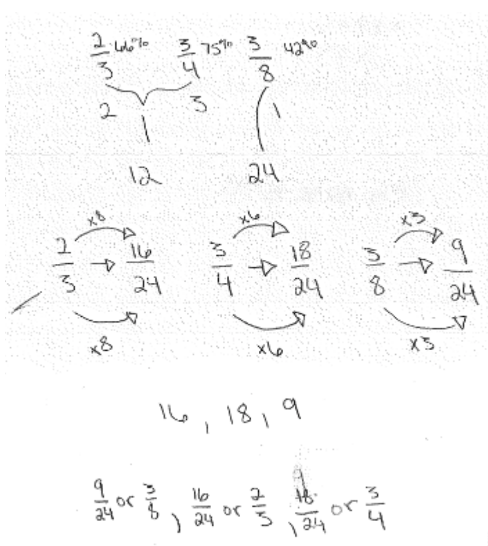
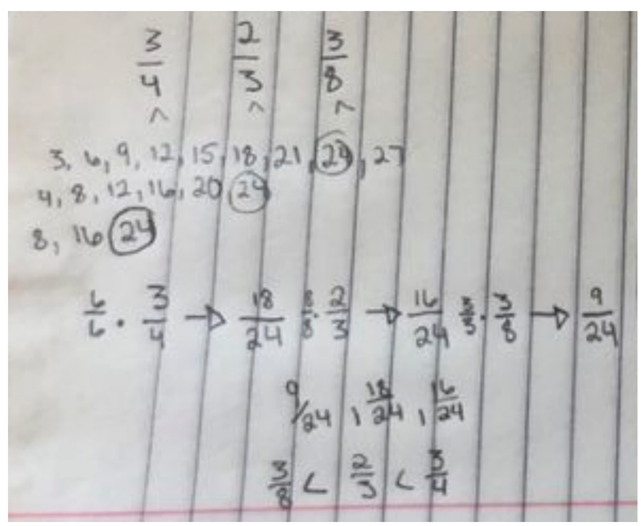
Setting and Participant

This study is part of a larger research project which aims to promote PSTs' fraction proficiency by engaging them in writing collective argumentations for each of the eight commonly used fraction comparison strategies (e.g., the common denominator strategy, the number line strategy, the fraction bar strategy) in an online setting (Liu, 2021). To assess the impact of the intervention on PSTs' fractional understanding, the larger project employed a pretest-post-test design. Each participant was asked to order 2/3, 3/4, and 3/8 from the smallest to the greatest using as many strategies as they could on a paper sheet and orally explain their solutions at the beginning and the end of the project (We refer to them as pre-and post-test). A reflective interview was also conducted at the end of the larger project.

We adopted Yin's (2014) selection procedure of typical cases (details see p. 46) and identified Hope as a typical case to focus on. Hope was a white female who engaged in a WCAO session using the common denominator strategy to order 2/3, 3/4, and 3/8. Hope was purposefully selected for this study because she expressed an interest in engaging in WCAO due to her fluency with fraction-related procedures but not confident with her conceptual understanding of fractions. Hope was in the second year of her undergraduate teacher education program and had completed two mathematics content courses (rational numbers and geometry) so far. We focused on examining Hope's understanding of the common denominator strategy because this strategy is the most wildly taught and used procedure-oriented strategy and was the strategy that Hope felt "the most comfortable with when we started this project" and "would just kind of default to it" before joining the project.

Data Source and Analysis

The data source used in this study includes the pre-and post-tests, where Hope used the common denominator strategy to order three fractions with explanations, and her reflective interview after the WCAO. The data has been video-recorded and transcribed for analysis (see Figure 3, for the sake of space, here, only show the beginning part of the transcript).

Pre-test	Post-test
	
Hope said while she gave the above solution in the pre-test, "I want to find a common denominator. So, this is, so between 3 and 4, the obvious one is, 2, is 12 to me. But then I also have to include eight, So, instead I'm going to use 24, and I should use a factor tree, but in my mind, the numbers just kind of pop out. And so, I'm going to set up my little tables to make these all out of 24. 24 is, yeah, it is divisible by three. So, to get to 24 from three, you have to multiply	Hope talked while she gave the above solution in the post-test, "I believe next I'm going to do the common denominator strategy, which this was the strategy that I was the most comfortable with when we started this project and like, I would just kind of default to it. Yes. Okay, So I'll start by listing out my fractions two thirds, three fourths, and three-eighths. So, the goal is to find a number where each denominator is a factor of it. And we want to find the least common multiple of these three fractions. So, like off the top my head obviously, I know what it is, but what younger

it by eight. So, I'm going to do the same thing on the top. So, two times eight becomes 16 and 24. And to get from four to 24, you have to multiply by six. So again, do the same thing on the top. And to get from eight to 24, you have to multiply by three. So, I'm going to multiply the top by three. And so, these two, this one [3/8] and this one [2/3] are kind of like the denominators are, I always think of them as like inverses of each other, because 8 times 3 is 24 and then 3 times 8 is 24. But now we can just compare the numerators. So, we have 16, 18, and 9…"

students will likely be taught to do is list out multiples of these denominators until they find one in common. So, 3,6, 9, 12, 15, 18, 21,24,27 And then we'll move on to 4, 8, 12, 16, 20, 24. So now I see that two of these have 24 in common. So, then I can check and see if eight would also go into 24, which evenly in which it will. So, 8, 16, 24. So that means that my least common multiple is 24. So now I'm going to setup fractions to convert them. So, 3 over 4 to something over 24, two thirds to something over 24. And what was last three eighths, something over 24. So, 3 over 4 to something… over 24, So, we want to multiply the denominators of each fraction by whatever factor it takes to make them into 24. But we also have to multiply that same number to the top in order to keep the fractions proportional. Because if we only multiply it a four by six to get it to 24. And we said it was 3 out of 24. That's not going to be the same ratio as three-fourths. Three-fourths isn't equivalent to this number. And so, what we're really doing is multiplying each number by one but by different versions of one to increase that number. So now I'm going to setup fractions to convert them... So, 3 over 4 to something over 24... So, to get four to 24, we have to multiply by six. So, six times three is 18. So, three-fourths becomes 18 over 24. So, we're really multiplying this fraction by six over six, which is the same as multiplying by one, but this just scales it up …"

Figure 3. Hope's Pre-and Post- Written Response and Partial of the Transcript of Her Verbal Response

We applied adapted Toulmin's model to explore Hope's understanding of the common denominator strategy in the pre-and post-tests through her participation in WCAO. Any differences in diagramming were discussed among researchers until a consensus was reached. We identified her talking about the common denominator strategy in her reflective interview after the WCAO to triangulate and contextualize our results.

Results

In this section, we report our results of Hope's understanding of the common dominator strategy in terms of structure and content changes from pre-test to post-test. The following Figure 4 shows the basic structure of Hope's argument from the pre-test and post-test. We found that she made two typical changes regarding an argument's structure and content. From the structure perspective, we saw Hope made two warrants from implicit to explicit, and added more warrants to justify her claims post-WCAO. Additionally, from the content perspective, Hope transferred from talking about the specific task only to talking about the common denominator strategy in general, especially the fundamental ideas. She used more than one warrant to support a claim after the WCAO. These changes suggest Hope has developed a more relational understanding of the specific strategy after engaging in the WCAO session.

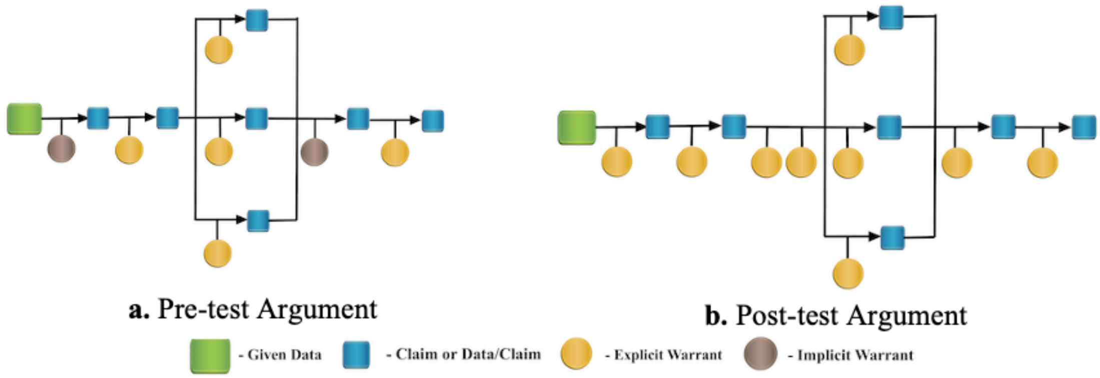


Figure 4. Toulmin's Model of Hope's Argument in the Pre-and Post-Test

Structure Changes

As Figure 4a and Table 1 show, Hope made seven claims and supported these claims with five explicit warrants and two implicit warrants in the pre-test. Hope made eight claims and supported the claims with nine explicit warrants in the post-test, as Figure 4b shows. Thus, Hope made one more claim backed up with two warrants and transferred two warrants from implicit to explicit after the WACO engagement. These results indicate that Hope increased the number of connections after engaging in WCAO.

Table 1. Summary of the Structure Changes

Argumentation Component	Pre-test	Post-test	Changes
Claims	7	8	**+1**
Explicit warrants	5	9	+2
Implicit warrants	2	N/A	Transferred two warrants from *implicit* to *explicit*

Content Changes

From comparing the content of specific argument components, we found that Hope made two essential changes: a) understanding the fundamental idea of the common denominator strategy and b) understanding the rationale of the procedure for finding equivalent fractions.

Understand the Fundamental Idea of the Common Denominator Strategy. One fundamental idea of the common denominator strategy is to understand why we want to apply it to compare 2/3, 3/4, and 3/8 rather than compare them directly. Hope developed her answers after WCAO. As we mentioned, the common denominator strategy was the fraction comparison strategy that Hope was "most comfortable with" and "just kind of default to" use. But she did not think about the why questions before WCAO. As Hope expressed in her reflective interview, "For common denominators, and I just thought of it as, oh, you know, we have the same denominator, so we can just add [compare] them." This expression suggests an implicit connection between the "same denominator" and "just [compare] them." However, Hope could articulate this connection explicitly after WCAO. She stated in the reflective interview, "It as we have a common denominator so that we only have to look at the numerator because before we were comparing two variables [denominator and numerator] and [now] we are only comparing one [numerator]." This statement shows Hope made connections between the number of variables, the affordance of the common denominator strategy, and the goal of fraction comparison, suggesting progress in understanding the fundamental idea of the common denominator strategy after WCAO.

The connections Hope made between the number of variables and the affordance of the common denominator strategy supported her in providing more warrants to her claim in the post-test. For instance, Hope claimed in the pre-test that "now we can just compare the numerators" without providing an explicit warrant of why we can only compare the numerator after she transferred all the fractions into the fractions with a common denominator 24. In contrast, in the post-test, Hope made a more precise claim that "By finding the common denominator we've made, so that we only need to look at the numerator," and provided an explicit warrant by stating that "Because every denominator is the same, so we know that all of the pieces are the exact same size... So, we only order the numerators from least to greatest." This example shows progress in Hope's relational understanding of the common denominator strategy, in which she transformed an implicit warrant (previous knowledge about only comparing the numerator) into an explicit warrant (using the common denominator strategy to make "all of the pieces are the exact same size," that is, to control one variable).

Understand the Rationale of the Procedures for Finding Equivalent Fractions. After WCAO, Hope also understood why we must multiply the same non-zero whole number by the numerator and denominator when finding equivalent fractions. Hope said in the reflective interview that she "knew you multiply by the numbers" but "wasn't understanding exactly how we create those equivalent fractions [works]" before WCAO. Nevertheless, she learned that "it [transferring to equivalent fractions] stays the same because you're really just multiplying by one and scaling it up by doing, like, four over four" from engaging in WCAO.

The new insight Hope gained that "you're really just multiplying by one" seemed to support Hope in providing two new explicit warrants to justify why the finding equivalent fraction procedure works in the post-test. Her first warrant used indirect reasoning, in which Hope stated in the post-test, "[For 3/4], because if we only multiply a four by six to get it to 24 [only multiply the denominator]. And we said it was 3 out of 24. That's not going to be the same ratio as three-fourths. Three-fourths isn't equivalent to this number." Here, Hope justified the rationale of multiplying the same number by the numerator by showing the mathematical conflict created by not doing so (3/24 "is not going to be the same ratio as" and "isn't equivalent" to 3/4).

The second warrant justified why the procedure of multiplying the same non-zero whole number to the numerator and denominator can maintain equivalence by connecting to the multiplicative identity property one that "a x 1 = 1 x a = a" and the fact that 1= n/n ($n \neq 0$). Hope stated this idea specifically in the post-test as follows,

What we're really doing [multiply the same number to the numerator or denominator] is multiplying each number [each fraction] by one but by different versions of one to increase that number [the denominator]. So [for fraction ¾] to get four to 24, we have to multiply by six. So, we're really multiplying this fraction by six over six, which is the same as multiplying by one, but this just scales it up. So, six times three is 18. So, three-fourths becomes 18 over 24.

Hope's above understanding could be expressed in an algebraical form that,

$$\frac{a}{b} = \frac{a \cdot n}{b \cdot n} = \frac{a}{b} \cdot \frac{n}{n} = \frac{a}{b} \cdot 1 = \frac{a}{b} \ = \frac{c}{d} \ (n \ \neq 0)$$

For the case of ¾, this reasoning process could be expressed as,

$$\frac{3}{4} = \frac{3 \cdot 6}{4 \cdot 6} = \frac{3}{4} \cdot \frac{6}{6} = \frac{3}{4} \cdot 1 = \frac{3}{4} = \frac{18}{24} \; (n \neq 0)$$

The above results showed that Hope had improved her relational understanding of the common denominator strategy by making connections between the procedure of finding equivalent fractions, indirect reasoning, ratio, different versions of one, and the property after engaging in WCAO.

Conclusions and Implications

We identified what relational understandings Hope developed from both the structural and content perspectives by examining Hope's use of the common denominator strategy in the pre-and-post-tests through Toulmin's (1958/2003) model. These findings exemplified that Toulmin's model could be a useful analytic tool to characterize the progress of a person's relational understandings by concerning both the content and structure lens. These two lenses provide a more comprehensive picture of understanding a person's development of relational understanding of mathematical concepts. Hope's case also shows that although a PST has mastered a strategy for solving tasks correctly, plentiful spaces for expanding their relational understandings exist. Collective argumentation (e.g., WCAO) could be an effective pedagogical approach because it allows PSTs to address their learning needs using their strength in mathematical procedures. Through engaging in collective argumentation, furthermore, WCAO provides a context for preservice teachers to understand collective argumentation better, which may support them in learning to implement collective argumentation in their future teaching. This study urges us to explore further how engaging in the WCAO session impacted Hope and other participants' development of rationale understanding of mathematical concepts.

We also notice this study's limitation in that we only examined one critical case study of Hope. We studied how Hope developed her conceptual understanding of fractions before and after participating in WCAO. Although we cannot claim that the findings will generalize to other learning of mathematical concepts with the WCAO approach, the findings may be generative for teaching and learning mathematics through an online collaborative learning community.

References

Association of Mathematics Teacher Educators. [AMTE] (2017). Standards for Preparing Teachers of Mathematics. https://amte.net/standards

Bailey, D. H., Hoard, M. K., Nugent, L., & Geary, D. C. (2012). Competence with fractions predicts gains in mathematics achievement. Journal of experimental child psychology, 113(3), 447-455.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content Knowledge for Teaching: What Makes It Special? Journal of Teacher Education, 59(5), 389–407.

Brown, G., & Quinn, R. J. (2007). Investigating the Relationship between Fraction Proficiency and Success in Algebra. Australian Mathematics Teacher, 63(4), 8-15.

Booth, J. L., & Newton, K. J. (2012). Fractions: could they really be the gatekeeper's doorman? Contemporary Educational Psychology, 37(4), 247–253.

Cobb, P. (1994). Where is the mind? Constructivist and sociocultural perspectives on mathematical development. Educational Researcher, 23, 13–20.

Carpenter, T. P. (1981). Results from the Second Mathematics Assessment of the National Assessment of Educational Progress. National Council of Teachers of Mathematics, Inc., 1906 Association Dr., Reston, VA 22091.

Cramer, K., Post, T., & DelMas, R. (2002). Initial Fraction Learning by Fourth- and Fifth-Grade Students: A Comparison of the Effects of Using Commercial Curricula with the Effects of Using the Rational Number Project Curriculum. Journal for Research in Mathematics Education, 33(2), 111-144. doi:10.2307/749646

Gómez, D. M., & Dartnell, P. (2019). Middle schoolers' biases and strategies in a fraction comparison task. International Journal of Science and Mathematics Education, 17(6), 1233-1250.
Hansen, N., Rinne, L., Jordan, N. C., Ye, A., Resnick, I., & Rodrigues, J. (2017). Co-development of fraction magnitude knowledge and mathematics achievement from fourth through sixth grade. Learning and Individual Differences, 60, 18-32.
Hatano, G., & Inagaki, K. (2003). When is conceptual change intended? A cognitive sociocultural view. In G. Sinatra & P. Pintrich (Eds.), Intentional conceptual change (pp. 407–427.) Mahwah, NJ: Lawrence Erlbaum Associates, Inc
Hiebert, J. (1997). Making sense: Teaching and learning mathematics with understanding. Heinemann, 361 Hanover Street, Portsmouth, NH 03801-3912.
Hill, H. C., Rowan, B., & Ball, D.L. (2005) Effects of teachers' mathematical knowledge for teaching on student achievement. American Educational Research Journal. 42, 371-406.
Holm, J., & Kajander, A. (2012). 'I Finally Get It!': developing mathematical understanding during teacher education. International Journal of Mathematical Education in Science and Technology, 43(5), 563-574.
Krummheuer, G. (1995). The ethnography of argumentation. In P. Cobb & H. Bauersfeld (Eds.), The emergence of mathematical meaning: Interaction in classroom cultures (pp. 229– 269). Hillsdale, NJ: Erlbaum.
Liu, J. (2021). Advancing Fraction Proficiency via Deliberate Fraction Comparison.[Dissertation]. Indiana University.
Liu, J., & Jacobson, E. (2022). Examining US elementary students' strategies for comparing fractions after the adoption of the common core state standards for mathematics. The Journal of Mathematical Behavior, 67, 100985. https://doi.org/10.1016/j.jmathb.2022.100985
National Council of Teachers of Mathematics [NCTM]. (2014). Principles to Actions: Ensuring Mathematical Success for All, Author.
National Mathematics Advisory Panel [NMAP]. (2008). Foundations for success: The final report of the National Mathematics Advisory Panel. Washington, DC: U.S. Department of Education.
Siegler, R. S., Duncan, G. J., Davis-Kean, P. E., Duckworth, K., Claessens, A., Engel, M., Susperreguy, M. I., & Chen, M. (2012). Early predictors of high school mathematics achievement. Psychological science, 23(7), 691-697.
Siegler, R. S., & Pyke, A. A. (2013). Developmental and individual differences in understanding of fractions. Developmental Psychology, 49(10), 1994.
Stafylidou, S., & Vosniadou, S. (2004). The development of students' understanding of the numerical value of fractions. Learning and Instruction, 14(5), 503-518.
Torbeyns, J., Schneider, M., Xin, Z., & Siegler, R. S. (2015). Bridging the gap: Fraction understanding is central to mathematics achievement in students from three different continents. Learning and Instruction, 37,5-13.http://dx.doi.org/10.1016/j.learninstruc.2014.03.002
Toulmin, S.E. (2003). The uses of argument (updated ed.). New York: Cambridge University Press. (First published in 1958).
Van de Walle, J. A., Karp, K. S., & Bay-Williams, J. M. (2020). Elementary and middle school mathematics. Pearson Education.
Yin, R. K. (2014). Case study research: Design and methods. Los Angeles, CA: Sage.
Zhuang, Y. & Conner, A. (2022). Secondary mathematics teachers' use of students' incorrect answers in supporting collective argumentation. Mathematical Thinking and Learning, 26, 1-24. https://doi.org/10.1080/10986065.2022.2067932

LITERATURA INFANTIL Y MODELACIÓN MATEMÁTICA

CHILDREN'S LITERATURE AND MATHEMATICAL MODELING

Samantha Rivera
Universidad Autónoma de Coahuila
Samanthaq.rivera@gmail.com

José David Zaldívar
Universidad Autónoma de Coahuila
david.zaldivar@uadec.edu.mx

La modelación matemática, vista como estrategia didáctica, busca entender la matemática como una herramienta que permite la resolución de problemas en contextos extra-matemáticos. Los tipos de contextos que se pueden utilizar en modelación matemática suelen ser diversos, siendo los principales aquellos relacionados con situaciones de la vida cotidiana y los relacionados con otras ciencias como la física, química, biología, entre otras. En la presente investigación se describe una situación cuyo contexto se relaciona con la literatura infantil, que desencadena un proceso de modelación matemática. La implementación de esta se llevó a cabo con profesores en formación de educación primaria. Los resultados muestran las representaciones informales que los profesores y las profesoras utilizaron y las diferentes estrategias para llegar a la respuesta correcta.

Palabras clave: Mathematical representations, Modeling, Elementary School Education

La modelación es un proceso que da inicio mediante la conceptualización de una situación problema, para posteriormente buscar su simplificación y estructuración, que culmine con un proceso de matematización donde se usa o desarrolla un modelo matemático. Los resultados matemáticos son trasladados de nuevo al dominio extra matemático donde el problema estaba localizado (Niss, et al., 2007). De acuerdo con Pollak (1969, 2007) las etapas que contempla el ciclo de modelación matemática son: Identificación de una pregunta del mundo real que se quiere entender; Selección de objetos particulares importantes para la pregunta hecha e identificar relaciones entre ellos; Decición cuáles son útiles e ignorar los que no lo son; Traslado de esta versión en términos matemáticos, para la obtenención de fórmulas matemáticas para esta pregunta determinada y resolver el problema.

De acuerdo con California State Board of Education (2015), las situaciones de modelación matemática comparten características como los siguientes: son desconocidas, memorables, relevantes, tienen más de una respuesta correcta, tienen un proceso de resolución extenso, son complejas, los estudiantes descubren procesos, no tienen límites fijos, son cíclinas y no aparecen en un lugar en particular. Para Lesh y Doerr (2003), los problemas planteados en una perspectiva de modelación, buscan que el estudiante descubra ideas y construya procesos y habilidades que se necesitan para dar respuesta a problemas de la vida real, a diferencia de una vista más tradicional donde se necesita conocer previamente los requisitos y herramientas en situaciones descontextualizadas.

En la investigación de Robledo (2019) se especifican los principales contextos que se utilizan en los examenes PLANEA (Plan Nacional para la Evaluación de los Aprendizajes). Los reactivos de dicha prueba, plantean problemas que involcuran contextos de la vida cotidoana del alumno (compras, rutas, tiendas), así como problemas relacionados con el estudio de otras ciencias como física, química o biología. Algunos estudios también describen entre los contextos principales, el uso de matemáticas lúdicas (Quiroz, 2015) como lo son problemas relacionados con juegos de azar, o desarrollos tencológicos o videojuegos. La investigación actualmente busca reconocer estas tipologías y describirlas en el uso y desarrollo de situaciones basadas en modelación matemática. La presente investigación tiene como objetivo el diseño de una situación basada en la modelación matemática donde se utilice un contexto desde la literatura infantil. Posterior a esto, es de interés reconocer las representaciones iniciales que surgen en el proceso de resolución de la situación, mediante su aplicación con la metodología ACODESA.

Representaciones iniciales

El estudio de representaciones matemáticas ha sido el objeto de la Teoría de las Representaciones Semióticas de Duval (año). Para esta teoría, el aprendizaje de las matemáticas se propicia mediante el tránsito entre diferentes representaciones entre las que destacan: la algebraica, la gráfica, la numérica, entre otras. Ahora bien, estas representaciones son en su totalidad aceptadas por la escuela como válidas en el estudio de las matemáticas, por lo que pueden clasificarse como representaciones institucionales. Sin embargo, estudios recientes han mostrado que los alumnos difícilmente establecen representaciones institucionales cuando dan respuesta a un problema basado en modelación.

Ante una situación no rutinaria, basado en un contexto matemático, se produce un pensamiento diversificado que lleva a los alumnos a dar respuestas expresadas mediante representaciones, que se alejan de aquellas representaciones descritas en la Teoría de las Representaciones Semióticas. Estas representaciones no institucionales han sido estudiadas por diversos autores como Hitt y Quiroz (2017).

La representación funcional espontánea (RFE) es aquella que emerge de la acción que realiza un objeto frente a una tarea no rutinaria. Las representaciones iniciales, poseen un carácter funcional y espontáneo, ya que estas surgen de manera natural y persiguen una meta de aprendizaje. Es importante el estudio de estas representaciones ya que nos ayudan a comprender el proceso de aprendizaje de conceptos matemáticos. Además de esto, también se debe prestar atención en cómo estas evolucionan hacia representaciones institucionales, mediante un proceso de objetivación inmerso en tareas de colaboración y comunicación entre los sujetos. Por lo cual, un concepto también importante son las representaciones grupales, las cuales se centran en la evolución de estas cuando son sometidas a un proceso de discusión, comunicación, argumentación, comparación y validación frente a una tarea que requiere el trabajo en equipo en el aula de matemáticas. (Hitt y Quiroz, 2017).

Metodología de la investigación

La investigación se enmarca en un paradigma fenomenológico. Existen diversas maneras en las que se pueden interpretar las experiencias, además de que el significado que se le dé a estas constituye la realidad. Es por esto por lo que se dice que la realidad es socialmente construida. (Valenzuela y Flores, 2012). Se hace uso de una metodología cualitativa, puesto que se busca describir y comprender el problema de investigación (Valenzuela y Flores, 2012). Las carcaterísticas de la población fueron docentes en formación inicial de educación primaria de cuarto semestre. La muestra estuvo compuesta por 15 docentes, de los cuales fueron 10 mujeres y 5 hombres.

La aplicación se llevó a cabo en una sesión de 90 minutos. Las técnicas utilizadas para la recolección y el análisis de los datos fueron el análisis de documentos y las observaciones participantes. Se apoyó de una videocámara para el llenado de los instrumentos diseñados.

Diseño de investigación y Situación propuesta

Para la realización de la investigación se siguió el siguiente diseño, el cual consta de 3 etapas:

Etapa 1. *Diseño*: En esta primera etapa, se llevó a cabo una revisión de literatura, esto primeramente para encontrar un tema interesante e innovador. Una vez encontrado este, se procedió a revisar artículos, libros, tesis, tanto en inglés como en español. Después se realizó un diseño preliminar de la situación de investigación el cual fue sometido a un juicio del expertos, para así recibir recomendaciones y hacer los cambios que fueran pertinentes.

Etapa 2. *Aplicación*: Una vez que la situación de investigación (SI) fue modificado y mejorado, se llevó a cabo la aplicación de este. Durante la aplicación del instrumento se utilizaron distintas herramientas, así como cámaras fotográficas y de video, hojas de máquina, cuadernos, plumas.

Etapa 3. *Análisis*: Una vez que se recolectaron los datos se llevó a cabo una base de datos para que a través de ella fuera más sencillo analizar la información. Durante esta etapa también se realiza el escaneo de información relevante.

La situación propuesta nace del libro infantil “El regreso del gato ensombrerado” del autor Dr. Seuss. El problema establece lo siguiente:

> *En el libro “El regreso del gato ensombrerado” del Dr. Seuss, el Gato, para apoyar a los niños a limpiar el desastre, saca de su sombrero a sus amigos. Primero sale de su sombrero Gatito A. Gatito A es del tamaño del sombreo del Gato (incluyendo el sombrero del Gatito A). Del sombrero de Gatito A, salió Gatito B. Así fueron apareciendo Gatitos hasta que llegó Gatito Z.*
> *¿Cuánto mide Gatito Z? (Ver figura 1)*

Figura 1. Imagen del libro de Dr. Seuss

Discusión actual

La investigación ctualmente se encuentra en fase de análisis de datos. Los resultados preeliminares muestran que los docentes en formación dieron respuesta a la situación planteada utilizando diferentes procedimientos. Los procedimientos fueron planteados mediante la creación de representaciones funcionales espontáneas, que contienen elementos pictográficos, pero también elementos matemáticos como números, figuras geométricas y algunos algoritmos.

El contexto utilizado fue muy interesante para los futuros profesores, quienes analizaron en un primer momento el cuento completo, y posteriormente mediante los cuestionameintos atendieron la problemática planteada.

Se busca que mediante el análisis se puedan identificar los elementos de cada una de las representaciones y organizar los tipos de procedimientos que se realizaron para la resolución de la situación planteada.

References

California State Board of Education. (2015). Appendix B: Mathematical modelling of the mathematics framework for California public schools: kindergarten through grade twelve. En California mathematics framework (págs. 792-817). Sacramento: California Departament of Education.

Hitt, F. y Quiroz, S. (2017). Aprendizaje de las matemáticas a través de la modelación matemática en un medio sociocultural ligado a la teoría de la actividad. Revista: Colombiana de Educación. Núm. 73. pp. (151-175). Colombia: Universidad Pedagógica Nacional.

Lesh, R., & Helen, D. (2003). Foundations of a models and modeling perspective on mathematics teaching, learning, and problem solving. En R. Lesh, & H. Doerr, Beyond constructivism. Models and modeling perspectives on mathematics problem solving, learning and teaching (págs. 3-33). Mahwah, New Jersey: Lawrence Erlbaum Associates, Publishers (LEA).

Niss, M., Blum, W., & Galbraith, P. (2007). Modelling and applications in Mathematics Education.

Pollak, H. (1969). How can we teach applications of Mathematics? Educational S, 2(2), 393–404. doi:10.1007/BF00303471

Pollak, H. (2007). Mathematical modelling- A conversation with Henry Pollak. Modelling and Applications in Mathematics Education, The 14th ICMI Study, 10(28), 109–120. doi:10.1007/97803872982219

Quiroz, S. (2015). Concepciones sobre modelación matemática de docentes en formación de educación primaria, Tesis de doctorado no publicada, México: Tecnológico de Monterrey.

Robledo, A. (2019). Análisis de principios de modelación matemática en reactivos de la prueba planea, Tesis de licenciatura no publicada, México: Universidad Autónoma de Coahuila.

Valenzuela, J., y Flores, M. (2012). Paradigmas de investigación. En J. Valenzuela, & M. Flores, Fundamentos de investigación educativa (Vol. 1, págs. 31- 49).

MATHEMATICS LEARNER AND MATHEMATICS EDUCATOR IDENTITIES IN A PROBLEM-SOLVING FOCUSED MATHEMATICS CONTENT COURSE

Stephen L. Caviness
Syracuse University
scavines@syr.edu

Joanna O. Masingila
Syracuse University
jomasing@syr.edu

In this study, we sought to understand what might be learned about preservice teachers' mathematics learner and mathematics educator identities in the context of a problem-solving focused mathematics content course. Twenty-two preservice teachers participated in this study as part of their undergraduate teacher preparation program. We implemented a narrative identity lens which operationalized stories as identity with a sociocultural understanding of learning. In our findings we discuss examples of how mathematics learner identity influenced mathematics educator identity, and vice versa. In addition, we emphasize the importance of the way that preservice teachers define mathematics and how this influences mathematics identities.

Keywords: Preservice Teacher Education, Problem Solving, Instructional Vision, Affect, Emotions, Beliefs, and Attitudes

The purpose of studying identity in mathematics education is to emphasize the whole person—inclusive of but expanding beyond one's mathematical skills and knowledge (Lerman, 2009). In this sense, mathematics identity research enables a broad view of "what counts as mathematics" and where mathematics learning occurs (Saxe, 1988, p. 14). Researchers of mathematics identity acknowledge the multiple identities that an individual holds and seek to understand how those identities interrelate (Berry 2008; Gresalfi & Cobb, 2011; Wenger, 1998). These multiple identities are enacted in the many roles that one performs (Darragh, 2015). In the context of teacher education, an undergraduate preservice teacher plays the roles of both *learner* and *teacher* while simultaneously enacting countless other identities, perhaps of roommate, artist, first-generation student, etc.

Undergraduate preservice teachers enroll in a university with already well-developed concepts of what it means to play the role of teacher (Lortie, 1975). Still, the formal action of enrolling in a teacher education program suggests an intentional step toward developing one's understanding of this role. This particular juncture of entering a teacher-education program seems an appropriate context to take up calls for more research understanding the link between learner and teacher identity (Graven & Heyd-Metzuyanim, 2019; Lutovac & Kaasila, 2018). By maintaining a whole-person-view of preservice teachers' multiple identities, we aim to contribute to further understanding of both mathematics learner and mathematics teacher identities.

A Problem-Solving Focused Mathematics Content Course

Preservice teachers can be expected to teach similarly to how they were taught without some means of intervention to create a "break with experience" (Ball, 1990, p. 11). The context of this study is a mathematics content course for primarily first-year undergraduate preservice teachers that attempts to intervene in this cycle of teaching. In doing so, instructors of this course not only teach content knowledge, but model pedagogical strategies that preservice teachers might adopt in their future practice to engage all students in learning.

The curriculum developers of this problem-solving focused course assume that enrolled students have generally experienced a traditional, teacher-centered mathematics classroom in which direct instruction of procedures to follow is the norm (Masingila et al., 2011). They push back on a traditional, or instructionist model of teaching (Sawyer, 2006), and have designed the course to encourage learning through collaboration and discussion. In this way, the course holds promise for interrupting previously developed conceptions of what it means to learn and to teach mathematics. Given this background, the course is an appropriate setting for understanding participants' existing and developing roles of both learner and educator. Our research was guided by the following research question: What might be learned about preservice teachers' mathematics identities of *learner* and *educator* in the context of a problem-solving focused mathematics content course?

Mathematics Identity

Researchers of mathematics identity have called on the field to be explicit in defining the construct to ensure theoretical and methodological coherence and also to prevent the term from becoming synonymous with other related but different constructs such as beliefs or attitudes (Darragh, 2016; Sfard & Prusak, 2005). In response to this call, we first describe key features of mathematics identity and then expand further on two sub-constructs of mathematics identity: mathematics learner identity and mathematics educator identity.

A key feature of mathematics identity is performance—identity is what we do (Darragh, 2015). In addition to this, mathematics identity can be viewed as storied—it is the stories we tell (Sfard & Prusak, 2005). One has multiple identities and may perform them at the same time (Berry 2008; Darragh, 2015). Not only this, but these multiple identities interact together and need not be confined to the mathematics classroom. For example, Berry (2008) found that alternative identities, such as religious identity or athletic identity, contributed to the success of African American boys by helping them to overcome barriers that may have otherwise limited their access to more advanced mathematics. From a socio-cultural perspective, these performances of identity are formed within one's social environment while simultaneously shaping that environment (Wenger, 1998). In addition to these key features, mathematics identity over the last few decades has split into sub-categories of learning and teaching (Darragh, 2016; Graven & Heyd-Metzuyanim, 2019; Graven & Lerman, 2020).

Mathematics Learner Identity

Mathematics learner identity can be thought of as a category under the "umbrella" of mathematics identity concerning the identities of those learning mathematics. Darragh and Radovic (2020) defined mathematics learner identity as

> a socially produced way of being, as enacted and recognized in relation to learning mathematics. It involves stories, discourse and actions, decisions, and affiliations that people use to construct who they are in relation to mathematics, but also in interaction with multiple other simultaneously lived identities. (p. 582)

Though researchers may use the more general term mathematics identity, the content of their studies have largely focused on mathematics learners. For example, Cribbs et al. (2021) developed a questionnaire intended to be a proxy for mathematics identity. They found that the constructs of recognition and interest were predictors of whether learners participating in their study were likely to pursue a STEM-related career. Similarly, Bishop (2012) used the term mathematics identity and focused on learners. Bishop's work displayed how discourse between two seventh-grade students not only revealed their mathematics learner identities but also how those identities influenced and shaped one another as learners.

Mathematics Educator Identity

Mathematics educator identity is the act of being or becoming a mathematics educator and a sense of belonging to mathematics education (Graven & Lerman, 2020; Skott, 2019). Many researchers adopt the term mathematics teacher identity as their studies involve practicing educators with the titles of "mathematics teacher" or "elementary teacher." For example, Drake et al. (2001) studied elementary teachers and found that mathematics teacher identity influences whether a teacher is likely to adopt reform-oriented mathematics instruction. For cases such as this, Lutovac and Kaasila (2018) suggested the term mathematics-related teacher identity to acknowledge the fact that not all mathematics teachers are specialists in mathematics (i.e., elementary vs. secondary school teachers). Building on this, we are proposing a shift in terminology from mathematics teacher identity or mathematics-related teacher identity to *mathematics educator identity*.

In using this term, we hope to expand what it means to be a mathematics teacher. In the context of a school, this shift in language acknowledges that counselors, librarians, administrators, and personnel indeed contribute to a mathematics learner's development. Beyond the school, this term acknowledges the mathematics educating role that family members and community stakeholders play. In addition, adopting mathematics educator identity carries implications for researchers of mathematics identity. This shift necessarily expands a currently limited research scope focused on professionals with titles similar to "mathematics teacher" to include the many contexts where mathematics learning occurs and the influence of a wide range of educators who are in fact *teaching* mathematics.

Regardless of terminology, some researchers investigating mathematics identity have considered both mathematics learner and educator identities. For example, Gujarati (2013) studied the mathematics identities of three elementary teachers through autobiographies, interviews, and observations. Though she did not explicitly describe learner and educator identities as subsets of mathematics identity, Gujarati found that her participants' previous negative experiences as mathematics learners positively influenced who they had become as mathematics teachers. In addition, Maloney and Matthews (2017) found that a teacher's care for students, mediated by a sense of connectedness in the classroom, is crucial for students' development of a positive mathematics learner identity and perceived relevance of mathematics, especially for traditionally disenfranchised students. These examples show how a distinction between mathematics learner and mathematics educator identities is helpful for understanding how these dual identities are enacted.

Narrative Identity

We approached this research with the theoretical lens of narrative identity. Drawing primarily on Sfard and Prusak's (2005) operationalization of stories as identity and Wenger's (1998) communities of practice, narrative identity emphasizes stories as a reification of identity while simultaneously viewing mathematics identity through a broader socio-cultural lens (Langer-Osuna & Esmonde, 2017). Sfard and Prusak (2005) viewed identity as the many stories that one says about themselves, and especially the first-person stories that one shares to themselves. Instead of viewing identity as an abstract and elusive "core" of one's person, narrative identity defines identity as the stories one tells. This definition makes identity accessible to researchers through methods that elicit stories of both learners and educators such as interviews or autobiographies (Langer-Osuna & Esmonde, 2017). These stories may take the form of actual or designated identities. Actual identity stories are told in the present tense while designated identity stories envision oneself in the future.

In the narrative identity framework, learning is inherently linked to identity because learning is defined as shifts in participation recognized by others (Esmonde, 2017; Lave & Wenger, 1991). Wenger (1998) discussed the importance of discourse in defining identity and emphasizes the lived experiences within a community that shape one's identity. This perspective views an individual's identities as constantly evolving and shaping one another through one's participation in a community. In other words, the stories one shares are reflections based on their experiences of participation in a community context.

These perspectives of identity-as-stories and learning in the context of a community are compatible in that an individual's acts of participation in a social setting inevitably become the stories they share about their experiences (Darragh 2013; Langer-Osuna & Esmonde, 2017). For the purpose of understanding preservice teachers' mathematics learner and mathematics educator identities, these stories allow insight into how participants simultaneously embody these dual identities. Furthermore, the problem-solving mathematics content course provides a setting to frame preservice teachers' experiences as stories. Applying the narrative framework in this study allows for operationalization of participants' identity-as-stories while contextualizing the influence of the problem-solving mathematics content course.

Data Collection and Analysis

This study took place at a private university in the northeastern United States. Participants in this study were recruited across three sections of the problem-solving focused mathematics content course. This course is a requirement for preservice teachers beginning in early childhood and elementary teacher-preparation programs. The course sequence covers two semesters for a total of six credits, however, data collected in this study focused solely on the first semester of the course.

The purpose of the study was explained to participants, and they were invited to volunteer by submitting three artifacts of data. The first two artifacts were a written mathematics autobiography, which was an assignment as part of the course, and an online questionnaire developed for this study. These two artifacts were collected within the first week of the semester to establish participants' mathematics identities at the beginning of the course. A third data artifact took the form of a semi-structured, in-person interview in the final weeks of the semester to understand how the course may have influenced participants' mathematics identities. In total, 22 preservice teachers participated in the study and were evenly spread across all three sections.

To organize data, interviews were transcribed and then all artifacts were uploaded into and analyzed via MAXQDA software (VERBI Software, 2021). Data were analyzed following Corbin and Strauss' (2008) approach to qualitative research. Analysis began with immersion in the data by reading through all data artifacts that were collected. Memos were written after reading through each artifact. Data were then coded using comparative analysis focused on similarities and differences between incidents and axial coding to understand how concepts broadly relate to one another (Corbin & Strauss, 2008). The researchers made passes through the data until conceptual saturation was achieved and themes were developed. The data excerpts displayed below were selected because of their richness in exemplifying the themes we generated when analyzing data; however, we do not intend to suggest that this is an exhaustive display of all such examples from the data set.

Results and Implications

Based on our review of relevant research literature, we predicted that the context of a problem-solving focused mathematics content course would produce valuable insights into understanding preservice teachers' mathematics learner and mathematics educator identities. Below we share the results from our study. Note that all names used are pseudonyms.

Mathematics Educator Identity Influences Mathematics Learner Identity

As preservice teachers in this study were not yet teaching in field placements outside of the university, their mathematics educator identities took the form of designated identities. One's designated identity "[consists] of narratives presenting a state of affairs which, for one reason or another, is *expected* to be the case, if not now then in the future" (Sfard & Prusak, 2005, p. 18, emphasis in original).

For example, Julie's personal background and experiences have resulted in a passion for educating elementary students with special needs. This designated identity as an educator directly influenced Julie's mathematics learner identity by generating motivation and a sense of purpose for her work in the problem-solving focused mathematics course. She maintained a clear focus on her role of teaching future students while enacting the role of mathematics learner. Julie shared:

> I know that, like, the point of doing this right now is not to learn how to find the least common multiple, or, like, what the factors of the prime factorization of a number is. It's more like understanding how to teach it, which is like what I want to do. So, it's more beneficial to me. … [I] see myself developing as, like, a better critical thinker, which I know will help me in teaching and like other ways of my life. I, like, kind of see the purpose now.

Julie maintained awareness of her designated identity of mathematics educator throughout the course which supported her current role as a mathematics learner. In another example, we asked participants at the end of the semester what they might title their time in the content course as if it were one chapter of their broader story related to mathematics. Julie titled her chapter as "math benefitting my future."

In contrast with Julie, some preservice teachers' mathematics educator identities negatively influenced their mathematics learner identities. For example, Sadie did *not* view mathematics as relevant to her designated role as an educator. She shared, "I want to be an early intervention specialist. So that's why, like, this class does nothing for me, not, like, not, like, nothing, but you know, I would rather not take it."

Though Sadie's goal is to be an educator, she did not see her future role specifically involving mathematics. When asked generally how the content course was going, Sadie shared,

> It's going fine but I don't like it. I'd say it's just, like, thinking about math in a different way, which is … I don't want to think about math in a different way. So that makes it difficult. But in general, I feel like I have an okay understanding of it.

With our definition of mathematics educator identity, we believe that Sadie would still very much play a role in educating her future students in mathematics, even if her specific title is "intervention specialist." In this role she may not directly teach mathematics but, at the very least, her mathematics identity will shape and be shaped by the future school community that she participates in. At most, it is quite possible that her desired role as an intervention specialist would involve directly working with students to facilitate and assess their learning in mathematics. Regardless of the specifics of Sadie's future role, it is clear that Sadie's designated mathematics educator identity shaped her role as a learner because she did not see mathematics' relevance to her goals. Sadie's story exemplifies how mathematics educator identity may shape mathematics learner identity in a negative way.

Mathematics Learner Identity Influences Mathematics Educator Identity

One unit in the course was intended to teach students about place value and required problem solving in bases other than base 10. This particular unit seemed to highlight how mathematics learner identity influences mathematics educator identity as many students articulated how their experience as learners in this context facilitated an awareness of how their future students might feel. For example, Jeff shared,

> Like, it was difficult, but I feel, like, that was the most valuable thing, like, or the not necessarily the most valuable, but it definitely sticks out the most in my mind in terms of when I had a moment where I was, like, Whoa, like, that was, like, an eight-year-old in my classroom. You know, I mean, like, me now, like, that's gonna be someone in my classroom. You know?

Jeff's experience in his role as a learner prompted him to consider his future role as an educator. Julie also had a similar experience and connected it to her role as an educator. She shared,

> **…** in the beginning, we learned in different base systems and, like, basically, the goal of it was for us to be as confused as our students will be when we're teaching them base 10, because they have no idea what we're talking about. And so, like, I think it definitely, like, shed light on, like, how, like, I literally have to start from zero when teaching when, like, I always forget that, like, when I go into a class, I have some form of, like, a background knowledge of an idea.

In the excerpts above, Jeff and Julie's remarks show how the problem-solving course created an experience that was challenging for them as learners, which in turn caused them to consider their future role as educators. Mathematics teacher educators may leverage this influence by intentionally creating similar learning experiences for preservice teachers and prompting reflection on their mathematics educator identities.

Defining Mathematics

In analyzing our data, we were consistently cognizant of how preservice teachers defined what it means to do mathematics and what it means be successful at mathematics. Our findings suggest that the context of the problem-solving focused mathematics content course intervened by challenging students' definitions of "mathematics" and, in their stories, exposed both their mathematics learner and mathematics educator identities.

Problem-solving definitions of mathematics. Some preservice teachers understood mathematics as problem-solving based and student-centered instead of procedural and lecture based. For example, Julie articulated her identity of mathematics learner by viewing the mathematics classroom as "a space where you're supposed to be able to…ask questions, develop, be curious." In her autobiography, she reflected on a previous positive experience in which a mathematics instructor "prioritized giving each student an opportunity to make mistakes in order to learn and grow."

Julie connected her current success in the course to the way that she defines mathematics as problem-solving based instead of her previous experiences defining mathematics as procedural. She said,

> You have to, like, genuinely understand the concepts in order to … get the right answer. And I think in math, like algebra, and calculus, and things like that… it's really just, like, plug and chug. And so that's definitely, like, changed. … And I just feel, like, I'm also, like, I'm doing well in the course. And so, like, that has definitely continued to, like, boost my confidence and … my understanding of, like, if I just, like, focus and put effort into it … I can do it.

Julie's problem-solving focused definition of mathematics shaped her mathematics learner identity by not only helping her to define herself as successful in mathematics but has also motivated her to engage in learning and doing mathematics.

Procedural definitions of mathematics. Other students defined mathematics as procedural and this negatively affected their mathematics learner identities. For example, Georgia shared,

> Yeah, no, math is, like, not my thing. I'm way more, like, English, history. But I've always, like, tried math; I've just never been good at it. I'm not good at, like, memorizing things because if you give me, like, a problem,, like, I'll solve it with an equation, but then, like, when you have a quiz or test and they don't give you all the equations, you have to memorize those.

Georgia viewed mathematics as an exercise of memorizing procedures for testing performances. Because she had historically struggled with performing well on tests, she did not view herself as good at mathematics. The way that Georgia defined mathematics ultimately became the measure that she used to assess her sense of belonging to mathematics, and in this case, negatively affected her mathematics learner identity and inhibited meaningful learning.

Not only did a procedural definition of mathematics negatively affect some students' learner identities, but it led to resistance in learning mathematics through the problem-solving methods that were implemented in the course. For example, Ainsley defined mathematics as sequential and building on itself. She said, "… everything builds upon each other in math. So, if you're, if you get one thing wrong, you're gonna keep getting it wrong throughout the course of the exam." She also, like Georgia, was concerned about understanding this sequence of mathematics for the purpose of achieving a grade. Ainsley became frustrated with the course because she felt she did not receive the supports that she needed to learn mathematics, which would be true if learning mathematics is following a set of procedures in a well-guided sequence of topics. Ultimately, Ainsley resisted learning mathematics as intended by the course instructors and outsourced her learning to online videos. "I've found that I've watched more YouTube videos to learn what we're doing than what the teacher is teaching us. So, I've been getting most of my information and knowledge off of YouTube videos."

This act of resistance to learning mathematics through problem solving based on one's definition of mathematics also appeared through the language participants used to talk about the course and the way they perceived it. For example, Elyse said,

> … this math hurts my head, it confuses me; it makes me hate math more than I have in the past, because math in the past for me has been, like, a lot of equations in formulas, like that, which is why I've learned to like it. That style. And this is just totally, like, totally different.

Elyse defined mathematics as procedural and resisted learning it in this course by compartmentalizing it as "this math," refusing to define what she was learning as actually being mathematics. Similarly, Denise shared, "I really liked math, okay. This, however, is not like normal math."

Redefining mathematics throughout the course. Some students discussed how their definitions of success in mathematics have changed over the course of the semester. At the beginning of the semester, Alyssa was asked if she considers herself a "math person," to which she responded, "I do not consider myself a math person because I am not very good at it and am more interested in/ better at language." In the post-interview, Alyssa described her experience reflecting on her evolving views:

> So, I've always enjoyed math, but I feel I've never been like good at it. Just because, like, I've struggled with it throughout my years of high school, I guess. But … I feel like this course has kind of, like, involved different interpretations of [solutions] too, which I thought I wouldn't enjoy. But I've started to like it because, like, it helps me learn … different ways to get it, like, what I've been saying so many times. … I guess, like, the process [of] getting there has kind of become, like, my definition of how to be good at math. Rather than, like, just knowing the solutions.

Alyssa voiced how the course helped her to redefine what it means to be good at mathematics as focused on engaging in the process rather than focused on correct answers.

In the same interview, Alyssa was asked if she now sees herself as "good" at mathematics. She said, "I think I'm approaching being good at math." When pressed farther and asked, "What if I asked if you're good at problem solving, what do you think?" Alyssa said, "Yeah, I think that, like, that would be more, like, I'd be like, yeah, I'm good at problem solving." Throughout the semester, there was a shift in the way that Alyssa defined what it means to be good at mathematics and how she was beginning to see herself as enacting that role of what it means to be good at mathematics. Interestingly, she *did* claim to be good at "problem solving" but not necessarily "mathematics." Implications for mathematics teacher educators may be to help preservice teachers unpack how they are defining mathematics and any baggage that is being carried with that definition. In this way, preservice teachers might begin to redefine what it means to do and to be successful at mathematics, which will result in shaping mathematics identities.

Concluding Remarks

A limitation to this study is that, though we each have familiarity with the course and experience teaching it, we did not observe students in this setting and instead relied solely on their stories as evidence. However, our research offers greater understanding of the ways in which preservice teachers' mathematics learner and mathematics educator identities are shaped. The context of a problem-solving focused content course created experiences that challenged participants' schemas of mathematics and prompted reflection of their roles as both learners and educators. The way in which preservice teachers in this study defined mathematics shaped their respective mathematics identities. This resulted in differences in the ways that participants engaged in learning mathematics and also how they envisioned their future role of teaching their students mathematics.

Future research on this influential power of defining mathematics is needed, including how mathematics teacher educators might support preservice teachers to *redefine* mathematics and, in doing so, shape their mathematics identities. In addition to this, we have presented several examples of how mathematics learner and mathematics educator identities interrelate and influence one another. Acknowledging the complexity of these interactions, it is possible that mathematics teacher educators might create specific interventions for learners to shape their educator identities. Similarly, it may benefit mathematics teacher educators to prime preservice teachers to maintain a focus on their future role as educators for the sake of engaging them in learning in the present moment.

References

Ball, D. L. (1990). Breaking with experience in learning to teach mathematics: The role of a preservice methods course. For the Learning of Mathematics, 10(2), 10–16. https://www.jstor.org/stable/40247978

Berry, R. Q. (2008). Access to upper-level mathematics: The stories of successful African American middle school boys. Journal for Research in Mathematics Education, 39(5), 464–488. https://doi.org/10.5951/jresematheduc.39.5.0464

Bishop, J. P. (2012). "She's always been the smart one. I've always been the dumb one": Identities in the mathematics classroom. Journal for Research in Mathematics Education, 43(1), 34–74. https://doi.org/10.5951/jresematheduc.43.1.0034

Cribbs, J., Hazari, Z., Sonnert, G., & Sadler, P. M. (2021). College students' mathematics-related career intentions and high school mathematics pedagogy through the lens of identity. Mathematics Education Research Journal, 33(3), 541–568. https://doi.org/10.1007/s13394-020-00319-w

Darragh, L. (2013). Constructing confidence and identities of belonging in mathematics at the transition to secondary school. Research in Mathematics Education, 15(3), 215–229. https://doi.org/10.1080/14794802.2013.803775

Darragh, L. (2015). Recognising 'good at mathematics': using a performative lens for identity. Mathematics Education Research Journal, 27(1), 83–102. https://doi.org/10.1007/s13394-014-0120-0

Darragh, L. (2016). Identity research in mathematics education. Educational Studies in Mathematics, 93(1), 19–33. https://doi.org/10.1007/s10649-016-9696-5

Darragh, L., & Radovic, D. (2020). Mathematics learner identity. In S. Lerman (Ed.), Encyclopedia of mathematics education (2nd ed., pp. 582–585). Springer.

Drake, C., Spillane, J. P., & Hufferd-Ackles, K. (2001). Storied identities: Teacher learning and subject-matter context. Journal of Curriculum Studies, 33(1), 1–23. https://doi.org/10.1080/00220270119765

Esmonde, I. (2017). Power and sociocultural theories of learning. In I. Esmonde & A. N. Booker (Eds.), Power and privilege in the learning sciences: Critical and sociocultural theories of learning (pp. 6-27). Routledge.

Graven, M. & Heyd-Metzuyanim, E. (2019). Mathematics identity research: The state of the art and future directions. ZDM, 51(3), 361–377. https://doi.org/10.1007/s11858-019-01050-y

Graven, M. & Lerman, S. (2020). Mathematics teacher identity. In S. Lerman (Ed.), Encyclopedia of mathematics education (2nd ed., pp. 597–600). Springer. https://doi.org/https://doi.org/10.1007/978-3-030-15789-0_109

Gresalfi, M. S. & Cobb, P. (2011). Negotiating identities for mathematics teaching in the context of professional development. Journal for Research in Mathematics Education, 42(3), 270–304. https://doi.org/10.5951/jresematheduc.42.3.0270

Gujarati, J. (2013). An "inverse" relationship between mathematics identities and classroom practices among early career elementary teachers: The impact of accountability. The Journal of Mathematical Behavior, 32(3), 633–648. https://doi.org/10.1016/j.jmathb.2013.08.002

Hodgen, J. & Marks, R. (2009). Mathematical "ability" and identity. In L. Black, H. Mendick & Y. Solomon (Eds.), Mathematical relationships in education (pp. 31–42). Routledge.

Langer-Osuna, J. M. & Esmonde, I. (2017). Identity research in mathematics education. In J. Cai (Ed.), Compendium for research in mathematics education (pp. 637–648). National Council of Teachers of Mathematics.

Lave, J., & Wenger, E. (1991). Situated learning: Legitimate peripheral participation. Cambridge, UK: Cambridge University Press.

Lerman, S. (2009). Pedagogy, discourse, and identity. In L. Black, H. Mendick & Y. Solomon (Eds.), Mathematical relationships in education (pp. 147–155). Routledge.

Lortie, D. (1975). Schoolteacher. The University of Chicago Press.

Lutovac, S., & Kaasila, R. (2018). Future directions in research on mathematics-related teacher identity. International Journal of Science and Mathematics Education, 16(4), 759–776. https://doi.org/10.1007/s10763-017-9796-4

Maloney, T., & Matthews, J. S. (2020). Teacher care and students' sense of connectedness in the urban mathematics classroom. Journal for Research in Mathematics Education, 51(4), 399–432. https://doi.org/10.5951/jresematheduc-2020-0044

Masingila, J. O., Lester, F. K., & Raymond, A. M. (2011). Mathematics for elementary teachers via problem solving: Instructor Manual (3rd ed.). Ann Arbor, MI: XanEdu Publishing.

Sawyer, R. K. (2006). Introduction: The new science of learning. In R. K. Sawyer, Handbook of the learning sciences (pp. 1–16). Cambridge University Press.

Saxe, G. B. (1988). Candy selling and math learning. Educational Researcher, 17(6), 14–21. https://doi.org/10.3102/0013189x017006014

Sfard, A. & Prusak, A. (2005). Telling identities: In search of an analytic tool for investigating learning as a culturally shaped activity. Educational Researcher, 34(4), 14–22. https://doi.org/10.3102/0013189x034004014

Skott, J. (2019). Changing experiences of being, becoming, and belonging: Teachers' professional identity revisited. ZDM, 51(3), 469–480. https://doi.org/10.1007/s11858-018-1008-3

VERBI Software. (2021). MAXQDA 2022 [computer software]. Berlin, Germany: VERBI Software. Available from maxqda.com.

Wenger, E. (1998). Communities of practice: Learning, meaning, and identity. Cambridge, UK: Cambridge University Press.

PRE-SERVICE ELEMENTARY TEACHERS' FRAMING OF MATHEMATICAL DISCUSSIONS AFTER PROBLEM-SOLVING THROUGH MURSION™ SIMULATION

Sezai Kocabas
Purdue University
Skocabas@purdue.edu

Signe Kastberg
Purdue University
Skastber@purdue.edu

Melva Grant
Old Dominion University
Mgrant@odu.edu

Hanan Alyami
Purdue University Fort Wayne
Alyamih@pwf.edu

Research on pre-service teachers' discussion practices has focused on decompositions of practice into subskills, while acknowledging the importance of the role of context, identity, and relationships between interactive moves. We focused on 66 elementary preservice teachers' (PSTs') framing-launching moves in discussions after problem-solving in a Mursion™ custom simulation. PSTs used five moves: gathering information about student processes, focusing on problem features, task and non-task oriented social interactions, and partner talk. Empirical findings of PSTs' intentions and tacit actions coupled with study findings of the diversity in PSTs' framing moves, highlight the complexity of teacher decision making involved in discussion subsills such as framing. We argue that PSTs' framing moves are motivated by an array of intentions including the mathematics aims of discussions.

Keywords: Preservice Teacher Education, Teacher Knowledge.

Inquiry into mathematics teachers' discussion practice has focused on relationships between teacher and student moves (Bishop et al., 2022; Leatham et al., 2015; Stockero et al., 2022). Yet inquiry into preservice teachers' (PSTs') discussion practices highlights discussion subskills within decompositions of practice (Grossman et al., 2009) including planning (Tyminski et al, 2014) and questioning (Moyer & Milewicz, 2002). Shaughnessy et al. (2021) hypothesized that discussion practice contains three areas: framing-launching, orchestrating, and record/representing content (p. 455). Framing-launching a discussion within decomposition of practice involves teacher moves including focusing attention on mathematics but differs from launching a task in that a goal of the launch is to prepare students to share ideas about strategies rather than prepare students for problem-solving. Mathematics teacher educators need a better understanding of specific moves teachers use for framing-launching mathematics discussions to develop pedagogical learning trajectories for teacher discussion practice needed to inform development activities for PSTs. We report on PSTs' framing-launching (Shaughnessy et al., 2021, p. 455) of discussion after problem-solving, which we will refer to as framing. We argue that PSTs' framing moves are motivated by factors that include but are not limited to mathematical aims for discussions.

Drawing from Steffe and D'Ambrosio's (1995) description of constructivist teaching, we view learning to teach as dynamic testing of practice. Such testing involves the use of tacit and explicit theories involved in educative interactions such as social and socio-mathematical norms (Yackel & Cobb, 1996). Generalizations of normative behavior can reproduce societal inequities, yet PST and student interactions can also challenge generalizations. PSTs' experience with unexpected student responses, can produce disequilibrium in understandings of norms. PSTs interact as model builders (Ulrich et al., 2014), creating practices by informal tacit testing of utility and productivity of interactive moves in mathematics discussions with students.

Research into PSTs' preparation for (and engagement in) discussions describes intentions (e.g. Tyminski et al., 2014) and moves (e.g. Shaughnessy & Boerst, 2018) PSTs use. PSTs design discussions to provide students with opportunities to share, learn from sharing, and/or compare strategies (Tyminski et al., 2014). PSTs also use intuitive skills to elicit students' problem-solving process or probing students' understanding about the problem (Shaughnessy & Boerst, 2018). Many PSTs launch interactions with students by asking what they did when solving the problem, "What was your first step when you solved the problem?" (Shaughnessy & Boerst, 2018, p. 52). Some PSTs probe to gather evidence of what students noticed about task features or understood about their problem solving process by asking, "So can you kind of tell me what this row that you wrote stands for or what that number is?" (p. 46). PSTs may also utilize other moves that support learning about students' thinking, such as encouraging students to write or facing students when asking a question (Shaughnessy & Boerst, 2018)

In-service teachers select student work for discussion using the mathematics potential of the work, but also historical patterns of student participation and mathematical understanding (Dunning, 2022). While decompositions of practice (Grossman et al., 2009) play an important role in developing PSTs' practice (Jacobs & Spangler, 2017), discussion is complex and contextual, it involves teachers' interactive moves influenced by factors including teachers' view of learning (Simon, 2008); teachers' identities (Drake, 2006), views of learners' participation (Dunning, 2022), learners' needs (Sztjan, 2003), learners' identities (Rubel et al., 2022), and learners' lived experiences (Van Es et al., 2022). These findings taken together suggest further inquiry into PSTs' subskills of discussion practice (e.g. Tyminski et al., 2014; Shaughnessy et al., 2021) using frameworks for analyzing interactions (e.g. Bishop et al., 2022) to describe PSTs' framing of mathematical discussions after problem-solving.

Studies of teacher discussion practice have focused on teacher moves in relation to students' mathematics. For example, Leatham et al. (2015) defined mathematically significant pedagogical opportunities to build on student thinking (MOSTs) as a tool to explore teachers' use of students' mathematics in discussions. Bishop et al., (2020) described three categories of teacher moves: confirm/correct, probe/publicize, engage, where confirm/correct moves "did not use students' mathematical contributions" (p. 15). Instead, teachers shared their thinking, acknowledged or evaluated student responses. Probe/publicize moves used probing questions to understand student thinking and revoicing to highlight student responses. Engaging moves directed students to engage with others' ideas by asking students to restate, reason about, or apply another students' approach. Bishop et al. (2022) also classified student contributions: participate (sharing of facts or procedures), explain (sharing strategies without justification), and "substantive reasoning" (p. 15). Less clear was how teachers developed discussion practices.

To unpack the development of discussion practices, we report findings from analysis of data from a larger study of PSTs' moves to facilitate discussions with students after problems solving in the context of an online virtual custom simulation of discussion. Based on coding of 66 PSTs' framing moves in a discussion after problem solving we identified five framing moves: (a) gathering information about student processes; (b) focusing on features of the problem; (c) task-oriented social interactions, (d) non-task oriented social interactions; and (e) partner talks. We argue that these initial moves were motivated by factors beyond mathematics.

Use of Mixed-Reality Simulations (MRSs) has increased in teacher education (REF). MRS including human-in-the loop Mursion™ simulations encompasses real and virtual environments

(Gundel et al., 2019). MRSs provide a low-risk environment (Dieker et al., 2014; Piro & O'Callaghan, 2019) for PSTs to develop teaching related theory, practice (e.g., Hudson et al., 2019), and beliefs (e.g., Bautista & Boone, 2015; Gundel et al., 2019). This study uses a Mursion™classroom custom simulation (Grant & Ferguson, 2021) as the context for PSTs to facilitate discussions of mathematics after problem-solving. Prior findings using this simulation include PSTs reports that after facilitating a mathematics discussion in the simulation, they felt more prepared, confident, and their teaching anxiety reduced (Grant & Ferguson, 2021). Our findings extend literature on MRSs' impact from efficacy to describing practice.

Methods

Participants were 23 PSTs in elementary mathematical method courses (fall 2018, spring 2019, fall 2019) at a Mid-Atlantic university and 43 PSTs in elementary methods courses (spring 2021) at a Midwestern university. Analysis of PSTs' framing of discussions after problem-solving is part of a larger study to describe PSTs' development of discussions after problem-solving.

In this study, we analyzed PSTs' framing moves in a Mursion™ classroom simulation. Mursion is a real-time software that creates realistic and interactive environment by blending artificial intelligence and trained simulation specialists who control the speech and movements of avatars (virtual students) in a virtual environment during the simulation (Dieker et al., 2014; Grant & Ferguson, 2021). PSTs sit in front of a screen, with a camera and microphone (Figure 5). Five avatars appear on screen, sitting behind a desk with name cards on the desk (e.g., Savannah, Ethan, Dev, Eva, Jasmin, Harrison, etc.). Avatars' actions and utterances are controlled by a simulation specialist (off screen) who can see and hear the PSTs (Figure 6). Avatars respond as elementary students providing answers, explanations, and asking questions. Further, the avatars reflect various physical and social student characteristics such as hair color, phenotype, quick to volunteer, and talkative through their actions or engagement. While the PSTs were interacting with avatars, we refer to these avatars as students, to highlight the realistic nature of the custom simulation for the PSTs.

Figure 5. Mursion™ Simulation Room

Figure 6. Avatars in Virtual Environment

Study Design

Mursion™ simulation provides an opportunity for PSTs to practice core practices (Jacobs & Spangler, 2017) of teaching. In a custom simulation (Grant & Ferguson, 2021), PSTs had opportunities to orchestrate a mathematical discussion about student's solutions to a problem solving task. In the elementary mathematics method courses, the simulation was used as a microteaching environment (Grossman et al., 2009; Ledger et al., 2019). PSTs were directed to elicit students' reasoning and justification, and to help students make meaningful connections across strategies generated through problem-solving to strengthen mathematical understanding (Graeber, 1999; Jacobs & Spangler, 2017; Smith & Stein, 2018). The mathematical discussion focused on student solutions of "Harry the Dog Problem" (Figure 7).

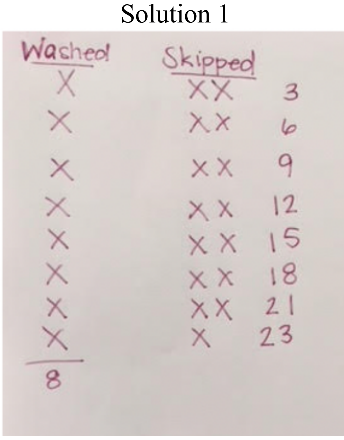
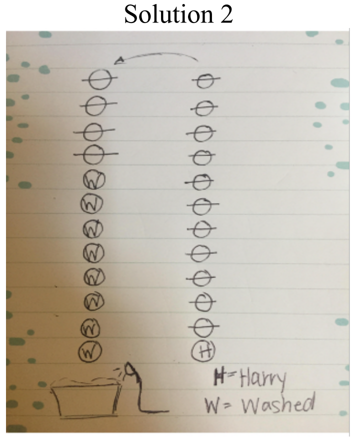
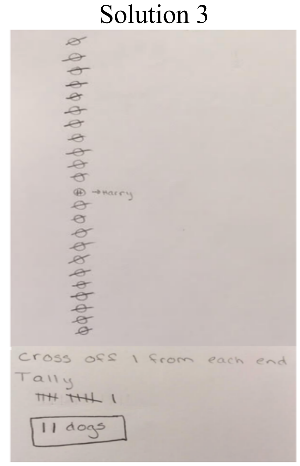
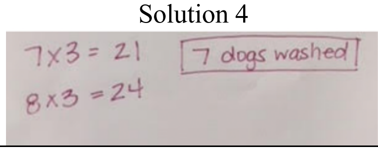
Harry, the dog, is standing in a line of 23 dogs, waiting to be washed. He is hot and impatient.

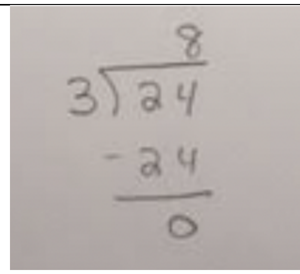
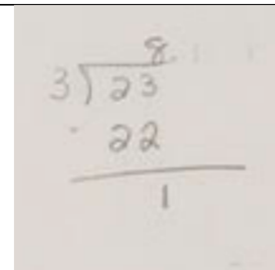
Each time a dog is washed and Becky, the groomer, turns to let the dog into the cool doggie run, Harry sneaks ahead of two dogs. How many dogs will be washed before Harry?

Figure 7. Harry the Dog Problem

We provided 6 solutions to "Harry the Dog Problem", and told the PSTs that the solutions were generated by the students during problem-solving (Table 2). PSTs did not know which avatar had which solution. The PSTs were given five to eight minutes to engage the avatars in the Mursion™ simulation with the goal of facilitating discussion after problem-solving to generate mathematical sense-making. We expected PSTs to ask questions to elicit students' thinking by encouraging them to use good mathematical explanations to make sense of problem solutions. PSTs were cautioned not to provide any hints or verify students' answers. Each PST's simulation was video recorded, and the de-identified transcripts of the recordings constitutes the data for this study.

Table 2: Students' Solutions Provided to PSTs Before Facilitating the Discussion

<table>
<tr><td>Solution 1

</td><td>Solution 2

</td><td>Solution 3

</td></tr>
<tr><td>Solution 4

</td><td>Solution 5</td><td>Solution 6</td></tr>
</table>

Coding and Analysis

The objective of the coding process was to describe how PSTs framed discussion after problem-solving. We coded the initial discussion interactions between PSTs and the students, drawing on Shaughnessy and Boerst's (2018) classification of PSTs' moves of eliciting students' problem-solving process and probing students' understanding about the problem. We also identified new moves PSTs used to support learning about students' thinking (Table 3). All PSTs' framing moves were coded by two members of the research team to guarantee consistency throughout the data (Braun & Clarke, 2006).

Table 3: PSTs' Discussion Moves in Mathematical Discussion

Framing Moves to Initiate Discussion	**Description**
Gathering students answers or strategies (see elicit students' thinking, Shaughnessy & Boerst, 2018)	PSTs initiate the discussion by posing questions to learn students' answers or strategies for the problem solving process (e.g., Can you tell me how you solved the problem?)
Focusing on features of the problem (see probing students' understanding; Shaughnessy & Boerst, 2018)	PSTs initiate the discussion by posing questions to learn students' understanding of key mathematical features of the problem or draw students' attention to key points of the problem (e.g., How many dogs are in the line?).
Other Moves (Social Moves)	**Description**
Task oriented social interactions (see also other moves to support student thinking; Shaughnessy & Boerst, 2018)	PSTs initiate the discussion by engaging in task oriented social interactions. They pose task oriented questions to elicit students' emotions or opinions regarding the task or problem solving process to prepare them for the discussion.(e.g., Do you remember the Harry the Dog problem?).
Non-task oriented social interactions (see other moves to support student thinking; Shaughnessy & Boerst, 2018)	PSTs initiate the discussion by engaging in non-task oriented social interaction to greet or gauge students' energy before digging into the problem solving process (e.g., How are you today?).
Partner Talk (see other moves to support student thinking; Shaughnessy & Boerst, 2018)	PSTs initiate partner talk, directing students to engage in a conversation with peers to talk about each other's process and answer (e.g., I want you to turn to your partner and talk about how you solved the problem?).

Findings

Overall, majority of the PSTs initiated the discussion after problem solving by gathering student answers or strategies (Table 4). Fifty seven of 66 PSTs (86%) framed discussions by gathering student answers or strategies, a framing move that involves the PSTs selecting *a volunteer or one student* to share. The student responded by providing an answer (e.g. eight) or a descriptor of their approach (e.g. multiplied, created a chart). For example, PST1 chose Savannah to begin the discussion by inviting her to share the problem solving process.

PST1: Hi guys! So, we are going to talk about the Harry the dog problem again. So, do you guys want to walk me through, kind of what your thought processes were when you were doing the problem? Savannah, do you want to go ahead and start?

Savannah: Okay! So, I did multiplication. And I did seven times three is twenty one and eight times three is twenty four but we only have twenty three dogs. So, I got seven.

Alternatively, some PSTs who gathered students' answers or strategies to initiate the discussion *solicited volunteers* to share their solution. Specifically, 30 of 57 PSTs (53%) who used the gathering student answers or strategies moves invited a volunteer rather than selecting a specific student. For example, PST2 affirmed Harrison's request that the class go over the problem and again asked for a volunteer.

PST2: Let's ask the question! Who can give me a solution to the problem?

Harrison: It was kind of hard. Yeah, I don't know. Yeah, it was a pretty tough one. Are we gonna go over it at all?

PST2: Yes. Can somebody give me what they got? And we'll talk about it.

Savannah: Oh, Well, I yeah, I think it was five.

Seven of 66 PSTs (11%) drew students' attention to problem features. PST3 selected a student and invited him to share key mathematical features of the problem, an important move to support problem solving discussions.

PST3: … So the first thing we're going to look at is who can raise their hand and tell me what keywords that you found when going into this that you saw immediately that stood out to you that would help you go about answering this question. Harrison

Harrison: Oh, well, I mean, just the 23 stood out to me right away, just because I mean, you know, I know it's gonna be a math problem. And you know, just set the scene of like, how many dogs were in the line ahead of Harry.

Some PSTs who attempted to draw students' attention to problem features (see Table 3), solicited volunteers rather than selecting a student to initiate the discussion. PSTs 4 and 5 used the number of dogs in line and invited a volunteer to discuss a key feature of the problem.

PST4: Okay, class, let's begin with going over the problem. We want to get some information. So how many dogs are in line?

PST5: So, we've been working on the Harry the dog problem, it's a math problem, but a dog and he's in line to get washed with a bunch of other dogs, right? And it says that Harry is the 23rd dog in line, or is he the 24th dog? Now, this question set up a little bit odd and it's a different interpretation. So I'm going to see how many of you, go ahead and raise your hand, has have Harry is the 24th dog in line?

Table 4: A Breakdown of the Discussion Moves in Mathematical Discussion Used by PSTs

	Framing moves		Other moves (Social Moves)		
	Gathering student answers or strategies n (%)	**Focusing on features of the problem n (%)**	**Task oriented n (%)**	**Non-task oriented n (%)**	**Partner talk n (%)**
Mid-Atlantic University (n=23)	21 (91%)	2 (9%)	1 (4%)	7 (30%)	---
	Selecting a volunteer 12 (57%)	**Selecting a volunteer** 2 (100%)			
Midwestern University (n=43)	36 (84%)	5 (12%)	10 (23%)	17 (40%)	5 (12%)
	Selecting a volunteer 18 (50%)	**Selecting a volunteer** 4 (80%)			

Other Moves

Twenty four of the 66 PSTs (36%) engaged in non-task oriented social interactions before framing the mathematical discussion. These teachers made an effort to greet or gauge student energy, readiness for discussion. For example, PST6 began with an emotionally focused question about feelings

PST6: How are we doing today?

Jasmine: Hi, really good. I like your scarf.

PST6: Thank you. I like your shirt.

Jasmine: Thank you.

PST6: So does anyone want to start off with their problem? Anyone want to explain to me what they did for Harry the dog?

Eleven of the 66 PSTs (17%) engaged in task oriented social interactions before framing the mathematical discussion. These PSTs concentrated on initiating the discussion by posing more task-oriented questions to elicit avatars' emotions or opinions regarding the task or solving process, and prepared them for the discussion. For example, PST7 began with a mathematically focused question about students' opinions as a social interaction for readiness of the discussion, i.e., *"Good morning class! Today, we are going to talk about the Harry the dog problem. Do you guys remember that?"*. These PSTs set the stage for student mathematics engagement by first initiating non-task or task-related social interactions attending to students as humans.

Five of 43 PSTs (12%) at the Midwestern University invited students to partner talk to discuss their problem-solving process and answers before initiating a whole class mathematics discussion. PST8 let the students engage in a small group mathematical conversation for readiness of whole class discussion.

> PST8: We are going to talk again, I know you might have already once, but I want to talk again about the Harry the dog problem. So, in the problem, it says, (reading the problem). I saw all of you do your problems and how you solved it. Could you partner talk really quick and talk about how you solved it and then we will meet together and about like 10 seconds, take 10 quick seconds. How did you solve it? Can you talk to your partner?

Some PSTs who used social interactions before framing the mathematical discussion (8 PSTs at Mid-Atlantic university, 32 PSTs at Midwestern University) used at least two social moves prior to initiating discussion of mathematics. One of the eight PSTs (13%) at the Mid-Atlantic university and eight of 32 PSTs (25%) at the Midwestern University engaged in at least two different social interactions before framing the mathematical discussion (e.g., using a non-task oriented social move with a task oriented social interaction; using a non-task oriented social move with partner talk).

Discussion

PSTs discussion practice has been studied using decompositions of practice (Grossman et al., 2009). As anticipated, PSTs at both universities typically framed discussion after problem-solving by gathering information about student processes (Shaughnessy & Boerst, 2018). PSTs' framing moves focused on gathering information about student processes, focusing on features of the problem, and social interactions, rather than focusing on mathematics reasoning of a selected work sample.

At first glance, PSTs' initial move may seem inefficient in that with student solutions available, PSTs could choose to begin with a student work based on mathematical potential (Leatham et al., 2015). Yet Dunning (2022) identified teachers' use of factors including student historical patterns of participation and mathematics understanding in selecting student work for mathematics discussions. Additionally, Tyminski et al. (2014) noted that PSTs' intention for discussions often focused on more general aims such as strategy sharing to enable students to learn from others' work. The complexity involved in teacher decisions associated with discussion subskills including framing-launching (Shaughnessy et al, 2019) suggests additional study of "interactive detail" and "teacher voice" is needed (Jacobs & Spangler, 2017, p. 704) to contribute to our understanding of PSTs' modeling of students as mathematics learners. Additionally, PSTs' assumptions about learning (Simon, 2008) coupled with details of interactions are needed to discern how PSTs are developing discussion practices.

PSTs' views of students' mathematics develop through interactions (Steffe & D'Ambrosio, 1995). PSTs as model builders take interactive moves (e.g., Bishop, 2022) that shape mathematical discussions (Ulrich et al., 2014). Interactive moves are influenced by factors beyond mathematics including teacher identities (Drake, 2006), learner identities (Rubel et al., 2022), and perceptions of learner participation (Dunning, 2022). In our study, PSTs' relied on hearing students' processes to frame discussions, yet framing moves also attended to students as humans and mathematicians. Taken together, the research findings support the argument that PSTs' framing moves are motivated by intentions beyond mathematical aims of discussions. Additional exploration of factors such as student identities that motivate PSTs' framing moves is needed to model PSTs' discussion practice.

References

Bautista, N. U., & Boone, W. J. (2015). Exploring the impact of TeachMETM lab virtual classroom teaching simulation on early childhood education majors' self-efficacy beliefs. Journal of Science Teacher Education, 26(3), 237–262. Doi:10.100710972-014-9418-8

Bishop, J. P., Hardison, H. L., & Przybyla-Kuchek, J. (2022). Responsiveness to students' mathematical thinking in middle-grades classrooms. Journal for Research in Mathematics Education, 53(1), 10-40. https://doi.org/10.5951/jresematheduc-2020-0188

Bishop, J. P., Hardison, H., Przybyla-Kuchek, J., & Hassay, E. (2020). Leveraging student thinking to foster productive discussions. Mathematics Teacher: Learning and Teaching PK-12, 113(12), 995-1002. https://doi.org/10.5951/MTLT.2019.0096

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. Qualitative Research in Psychology, 3(2), 77-101. https://doi.org/10.1191/1478088706qp063oa

Dawson, M. R., & Lignugaris/Kraft, B. (2017). Meaningful practice: Generalizing foundation teaching skills from TLE TeachLivE™ to the classroom. Teacher Education and Special Education, 40(1), 26-50. https://doi.org/10.1177/0888406416664184

Dieker, L., Rodriguez, J., Lignugaris/Kraft, B., Hynes, M, & Hughes, C. (2014). The potential of simulated environments in teacher education: Current and future possibilities. Teacher Education and Special Education, 37(1), 21– 33. https://doi.org/10.1177/0888406413512683

Dixon, R. A., Hall, C., & Shawon, F. (2019). Using virtual reality and web conferencing technologies: Exploring alternatives for microteaching in a rural region. Northwest Journal of Teacher Education 14(1), 1–18. https:// doi.org/10.15760/nwjte.2019.14.1.4

Drake, C. (2006). Turning points: Using teachers' mathematics life stories to understand the implementation of mathematics education reform. Journal of Mathematics Teacher Education, 9(6), 579-608. Doi: 10.1007/s10857-006-9021-9

Dunning, A. (2022). A framework for selecting strategies for whole-class discussions. Journal of Mathematics Teacher Education, 1-22. https://doi.org/10.1007/s10857-022-09536-5

Graeber, A. O. (1999). Forms of knowing mathematics: What preservice teachers should learn. In D. Tirosh (Ed.). Forms of mathematical knowledge (pp. 189–208). Springer. https://doi.org/10.1007/978-94-017-1584-3_9

Grant, M., & Ferguson, S. (2021). Virtual microteaching, simulation technology & curricula: A recipe for improving prospective elementary mathematics teachers' confidence and preparedness. Journal of Technology and Teacher Education, 29(2), 137-164.

Grossman, P., Compton, C., Igra, D., Ronfeldt, M., Shahan, E., & Williamson, P. W. (2009). Teaching practice: A cross-professional perspective. Teachers College Record, 111(9), 2055-2100.

Gundel, E., Piro, J. S., Straub, C., & Smith, K. (2019). Self-efficacy in mixed reality simulations: Implications for preservice teacher education. The Teacher Educator, 54(3), 244-269. https://doi.org/10.1080/08878730.2019.1591560

Hartle, L., & Kaczorowski, T. (2019). The positive aspects of Mursion when teaching higher education students. Quarterly Review of Distance Education, 20(4), 71-100.

Hudson, M. E., Voytecki, K. S., Owens, T. L., & Zhang, G. (2019). Preservice teacher experiences implementing classroom management practices through mixed reality simulations. Rural Special Education Quarterly, 38(2), 79–94.

Jacobs, V. R., & Spangler, D. A. (2017). Research on core practices in K–12 mathematics teaching. In J. Cai (Ed.), Compendium for research in mathematics education (pp. 766–792). National Council of Teachers of Mathematics.

Judge, S., Bobzien, J., Maydosz, A., Gear, S., & Katsioloudis, P. (2013). The use of visual-based simulated environments in teacher preparation. Journal of Education and Training Studies, 1(1), 88-97. doi: 10.11114/jets.v1i1.41

Leatham, K. R., Peterson, B. E., Stockero, S. L., & Van Zoest, L. R. (2015). Conceptualizing mathematically significant pedagogical opportunities to build on student thinking. Journal for Research in Mathematics Education, 46(1), 88-124.

Ledger, S., Ersozlu, Z., & Fischetti, J. (2019). Preservice teachers' confidence and preferred teaching strategies using TeachLivE™ virtual learning environment: A two-step cluster analysis. Eurasia Journal of Mathematics, Science and Technology, 15(3) em1674. https://doi.org/10.29333/ejmste/102621

Moyer, P. S., & Milewicz, E. (2002). Learning to question: Categories of questioning used by preservice teachers during diagnostic mathematics interviews. Journal of Mathematics Teacher Education, 5(4), 293-315.

Rubel, L. H., Ayalon, M., & Shahbari, J. A. (2022). Sequencing & selecting solutions in a gendered world. Mathematical Thinking and Learning, 1-23. https://doi.org/10.1080/10986065.2022.2062641

Piro, S. J., & O'Callaghan, C. (2019). Journeying towards the profession: Exploring liminal learning within mixed reality simulations. Action in Teacher Education, 41(1), 79-95. https://doi.org/10.1080/01626620.2018.1534221

Shaughnessy, M., & Boerst, T. A. (2018). Uncovering the skills that preservice teachers bring to teacher education: The practice of eliciting a student's thinking. Journal of teacher Education, 69(1), 40-55.

Shaughnessy, M., Garcia, N. M., O'Neill, M. K., Selling, S. K., Willis, A. T., Wilkes, C. E., ... & Ball, D. L. (2021). Formatively assessing prospective teachers' skills in leading mathematics discussions. Educational Studies in Mathematics, 108(3), 451-472. https://doi.org/10.1007/s10649-021-10070-z

Shaughnessy, M., Ghousseini, H., Kazemi, E., Franke, M., Kelley-Petersen, M., & Hartmann, E. S. (2019). An investigation of supporting teacher learning in the context of a common decomposition for leading mathematics discussions. Teaching and Teacher Education: An International Journal of Research and Studies, 80(1), 167-179. https://doi.org/10.1016/j.tate.2019.01.008

Simon, M. (2008). The challenge of mathematics teacher education in an era of mathematics education reform. In B. Jaworski & T. Wood (Eds.) International Handbook of Mathematics Teacher Education: The Mathematics Teacher Educator as a Developing Professional (pp. 17-29). Sense.

Smith, & Stein, M. K. (2018). 5 Practices for orchestrating productive mathematics discussion. National Council of Teachers of Mathematics.

Steffe, L. P., & D'ambrosio, B. S. (1995). Toward a working model of constructivist teaching: A reaction to Simon. Journal for Research in Mathematics Education, 26(2), 146-159. https://www.jstor.org/stable/749206

Stockero, S. L., Leatham, K. R., Ochieng, M. A., Van Zoest, L. R., & Peterson, B. E. (2020). Teachers' orientations toward using student mathematical thinking as a resource during whole-class discussion. Journal of Mathematics Teacher Education, 23(3), 237-267. https://doi.org/10.1007/s10857-018-09421-0

Stockero, S. L., Van Zoest, L. R., Freeburn, B., Peterson, B. E., & Leatham, K. R. (2022). Teachers' responses to instances of student mathematical thinking with varied potential to support student learning. Mathematics Education Research Journal, 1-23. https://doi.org/10.1007/s13394-020-00334-x

Sztajn, P. (2003). Adapting reform ideas in different mathematics classrooms: Beliefs beyond mathematics. Journal of Mathematics Teacher Education, 6(1), 53-75.

Tyminski, A. M., Zambak, V. S., Drake, C., & Land, T. J. (2014). Using representations, decomposition, and approximations of practices to support prospective elementary mathematics teachers' practice of organizing discussions. Journal of Mathematics Teacher Education, 17(5), 463-487. DOI 10.1007/s10857-013-9261-4

Ulrich, C., Tillema, E. S., Hackenberg, A. J., & Norton, A. (2014). Constructivist model building: Empirical examples from mathematics education. Constructivist Foundations, 9(3), 328-339. http://constructivist.info/9/3/328

Van Es, E. , Hand, V. , Agarwal, P. , & Sandoval, C. (2022). Multidimensional noticing for equity: Theorizing mathematics teachers' systems of noticing to disrupt inequities. Journal for Research in Mathematics Education, 53(2), 114–132.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. Journal for Research in Mathematics Education, 27(4), 458-477. https://www.jstor.org/stable/749877

PRESERVICE TEACHERS' LEARNING TO TEACH WITH ONLINE TECHNOLOGIES DURING AN INITIAL METHODS OF TEACHING COURSE

Maria L. Fernandez
Florida International University
mfernan@fiu.edu

Keywords: Preservice Teacher Education, Teacher Knowledge, Technology

The use of technology for teaching and learning mathematics and science has been advocated for many years in standards documents such as NCTM (2000) *Mathematics Principles and Standards*, the NRC (1996) *National Science Education Standards,* and the AMTE (2017) *Standards for Preparing Teachers of Mathematics.* In particular, AMTE envisions that well-prepared beginning teachers are proficient with the use of technology to support students' mathematics sense making and reasoning in the learning of mathematics. Additionally, research has shown that online tools and manipulatives support teachers' differentiation of instruction to meet the needs of diverse students (Bouck, Flanagan, & Bouck, 2015; Shin et al., 2017). However, research has shown that preservice teachers (PSTs) demonstrate low levels of technology literacy with respect to knowledge and skills (e.g., Dincer, 2018). While teachers report being unprepared and unsure about how to use technology to promote students' learning with understanding (e.g., Albion, Tondeur, Forkosh-Baruch, & Peeraer, 2015; Ertmer, 2005).

Research suggests that integrating pedagogy, content, and technology is an effective approach for preparing teachers to use technology for teaching (Lee & Hollebrands, 2008; Niess, 2005). In its standards, AMTE (2017) recommends the use of practice-based experiences that provide PSTs opportunities to learn from their own teaching, as well as the teaching of others. This study investigated 34 PSTs' development of TPACK (Mishra & Koehler, 2006) during a technology-focused initial methods of teaching secondary mathematics and science course incorporating practice-based experiences using online technologies for student learning.

Throughout the initial methods course, the use of online technologies for teaching and learning was modelled by the teacher as well as the students. The students worked in pairs on a microteaching lesson study project that required them to teach with online mathematics or science technologies. This allowed the PSTs to learn to teach with online technologies through their own teaching, as well as the teaching of their peers. The microteaching project involved each pair's development of two technology-based, inter-related lessons, including bridging activities and formative assessments; the teaching (and video recording) of each lesson (one by each partner); feedback from their instructors and peers; and a report including self-analysis of their video and the feedback they received, as well as revisions of their original lesson plans.

Data sources for the investigation of PSTs' TPACK included lesson plans, self-analysis reports, and lesson revisions, as well as self-report surveys about their learning. Descriptive and qualitative data analysis revealed that the PSTs developed their knowledge of online technologies for teaching (e.g., GeoGebra Classroom, Teacher Desmos, simulations) and pedagogical ways of using these technologies to facilitate student learning, in some cases from having no knowledge of a technology to using it proficiently for students' learning. Survey results revealed high mean scores for PSTs' development of TPACK, including pedagogical knowledge (1.35 on a scale of 1 high to 5 low), technological knowledge (1.23) and technological pedagogical content knowledge (1.76). The results highlight that one methods course can facilitate substantive growth in PSTs' preparation to teach with online technologies.

References

Albion, P. R., Tondeur, J., Forkosh-Baruch, A., & Peeraer, J. (2015). Teachers' professional development for ICT integration: Towards a reciprocal relationship between research and practice. Education and Information Technologies, 20(4), 655-673.

Association of Mathematics Teacher Educators. (2017). Standards for Preparing Teachers of Mathematics. Available online at amte.net/standards.

Bouck, E. C., Flanagan, S., & Bouck, M. (2015). Learning area and perimeter with virtual manipulatives. Journal of Computers in Mathematics and Science Teaching, 34(4), 381-393.

Dincer, S. (2018) Are preservice teachers really literate enough to integrate technology in their classroom practice? Determining the technology literacy level of preservice teachers. Education and Information Technologies (2018) 23:2699–2718

Ertmer, P. (2005). Teacher pedagogical beliefs: The final frontier for technology integration? Educational Technology Research and Development, 53(4), 25-39.

Lee, H. S., & Hollebrands, K. (2008). Preparing to teach mathematics with technology: An integrated approach to developing technological pedagogical content knowledge. Contemporary Issues in Technology and Teacher Education, 8(4). Retrieved from http://www.citejournal.org/vol8/iss4/mathematics/article1.cfm

Mishra, P., & Koehler, M. J. (2006). Technological pedagogical content knowledge: A framework for teacher knowledge. Teachers College Record, 108(6), 1017-1054.

National Council of Teachers of Mathematics (NCTM). (2000). Principles and standards for school mathematics. Reston, VA: Author.

National Research Council. (1996). National Science Education Standards. Washington, DC: National Academy Press.

Niess, M. L. (2005). Preparing teachers to teach science and mathematics with technology: Developing a technology pedagogical content knowledge. Teaching and Teacher Education, 21, 509-523.

Shin, M., Bryant, D.P., Bryant, B.R., McKenna, J.W., Hou, F., & Ok, M.W. (2017). Virtual manipulatives: Tools for teaching mathematics to students with learning disabilities. Intervention in School and Clinic, 52(3), 148-153.

PROFESSIONAL NOTICING: THE INTERRELATED SKILLS OF ATTENDING TO AND INTERPRETING STUDENT MATHEMATICAL THINKING

J. Matt Switzer
Texas Christian University
j.switzer@tcu.edu

Dawn Teuscher
Brigham Young University
dawn.teuscher@byu.edu

Keywords: Teacher noticing, professional noticing, preservice teacher education, student mathematical thinking

Teachers tend to notice a variety of classroom information (e.g., teacher and student actions, classroom management); therefore, guidance on noticing specific aspects of teaching, especially for preservice teachers (PSTs), is essential (Carlson et al., 2017; Sherin & van Es, 2005; Star et al., 2011; Star & Strickland, 2007; Stockero, 2014). To help guide teachers' noticing, van Es and Sherin (2002) proposed three key aspects of noticing:

> a) identifying what is important or noteworthy about a classroom situation; (b) making connections between the specifics of classroom interactions and the broader principles of teaching and learning they represent; and (c) using what one knows about the context to reason about classroom interactions. (p. 573)

Jacobs, Lamb, and Philipp (2010) extended the construct of teacher noticing to professional noticing of student's mathematical thinking (SMT) "as a set of three *interrelated* skills: attending to children's strategies, interpreting children's understanding, and deciding how to respond on the bases of children's understanding" (Jacobs et al., p. 172, emphasis added). Researchers have begun studying the interrelatedness of professional noticing skills (e.g., Hine & Lesseig, 2021; Thomas et al., 2021; Vogler & Prediger, 2017; Walkoe et al., 2019).

The research question guiding our study was how do PSTs' individual skills of attending to and interpreting SMT differ from their interrelated noticing of SMT? The interrelatedness of the three professional noticing skills may be conceptualized as a linear progression of attending, then interpreting, then responding to SMT. We hypothesize that the interrelatedness of professional noticing skills is not linear but co-develop as illustrated in figure 1.

In this poster, we share our frameworks for analyzing our PSTs' written justifications for the individual subskills of attending to and interpreting SMT, nine potential subtypes of professional noticing (Authors, under review) addressing the interrelatedness of attending to and interpreting SMT and results of analyzing our PSTs' written justifications for the individual skills of attending to and interpreting SMT and subtypes of professional noticing.

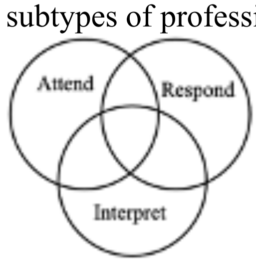


Figure 1: Conceptualization of interrelatedness of professional noticing skills

We provide evidence that the subtypes of professional noticing provide insights into our PSTs professional noticing that are masked by analyzing their written justifications for the individual skills of attending to and interpreting SMT and that the linear development of professional noticing may actually not benefit PSTs' learning to notice student mathematical thinking. We hypothesize that some subtypes of professional noticing are not possible given the interrelated nature of attending to and interpreting SMT.

References

Authors (under review)

Authors (2016).

Carlson, M. A., Heaton, R., & Williams, M. (2017). Translating Professional Development for Teachers Into Professional Development for Instructional Leaders. Mathematics Teacher Educator, 6(1), 27-39. https://doi.org/10.5951/mathteaceduc.6.1.0027

Hine, G., & Lesseig, K. (2021). Supporting preservice secondary mathematics teachers' professional noticing of student thinking. Australian Journal of Teacher Education, 46(8), 96-116.

Jacobs, V. R., Lamb, L. L., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. Journal for Research in Mathematics Education, 41(2), 169-202.

Sherin, M., & van Es, E. A. (2005). Using video to support teachers' ability to notice classroom interactions. Journal of Technology and Teacher Education, 13(3), 475-491. <Go to WoS>://WOS:000267958900002

Star, J. R., Lynch, K., & Perova, N. (2011). Using video to improve preservice mathematics teachers' ability to attend to classroom features: A replication study. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), Mathematics teacher noticing. Routledge.

Star, J. R., & Strickland, S. K. (2007). Learning to observe: Using video to improve preservice mathematics teachers' ability to notice. Journal of Mathematics Teacher Education, 11(2), 107-125. https://doi.org/https://doi.org/10.1007/s10857-007-9063-7

Stockero, S. L. (2014). Transitions in prospective mathematics teacher noticing. In J. J. Lo, K. R. Leatham, & L. R. Van Zoest (Eds.), Research trends in mathematics teacher education (pp. 239-259). Springer International. https://doi.org/https://doi.org/10.1007/978-3-319-02562-9_13

Stockero, S. L., & Rupnow, R. L. (2017). Measuring noticing within complex mathematics classroom interactions. In E. O. Schack, M. H. Fisher, & J. A. Wilhelm (Eds.), Teacher noticing: Bridging and broadening perspectives, contexts, and frameworks (pp. 281-301). Springer International Publishing. https://doi.org/https://doi.org/10.1007/978-3-319-46753-5_17

Thomas, J., Dueber, D., Fisher, M. H., Jong, C., & Schack, E. O. (2021). Professional noticing coherence: Exploring relationships between component processes. Mathematical Thinking and Learning, 1-19. https://doi.org/10.1080/10986065.2021.1977086

van Es, E. A., & Sherin, M. G. (2002). Learning to notice: Scaffolding new teachers' interpretations of classroom interactions. Journal of Technology and Teacher Education, 10(4), 571-596.

Vogler, A.-M., & Prediger, S. (2017). Including students' diverse perspectives on classroom interactions into video-based professional development for teachers [journal article]. Journal of Mathematics Teacher Education, 20(5), 497-513. https://doi.org/https://doi.org/10.1007/s10857-017-9382-2

PROJECT CRAFTED: AN ADAPTED LESSON STUDY FOR PRE-SERVICE MATHEMATICS TEACHERS

Michael S. Meagher
Brooklyn College - CUNY
mmeagher@brooklyn.cuny.edu

Christopher Nazelli
Wayne State University
nazelli@wayne.edu

Asli Özgün-Koca
Wayne State University
aokoca@wayne.edu

Michael Todd Edwards
Miami University
edwardm2@MiamiOH.edu

Reports on a research project designed to implement an adapted Lesson Study cycle whereby pre-service mathematics teachers co-create a lesson with an experienced expert instructor and observe the instructor teach the lesson. Results show development in the pre-service teachers' attention to elements of planning such as anticipation and pedagogical choices during instruction.

Keywords: Teacher Education - Preservice, Teacher Knowledge

Introduction

This paper reports on a study designed to show how an adapted lesson study (Isoda, 2007) can enhance pre-service teachers' early experiences in constructing mathematical tasks for use in the classroom. Teachers engaged in lesson study assume shared responsibility for the teaching lessons they create, observe the effectiveness of these materials as they are taught, and then work to revise lessons for future use (Lewis, 2002). For the purposes of this study, *lesson study* refers to an improvement cycle in which teachers collaborate to set learning goals, study curricular materials, plan a lesson that will help students meet those goals, see the lesson taught by an experienced expert teacher, and reflect on its effectiveness using observational data and artifacts of student learning.

We have created and studied communities of planning and practice (Wenger, 1999), in a number of iterations, by implementing and studying a cycle of pre-service teachers designing lessons which promote inquiry and learning for understanding; the implementation of those lessons by a master teacher (observed by the pre-service teachers); and co-reflection on the lesson by the inservice and pre-service teachers (see e.g. Meagher, Özgün-Koca & Edwards, 2009; Meagher, Edwards & Özgün-Koca, 2011). Drawing on the different steps of the original instantiation of this process, we call this the CRAFTeD (Call for lesson; Referendum, Advising session; Fine-tune; Teach-Experience; Debrief) cycle. Based on a number of iterations of the process, we have found that the key component of this cycle is that pre-service teachers see their own Lesson Plan implemented by a master teacher which (a) gives them a different level of investment in the lesson than if they watched exemplary lessons, and (b) they see their lesson plan implemented without having the pressure of teaching the lesson themselves whereby their concentration may be alerted to such aspects as classroom management, teaching style, and their interactions with the students.

In studies undertaken in the past (Meagher, Özgün-Koca & Edwards, 2009; Meagher, Edwards & Özgün-Koca, 2011), we have studied the implementation of the CRAFTeD cycle with pre-service secondary mathematics teachers working with an experienced expert teacher to design and implement a lesson for high school students. In the current study we report on 3 years of implementation of the cycle with pre-service secondary mathematics teachers, working with an experienced university professor, to design and implement a lesson for pre-service elementary mathematics teachers in a university content class for the latter group.

As in the previous studies, the research questions guiding the research and the analysis are:
(i) how do pre-service teachers co-create a lesson with an experienced expert instructor, and
(ii) what is the particular impact of ownership of the lesson on pre-service teachers' learning.

Literature Review and Relationship to Research

There is now a large body of literature (Fernandez, Cannon & Chokshi, 2003; Fernandez & Yoshida, 2004; Huang, Takahashi & da Ponte, 2019; Isoda, 2007; Lewis, Perry, & Murata, 2006) on the importance and effectiveness of lesson study approaches in improving teaching, curricular content, and instructional sequences. Hart, Alston & Murata (2009) draw on a plethora of research studies to argue that while many professional development models such as action research place teachers at the center of the research, lesson study is unique in the focus that is brought to bear on a "live lesson." They assert that, during lesson study "teachers notice certain aspects of teaching and learning, and this implicit and organic noticing does not happen in artificially replicated professional development settings" (p.1). Lesson study approaches can also provide for more focused professional development than many traditional professional development models (Gersten et al., 2014; Lewis, 2002). Furthermore, developing communities of practice (Wenger, 1999) and lesson study groups (Fernandez, 2002) can help teachers to adopt a more research-based focus in their lesson planning and to develop a shared repertoire of communal resources which can transcend individual contributions. Most research thus far has focused on inservice teachers: our research involves implementing an adaption of the lesson study approach for pre-service teachers.

Our research applies lessons learned from lesson study approaches to address the problem that teaching methodologies advocated by methods instructors in teacher preparation programs are not readily observed by pre-service teachers in actual classroom settings, a disconnect that has become more pronounced in the age of high-stakes standardized testing. While university methods instructors laud the merits of student-led inquiry, exploration, and discovery-based teaching methods, secondary mathematics teachers in too many schools "set aside" such teaching in favor of instruction directly focused on student preparation for high-stakes, multiple choice state tests (Seeley, 2006). In an age where testing dominates the landscape of too many classrooms, it becomes increasingly difficult to provide teachers-in-training with models of high-quality mathematics instruction in school environments. The proposed study explores a possible response to this situation. As secondary-level math PSTs construct lessons to be taught to prospective elementary school teachers with the guidance of their mathematics education professor, all parties benefit. The secondary-level teachers learn nuances of planning, implementing, and assessing instruction. The prospective elementary teachers gain a deeper understanding of the content that they will soon explore with their own students. By providing pre-service teachers with opportunities to collaborate across grade levels, we create synergies that lead to deeper understanding of content and pedagogy.

Methods and Methodologies

The cycle

The design of the CRAFTeD cycle is as follows:

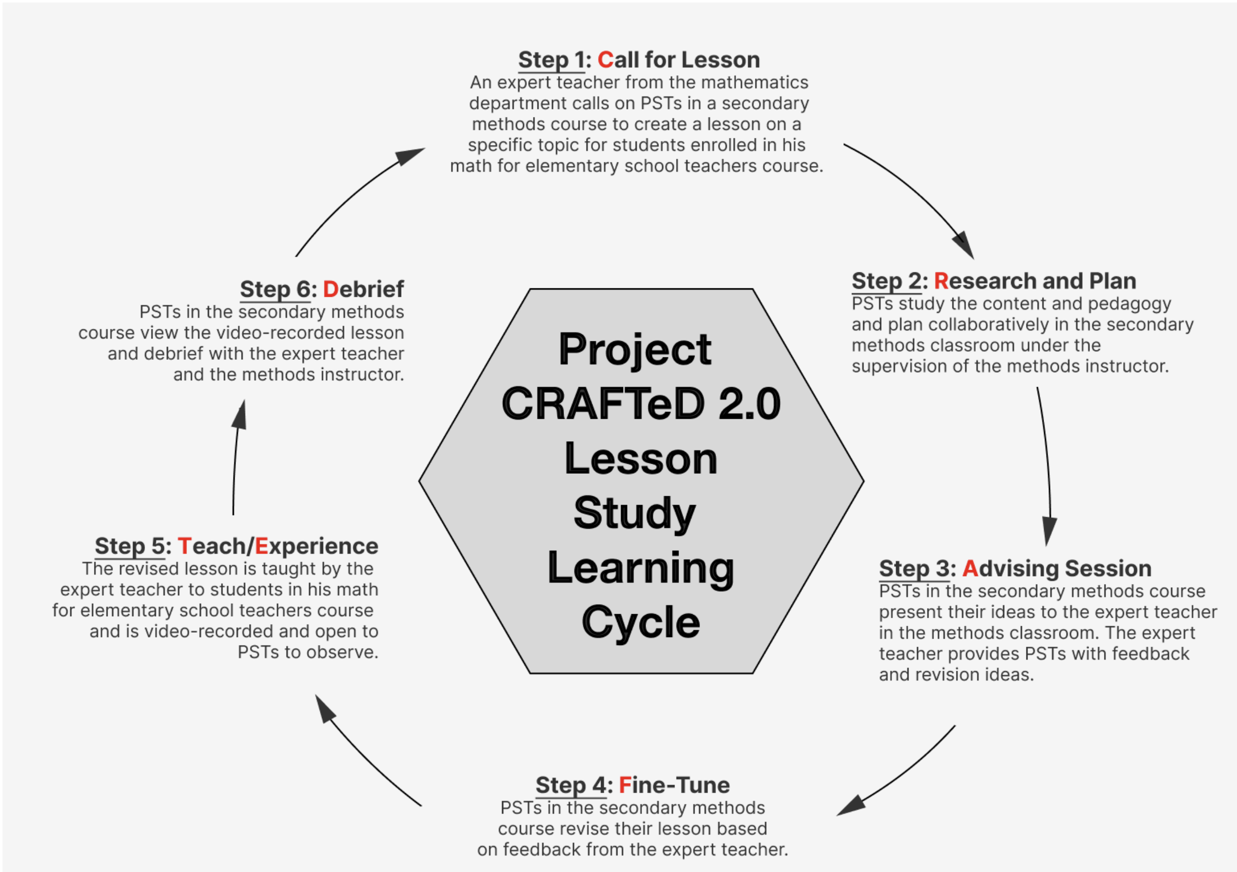


Figure 1: The CRAFTeD Cycle

The basic sequence of the original cycle and the revised cycle is (i) A class of pre-service high school teachers write Lesson Plans on a given topic and then worked together to develop improved lessons/short units designed often for technology-rich environments; (ii) an experienced expert instructor reviews the lessons/short units and presented an initial redesign; (iii) the experienced expert instructor teaches the lesson, observed in person or on video by the pre-service teachers; (iv) the pre-service teachers and the experienced expert instructor meet together to reflect on and redesign the lesson based on their experiences in the classroom.

In changing the target population of the lessons from high school students to pre-service elementary teachers and drawing on previous experiences with the cycle, there were a number of noteworthy changes in our approach.

(a) The content of the lesson was now to be taken from the mathematics curriculum for elementary school teachers (e.g. comparing partitive and quotitive models of division for whole numbers). As we will see below in the data analysis this provided some interesting challenges for the secondary pre-service teachers as they thought about elementary content in a deep conceptual manner.

(b) Rather than planning multiple lessons and then voting to select a single lesson during what had been called the Referendum stage, the class, which tended to be relatively small, worked as a whole group on the lesson.

(c) The new version of the cycle placed greater emphasis on researching and studying curriculum with a bank of research papers related to the topic created for the class to work with.

The purpose of the cycle is to examine:

(i) how do pre-service teachers co-create a lesson with an experienced expert instructor, and

(ii) what is the particular impact of ownership of the lesson on pre-service teachers' learning.

The pre-service secondary teachers (i.e., research lesson planners)

The pre-service secondary mathematics teachers (c. n=12 for each of three years) were engaged in routine activities that comprise a mathematics teaching methods course, which met for two 75-minute sessions each week for 15 weeks, at a large public Midwestern university. The methods course was one of a two-course methods sequence; these courses being the pre-service teachers' only methods courses in the program. Prior to taking or along with the methods courses the pre-service teachers take foundations of education and mathematics content courses. The course was designed specifically for pre-service secondary school mathematics teachers, who engaged in activities focused on pedagogical issues (e.g., constructing lesson plans and grading rubrics, creating technology-oriented math activities) and content issues (e.g. solving mathematics problems, assessing student work). As part of the course the pre-service secondary teachers engaged in one iteration of the CRAFTeD cycle.

The pre-service elementary teachers (i.e., students experiencing the research lesson)

The pre-service elementary teachers (c. n=12 for each of the three years) were engaged in a mathematics content course which met for four 50-minute sessions each week for 15 weeks, at the same university. This course was one of a two-course sequence which all pre-service elementary teachers, regardless of specialization (e.g. Language Arts, Early Childhood), are required to complete. The students, typically first years and sophomores, take these courses early in their degree programs. The course focused on developing the pre-service elementary teachers' mathematical knowledge for teaching (Ball et al., 2008), and grounded that development in the core elements of the elementary school curriculum (e.g. operations on the real numbers). Each week, the pre-service elementary teachers worked in small groups on problem sets that required them to push beyond "the tacit understanding that characterizes and is sufficient for personal knowledge and performance" (Ball, 2000, p. 245). The lesson created by the secondary pre-service teachers, therefore, was planned to fit into this weekly problem-set activity schedule.

The teacher of the research lesson

The teacher of the lesson was a professor in the mathematics department of the large public Midwestern university. He and the instructor of the secondary mathematics teaching methods course had nearly a decade of experience co-facilitating lesson study cycles with inservice K-12 teachers from local school districts. In addition, the two instructors collaborated on the development of the mathematics content course and its weekly problem-set activities.

Data Collection

The data collected during each CRAFTeD cycle in order to answer the study's research questions were:

- Field notes from an in-class visit by the expert instructor who initiated the cycle
- Curriculum study exchanges via an online discussion board
- Lesson Plan created by the pre-service teachers
- The revisions suggested by the expert instructor via an in-class visit
- Revisions of the lesson plan by the pre-service teachers

- The final implemented Lesson Plan
- Videotape and fieldnotes from the lesson as taught by the expert instructor
- Field notes from the debriefing sessions between the pre-service and expert instructor.
- Reflection papers by pre-service teachers on the entire cycle.

Analysis

Using the coding book from the previous versions of the project, the data was initially coded through a deductive coding process for direct answers to the research questions with two basic codes: "cocreation" (CC) and "their own lesson" (TOL) as well as codes for each stage of the cycle. The data was then re-analysed using open coding and the constant comparative method (Cresswell & Poth, 1998) to examine emerging patterns within the broad categories of the first round of coding. Sub codes such as Classroom Management, and Pedagogical Choices emerged at this stage. Quotes exemplifying the quotes were organized and exemplary quotes for each of the codes chosen to support the analysis. The analysis is presented in the order of the cycle.

Results

Call for Lesson

The Call for a Lesson occurred during the second week of the university semester. The call was for a lesson to be implemented in a mathematics content class for elementary education students, not for 6-12 grade students. As noted earlier, the mathematics content class focuses on developing the pre-service elementary teachers' Mathematical Knowledge for Teaching (MKT) (Ball et al., 2008); for example, the curriculum focuses on comparing partitive and quotitive models of division for whole numbers rather than learning to perform the standard algorithm for division. The expert instructor, who was from the mathematics faculty, visited the class in-person or virtually to inform the secondary PSTs about the nature of the class, its curriculum and students and share the topic of the lesson. He also shared what would be covered by the day of the lesson and related textbook pages. In each iteration of the cycle, the most striking aspect of the Call for a Lesson was the pre-service teachers' reaction to the content which they expected to be trivial for them. As the cycle progressed we saw the challenge the content presented for the secondary pre-service teachers. We see this exemplified in the following PST quote from 2019: "The idea of building a lesson plan based on subtraction at first seemed easy, but there is a lot of thinking involved when explaining how to subtract two numbers. Because we are math educators doing mental subtraction in our head or solving a subtraction problem on paper seemed simple but when breaking down the math problem we must take into account that we are trying to explain to younger students."

Research and Plan

Following the Call for the Lesson the secondary PSTs take 20-25 minutes in the next series of lessons (5-6 weeks) to work as a whole class on the lesson. The secondary PSTs learned from each other, as they synthesized different ideas and experiences, and this process began to normalize collaboration in teaching. There were also inherent challenges, such as limited time and space for individual contribution. During the three iterations of the cycle the number of PSTs has been approximately 12 so this was manageable as a whole class activity.

A key aspect to the Research and Plan step of the cycle was the time spent on engaging in research on best practices for teaching the content of the lesson. An online` discussion forum was created which was used to locate/share resources and to allow asynchronous brainstorming about the lesson. This discussion centered on both content and pedagogical aspects of the lesson including the study of materials from practitioner journals, youtube videos, internet resources, and selections from the research literature that focus on elementary teachers' mathematical knowledge on the topic of the research lesson.

The research phase, in each iteration, was an eye-opening experience for the PSTs in the richness of the seemingly straightforward topic for the lesson. They might think of an operation, e.g. subtraction or division of whole numbers, from a purely procedural perspective; but, when they began researching the conceptual underpinnings, they were challenged and humbled as they sought profound understanding of fundamental mathematics (Ma, 2010). As one PST in the 2019 cohort put it "Going into the lesson study, I thought subtraction was a fairly simple topic and that just reading the material provided by [the instructor] was simple enough. It wasn't until after we researched the topic and found articles such as "Subtraction: More Than Just Take Away", by Snow, that I realized how important the wording of problems can be."

This phase also engaged the PSTs in developing their MKT as they began to consider more closely how someone learns about the meaning of subtraction and where they might have difficulty. This PST Quote from 2019 reflects a locus of learning for many of the PSTs: "By investigating more about the topic, we learned what students typically struggle with, and we were able to incorporate this . . . into our lesson plan."

Advising Session and Fine Tune

An advising session with the expert instructor to discuss the draft lesson plan was scheduled for the middle of the semester (roughly the seventh week). This 30-45 minute session took place during a regular methods class meeting time. The draft lesson plan was shared with the expert instructor before this in-class visit, so that he can review it and formulate initial questions and feedback. When the expert instructor visited, one of the PSTs introduced the research lesson plan including the reasoning behind some of the design decisions. Following that, the expert instructor shared his critique, suggesting improvements orally as the PSTs took notes.

Receiving critical feedback was not easy at first. This PST Quote from 2019 typifies the PSTs reaction: "Once we had finished our rough draft of the lesson and presented it to [the expert instructor], I was completely taken aback by the amount of constructive criticism he had towards our lesson. I thought we had put together this amazing lesson where we not only required our students to think but also have explanations and representations for their thinking processes, only for [the expert instructor] to come in and nitpick all of the details we had so carefully constructed. He asked us questions regarding our wording and our timeline of the lesson that I had not even thought about, but once we took those questions into account our lesson came out even stronger." By training the attention of the PSTs to details such as the clear wording of questions and the appropriate amount of time needed for a productive, inclusive, whole-class discussion, they began to appreciate the amount of detailed, thoughtful, behind-the-scenes work involved in effective lesson planning.

The Fine Tuning of the lesson was done collaboratively and occupied approximately 25 minutes of each class in the weeks leading up to the teaching of the research lesson. The revised research lesson plan was shared with the expert instructor a week before implementation, providing an opportunity for him to share any last minute questions or concerns and make any needed adjustments.

Teach/Experience

The research lesson was taught by the expert instructor; and was recorded using Swivl. The secondary PSTs were invited to observe the research lesson in-person and take field notes. Usually 2-4 PSTs observed the lesson in person and those who were unable to attend the live lesson, watched the recording. As is the custom in a research lesson, the PST observers did not engage with the students but concentrated on collecting evidence of their learning (e.g. through their work and peer discussions). Even though the expert instructor enacted the planned research lesson faithfully most of the time, there were times when he deviated from the plan, typically through in-the-moment decisions such as choosing to explore a novel, unanticipated student idea or suggestion.

The key aspect of the teaching stage of the lesson is the level of investment the PSTs bring to the experience. In teacher education programs lesson plans are typically abstract exercises and are never taught. The CRAFTeD cycle provides the opportunity to see the lesson taught and it is a deeply meaningful experience for the PSTs. As one PST in a Quote from 2019 put it: "For most of the education classes I’ve taken thus far, I have created lesson plans, but I have never been able to actually see one through, so being able to watch this lesson come to life was very exciting. While it was cool to see certain things in the lesson go well, it was also helpful to see the snags in the lesson as well."

In planning for the lesson the PSTs picture how it might be taught and, therefore, they are able to notice and value the pedagogical choices made by the expert instructor. For example, this PST Quote from 2019: "One thing I had not thought much about were transitions from one activity or thought process to the next. [The expert instructor] did this so well.” The PSTs also saw value in how the expert instructor was able to implement research-based best practice strategies they had discussed in the research phase of the cycle. This PST Quote from 2021 is typical of the connections the PSTs made: "From our in-class discussion of Arbaugh (2010): ‘A teacher who revoices students’ ideas can clarify a mathematical relationship, identify or insert important mathematical vocabulary, or allow a misconception to be a place for learning’ (p. 46). [The expert instructor] took special care in revoicing the student ideas that were presented into solid information for everyone. He did not create the knowledge, the students did that, but he made sure to repackage it for everyone to grasp it and walk away with the same understanding." Other PSTs reflected other aspects of the teaching such as in this PST Quote from 2020: “In the actual lesson, [the expert instructor] told the students to “represent” multiple ways. I think that was a really good word for this because the students ended up trying to visually show the different ways they solved the problem. This reminded me of our “Developing Symbolic Meaning” (Lapp et al., 2013) reading where they discussed that providing representations of mathematical ideas internalizes those ideas for students.”

Debrief

During the first class after the research lesson, the expert instructor visited the methods class and participated in a 45-minute debrief (i.e. a post-lesson discussion). Staying true to the typical lesson study structure, the enacting teacher, in this case the expert instructor, shared his thoughts on the research lesson first. Following that we opened up the floor to PSTs to share their observational data and takeaways. The discussion usually centered around positives (what worked), what went according to plan, what could be improved, and what changes would lead to that improvement. Reflection on the lesson and plans for improvement are an important aspect of the lesson studies and its importance was recognised by the PSTs, for example in this Quote from a PST in 2021: "Lastly, this study showed me the importance of reflection. Upon reflecting on this lesson, I am able to see what could have gone more smoothly and what went well. This reflection can inform future lessons with my students and can allow me to make changes if I teach this lesson again. Reflecting can also allow me to see what I should have anticipated and can help me better predict my students’ responses."

Analysis & Discussion/Conclusion

This study was designed to implement the six stage CRAFTeD cycle we developed to provide pre-service teachers enrolled in a mathematics methods class a rich and meaningful experience in writing lesson plans and to answer the following research questions: (i) how do pre-service teachers co-create a lesson with an experienced expert instructor? and (ii) what is the particular impact of ownership of the lesson on pre-service teachers' learning?

This study was not on a large scale but the data presented above provides evidence that there is a tangible development in understanding of lesson planning as well as many basic and more nuanced issues of teaching when the CRAFTeD cycle is implemented.

The key benefits of the CRAFTeD cycle, and the findings in response to research question (i), as evidenced in the three iterations reported on in this paper and in answer to the research questions are (a) particularly in the collaboration with the expert instructor, the development of the PSTs' MKT as they research the content of the lesson in the framework of elementary content from an advanced viewpoint (b) the investment the PSTs make in the lesson knowing that it will be actually taught and the enhanced ability for writing a lesson plan to be a learning experience because of the full CRAFTeD cycle.

In regards to the relationship with the expert instructor, as was seen in the quotes above that were typical of the PSTs experience, the instructor's framing of his class to emphasize MKT was very beneficial to the PSTs. This framing, and their research on their lesson topic, developed key components of MKT such as knowing multiple representations of a concept, knowing common misconceptions for students learning the topic, and knowing how a particular topic fits with learning trajectories for that topic. The other main positive aspect arising from the relationship with the expert instructor was the quality of the feedback on their work and the necessity of working with the feedback since the lesson will actually be taught.

In regards to the impact of the ownership of the lesson on pre-service teachers' learning i.e. the findings in response to research question (ii), as can be seen in the quotes in the results section the PSTs clearly had a very different level of investment in the quality and impact of the lesson since it would be a live lesson. This level of investment manifested both in the construction and writing of the lesson but, perhaps more importantly, in observing the teaching of the lesson: the PSTs were highly attuned to the pedagogical choices of the expert instructor as well as being highly attuned to the learning of the students engaged in the lesson.

We believe the CRAFTeD cycle provides a variant model on traditional Lesson Study that can be very effective in the development of pre-service teachers.

References

Arbaugh, F. (2010). Secondary school mathematics teachers' classroom practices. Teaching and learning mathematics: Translating research for secondary school teachers, 452-456.

Ball, D. L. (2000). Bridging practices: Intertwining content and pedagogy in teaching and learning to teach. Journal of Teacher Education, 51(3), 241-247. http://dx.doi.org/10.1177/0022487100051003013

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? Journal of Teacher Education, 59(5), 389-407.

Creswell, J. W., & Poth, C. N. (2016). Qualitative inquiry and research design: Choosing among five approaches. Sage.

Fernandez, C. (2002). Learning from Japanese approaches to professional development: The case of lesson study. Journal of teacher education, 53(5), 393-405. http://dx.doi.org/10.1177/002248702237394

Fernandez, C., Cannon, J., & Chokshi, S. (2003). A US–Japan lesson study collaboration reveals critical lenses for examining practice. Teaching and teacher education, 19(2), 171-185. http://dx.doi.org/10.1016/S0742-051X(02)00102-6

Fernandez, C., & Yoshida, M. (2012). Lesson study: A Japanese approach to improving mathematics teaching and learning. Routledge.

Gersten, R., Taylor, M. J., Keys, T. D., Rolfhus, E., & Newman-Gonchar, R. (2014). Summary of research on the effectiveness of math professional development approaches. (REL 2014–010). Washington, DC: U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance, Regional Educational Laboratory Southeast. Retrieved from http://ies.ed.gov/ncee/edlabs.

Hart, L. C., Alston, A. S., & Murata, A. (2011). Lesson study research and practice in mathematics education. Springer. http://dx.doi.org/10.1007/978-90-481-9941-9

Huang, R., Takahashi, A., & da Ponte, J. P. (2019). Theory and practice of lesson study in mathematics. Springer. http://dx.doi.org/10.1007/978-3-030-04031-4

Isoda, M. (2007). Where did Lesson study begin, and how far has it come? In M. Isoda M.Stephens Y. Ohara & T. Miyakawa (Eds.). Japanese lesson study in mathematics: Its impact, diversity and potential for educational improvement. pp. 8-15. http://dx.doi.org/10.1142/9789812707475_0002

Lapp, D. A., Ermete, M., Brackett, N., & Powell, K. (2013). Connecting Research to Teaching: Linked Representations in Algebra: Developing Symbolic Meaning. The Mathematics Teacher, 107(4), 306-312. http://dx.doi.org/10.5951/mathteacher.107.4.0306

Lewis, C. C. (2002). Lesson study: A handbook of teacher-led instructional change. Research for better schools.

Lewis, C., Perry, R., & Murata, A. (2006). How should research contribute to instructional improvement? The case of lesson study. Educational researcher, 35(3), 3-14.

Ma, L. (2010). Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States. Routledge. http://dx.doi.org/10.4324/9781003009443

Meagher, M., A. Ozgun-Koca and M.T. Edwards. "Pre-service Teachers Experiences With Advanced Digital Technologies: The Interplay Between Technology in a Pre-service Classroom and in Field Placements." 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Georgia State University. Atlanta, October.

Meagher, M., M.T. Edwards and A. Koca. "Project CRAFTeD: An Adapted Lesson Study Partnering Preservice Mathematics Teachers With a Master Teacher." 33rd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Reno, Nev., October.

Wenger, E. (1999). Communities of practice: Learning, meaning, and identity. Cambridge University Press. http://dx.doi.org/10.1017/CBO9780511803932

PROSPECTIVE MATHEMATICS TEACHERS' EXPERIENCES ENGAGING IN AN OPPORTUNITY FOR PRODUCTIVE STRUGGLE

Nitchada Kamlue	Laura R. Van Zoest	Yaronn James Arciaga
Western Michigan University	Western Michigan University	Western Michigan University
nitchada.kamlue@wmich.edu	laura.vanzoest@wmich.edu	yaronnjames.arciaga@wmich.edu

This study investigated what 12 prospective mathematics teachers (PTs) in a middle school mathematics method course reported during a video-stimulated recall interview about their experiences when they were engaged in a doing math task that yielded an Opportunity for Productive Struggle (OPS). We investigated their reported feelings during the OPS, what mathematics they made sense of as a result of it, and the relationships between their feelings and sense making. We found that PTs' feelings did not predict the nature of their sense making and that regardless of how they felt during the OPS, the majority of them (66.67%) reported that engaging in the OPS resulted in mathematical sense making. Other PTs reported pedagogical sense making. We suggest future research to expand on our findings.

Keywords: Instructional Activities and Practices, Preservice Teacher Education

Productive struggles are opportunities for learners to make sense of mathematics within their *zone of proximal development* (Vygotsky, 1978) and deepen their understanding of mathematical ideas and the relationships among those ideas (e.g., National Council of Teachers of Mathematics [NCTM], 2014; Peterson & Viramontes, 2017). Hence, this concept is crucial for all learners at all learning levels, including prospective mathematics teachers (PTs) who will be charged with enacting the teaching practice *support productive struggle in learning mathematics* (NCTM, 2014) in their future classrooms. Research about productive struggle has identified ways PTs struggle as learners and what might make their struggles productive. For example, researchers (e.g., Zeybek, 2016) who have investigated PTs' productive struggles when engaging them in *high cognitive demand tasks* (Stein et al., 1996) have highlighted the important role of such tasks in supporting PTs to gain a deeper mathematical understanding. Thus, it seems that the use of high cognitive demand tasks might generate opportunities for productive struggle. Past researchers have focused on better understanding PTs' struggles (productive or otherwise) once they have been identified (e.g., Ducloux et al., 2018). Existing studies have provided some information about how to support PTs to engage in productive struggle themselves (e.g., Rahman, 2022) and to support their students' productive struggle (e.g., Anthony, 2021). Based on prior work, we can anticipate which aspects of instruction provide rich opportunities for productive struggle, but little is known about how the various PTs in a class respond to that same opportunity. Better understanding how PTs experience such opportunities—what they report having felt and made sense of—would provide useful information for better supporting productive struggle. Thus, we investigated the research question *What do prospective mathematics teachers report having experienced when reflecting on their engagement in the same opportunity for productive struggle?*

Literature Review

Researchers who study struggle have used the term *productive struggle* in different ways. Some have used it broadly to encompass when learners engage in a task that has an unclear path for them to solve, as long as they work towards the goal of the task without the teacher decreasing the level of cognitive demand of the task (e.g., Warshauer, 2015; Zeybek, 2016). Others have used the term to reflect a specific research situation, such as “a student persisting in a digital learning task while maintaining a likelihood of future success” (Krumm et al., 2022, p. 514). Kamlue and Van Zoest (2022b) defined *mathematically productive struggle* as the type of struggle that occurs when students “delv[e] more deeply into understanding the mathematical structure of problems and relationships among mathematical ideas, instead of simply seeking correct solutions” (NCTM, 2014, p. 48). Across all these uses is the idea that productive struggle supports learning and that other types of struggles do not.

Two themes arise across studies that have focused on productive struggle with PTs. Some researchers have explored how PTs facilitated productive struggle in their classes. For example, Anthony (2021) found that 10 middle grades mathematics and science pre-service teachers who learned about productive struggle in their mathematics methods courses still struggled with creating and sustaining productive struggle with students in their field experience. Similarly, Rahman (2022) investigated the learning opportunities of seven PTs enrolled in a middle school mathematics methods course who were teaching high cognitive demand tasks to their peers. They reported that these PTs had opportunities to learn about productive struggle through the process of selecting tasks and responding to students’ struggles.

Other researchers have investigated PTs’ productive struggle when engaging with high cognitive demand tasks. For example, Ducloux et al. (2018) directly interviewed 32 prospective elementary, middle, and secondary teachers from three different mathematics content courses for teachers to investigate their struggles after engaging in a non-routine problem-solving task. The participants in the study reported experiencing both negative aspects of struggles (e.g., frustration, too challenging) and positive aspects of struggles (e.g., perseverance, collaborative struggle). Zeybek (2016) investigated 48 middle grades pre-service teachers’ struggles in a geometry class and stated that for her participants’ struggle to be productive, the participants needed to engage in a high cognitive demand task similar to what Warshauer’s (2015) study suggested. Finally, Kamlue and Van Zoest (2022a) investigated 18 PTs’ struggles when engaging with a *doing math* task (Stein et al., 1996) and noted that mathematically productive struggle could occur outside the intended learning goal. That is, since *doing math* tasks do not have an obvious solution path, the authors observed the PTs introducing ideas that did not help them solve the problem, but that they came to better understand as a result of trying to use these ideas.

From the above, we notice that high cognitive demand tasks provide learners with opportunities to struggle productively because of the nature of these tasks—they require students to engage with conceptual ideas and often do not have an obvious path to the solution (Stein et al., 1996). Warshauer (2015) also pointed out that if teachers maintain the high level of cognitive demand of a task when noticing students’ struggles, those struggles can become productive because the teachers let the students make sense of mathematics with which they are struggling. However, what we as a field do not know is how various PTs in a class respond to the same opportunity for productive struggle. Thus, the purpose of this study was to answer this research question: *What do prospective mathematics teachers report having experienced when reflecting on their engagement in the same opportunity for productive struggle?*

Theoretical Perspectives

Our research is based on a participationist perspective (Vygotsky, 1987). That is, we see learning as taking place through learners’ interactions with more knowledgeable others, such as their teacher and their peers. Drawing on this perspective and the research described above, we defined an Opportunity for Productive Struggle to be a situation where (a) the PTs were engaged in a *doing math* task (Stein et al., 1996) and (b) the teacher had positioned the class to engage in, and was facilitating a discussion about, collaborative sense making of a peer's high-leverage contribution (Leatham et al., in press).

To further unpack PTs experiences, we drew on Goldin's (2000) theoretical framework that described the relationships between affect and heuristics. Hannula et al. (2004) provided an example of how Goldin illuminated interactions between students' feelings and their cognitive processes:

> The feeling of *bewilderment* in approaching a problem in mathematics may simultaneously suggest that certain standard problem interpretations or problem-solving strategies do not work… [a]ffective states may evoke heuristic strategies; thus frustration may evoke a major change in strategy. (p. 110)

This framing suggests the importance of asking PTs about their feelings when engaging in an Opportunity for Productive Struggle.

Methodology

The participants in this study were twelve (of nineteen) PTs in a middle school mathematics methods course who agreed to be interviewed. Five (of seven) PTs were taking it as the first mathematics methods course in a program leading to a secondary mathematics education degree. Seven (of twelve) were taking it as their second mathematics methods course as middle school mathematics majors in an elementary teacher education program. We used video-stimulated recall interviews to ask the PTs about what they had experienced when engaged in an Opportunity for Productive Struggle. The context was a lesson centered on the *doing math* (Stein et al., 1996) Frog Problem (see Figure 1; for more details, see Andrews, 2000, and Dixon and Watkinson, 1998). The PTs engaged in activities around the Frog problem for three ninety-minute sessions, beginning with two teams of PTs physically modeling their peers' suggested ways to achieve the fewest number of moves using chairs set up at the front of the room. They had opportunities to develop their ideas about the mathematics of the Frog Problem and the *mathematical practices* (NCTM, 2014; NGACBP & CCSSO, 2010) they used in solving the problem by participating in whole-class discussions, small-group discussions, and written reflections.

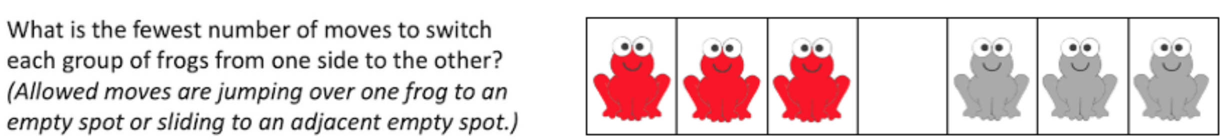


Figure 1: The Frog Problem Prompt and Representation of Two Frog Teams of Size Three

The researcher-identified Opportunity for Productive Struggle discussed in this report occurred at the beginning of the third session and was six minutes long. It was chosen because it fit the two criteria of an OPS identified in the Theoretical Perspective section above: (a) the PTs were engaged in the Frog Problem, a task that met the *doing math* criteria (for more details, see Kamlue & Van Zoest, 2022b) and (b) the teacher had identified an incorrect PT's explanation (See Figure 2) as a high-leverage contribution and positioned the class to engage in mathematical sense making about it. This OPS provided the opportunity to better understand the difference between exponential and quadratic equations and included multiple PTs' contributions to the whole-class discussion about the PT explanation in Figure 2.

The equation [$n^2 + 2n = m$] works because the number of moves [m] increases exponentially each time the amount of people [on a team; n] increases by *2*. And the *2n* works because the difference in the number of moves increases by *2* each time.

Figure 2: The (Incorrect) Explanation that Initiated the Opportunity for Productive Struggle

The audio-taped and transcribed video-stimulated recall interviews began by showing the PT a video of the selected Opportunity for Productive Struggle (OPS) to help them to recall the experience. This report analyzed the responses to these questions (and follow-up probing):

1. Please describe what you were experiencing in this [OPS].
    a. Do you remember what kind of thoughts you had?
    b. Do you remember how you felt at that time?
2. What mathematics did you make sense of as a result of that [OPS]?

The first and third author independently read through the transcripts and holistically identified a word that captured how each PT described feeling during the Opportunity for Productive Struggle and a phrase that captured what they described making sense of as a result of their experience. They then discussed their coding, developed code names and definitions, and refined them with the help of the second author. The data was then re-coded using the refined codes (see Figure 3 and 4 in Results & Discussion section for the resulting code names and definitions).

Results & Discussion

We report here on the PTs' reflections on what they experienced as they engaged in the interaction that we had identified as an OPS. (See Figure 2 for the explanation that initiated the OPS). We first report on the code names and definitions that we developed in our first level analysis of the data (Figures 3 and 4). We then report on our second level analysis of those codes (Figure 5).

Figure 3 shows code names and definitions that arose from our analysis of the PTs' responses to the first interview question—*Do you remember how you felt at that time?*

Code Name	Definition	Illustrative quote from our data
unsettled	When PTs said "nervous", "confused", or "frustration"	"I think I might have been a little **confused** on where [the classmate's idea] was coming from…"
neutral	When PTs expressed no feeling or said "bored" or "fine"	"Honestly, kind of **bored**…I expected like way more math because [this class] seems to me more like education class…"

settled	When PTs said “acceptable”, “connected”, “validated, or “good”	“I felt **good** because I was able to connect it back to something I already knew…”

Figure 3: Code Names and Definitions for PTs’ Feelings during the Opportunity for Productive Struggle

Figure 4 provides code names and definitions that arose from our analysis of the PTs responses to the second interview question—*What mathematics did you make sense of as a result of that [OPS]?*

Code Name	Definition
Mathematics	An insight or outlook that provides definitions, justifications, questions, explanations of the logical concepts, techniques, tools, or skills identified when applied to a problem.
Teaching	The teacher’s use of techniques, tools, or skills in an articulate form intended to accomplish a goal for students.
Other	A characteristic that belongs to a person (Trait) and responses where what the PT made sense of could not be inferred (CNI)

Figure 4: Code Names and Definitions for What PTs Made Sense of from the Opportunity for Productive Struggle

Figure 5 provides the following information for each PT: the PT’s reported feelings during the OPS (column 2), their reported sense making as a result of the OPS (column 3), and the type of sense making (column 4). We noticed that the PTs experienced the same OPS differently in two ways: different feelings and different types of sense making. First, they felt differently when engaging in the same task. Six PTs felt unsettled, three PTs felt neutral, and three PTs felt settled. Second, the PTs responded differently when they were asked to answer the same question: *what mathematics did you make sense of as a result of that [OPS]?* Eight PTs made sense of mathematics, three PTs made sense of teaching, and two PTs made sense of other categories than mathematics and teaching—all as a result of engaging in the same OPS.

There were no clear patterns between PTs in the elementary and secondary programs (column 1). There also were no clear patterns related to the feelings the PTs expressed across the types of their sense making; it is noteworthy that each of the sense-making categories had one PT categorized as having a *neutral* feeling. Two PTs did not provide clear evidence of something they had made sense of, and thus their responses were categorized as *other*. PT11 talked about increasing their “confidence” and PT12 simply described what happened during the OPS.

There were three PTs’ whose sense making was categorized as *teaching*, one in each feeling category—*settled*, *neutral*, and *unsettled*. Although each of these PTs talked about a different aspect of teaching, they all focused on supporting students. PT8 expressed making sense of the critical role of clarifying questions to support students’ justifications, PT9 gained insight into the selection of tasks that support student learning, and P10’s response was about supporting students as they struggle to communicate their mathematical thinking. The fact that all three of the PTs in this category thought deeply about a critical aspect of teaching as a result of participating in this OPS suggests that engaging PTs in making sense of their peer’s thinking about a high cognitive demand task in a methods course has both pedagogical as well as mathematical benefits.

PT	Reported feeling	Reported sense making as a result of the OPS	Type of sense making
1	unsettled	"I know when I said increased by two each time I was talking about there [CNI*] and the problem was like a pattern inside a pattern...[the little pattern], was the little pattern in [the big pattern], was by twos and then the big obvious pattern wasn't by twos..."	mathematics
2**		"...I kept thinking this is an exponential because there needs to be an *x* in the exponent…"	
3		"I would just say I have learned about the importance of variables in an equation and like the importance of explaining your variables."	
4		"I would say that everything has structure. You just need to like dig a little deeper and find [the answer]. And then I, the problem will be easier."	
5	neutral	"...look carefully into the equations…"	mathematics
6	settled	"Like breaking apart student work and understanding the different parts"	mathematics
7		"...I was able to understand, um, what, like what the numbers meant in the equation, like what they represented"	
8a***		"I think like justification in mathematics, like an explanation. There needs to be a proper justification of like why it [CNI] works…"	
8b	settled	"...So I think thinking really deeply about like, clarifying questions I could ask if this was like a student of mine, or um, like asking questions that might get them to provide me with a proper justification…"	teaching
9	neutral	"...it's more like just good teaching practices than math itself…[for example:] What makes a good problem? What makes a bad problem?"	teaching
10	unsettled	"...[difficulty explaining equations to peers] is testing my patience a lot, which is good because I know that that will be pushed when I'm a teacher [because my students will also struggle to explain equations]"	teaching
11	unsettled	"...So [the discussion] was very nice; hearing other people's thought processes and ideas and trying to make sense of something that confuses me as well. … when the entire class and the teacher agree[ed] with [me] [it increased my confidence]"	other
12	neutral	"...I would say [the course instructor] put a lot of emphasis on us knowing about exponential functions, and [the instructor] had us do an assignment…"	other

Notes: *CNI means cannot be inferred. **Bold indicates a PT in the secondary education program. ***PT8 reported two explicit pieces of information.

Figure 5: PTs' Reflections on their Engagement with the Opportunity for Productive Struggle

For the eight PTs whose sense making was categorized as *mathematics*, four expressed feelings that were categorized as *unsettled*, three *settled*, and one *neutral*. There were no noticeable patterns related to the feelings the PTs expressed. What is noteworthy was the range of mathematics that the PTs reported making sense of during the OPS. PT1 described “a pattern inside a pattern,” reflecting their increasing awareness of nonlinear situations. PT2 discussed the structure of an exponential equation and their realization that the Frog Problem did not fit that structure. PT3 and PT7 reported making sense of representations, such as “variables” and “numbers,” respectively. PT4 and PT6 emphasized techniques for solving challenging mathematics problems, and PT8 focused on the importance of justification. Thus, it appears that regardless of how they felt during the interaction, the majority (66.67%) reported that discussing a PT’s high-leverage contribution during a *doing math* task resulted in mathematical sense making.

Conclusion

This study discussed what 12 prospective mathematics teachers (PTs) reported having experienced when reflecting on their engagement in an Opportunity for Productive Struggle in their middle school mathematics methods course. The way in which the PTs reported making sense of mathematics suggests that most of them were engaged in *mathematically productive struggle* (Kamlue & Van Zoest, 2022b). Thus our work provides further evidence that *doing math* tasks (Stein et al., 1996) support productive struggle.

Our finding that the PTs’ feelings did not predict the nature of their sense making suggests that even when PTs feel unsettled, they can make sense of important mathematical and pedagogical concepts. This finding supports the idea that rushing in to relieve their struggle may undermine the benefits of high cognitive demand tasks (Warshauer, 2015). Our findings also suggest that generating Opportunities for Productive Struggle in mathematics methods courses may support PTs to learn about teaching as well as about mathematics. Rahman (2022) found that PTs learned about productive struggle through selecting tasks. It may also be possible that engaging PTs in worthwhile tasks in a way that models effective pedagogy can lead to *pedagogically productive struggle*—the type of struggle that occurs when delving deeply into understanding the relationship between ideas about teaching and student learning—as well as *mathematically productive struggle* (Kamlue & Van Zoest, 2022b).

This study provided some insight into how different PTs experience the same Opportunity for Productive Struggle. The variety of ways that the PTs in the class experienced this same opportunity illustrates the complicated nature of teaching—teachers need to generate learning opportunities for each of their students knowing that the students will not all respond to that opportunity in the same way. Better understanding the different ways PTs experience rich tasks, such as the Frog Problem, can inform teacher educators’ preparation for using these tasks with their PTs.

Extending the research reported here to a larger participant group might reveal additional patterns. Our data was based on PTs’ reports of what they made sense of as a result of an Opportunity for Productive Struggle; it would also be useful to have a measure of whether PTs actually gain the better understanding of a mathematical idea that a given Opportunity for Productive Struggle provides.

Acknowledgements

This work was supported by the Graduate Student Association Data Collection Grant, Western Michigan University. We would like to thank the Western Michigan University Mathematics Education Community for providing valuable feedback on this project.

References

Andrews, P. (2000). Investigating the structure of Frogs. Mathematics in School, 29(2), 7-9.

Anthony, M. (2021). Middle grades pre-service teachers' struggle to create productive struggle. In D. Olanoff, K. Johnson, & S. M. Spitzer (Eds.), Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (pp. 989-993). PME-NA.

Dixon, J., & Watkinson, R. (1998). The investigating experience: Frogs. Mathematics in School, 27(5), 42-43.

DuCloux, K., Gerstenschlager, N., Marchionda, H., & Tassell, J. (2018). Characterizing prospective mathematics teachers' productive struggle. In Venenciano, L. & Redmond-Sanogo, A. (Eds.), Proceedings for the 45th Annual Meeting of the Research Council on Mathematics Learning (pp. 9-16). Baton Rouge, LA.

Goldin, G. A. (2000). Affective pathways and representation in mathematical problem solving. Mathematical thinking and learning, 2(3), 209-219. http://doi.org/10.1207/S15327833MTL0203_3

Hannula, M., Evans, J., Philippou, G., & Zan, R. (2004). Affect in mathematics education – Exploring theoretical frameworks. Research forum. In M. Johnsen Høines & A. Berit Fuglestad (Eds.), Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education (Vol. 1, pp. 107–136). PME.

Kamlue, N. & Van Zoest, L. R. (2022a). A preliminary look at prospective mathematics teachers' productive struggle. In A. E. Lischka, E. B. Dyer, R. S. Jones, J. N. Lovett, J. Strayer, & S. Drown (Eds.), Proceedings of the forty-fourth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (pp. 1224-1228). Middle Tennessee State University.

Kamlue, N., & Van Zoest, L. R. (2022b, November 9). Productive struggle: An opportunity for in-depth mathematics learning. Advancing Research & Innovation in the STEM Education of Preservice Teachers in High-Need School Districts (ARISE). https://aaas-arise.org/2022/11/09/productive-struggle-an-opportunity-for-in-depth-mathematics-learning/

Krumm, A., Coulson, A., & Neisler, J. (2022). Defining productive struggle in ST math: Implications for developing indicators of learning behaviors and strategies in digital learning environments. ACM international conference proceeding series (pp. 514-520). Association for Computing Machinery. https://doi.org/10.1145/3506860.3506901

Leatham, K. R., Van Zoest, L. R., Peterson, B. E., & Stockero, S. L. (in press). A decomposition of the teaching practice of building. Proceedings of the 2022 NCTM Research Conference.

National Council of Teachers of Mathematics [NCTM]. (2014). Principles to actions: Ensuring mathematical success for all. Author.

National Governors Association Center for Best Practices & Council of Chief State School Officers [NGACBP & CCSSO]. (2010). Common core state standards for mathematics. Washington, DC: Author. Retrieved from https://learning.ccsso.org/wp-content/uploads/2022/11/Math_Standards1.pdf

Peterson, B. E., & Viramontes, R. (2017). Key questions to guide teachers in supporting productive struggle in learning mathematics. In D. A. Spangler, & J. J. Wanko (Eds.), Enhancing classroom practice with research behind principle to action (pp. 73-87). NCTM.

Rahman, Z. G. (2022). Pre-service mathematics teachers' experience with productive struggle. The Educational Forum, 1-19. https://doi.org/10.1080/00131725.2022.2072033

Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. American Educational Research Journal, 33(2), 455–488. https://doi.org/10.3102/00028312033002455

Vygotsky, L. S. (1978). Mind in society: The development of higher psychological processes. M. Cole, V. John–Steiner, S. Scribner, & E. Souberman (Eds.), Reprints of original writings. Harvard University Press. [pp. 15 – 16, 11 – 14, 79 – 91.]

Warshauer, H. K. (2015). Productive struggle in middle school mathematics classrooms. Journal of Mathematics Teacher Education, 18(4), 375-400.

Zeybek, Z. (2016). Productive struggle in a geometry class. International Journal of Research in Education and Science, 2(2), 396-415.

RECONCEPTUALIZING MATHEMATICAL PROBLEM SOLVING FOR MIDDLE/SECONDARY PRESERVICE TEACHERS

Pingping Zhang
Winona State University
pzhang@winona.edu

Brooke Krejci
University of Wisconsin – River Falls
brooke.krejci@uwrf.edu

Keywords: Problem Solving, Preservice Teacher Education, Middle School Education, High School Education.

Mathematical problem solving is defined as a "situation in which a goal (i.e. the answer) is to be attained and a direct route to the goal is blocked" (Kilpatrick, 1985, p.2). As the focus shifts to how to nurture a good problem solver (Schoenfeld, 2007), distinct challenges have emerged when the area of mathematical problem solving met the area of teacher education.

Son and Lee (2021) suggested five categories for conceptions of problem solving: (1) problem solving as a means to achieve other ends; (2) problem solving as a means to finding a solution; (3) problem solving as a skill requiring steps/procedures; (4) problem solving as employing multiple approaches/strategies; (5) problem solving as art. This poster examines the conceptions of problem solving on middle/secondary preservice teachers using the categories from Son and Lee (2021), as well as how a selected problem-solving activity potentially challenged their conceptions.

Ten middle/secondary preservice teachers from a Midwestern public university participated in this study. They completed a pre-questionnaire on the first day of class, including two questions: 1. *In your own opinion, what are mathematical problems? Give an example of a mathematical problem.* 2. *What does mathematical problem solving mean to you? Tell me one of your most recent mathematical problem-solving experiences*. At the end of the first week, each participant individually solved the Census Problem.

Each participant's answers in the pre-questionnaire were analyzed and coded into one of the five categories suggested by Son and Lee (2021). Each of their documentations on solving the Census problem was examined to identify the situations where they experienced a *stuck* moment. At the end, we looked at how these moments potentially challenged their previously held conceptions on mathematical problem solving.

Echoing what was reported by Son and Lee (2021), Category 2 (problem solving as a means to finding a solution) has the largest number of participants. Table 1 below summarizes the *stuck* moments reported by participants under each type of category.

Table 1: Stuck Moments Reported Under Each Category

	Category 1	Category 2	Category 3
How is chocolate pudding related	1	4	3
I don't know the house number	1	3	2
Could there be multiple answers	1	0	1
Not enough information to solve	1	0	2

The reported *stuck* moments directly challenged the conception in Category 3. The participants under Category 2 were more likely to settle with any answer they deduced without questioning whether there could be no answer or multiple answers. Hence only the first two *stuck* moments were reported by them. A problem designed with multiple solutions, or no solution may better challenge this type of conception.

References

Kilpatrick, J. (1985). A retrospective account of the past 25 years of research on teaching mathematical problem solving. Teaching and Learning Mathematical Problem Solving: Multiple Research Perspectives, 1-15.

Schoenfeld, A. H. (2007). Problem solving in the United States, 1970–2008: Research and theory, practice and politics. ZDM Mathematics Education, 39, 537-551.

Son, J. W., & Lee, M. Y. (2021). Exploring the relationship between preservice teachers' conceptions of problem solving and their problem-solving performances. International Journal of Science and Mathematics Education, 19(1), 129-150.

REFLECTING ON AND EVALUATING TASKS: WHAT HAPPENS WHEN LEARNING GOALS ARE NOT MET?

Dana Olanoff
Widener University
dolanoff@widener.edu

Neet Priya Bajwa
Illinois State University
nbajwa@ilstu.edu

Jennifer M. Tobias
Illinois State University
jtobias@ilstu.edu

Rachael M. Welder
University of Nevada, Reno
rwelder@unr.edu

We (a team of mathematics teacher educators) engaged in designing a series of multiplication tasks with the goal of deepening prospective teachers' (PTs') specialized content knowledge of multiplication. In this paper, we discuss the learning goals and design elements from one of the tasks in our sequence. We highlight how we evaluated PT responses to the initial version of the task to determine that the task was not meeting our learning goals. Then we detail task modifications that were made after reflecting on what we learned from implementing the task across three semesters at three universities.

Keywords: Number Concepts and Operations, Preservice Teacher Education

Introduction and Theoretical Background

Mathematics teacher educators (MTEs) play an important role in helping prospective teachers (PTs) become well prepared beginning teachers of mathematics (Association of Mathematics Teacher Educators, 2017). Research has shown that engaging in meaningful, cognitively demanding tasks during their teacher education will improve PTs' instruction when they enter the classroom (Watson and Mason, 2007). In order to develop these types of tasks, MTEs must have clear, shared, and targeted learning goals (Jansen et al., 2009). In addition to defining these learning goals, MTEs must assess the implementation of their tasks to determine if the learning goals are being met, and then make progressive modifications to the tasks in the cases where the learning goals are not met (Berk and Hiebert, 2009). In this paper, we share our work designing, modifying, and implementing one task in a sequence focused on PTs' understanding of the array model of multiplication. We discuss our learning goals for the task, and how we modified the task when it was apparent that the learning goals were not being met.

We began our work by engaging in a task design cycle that we designed (Tobias et al., 2014), in which a task from an elementary curriculum is modified into a task for PTs. Our task design cycle includes an iterative process involving six steps. The process begins by selecting a children's task, followed by modifying that task for PTs to align with our learning goals. Next, the modified task is implemented with PTs and data are collected. This is followed by analyzing the data and reflecting on the implementation, and re-designing the task based on the reflection on our implementation and analyses. The third through sixth steps are repeated with further implementation, analysis, reflection, and redesign.

We designed a four-task sequence revolving around the array model of multiplication. Our overall goals for the entire sequence were for PTs to:

1. Understand how the array model can be used to represent multiplication,
2. recognize the distributive property as a driving force behind multiplication algorithms,

3. connect base 10 block models to the partial products and standard algorithms, and
4. recognize the standard multiplication algorithm as a partial products algorithm.

With these goals in mind, we began with a third-grade multiplication task which asked how many mailboxes were contained in a 12-by-12 array. Our first task presented PTs with a 29-by-23 grid, representing a rectangular array of mailboxes in a mailroom. We asked PTs to use the picture to find three different methods for determining the total number of mailboxes or unit squares (1-by-1) in the grid (e.g., by decomposing the provided grid-array into different combinations of smaller regions). Our second task, the focus of this paper, asked PTs to make connections between a base 10 block model of 29 x 23 and the mailbox problem from task 1. In task 3, we asked PTs to relate a base 10 block model to symbolic representations of the standard and partial products algorithms for whole-number multiplication. In our final task, we asked PTs to solve 16x25 in three different ways.

Methodology

A total of 11 sections of elementary content courses across three universities participated in our task sequence over three semesters (semester 1: 5 sections, semesters 2 and 3: 3 sections each). The courses were designed such that PTs were introduced to a task first then given time to work on the task individually and in small groups. This was followed by whole class discussions focused on solutions and solution strategies.

The data collected included PTs' written work on every task and instructor field notes. Since not every instructor taught the task sequence at the same time, field notes and reflections were used to make minor modifications to the tasks during the semester, prior to the next instructor teaching. After every instructor completed the task sequence, we examined PTs' work for themes which then informed larger task modifications prior to the reimplementation of the task sequence the next semester. In this paper, we focus on our second task, because this task was modified considerably for two reasons: a) the learning goals for this specific task as well as our task sequence overall were not being met by this task, as students were making surface level connections as opposed to those we intended for them to make through the task, and b) we expanded the original task with each iteration as we considered things students attended to or missed, leading us to critically examine our learning goals for this task as we engaged in the task design cycle.

Results

What we report here are our findings from the versions of our second task. We discuss PTs' work and reasoning for each iteration, as well as task modifications that we made across iterations.

Task 2 Version 1

Our original version of task 2 asked PTs to analyze a base 10 block representation (see Appendix for a full picture of the task) to determine if the base 10 blocks represented the original 29-by-23 grid and explain why or why not. In analyzing the data from this version of the task, we determined that PTs were successful in recognizing that the representation matched (97% said yes it matches). However, 38% of those focused solely on the fact that both representations contained 667 unit squares, rather than connecting the base 10 blocks to the original grid in any meaningful way. In addition, 27% reasoned that the base 10 block representation matched because it shows 29 rows with 23 columns, which was the same as in the mailbox problem, but did not account for how the base 10 blocks were used in the model. Only a few PTs justified that the base 10 blocks match the mailbox problem because it is a way of chunking the grid to assist in the original multiplication. Table 1 provides some examples of each of these types of PTs' responses on Task 2.

Table 1: PTs' Reasoning on Task 2

#	Does this model [base-ten block representation of 29 x 23 grid] represent the mailbox problem? If so, how? Please provide your reasoning.
1	"Yes, this model represents the mailbox problem because all of the boxes added together provides the same outcome as the other block representations are. Because they are the same outcome this means that each model represents the answer for this situation."
2	"667 is the same amount as the mailboxes problem. Same amount of boxes. It shows 23 x 29 as well. 29 rows, 23 columns."
3	"Yes, this model represents the mailbox problem. The way it was broken down into base 10 blocks shows how you could have broken the mailbox problem. Breaking it down into the base 10 blocks allows it to be easier to add it all together problem."

Task 2 Version 2

As discussed above, our results indicated that almost 40% of PTs responded that the representations were the same because they both resulted in an answer of 667. We felt that these responses were not in line with our learning goals for either the task sequence, or for Task 2 specifically. We intended for this task to be an opportunity for PTs to make connections between the grid model and the base 10 block model, seeing the base 10 block model as an efficient way to chunk the grid to make the multiplication easier. This would in turn act as a segway into making sense of the distributive property, and our follow up tasks that focused on the property's use in the standard and partial products multiplication algorithms. In our task analysis, we decided that by providing a correct base 10 block partition of a 29-by-23 grid, and asking if this matched the original grid, despite asking for justification, we allowed answers that only focused on the product, rather than understanding how both models showed the multiplication of 29-by-23, and different ways of figuring out this product. In modifying the task, we focused explicitly on the goal for PTs to see the issue with only matching the answer to the multiplication problem. The new task directions became: *"A student said that you can create a replica of the 29-by-23 grid with 6 flats, 6 rods, and 7 small cubes. Is this student correct? Why or why not? Support your reasoning by drawing the grid with these given pieces below."* Part two of this task asked PTs to create the 29-by-23 grid with the fewest number of base 10 blocks possible.

In analyzing this version of Task 2, we found that PTs were successful at answering the yes/no question, i.e., recognizing that you cannot create a replica of the grid with 6 flats, 6 rods, and 7 small cubes (91.5% said no you cannot create a replica). However, when compared to the first version of our task, we found differences in how PTs' reasoned about the replica. In our modified task, the majority of PTs either focused on the fact that 6 flats will create a row of 30, and thus be too big, that at most 4 flats would be able to fit in the grid, that the given blocks cannot create a rectangle, or the grid itself is not "big enough" for the blocks to fit without breaking a block apart (72.6% of PTs). Only two PTs (2.2%) focused solely on the answer of 667 and incorrectly concluded that a replica could be made. Thus, in our modified task we found that PTs were largely focusing on an aspect of the array not related to the total of 667 to determine if the blocks could be used to create a replica.

Discussion

The purpose of our second task was to introduce an array representation with base 10 blocks to model the mailbox problem with the underlying goal of moving PTs towards developing and understanding the distributive property and its role in whole number multiplication algorithms. Our second task specifically was designed for PTs to explore how base 10 blocks can be used to represent an array situation as well as begin to recognize partial products and how they can be combined to find a total. In the first version of this task we found that by giving PTs a correct base 10 block representation, over half (65%) either only focused on the total of 667 being the same or only focused on the number of rows and columns being the same in both representations. Thus, the majority of PTs did not reason about base 10 blocks, nor need to, to determine that the base 10 block representation matched the mailbox problem. This indicated that providing a correct base 10 block representation did not compel PTs to reason about multiplication in ways we were intending. Results from our modified task indicated that over 90% of the PTs determined that they could not create the array with 6 flats, 6 rods, and 7 small cubes and all but two of them provided reasoning that was not based on the answer, and instead focused on an aspect of the array itself (e.g. length of one side, or the fact that the given base 10 blocks would not create a rectangle). Thus, our modified task was much better at reaching our learning goals and focused PTs' attention towards how base 10 blocks could be used to represent and partition the grid, rather than draw their attention towards the total amount.

Conclusion

As part of engaging in our cyclical task design cycle, we considered modifications to our task sequence after the first round of implementation. These modifications were based on the fact that the original task did not help PTs meet the learning goals for the task sequence. Overall, through reflecting on our implementation and the above-mentioned results, we learned that we needed to consider ways to help PTs focus more explicitly on the two numbers being multiplied in an array, rather than only the total number of units. We found one way to do this was by modifying our task to ask PTs to design their own model, rather than reasoning about one that we provided.

References

Association of Mathematics Teacher Educators (2017). Standards for preparing teachers of mathematics. Available online at amte.net/standards.

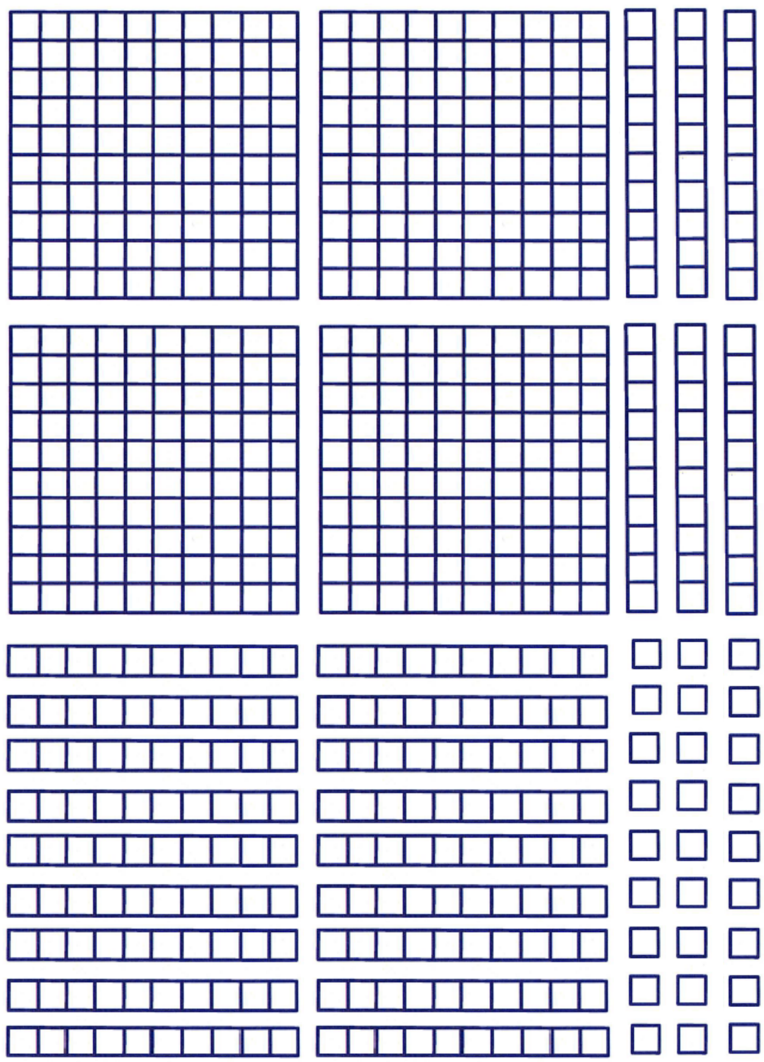
Berk, D., & Hiebert, J. (2009). Improving the mathematics preparation of elementary teachers, one lesson at a time. Teachers and Teaching: theory and practice, 15(3), 337-356.

Jansen, A., Bartell, T., & Berk, D. (2009). The role of learning goals in building a knowledge base for elementary teacher education. The Elementary School Journal, 109(5), 525-536.

Tobias, J.M., Olanoff, D., Hillen, A., Welder, R., Feldman, Z., & Thanheiser, E. (2014). Research-based modifications of elementary school tasks for use in teacher preparation. In Karen Karp (Ed.) Annual perspectives in mathematics education 2014: Using research to improve instruction. Reston, VA: National Council of Teachers of Mathematics.
Watson, A., & Mason, J. (2007). Taken-as shared: A review of common assumptions about mathematical tasks in teacher education. Journal of Mathematics Teacher Education, 10(4-6), 205-215.

Appendix: Original Version of Task 2

2. **Does this model represent the mailbox problem? If so, how? Please provide your reasoning.**



Reasoning:

RELATIONSHIPS BETWEEN COGNITIVE AND AFFECTIVE FACTORS AND NUMBER OF TEACHING MOVES: AN EXPLORATORY STUDY

Sam Rhodes	Montana Smithey	Rick Bryck
Georgia Southern University	Georgia Southern University	Landmark College
srhodes@georgiasouthern.edu	msmithey@georgiasouthern.edu	rickbryck@landmark.edu

Keywords: Preservice Teacher Education; Affect, Emotion, Beliefs, and Attitudes; Mathematical Knowledge for Teaching; Cognition

The interactions that teachers have with students through the questions and statements that teachers make—referred to herein as teaching moves (Jacobs & Empson, 2016)—can have profound impacts on student success (e.g., Carpenter et al., 1989). Given this, it is vital to train preservice teachers (PSTs) to teach in ways that are responsive to children's mathematical thinking (CMT) and that use CMT to engage learners and target their needs (e.g., Jacobs & Spangler, 2017; NCTM, 2014). Instructional practices are influenced by factors such as pedagogical beliefs (e.g., Peterson et al., 1989; Staub & Stern, 2002), and little is known about what factors influence the teaching moves PSTs use in their work with students.

This poster presents findings from an exploratory mixed-methods study that investigated what cognitive and affective factors were related to the number of teaching moves that 40 elementary PSTs made when engaging with six different student work samples—drawn from a noticing instrument (Jacobs et al., 2022). Specifically, quantitative data was collected through measures of mathematic anxiety (MA; Ganley et al., 2019), instructional beliefs (Schoen et al., 2019), executive functions (EFs; Younger et al., 2022), and pedagogical content knowledge (PCK; see Learning Mathematics for Teaching Project (LMT), 2011). Qualitative data was collected from written responses to student work produced by PSTs and follow-up interviews.

Teaching moves within written responses and follow-up interviews were coded using provisional coding (Miles et al., 2014) and analyzed through constant comparative analyses (Glaser & Strauss, 1967). Correlational analyses were then used to explore the relationships between the aforementioned factors and the number of teaching moves PSTs made.

After correlating factors with the number of teaching moves PSTs make, results suggest that working memory, as measured by filtering accuracy ($r(36) = .364, p<.05$) and mean reaction time ($r(36) = .364, p<.05$), facts-first beliefs ($r(38) = -.474, p<.01$), the number of collegiate math courses PSTs have taken ($r(38) = .466, p<.01$), and scores on the Number Concepts and Operations ($r(38) = .454, p<.01$) and Rational Number LMT ($r(36) = .382, p<.01$) tests, are all significant. In addition, the emotionality ($r(38) = -.313, p <.05$) and worry ($r(38) = -.324, p<.05$) subscales of MA were significant for rational number work samples but not for whole numbers.

The results of the present study suggest that facts-first beliefs, EF filtering, rational number LMT scores, and the number of collegiate mathematics courses PSTs had taken are significantly related to the number of teaching moves that PSTs make when engaging with both sets of student work samples. In addition, MA and rational number LMT scores were specifically related to the number of teaching moves PSTs made when engaging with work samples related to rational numbers. While we acknowledge that the work of understanding these relationships transcends number of teaching moves alone, these findings draw attention towards factors that predict the number of teaching moves PSTs make, thereby laying the foundation for future research to consider whether these factors also influence the types of teaching moves.

References

Carpenter, T. P., Fennema, E., Peterson, P. L., Chiang, C. P., & Loef, M. (1989). Using knowledge of children's mathematics thinking in classroom teaching: An experimental study. American educational research journal, 26(4), 499-531. https://doi.org/10.3102/00028312026004499.

Ganley, C. M., Schoen, R. C., LaVenia, M., & Tazaz, A. M. (2019). The construct validation of the math anxiety scale for teachers. Aera Open, 5(1). https://doi.org/10.1177/2332858419839702

Glaser, B. G., & Strauss, A. L. (1967). The discovery of grounded theory: Strategies for qualitative research (Ser. Observations). Aldine Publishing.

Jacobs, V. R., & Empson, S. B. (2016). Responding to children's mathematical thinking in the moment: An emerging framework of teaching moves. ZDM Mathematics Education, 48(1-2), 185-197.

Jacobs, V. R., Empson, S.B., Jessup, N.A., Dunning, A., Pynes, D., Karuse, G., & Franke, T.M. (2022). Profiles of teachers' expertise in professional noticing of children's mathematical thinking. Journal of Mathematics Teacher Education. https://doi.org/10.1007/s10857-022-09558-z

Jacobs, V. R., & Spangler, D. A. (2017). Research on core practices in K-12 mathematics teaching. In J. Cai (Ed.). Compendium for Research in Mathematics Education. National Council of Teachers of Mathematics.

Learning Mathematics for Teaching Project. (2011). Measuring the mathematical quality of instruction. Journal of Mathematics Teacher Education, 14(1), 25-47. https://doi.org/10.1007/s10857-010-9140-1

Miles, M. B., Huberman, A. M., & Saldana, J. (2014). Qualitative data analysis: A methods sourcebook (3rd ed.). SAGE Publications.

National Council of Teachers of Mathematics. (2014). Principles to actions: Ensuring mathematical success for all. National Council of Teachers of Mathematics.

Peterson, P. L., Fennema, E., Carpenter, T. P., & Loef, M. (1989). Teachers' pedagogical content beliefs in mathematics. Cognition and Instruction, 6, 1-40. https://doi.org/10.1207/s1532690xci0601_1

Schoen, R. C., LaVenia, M., & Ozsoy, G. (2019). Teacher beliefs about mathematics teaching and learning: Identifying and clarifying three constructs. Cogent Education, 6(1). https://doi.org/10.1080/2331186X.2019.1599488

Staub, F. C., & Stern, E. (2002). The nature of teachers' pedagogical content beliefs matters for students' achievement gains: Quasi-experimental evidence from elementary mathematics. Journal of Educational Psychology, 92 (2), 344-355. https://doi.org/10.1037/0022-0663.94.2.344

Younger, J., O'Laughlin, K., Anguera, J. A., Bunge, S., Ferrer, E. E., Hoeft, F., McCandliss, B., Mishra, J., Rosenberg-Lee, M., Gazzaley, A., & Uncapher, M. (2022). Better together: Novel methods for measuring and modeling development of executive function diversity while accounting for unity. PsyArXiv. https://doi.org/10.31234/osf.io/ctzwx

RELIABILITY AND VALIDITY OF DOCUMENTING ELEMENTARY PRE-SERVICE TEACHERS' PERFORMANCE IN A TEACHING SIMULATION

Daniel Heck
Horizon Research, Inc.
dheck@horizon-research.com

Evelyn M. Gordon
Horizon Research, Inc.
egordon@horizon-research.com

Meghan Shaughnessy
Boston University
mshaugh@bu.edu

Tim Boerst
University of Michigan
tboerst@umich.edu

Nicole Garcia
University of Michigan
nmgarcia@umich.edu

SimulaTE is studying teaching simulations as formative assessments of pre-service teachers' (PST) practice of eliciting and interpreting students' mathematical thinking. Preparation and protocols that promote reliability and validity of the simulations as formative assessments will enhance their effectiveness and generalizability. Teacher educators who use the simulations document each PST's performance to generate relevant feedback for the PST. As part of a coordinated set of validity studies, six researchers were prepared on the documentation protocol. Consistency of documentation within the group and with the simulation developers' judgments provided evidence supporting reliability and validity of the documentation protocol.

Keywords: Assessment, Mathematical Knowledge for Teaching, Preservice Teacher Education, Teacher Educators

Framing and Purpose of the Study

Mathematics teacher preparation ideally produces skillful and capable professionals whose classroom teaching will promote ambitious goals for student learning and counter chronic disparities in educational outcomes for students. Achieving this goal requires early and frequent engagement in practices of teaching with formative feedback to develop sophisticated knowledge, skills, and dispositions necessary for nurturing young learners of mathematics.

Formative assessment is a crucial component in teacher preparation (Darling-Hammond et al., 2005; AMTE, 2017) because it provides pre-service teachers (PSTs) with feedback they need to improve their practice (Grossman, 2010). It requires seeing teaching practices in action, but traditional field settings are limited in terms of frequent accessibility and the opportunity to work on specified facets of teaching. Simulations are an approximation of practice that can provide early, frequent, and substantive formative assessment opportunities while engaging PSTs in selected mathematics teaching practices.

PSTs enter preparation with knowledge, skills, and dispositions toward teaching that need to be surfaced, refined, or in some cases, counteracted (Boerst et al., 2020; Shaughnessy & Boerst, 2018a; Shaughnessy et al., 2020). To this end, efforts at the University of Michigan have resulted in a library of teaching simulations for elementary PSTs. The underlying premise is that PSTs' learning will be enhanced by performances of teaching practices that reveal the current state of their knowledge, skills, and dispositions and informing actions that facilitate growth (Shute, 2008; Hattie & Timperley, 2007). This study's purpose is to investigate reliability (consistency) and validity (accuracy) in the process for documenting performances to generate timely, interpretable, and actionable feedback.

Teaching Simulations as Formative Assessments

Using the teaching simulations as formative assessments involves three interacting roles:

1. The PST prepares for, engages in, and debriefs what they learn via the teaching practice of eliciting and interpreting student thinking with a Simulated Student.
2. The Simulated Student is an adult prepared to follow a provided profile and to respond in specific ways to anticipated questions and prompts.
3. The Teacher Educator (TE) documents the PST's performance during the simulation and debriefing interview and provides formative feedback based on the performance.

Figure 1 illustrates the full formative assessment process. Two components are underlined in the figure to indicate the parts of the process investigated in this reliability and validity study.

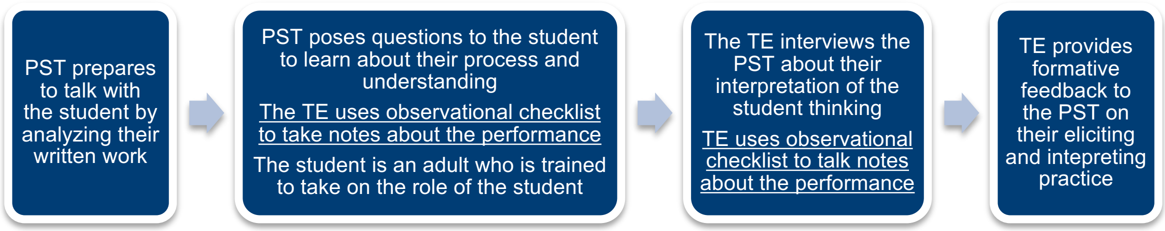
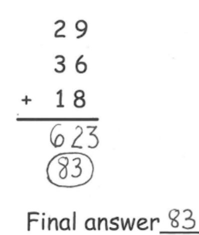


Figure 1: Structure of Teaching Simulations as Formative Assessments

Mathematics content in the simulations is high-leverage for elementary mathematics teaching (Shaughnessy et al., 2012) in that the tasks represent core disciplinary content and the student work and specifications of the role depict evidence-based recreations of student thinking about that content. Figure 2 is an excerpt of key elements of one simulation assessment.

Mathematics topic: Multi-digit addition



- **The student's process:** The student is using the column addition method for solving multi-digit addition problems, the student is working from left to right.
- **The student's understanding of the ideas involved in the problem/process:** The student has conceptual understanding of the procedure including why combining is necessary (and when and how to combine).
- **Other information about the student's thinking, language, and orientation in this scenario:** The student talks about digits in columns in terms of the place value of the column. The student uses the term "combining" to refer to trading/carrying/regrouping.

Sample PST prompts	Sample Responses
What did you do first?"	"I added the tens: 2 + 3 + 1 and I got 6."
"How did you get from 623 to 83?"	"I had to combine the 6 and the 2."
"Why did you need to combine those numbers?"	"Because they're both tens."

Figure 2: Excerpts from a Sample Teaching Simulation Protocol

By design, the student discloses aspects of their process and understanding only when asked. Consequently, the PST must prompt deliberately for each piece of information about the student's process and understanding. In the debriefing interview, the PST is asked to recount what they learned about the student's process and understanding in similar detail, including the particular evidence that supports their claims. In the interview, the PST is also asked about the mathematical underpinnings of the student's process and understanding; their responses provide evidence of their mathematical knowledge for teaching (MKT) the targeted content.

Documentation of performances is undertaken at a very fine grain size to provide information about multiple components of the teaching practice. For each performance, documentation involves judgments for about 75 items, varying slightly depending on the particular content and task. Most judgments for the simulation documentation indicate whether or not the PST elicited or probed for specific pieces of information, and whether or not they took various actions (e.g., asking the student to write, posing a new problem) in doing so. Items regarding respect for the student's thinking are also included. In the debriefing interviews, items about process and understanding are judged in terms of whether correct/incorrect claims are offered and whether evidence for correct claims is provided or acknowledgement of lack of evidence and a proposed means of asking for it. The interview documentation includes judgments of correct or incorrect statements about the mathematics of the task and the student work. This documentation results in ratings for eight performance categories, listed in Table 1. These ratings generate feedback for the TE to discuss with the PST about strengths and areas for growth in their performance.

Reliability and Validity of Documenting PST Performances

Method for Producing Documentation Data

A crucial step in the process is documentation of the performance which links the enactment of the assessment to formative feedback. To use the simulation assessments effectively, TEs must consistently and accurately document performances so that the feedback is relevant to experience of the situation and tailored to support PSTs' growth. To examine this step, six mathematics education researchers (2 PhDs, 1 PhD student, 2 MAs, 1 BA) who work with multiple teacher education programs were prepared on the documentation procedure for four assessments (2 on multidigit operations, 2 on methods for comparing fractions). For each, the preparation included a meeting in which the developers introduced and provided practice with the content, student protocol, and documentation; independent documentation of performances from video samples of PST performances, then comparison and negotiation of judgments; and a follow-up meeting with the developers to discuss discrepancies and remaining questions.

The researchers next independently documented 16 videorecorded performances, four from each of the four assessments. These recordings were made as various members of the development group had enacted the simulation assessments in past years with 16 participants who were mostly PSTs and a few early career elementary teachers (to generate a range of performances). To mirror the expectation that the documentation occurs in real time, the researchers documented both the simulation and interview portions by watching the recordings uninterrupted. The researchers' documentations were then analyzed in two ways.

Method of Analysis

First, to investigate inter-rater reliability, researchers' documentations were compared within the group using Fleiss's Kappa, which measures the degree of agreement among multiple raters on multiple items, accounting for the probability of chance agreements. Values range from 0 (no agreement) to 1 (perfect agreement), with values above 0.6 considered substantial agreement, and above 0.8 near perfect agreement. See the third column of Table 1 for reliability results.

Second, to investigate validity, researchers' documentations were compared to a standard documentation, which was the collective judgment of the development team to document the same 16 performances. Unlike the research team, the development team had produced their documentation from multiple viewings and collective negotiation to determine appropriate documentation. In the validity analysis, each researchers' judgments were compared to the developers' standard documentation using Cohen's Kappa. The six researchers' Cohen's Kappa values were then averaged. Ranging from -1 to 1, negative values represent greater non-agreement and positive values greater agreement. Values above 0.6 indicate substantial agreement and above 0.8 near perfect agreement. See the fourth column of Table 1 for results.

Results and Interpretation

Table 1: Fleiss's and Cohen's Kappa Values by Performance Category

Performance Category	Total Items	Fleiss's Kappa	Cohen's Kappa
Eliciting Process	28	0.86	0.86
Interpreting Process	38	0.86	0.86
Probing Understanding	26	0.75	0.72
Interpreting Understanding	112	0.83	0.78
Applying MKT	39	0.95	0.96
Other Math Knowledge/Skills	22	0.89	0.80
Attending to Student Thinking	16	0.90	0.94
Respecting the Student	16	0.96	0.90

As can be seen in the Fleiss's Kappa results in Table 1, inter-rater agreement among the six researchers was at least substantial (Landis & Koch, 1977) for all performance categories and can be considered near perfect for all but one. These results offer evidence that the researchers documented the performances quite similarly to one another, with the most common differences for Probing Understanding during the PST's interaction with the student.

Cohen's Kappa results in Table 1 indicate strong agreement (Landis & Koch, 1977) between researchers' documentation and the development team's standard judgments, characterized as at least substantial for all eight performance categories and near perfect for six. The most common differences between researchers and the development team were in Probing Understanding during the PST's interaction with the student and Intepreting Understanding in the interview.

Conclusions and Next Steps

The preparation researchers received and structured guidance the protocol provided appear sufficient to support documentation of performances that is reliable across different individuals for a variety of the simulation assessments. Additionally, the preparation and guidance supported valid judgments in documentation in that the researchers' and developers' judgments matched to a high degree. These results provide promise that teacher educators can take up these complex formative assessment tools for use in their own programs. The performance areas of lesser reliability and validity (Probing Understanding, Interpreting Understanding) suggest that preparation should emphasize and guidance highlight key decisions on items in these categories. Important next steps in validation studies of the simulation assessments will examine, via stimulated recall interviews, how TEs in the field document performances and investigation of the relevance of feedback generated from the assessments that TEs share and discuss with PSTs to support their growth in the teaching practice of eliciting and interpreting student thinking.

Acknowledgments

This material is based upon work supported by the National Science Foundation (NSF) Grant No. 2101343. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the NSF.

References

Association of Mathematics Teacher Educators. (2017). Standards for Preparing Teachers of Mathematics. Available online at amte.net/standards.

Boerst, T. A., Shaughnessy, M., DeFino, R., Blunk, M., Farmer, S. O., Pfaff, E., & Pynes, D. (2020). Preparing teachers to formatively assess: Connecting the initial capabilities of preservice teachers with visions of teaching practice. In C. Martin, D. Polly, & R. Lambert (Eds.), Handbook of Research on Formative Assessment in Pre-K through Elementary Classrooms (pp. 89–116). IGI Global. https://doi.org/10.4018/978-1-7998-0323-2

Darling-Hammond, L., Pacheco, A., Michelli, N., LePage, P., Hamerness, K., & Youngs, P. (2005). Implementing curriculum renewal in teacher education: Managing organizational and policy change. In L. Darling-Hammond & J. Bransford (Eds.), Preparing teachers for a changing world: What teachers should learn and be able to do (pp. 442–479). John Wiley & Sons. https://doi.org/10.5860/choice.43-1083

Grossman, P. (2010). Learning to practice: The design of clinical experience in teacher preparation. AACTE & NEA policy brief.

Hattie, J., & Timperley, H. (2007). The power of feedback. Review of Educational Research, 77(1), 81–112. https://doi.org/10.3102/003465430298487

Landis, J. R. & Koch, G. G. (1977), The measurement of observer agreement for categorical data. Biometrics, 33, 159–174. https://doi.org/10.2307/2529310

Shaughnessy, M., & Boerst, T. (2018). Uncovering the skills that preservice teachers bring to teacher education: The practice of eliciting a student's thinking. Journal of Teacher Education, 69(1), 40–55. https://doi.org/10.1177/0022487117702574

Shaughnessy, M., Boerst, T., Sleep, L., & Ball, D. L. (2012, April). Exploring how the subject matters in pedagogies of practice. Paper presented at the annual meeting of the American Educational Research Association, Vancouver, BC.

Shaughnessy, M., DeFino, R., Pfaff, E., & Blunk, M. (2020). I think I made a mistake: How do prospective teachers elicit the thinking of a student who has made a mistake? Journal of Mathematics Teacher Education. https://doi.org/10.1007/s10857-020-09461-5

Shute, V. (2008). Focus on formative feedback. Review of Educational Research, 78(1), 153–189. https://doi.org/10.3102/0034654307313795

SCAFFOLDING PRESERVICE TEACHERS' THINKING ABOUT EQUITY-BASED MATHEMATICS TEACHING PRACTICES

Alesia Mickle Moldavan
Georgia Southern University
amoldavan@georgiasouthern.edu

Laurel Dias
Utah Valley University
laurel.dias@uvu.edu

This study examines how preservice teachers recognize and justify equity-based teaching practices within a mathematics task incorporating a multicultural children's story. Referencing Aguirre et al.'s (2013) equity-based teaching practices, the researchers designed a reflection guide to assist preservice teachers in recognizing ways tasks can be used to affirm students' mathematics identities, draw on multiple resources of knowledge, and challenge spaces of marginality while going deep with mathematics to leverage multiple mathematical competencies. Findings indicate that the preservice teachers effectively used the reflection guide to recognize the practices and reference various components to justify each practice. Recommendations suggest that preservice teachers have explicit opportunities to use similar tools alongside strategic tasks to think deeply about equity-based teaching practices to develop their instruction.

Keywords: Preservice Teacher Education; Elementary School Education; Instructional Activities and Practices; Equity, Inclusion, and Diversity

Purpose

Bridging theory into practice can be challenging for preservice teachers (PSTs) in developing equity-based teaching practices. Although PSTs learn about theoretical approaches to promoting equity, they often struggle to recognize and justify specific practices in classroom contexts (Hollins, 2015; Zeichner, 2010). Recommendations for mathematics teacher education suggest that PSTs have structured learning opportunities to plan for and use equity-based pedagogy that elicits access, support, and challenge to advance students' learning (Association of Mathematics Teacher Educators [AMTE], 2017). It is when PSTs have opportunities to develop their disposition, knowledge, and skills to recognize and justify such pedagogy that they can begin to understand what equity-based mathematics instruction looks like, particularly instruction that acknowledges students' assets respecting the whole learner and their world (AMTE, 2022). One way to position and empower PSTs with the tools to understand equity-based teaching is by exploring example mathematics tasks through contexts of reflective teaching (Zeichner & Liston, 2010). We posit that mathematics tasks that strategically reference multicultural children's stories with connections to mathematical concepts can serve as a catalyst for eliciting equity-based teaching practices. More specifically, stories representing mathematics learners from linguistically, ethnically, and socioeconomically diverse settings can affirm students' mathematics identities, draw on multiple resources of knowledge, and challenge spaces of marginality to provide students opportunities to go deep with mathematics as their multiple mathematical competencies are leveraged, the foundation of Aguirre et al.'s (2013) equity-based teaching practices. This study reports on one aspect of PSTs' work engaging with mathematics tasks using multicultural children's stories to examine equity-based teaching practices during elementary mathematics methods courses. Our guiding research question was: How do PSTs make use of a reflection guide to recognize and justify equity-based teaching practices within a mathematics task incorporating a multicultural children's story?

Theoretical Framework

We approach this research with equity-based teaching, as defined by Aguirre and colleagues (2013), to cultivate positive mathematics identities and strengthen student learning. The five equity-based practices noted in Table 1 (i.e., P1-P5) are recognized as key intentional and complex elements. While the practices are noted individually, they may occur simultaneously or in various combinations; yet it is recommended that integrating all practices is critical to empower students mathematically. To support PSTs' understanding of the practices, we developed a Reflection Guide for Equity-Based Teaching Practice with key components of each practice noted to guide PSTs' understanding as they reflected on each practice (see Table 1). Our prior experiences observing PSTs as they recognized and justified equity-based teaching practices in a mathematics task, or designed their own tasks, indicated the need for a tool to scaffold PST understanding. We do acknowledge that the noted components do not constitute the only components for reflection of each practice. Instead, they serve as starting points from which deeper conversations must be had. We posit that such conversations can be enhanced with tasks grounded in multicultural children's literature.

Table 1: Reflection Guide for Equity-Based Teaching Practices

Equity-Based Teaching Practices	Key Components for Reflection
P1. Going Deep with Mathematics	a. Multiple mathematical strategies/representations included b. Opportunities to engage in conversation about the mathematical concept and explain/justify thinking c. Opportunities to "make sense" of mathematics and problem solve
P2. Leveraging Multiple Mathematical Competencies	a. Multiple entry points are offered for how to approach the task allowing students with different strengths and levels of confidence to make mathematical contributions b. Opportunities for student collaboration through mathematical discussions or solving complex problems
P3. Affirming Mathematics Learners' Identities	a. All students are encouraged to make mathematical contributions and build confidence as problem solvers b. Mathematical identities are recognized and validated c. Student persistence and reasoning are promoted
P4. Challenging Spaces of Marginality	a. Opportunities for students to demonstrate and connect their experiences and mathematics knowledge to the mathematics concept b. Students are encouraged to generate mathematics questions to probe issues
P5. Drawing on Multiple Resources of Knowledge	a. Opportunities for students to learn about their own and others' backgrounds/experiences during mathematics b. Students' multilingual and multicultural abilities are affirmed and supported while learning mathematics c. Opportunities to explore the impact of race and racism in mathematics learning Opportunities to learn about mathematics from family and community members

Research Methods

We used a qualitative case study (Yin, 2014) that encompassed two elementary mathematics methods courses across two urban, public four-year institutions, one in the Mountain West and the other in the Southeast of the United States. Participants included 38 PSTs who self-identified as female (n=38) seeking elementary teaching certification. Their demographics could be further described as Hispanic/Latinx (n=3), White (n=25), Black (n=3), and Asian (n=1). Any PSTs who selected multiple races were counted as Multiracial (n=6).

Task Design

We designed a mathematics task using the multicultural children's story "Just a Minute: A Trickster Tale and Counting Book," written and illustrated by the Mexican-American author Yuyi Morales (2003). The task connected to the K.CC.B4 content standard, "Understand the relationship between numbers and quantities; connect counting to cardinality." All but components P3c, P4b, P5c, and P5d were identified when we self-evaluated the task using the Reflection Guide for Equity-Based Mathematics Teaching Practices.

Our PSTs had the opportunity to engage in the task during their respective elementary mathematics methods courses. The task was assigned at the beginning of the semester and accompanied by an introductory overview of equity-based teaching practices where excerpts of Aguirre et al. (2013) were consulted as assigned readings to guide class discussion. The task began with the reading of the multicultural children's story. In the story, Grandma Beetle delays leaving with Señor Calavera as she counts various objects numbered one to ten in preparation for her birthday celebration. After reading the book, PSTs shared their noticings and wonderings about the story and were tasked with preparing their own party favor for Señor Calavera. Each PST chose an item that described their culture, language, or interests to bring to the party and a number between one and ten. Then, PSTs used a graphic organizer to create various representations of their number. PSTs met in small groups to discuss the representations of each person's chosen item and reflect on the similarities and differences between the items and representations. Key takeaways from the small groups were shared in a whole group discussion where PSTs also considered how mathematics can be used to learn about students' cultures, languages, and interests. Finally, each PST independently completed a survey created using the Reflection Guide for Equity-Based Mathematics Teaching Practices.

Data Sources and Analysis

Qualitative data was collected and coded for analysis. The data sources include the PSTs' responses to the survey and our memos as facilitators of the task. Responses to the survey were analyzed for patterns and themes (Gibbs, 2007), providing insights into how PSTs thought about equity-based teaching practices in the task. We used descriptive and in vivo coding (Miles et al., 2018) to examine the PSTs' response justifications, coding for the components outlined in our Reflection Guide for Equity-Based Mathematics Teaching Practices. The codes informing the themes were organized in a coding chart and cross-checked for coder reliability to safeguard the trustworthiness of the analysis (Grbich, 2013). Then, we calculated our codes to find the percentage of PSTs who identified individual components of each practice. This percentage provided insight into what PSTs recognized and how the reflection guide scaffolded PSTs to elaborate on different components in their justifications. Analytic memo-writing was also used to assist with extracting and synthesizing relevant data to guide the findings.

Findings

We found the Reflection Guide for Equity-Based Mathematics Teaching Practices to be an effective tool for scaffolding PSTs' recognition and justification of equity-based teaching practices. Here we share sample PST survey responses to demonstrate how PSTs recognized each practice in the context of the task.

For P1, typical responses mentioned P1a and P1b but not P1c as follows: "The number we chose was represented in multiple ways (pictures, numbers, word form). We had the opportunity to converse with our peers, explaining the representation we chose and why." Interestingly, P1c was the component PSTs failed to justify the most in their responses. Perhaps adult learners did not experience sense-making in the task because it was designed for kindergarten students.

In P2, many PSTs recognized both components of the task. Most PSTs' justifications focused on language accessibility; for example, "This task doesn't really require a lot of language/writing proficiency and is a task that has a low threshold, high ceiling. There are so many ways students could represent numbers and then talk about why they chose their representations."

Most PSTs recognized components P3a and P3b in alignment with the research team. One response typical of the justifications stated: "Identities were validated in this task by being able to choose a "party favor" that represented one's culture/interests. Students engaged in the task, although their representations looked different." PSTs frequently justified the choices given to students in the task as how they built confidence and had their mathematical identities validated.

P4a was included in 82% of PSTs' justifications. Once again, many PSTs identified student choice as a mechanism for elevating this practice. One student wrote, "Having students choose an item related to their culture or experience allows students to dig deeper into what they can use to represent this number." When justifying this practice PSTs highlighted choice as a way for students to connect with their experiences and prior knowledge.

Finally, P5 was more often justified by component P5a than P5b. PSTs tended to focus more on learning about students' backgrounds than how multilingual or multicultural abilities were affirmed. For example, "I enjoyed learning about everyone's item they would bring. It taught me a lot about them and things I did not know before. This is a fun way to connect our backgrounds, learn math, and learn about each other!" Many more PSTs recognized this practice as getting to know their peers' backgrounds rather than seeing the abilities and strengths their peers bring to the classroom.

Most PSTs' justifications focused on the same ten components of equity-based teaching practices that the research team identified during their self-evaluation (see Figure 1). The four components not identified (P3c, P4b, P5c, and P5d) by the research team were all mentioned by less than 20% of the PSTs in their justifications. This alignment demonstrates the tool's effectiveness in scaffolding PSTs' recognition of equity-based mathematics teaching practices.

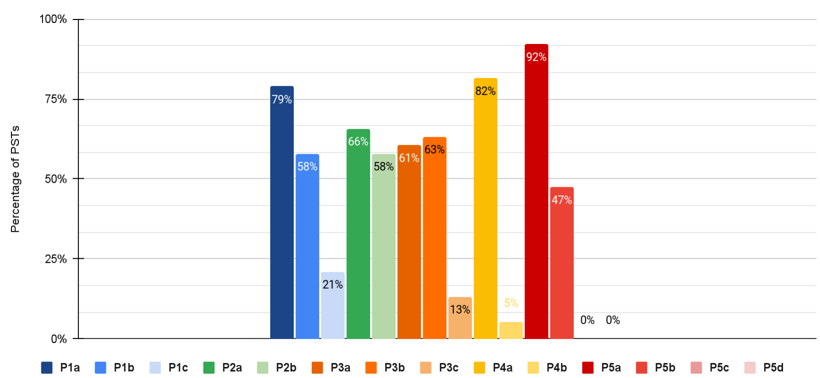


Figure 1: Equity-Based Mathematics Teaching Practices in PSTs' Justifications Conclusion

Mathematics teacher education must provide opportunities for PSTs to develop their understanding of equity-based teaching practices. Innovative tools like the Reflection Guide for Equity-Based Mathematics Teaching Practices can help PSTs recognize and justify each practice in the context of mathematics tasks. It is when PSTs can think deeply about each practice that they can begin to make strategic choices that lead to integrating all equity-based teaching practices in their instruction (Aguirre et al., 2013). Also, mathematics teacher educators can make use of such tools to teach to PSTs' strengths in specific practices while also acknowledging practices needing additional work. Then, particular components can be explored by implementing tasks that elicit such practices, like those connecting multicultural children's stories, to ensure equitable teaching that engages all learners.

References

Aguirre, J., Mayfield-Ingram, K., & Martin, D. (2013). The impact of identity in K-8 mathematics: Rethinking equity-based practices. The National Council of Teachers of Mathematics.

Association of Mathematics Teacher Educators. (2017). Standards for preparing teachers of mathematics. https://amte.net/standards

Association of Mathematics Teacher Educators. (2022). AMTE statement on equitable and inclusive mathematics teaching and learning. https://amte.net/sites/amte.net/files/AMTE%20Statement%20April%202022.pdf

Gibbs, G. R. (2007). Analyzing qualitative data. SAGE.

Grbich, C. (2013). Qualitative data analysis: An introduction (2nd ed.). SAGE.

Hollins, E. R. (2015). Rethinking field experiences in preservice teacher preparation: Meeting new challenges for accountability. Routledge.

Miles, M. B., Huberman, A. M., & Saldaña, J. (2018). Qualitative data analysis: A methods sourcebook. SAGE.
Morales, Y. (2003). Just a minute: A trickster tale and counting book (Y. Morales, Illus.). Chronicle Books.

Yin, R. K. (2014). Case study research: Design and methods (5th ed.). SAGE.

Zeichner, K. (2010). Rethinking the connections between campus courses and field experiences in college- and university-based teacher education. Journal of Teacher Education, 61(1–2), 89–99. https://doi.org/10.1177/0022487109347671

Zeichner, K. M., & Liston, D. P. (2013). Reflective teaching: An introduction (2nd ed.). Routledge. https://doi.org/10.4324/9780203771136

SIGNIFICANT FEATURES OF NUMBER TALKS AS PERCEIVED BY PROSPECTIVE TEACHERS

Byungeun Pak	Jillian M. Cavanna	Brent Jackson
Utah Tech University	University of Hartford	Wested
b.pak@utahtech.edu	cavanna@hartford.edu	bjackso@wested.org

Number talks, a popular mathematics teaching routine in the United States, may offer early support for prospective teachers (PSTs) to engage in ambitious instruction. Studies on teacher noticing, particularly attention, show a need for understanding what PSTS attend to as significant in teaching. There are few studies on what PSTs attend to in their enacted NTs. In this study, we analyzed interview transcripts of 11 PSTs to identify themes that show what they attended to as significant in their enacted NTs. We found three themes in relation to establishing a safe learning environment, allowing the PSTs to focus on students' conceptual understanding, and inviting students to see multiple strategies of others. Discussions and implications are offered in relation to mathematics teacher education and research.

Keywords: Elementary School Education, Instructional Activities and Practices, Teacher Educators

Purpose

Prospective teachers (PSTs) are expected to learn how to engage in ambitious mathematics instruction (Kazemi, Franke, and Lampert, 2009; Lampert et al., 2010) that is also equitable for all students (e.g., Jackson & Cobb, 2010). This kind of teaching is inherently challenging for teachers, particularly PSTs. Number talks (NTs), instructional routines in which students use mental mathematics to solve computational problems (Humphreys & Parker, 2015; Parker and Humphrey, 2018; Parrish, 2014), may offer one way for PSTs to engage in ambitious and equitable instruction; that is, instruction that provides opportunities to develop deep conceptual understanding for all students. NTs create a participation structure that may support "students to take back the authority of their own reasoning" (Humphreys & Parker, 2015, p. 1). The *routinized* nature of NTs can offer support for PSTs to engage in aspects of ambitious instruction as novices.

In our recent work (Authors, 2021a, 2021b, and 2021c), we reported our observations of variation in beginning teachers' enacted NTs in terms of what made novice teachers' NTs more or less ambitious. We found that features of ambitious NTs include going beyond simply asking students to share their strategies to evaluate or compare strategies of peers, and using mathematical errors to help students make sense of how strategies work. Our findings offer insights about how to support novice teachers in learning to implement ambitious NTs. Similar to the need identified by Matney, Lustgarten, and Nicholson (2020) for more empirical evidence regarding the effects of NTs for K-12 students, we suggest that the field also needs empirical evidence with respect to NTs enacted by PSTs to better understand how NTs may be used to support novice teachers to develop their toolkit for ambitious instruction. In this paper, we draw on theories of teacher noticing to explore what PSTs notice in their enacted NTs. There is a rich body of literature on teacher noticing in mathematics education, which offers perspectives into what teachers notice in their own and others' instruction, and how such noticing influences teachers own interactions with students (e.g., Jacob, Lamb, & Philipp, 2010; Star & Strickland, 2008). Taken together, PSTs' noticing within the space of enacted NTs may be a targeted opportunity for PSTs to develop understanding of ambitious and equitable instruction. As such, this paper aims to understand what PSTs notice in their enacted NTs.

Theoretical Perspectives

We draw on teacher noticing, more specifically *attention*, as our perspective to examine what PSTs notice in their enacted NTs. Researchers take different approaches to teacher noticing (e.g., Jacob et al., 2010; Star, Lynch, & Perova, 2011; Star & Strickland, 2008; van Es & Sherin, 2002, 2021). One of the common components of those approaches is *attention*, which we take to mean "identifying what is important or noteworthy about a classroom situation" (van Es & Sherin, 2002, p. 573). Building on the work of Star and colleagues' (2008, 2011), we focus on *attention* as a key first step of teacher noticing. If PSTs are unable to identify that certain classroom events have occurred, then they are likely to fail to engage in subsequent components of teacher noticing, such as deciding how to respond (Jacob et al., 2010) or making connections between particular events and broad principles of teaching and learning (van Es & Sherin, 2002). Therefore, we propose that PSTs will have difficulty in learning to implement ambitious NTs if they are not able to identify significant features of their own NTs.

Star and colleagues (2008, 2011) examined PSTs' *attention* in mathematics methods courses designed to support PSTs in developing their *attention* skills. In the data collection, the participants watched two different videos of experienced teachers in the beginning (pre-assessment) and at the end of the courses (post-assessment), respectively. After watching the videos, participants were asked to recall classroom features and events in observation categories, including classroom environment, mathematical content, and communication. As a result of comparing both assessments, the researchers found that the PSTs experienced improvements, high or modest, on all observation categories. Their studies suggest that PSTs can improve their noticing skills with support. The suggestion raises a question of whether and how NTs could support PSTs in improving their noticing skills.

In a recent study, Wood (2021) worked with PSTs in her elementary mathematics methods course to understand what they could notice about students. The PSTs explored, planned, and rehearsed two separate NTs during simulations in which they interacted with a group of avatar-students, followed by reflection on their NTs. The researcher found that NT simulations provided the PSTs with opportunities to learn to notice about students, including recording representations of students' thinking and probing students to make mathematics visible. Wood's study suggests that NT simulations could help PSTs improve noticing skills when the simulations take place within a cycle of enactment and reflection. Taken together, Star et al. (2008, 2011) and Wood (2021) suggest that NT may support PSTs to develop their noticing skills, especially *attention*, within the cycle in actual classrooms. However, the field knows little of PSTs' *attention* in implementing NTs working with students in classrooms. These perspectives from these studies on teacher noticing led us to investigate what PSTs attend to as significant in their enacted NT.

Methods

Context, Participants, and Data Sources

The research site was in mathematics methods courses taught by the first author for elementary school teachers at a small southwestern university. Data was gathered in Spring 2021 as part of a research project that investigated how to support PSTs in learning to implement the eight effective mathematics teaching practices recommended by National Council of Teachers of Mathematics (NCTM; NCTM, 2014). Twenty-one PSTs agreed to participate in data collection, including electronic submissions of course assignments, including their Number Talk Project. The Number Talk Project consisted of PSTs engaging in a learning cycle to plan, rehearse and implement their NT plan. At the end of the learning cycle, PSTs video-recorded themselves enacting their revised NT in their field placement classroom. PSTs also submitted a written reflection paper based on watching their NT video. Eleven of the 21 PSTs participated in 30-minute semi-structured interviews in which they reflected on their enacted mathematics instruction as well as NTs from the perspectives of the NCTM's teaching practices (e.g., posing purposeful questions). The interview protocols included asking PSTs to identify at least three moments in their NT video that were significant and elaborating on why those moments were significant. We obtained 33 episodes that were described by the PSTs as significant. This paper used the NT-related parts of the interview transcripts of 11 PSTs as data sources.

Data Analysis

There were three steps we took to analyze the interview transcripts using a thematic analysis (Saldaña, 2015). First, the authors coded interview transcripts of four PSTs to create initial analytic codes of what the PSTs perceived as significant in their enacted NTs. We began with codes from Star and colleagues' categories- classroom environment and communication and our prior works (Authors, 2021a, 2021b, and 2021c) and remained open to any emerging codes (e.g., encouraging flexible approaches to). We coded the transcripts independently and met together to compare independent codes and resolved the disagreements by consensus. Second, we applied the initial codes to the other seven interview transcripts to refine the codes. The refined codes include classroom environment, the benefits for the PSTs (e.g., supporting students' conceptual understanding), and the benefits for their students (e.g., learning from multiple strategies). Third, we compared and contrasted codes within and across the 11 PSTs to identify salient themes. As a preliminary result, we identified three salient themes regarding classroom environment, PSTs, and their students, respectively, which will be detailed in the next section.

Results

We identified three themes through our data analysis. These themes are related to classroom environment, conceptual understanding, and students' strategies. We present a few excerpts as an example of each theme. Each excerpt came from a particular moment perceived by the PSTs as significant in their enacted NTs. The first theme we observed was that PSTs attended to the classroom environment created during NTs. PSTs noticed NTs must be spaces in which students should feel safe enough to share their strategies and take intellectual risks with their peers. One PST said,

> It was the fourth student sitting across from me. Um, I feel this was significant only because she was really shy to share throughout, and then at that point was when she tried to share something, … but she was really, really quiet so when I asked her to explain again, she got really shy and she just, like, shut down… To those kinds of students, it was sad because she wanted to share, but when I didn't hear her she didn't want to repeat herself again, and so I just feel like it was like a learning lesson for me for the future when working with shy students, and developing classroom culture. (PST09).

In this excerpt, the PST described an event in which a female student tried to share her ideas but she shut down because "she was really shy to share throughout." The PSTs attended to a classroom culture that NTs might foster when working with shy students. The same PST said when identifying another moment that in NTs, students "have a growth mindset" and they are not “afraid to share their answers even though it could be wrong.” This short excerpt shows the PST saw NTs as an environment for students to take intellectual risks.

The second theme we observed was that PSTs noticed that NTs help them focus on students’ conceptual understanding. PSTs attended to ways to support students’ conceptual understanding during NTs by going beyond how students solved problems. One PST said,

> I had a student who asked me to stack the numbers. She wanted me to stack them, and I just thought that was super interesting because that’s what her brain instantly went to…That specific student wasn’t like, I wouldn’t say that she wasn’t conceptually thinking, like, wasn’t about to conceptually think, but that was something, like, a rule or procedure that she instantly, her brain instantly went to that she has been taught to do. And so, this is the line of thinking that she might be going down. So, it was important to me because I kind of noted that she might not be that conceptual (PST07).

This excerpt highlights how PSTs focused their attention on conceptual understanding as a part of NTs. The PST observed a student “instantly” using a conventional algorithm (“She wanted me to stack them”). We noticed similar kinds of attention across PSTs in the ways NTs N helped PSTs attend to students’ conceptual understanding.

The third theme we observed was that PSTs noticed that NTs increased students’ opportunities to share multiple strategies. PSTs attended to how students sharing strategies benefitted the class overall. One PST said,

> It was good to do because there were a lot of people [students] who came up with different ways to find the answers, and then that helped the students who didn’t find the answers to think of a new technique that they could use (PST10).

This excerpt shows that the PST attended to how the NT was beneficial for students: Students learned from their classmates’ strategies (“that helped the students who didn’t find the answers to think of a new technique”).

Discussions and Conclusion

We discuss two points in relation to the results. First, PSTs attended to features of their enacted NTs that were helpful (or challenging) to them as novice teachers. In doing so, PSTs attended to important elements of learning to teach ambitiously and equitably including developing a safe classroom environment, supporting students to develop conceptual understanding, and supporting students to learn from one another. While we did not analyze their enacted NTs, this provides evidence that NTs might be a "useful container" for PSTs to develop and refine teaching practices related to ambitious and equitable instruction.

Second, PSTs need support to develop their noticing, particularly *attention*, during teacher preparation programs. In our study, the PSTs were able to hone significant features of their own NTs related to ambition and equitable instruction with the support of engaging in the learning cycle in the mathematics methods courses. Like what Wood (2021) suggested in the context of NTs with simulation, PSTs need to engage in planning, rehearsing, enacting their plans, and reflecting on their enacted NTs in a supportive environment such as a mathematics methods courses to develop a sense of how to do NTs ambitiously.

Acknowledgments

This material is based upon work supported by the Center for Teaching and Learning and the Faculty Research Grant Project at Utah Tech University. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the funders.

References

Humphreys, C., & Parker, R. (2015). Making number talks matter: Developing mathematical practices and deepening understanding, grades 4-10. Stenhouse Publishers.

Kazemi, E., Franke, M., Lampert, M. (2009). Developing pedagogies in teacher education to support novice teachers' ability to enact ambitious instruction. In Hunter, R., Bicknell, B., Burgess, T. (Eds.), Crossing divides: Proceedings of the 32nd annual conference of the Mathematics Education Research Group of Australasia (Vol. 1, pp. 12-30). Palmerston North, NZ: MERGA.

Jackson, K., & Cobb, P. (2010). Refining a vision of ambitious mathematics instruction to address issues of equity. Paper presented at the annual meeting of the American Educational Research Association, Denver, CO.

Jacobs, V. R., Lamb, L. L., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. Journal for Research in Mathematics Education, 41(2), 169–202.

Lampert, M., Beasley, H., Ghousseini, H., Kazemi, E., & Franke, M. (2010). Using Designed Instructional Activities to Enable Novices to Manage Ambitious Mathematics Teaching. In M. K. Stein & L. Kucan (Eds.), Instructional Explanations in the Disciplines (pp. 129–141). Boston, MA: Springer.

Matney, G., Lustgarten, A., & Nicholson, T. (2020). Black holes of research on instructional practice: The case of number talks. Investigations in Mathematics Learning, 12(4), 246-260.

National Council of Teachers of Mathematics (NCTM). (2014). Principles to Actions: Ensuring Mathematical Success for All, Reston, VA: Author.

Pak, B., Cavanna, J. M., and Jackson, B. (2021a, October) Investigating what makes beginning teachers' enactment of number talks more ambitious. Presented for a research report session at the 43rd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Philadelphia, Massachusetts.

Pak, B., Jackson, B., and Cavanna, J. M. (2021b, April) Exploring about beginning teachers' equitable and ambitious teaching practice with number talk routine. Presented for a research report session at the 2021 virtual annual meeting of the American Educational Research Association, Orlando, Florida.

Pak, B., Jackson, B., and Cavanna, J. M. (2021c, May) Beginning teachers' equitable and ambitious number talks. In A.I. Sacristán, J.C. Cortés-Zavala, & P.M. Ruiz-Arias (Eds.) Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, (pp. 2158-2159). Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

Parker, R. E., & Humphreys, C. (2018). Digging deeper: Making number talks matter even more, grades 3-10. Stenhouse Publishers.

Parrish, S. (2014). Number Talks: Helping Children Build Mental Math and Computation Strategies, Grades K-5, Updated with Common Core Connections. Math Solutions.

Saldaña, J. (2015). The coding manual for qualitative researchers (3rd dd.). Sage.

Star, J. R., & Strickland, S. K. (2008). Learning to observe: Using video to improve preservice mathematics teachers' ability to notice. Journal of mathematics teacher education, 11(2), 107–125.

Star, J. R., Lynch, K. H., & Perova, N. (2011). Using video to improve mathematics' teachers' abilities to attend to classroom features: A replication study. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), Mathematics teacher noticing: Seeing through teachers' eyes (pp. 117-133). New York, NY: Routledge.

Van Es, E. A., & Sherin, M. G. (2002). Learning to notice: Scaffolding new teachers' interpretations of classroom interactions. Journal of technology and teacher education, 10(4), 571-596.

van Es, E. A., & Sherin, M. G. (2021). Expanding on prior conceptualizations of teacher noticing. ZDM–Mathematics Education, 53, 17-27.

Wagganer, E. L. (2015). Creating math talk communities. Teaching Children Mathematics, 22(4), 248–254. https://doi.org/10.1007/BF02985835

Woods, D. M. (2021). Enacting number talks in a simulated classroom environment: What do preservice teachers notice about students. International Journal of Technology in Education (IJTE), 4(4), 772-795.

THE ROLE OF BELIEFS, VISUALIZATION AND TECHNOLOGY IN TEACHING AND LEARNING PROOF: THE CASE OF SKYLAR

Shahabeddin Abbaspour Tazehkand, Ph.D.
University of Central Florida
Shaha8@ucf.edu

Farshid Safi, Ph.D.
University of Central Florida
Farshid.Safi@ucf.edu

Bramlett and Drake (2013) suggest that the ability of teachers to teach proof is crucial for students to learn and develop formal and informal proofs. Teachers need to be involved in the process of proving and have a firm understanding of the critical role of proofs in order to effectively engage their students in proving activities. It is unrealistic to expect K-12 teachers to educate students on proof if they themselves are not given opportunities to engage deeply in the process of proving and understand its significance (Bramlett & Drake, 2013). According to McClain (2010), it is important for teachers to help students understand proof and engage in proving tasks. However, there has been little research on how to teach proof in secondary school mathematics. Steele (2012) suggests that teachers should have the knowledge and skills to identify whether mathematical arguments constitute proofs or not and determine what counts as proof across different representations. Pedagogical beliefs toward proof may be influenced by teachers' beliefs about the nature and role of proof in mathematics, its role in school mathematics, and their beliefs about themselves as mathematical thinkers. In this case study, the researchers leveraged a qualitative method – case study – to study a pre-service teachers' belief about the process of proof and the role of visualization and technology in teaching proof.

Keywords: Reasoning and Proof, Calculus, Preservice Teacher Education, Technology

Introduction

This study aims to investigate the ways in which the integration of dynamic geometry software (DGS) and visual representations can enhance the understanding of proof in calculus concepts among pre-service secondary mathematics teachers. The study focuses on the teaching and learning of proof by using technology and aims to contribute to the field of mathematics education. The research question guiding the study focuses on the ways that dynamic technology integration and visual representations can support teachers' experiences and beliefs regarding the process of proving calculus concepts. The study is inspired by the works of Raman (2003), Stylianides (2009), Abbaspour (2022), and the National Council of Teachers of Mathematics (NCTM) *Catalyzing Change in High School Mathematics* (2018) and aims to investigate the effect of visual representations connected to key ideas through technology on teachers' understanding of the proof process. (Raman, 2003; Stylianides, 2009; Abbaspour & Safi, 2022, NCTM, 2018).

Rationale of the Study

The National Council of Teachers of Mathematics (NCTM) has consistently emphasized the importance of proof in mathematics education in its *Principles and Standards for School Mathematics* (2000) and *Catalyzing Change in Middle School Mathematics* (2020). However, the Association of Mathematics Teacher Educators (AMTE) Standards for Preparing Teachers of

Mathematics (SPTM) does not provide clear guidance on how to teach proof in school mathematics, and in fact only mentions proof six times. Although the document highlights the importance of argumentation, justification, and teachers' roles in helping students understand the limitations of notions, it does not provide specific standards related to proof. Generally, additional research is needed in ways to engage current and future teachers more intentionally (Safi, 2020) more effectively. There is a need for more research on this topic as the current standards, suggestions, and current practices may not be sufficient for preparing teachers to learn and teach proof (AMTE, 2017; NCTM, 2000, 2020). After considering the guidelines and practices related to teaching proof in mathematics, it becomes apparent that there is a need to improve the content of teacher preparation programs in this area. Since there is a lack of specific recommendations for preparing teachers to teach and learn proof, it is reasonable to conclude that further research is required to identify how to address this issue.

Methodology

This research study aims to investigate the impact of using DGS as an intervention to enhance the understanding and construction of proofs related to calculus concepts and theorems at the secondary school level (Abbaspour Tazehkand, 2022). Since this topic has not been explored before, the study employs a qualitative research approach (Creswell, 2014) using a multiple case study design (Merriam, 1998; Creswell, 2013). This approach allows the researcher to achieve an in-depth understanding of the subject matter by examining a variety of cases. To ensure that the study captures a range of perspectives, the maximum variation method is used to select participants (Creswell, 2013;Abassian, 2018). Multiple sources of information are collected from each participant to obtain a comprehensive understanding of each case.

The study planned to use various methods to collect data, including questionnaires, audio and video recording, artifact collection, and interviews. Participants provided consent and were asked to participate. Data collection consisted of one *pre*-interview and three *post*-interviews, as well as recorded videos of students working on assignments, student artifacts submitted online, and audio and video recordings of whole-class and group discussions. Three assignments were required for the study, and *post*-interviews were conducted after each assignment. The interviews were semi-structured, allowing for flexibility in question selection and modification based on student responses to class activities and assignments.

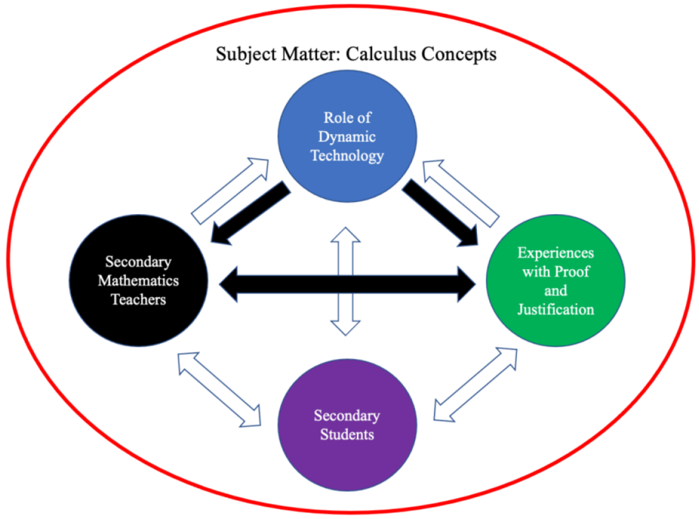


Figure 1: The Conceptual Framework

This study focuses on the interplay between preservice and in-service secondary mathematics teachers, dynamic interactive technology such as GeoGebra, and exploring calculus concepts and theorems. Teachers are the main focus of the study because achieving the goal of helping students make sense of calculus concepts requires teacher preparation programs and pedagogical support. The study fuses different aspects of mathematical teaching and learning to investigate secondary mathematics teachers' understanding of proof and reasoning, including justification and proof, teacher content knowledge, and technology and dynamic visual representations. Technology-enabled peer feedback activities, coupled with socialization opportunities, promote enhanced interaction and feedback generation (Cuocci et al., 2023). The study will analyze the interplay between technology and participants' experiences with proof and justification, as shown in Figure 1, to dig deeper into the dynamic setting of teaching proof. The study's definition of mathematical reasoning and proof is directly linked to mathematical activities, including identifying patterns, making conjectures, checking the validity of the conjectures, and proving arguments. The framework developed by Stylianides (2008) is suitable for the study's theoretical framework, which guides the selection and design of tasks and interventions. The study's focus on pre-service and in-service secondary mathematics teachers benefits from the pedagogical component of the framework. Additionally, the study examines the distinction between inductive and deductive reasoning and the relationship between the two. The framework provides a way to connect different modes of reasoning to the process of proving. Stylianides (2005) has analyzed the curriculum to identify opportunities for reasoning and proving tasks, categorizing them into two groups: those that begin with widely accepted truths and proven arguments using inductive reasoning, and those that begin by observing limited cases, identifying patterns, making conjectures, and ultimately generalizing mathematically.

The case of Skylar

Skylar is a recent graduate student who is new to teaching mathematics. Prior to this study, she had only supervised AP classes and had no formal teaching experience. Unlike some of her peers in the Mathematics Education master's program, Skylar did not have any familiarity with NCTM's Principles and Standards for School Mathematics. However, Skylar's mother is a mathematics teacher who specializes in Calculus and Algebra for college-bound seniors. Skylar shared that she has always enjoyed math and consistently earned high marks in advanced courses. When asked about her experience with algebra, Skylar expressed a fondness for the subject's logical structure and the satisfaction of finding solutions to algebraic problems.

During the interview, Skylar was asked about her understanding and beliefs regarding mathematical proofs. Initially, she defined proof as "showing how you get the answer." However, when asked about the difference between justification and proof, Skylar seemed uncertain and used the terms interchangeably, with no clear distinction between them in her explanation. Despite providing definitions, her descriptions of the two terms were very similar:

> *I think proof is more numbers, more number based and then the justification is okay well how did you use those numbers and why did you use those numbers, why did you use those steps to get the answer… and you're proving why you got those numbers.*

Skylar's view on mathematical proof was discussed in the second half of the interview, where she defined it as explaining the reasons for arriving at a solution to a numerical problem, but struggled to distinguish between proof and justification:

Researcher:	*You use the word "because".*
Skylar:	*Uhum.*
Researcher:	*So, we are talking about the whys?*
Skylar:	*Right.*
Researcher:	*But in the beginning you had a different definition. You told me that proof means that we need to know the hows… Has something changed in the past 20-25 minutes?*
Skylar:	*I think it is the same thing. The hows and the whys are closely related… This is how I did it, and this is why I did it. If you answer both questions, you are proving yourself.*

To understand Skylar's perspective on mathematical proofs, she was asked about the definition of proof and when it can be considered complete. Skylar demonstrated a strong understanding of proof construction and validity, and recognized the social element of evaluating proofs. She believed that all students can learn mathematical proofs and suggested that middle school is the ideal time to start learning. Skylar had encountered more proofs in Calculus, but as a K-12 student, she worked with proofs more in Geometry and had to memorize several of them. Although proving required a lot of memorization, she emphasized the importance of understanding the reasoning behind each proof. Skylar preferred symbolic representations over visual ones, but acknowledged the usefulness of visual representations in verifying statements or reasoning. She believed that students are often taught to rely more on numbers and variables than diagrams, and admitted that she was not well-prepared to help students work with visual representations:

> *I think it definitely needs to be practiced, but I don't know. I think it's hard, because I know for me like if I was told I have to teach visually I wouldn't necessarily even know how… I would definitely need help.*

Skylar was asked about generalization and its relationship to mathematical proofs. She displayed a solid grasp of what generalization means, describing it as the process of making a statement more widespread and applicable to a broader range of categories. However, Skylar's response to the question surprised the researcher as it contradicted the study's framework. Skylar believed that generalization is the opposite of proof, as it involves creating an overarching umbrella statement, while proofs require finding specific details. While Skylar did not recall engaging in any activities related to constructing mathematical conjectures, they stressed the importance of understanding proofs as a matter of justifying why one is undertaking a particular approach, with a proof being considered complete if all elements of a statement can be justified.

Skylar's work on the geometric series assignment showed a reliance on external resources such as course materials and the internet. She did not provide any visual representations and substituted numbers into formulas without explaining why they worked. However, she demonstrated an ability to construct conjectures by connecting the formula for the sum of the geometric series with the series given in the assignment. Skylar was less confident in her ability to construct a conjecture for the general case, but she identified the denominator of the first fraction in the assignment and wrote that the sum could be calculated by plugging in the denominator as *x*. Her reliance on memorization and external resources demonstrated her perception of how mathematics was taught in colleges and universities.

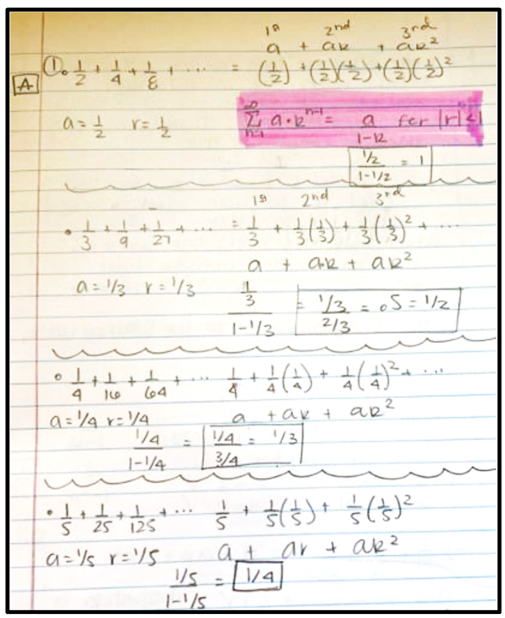


Figure 2: Skylar's response to the conjecture task

The students were given a DGS to interact with, connecting it to the general form of the geometric series to calculate the series sum. Skylar understood the DGS and interpreted the sliders, connecting it to the general form and deepening their understanding. During an interview, Skylar explained that they relied on memory for the conjecture problem and found the formula by going back to notes and looking online. Skylar initially struggled with visualizing the series but eventually understood after group discussions. They expressed a desire to teach geometric series visually to their future students.

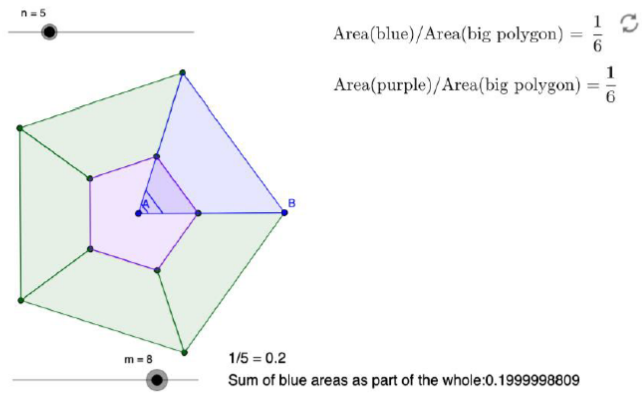


Figure 3: The conjecture GeoGebra task

Skylar found it difficult to visualize the geometric series and was unsure how visual representations were created. After discussing with her class group, she began to understand the visuals for other examples. During the interview, she expressed that she would use visual representations to teach the geometric series to her future students.

> *I think I would start with the visual and be like okay so what do we think… how do we solve that… You can either give them the equation at that point and be like Okay, this is the equation that most people use let's see how they compare, or you can try and work through it, I would need more practice with how to work through… I think it's important as a teacher to give both (numerical and visual). I know that's definitely something where I always am like let's just get to the math, but I think as a teacher, you always have to, try to do both mathematics, or the numerical values and then also a visual representation… I think, also with series and how numbers are formed, I think giving both options give every different type of learner a different way of finding that example or finding out how. It would be so I think that I would start with the visual. And then maybe teach the numerical side of it and then go back to the visual.*

During the investigation, the researcher asked Skylar why she didn't describe visual representations as mathematical. Skylar explained that she always thought of numbers and equations built by variables when working on a problem. She emphasized that this was because she was taught math that stressed the use of representations other than visuals. She noted that in school, students were required to prove everything symbolically and then might be given a visual representation to certify their work. While Skylar recognized the benefits of working with visual representations, her experiences as a student and the emphasis on symbols were hindering her ability to visualize mathematical ideas:

> *I think mathematics, in my mind is numerical. There's no other way… I was taught and there is no other, like I was taught numerically… when I think mathematics, I think numbers, paper and pencil.*

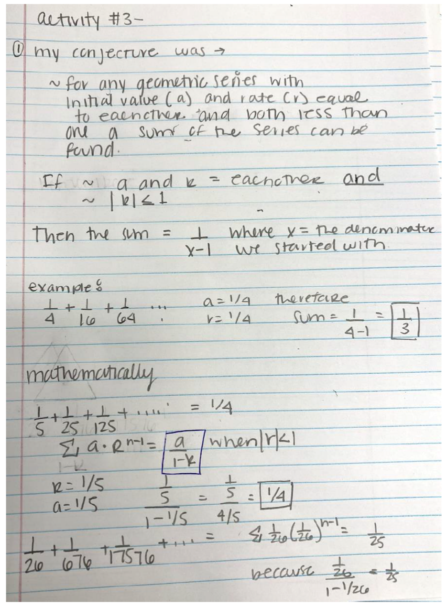
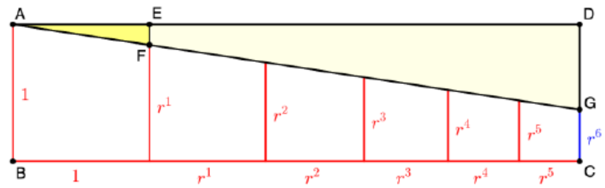


Figure 4: Skylar's response to the proof activity

Skylar's submitted work on proof did not focus on proving her conjecture, instead, she simply repeated what she had previously written in response to task 1 of assignment II. The work only demonstrated that the formula she presented worked for a specific numerical case. Skylar used two formulas, one of which she discovered by referring to her notes from Calculus II, and the other she developed by working with DGS and connecting it to the denominator of the first term of the geometric series. In attempting to prove the validity of the formula she remembered; Skylar provided an example that worked. The researcher noted that Skylar needed to be reminded about the connection between generalizations and proofs and connected her work to her original perception that proof was the opposite of generalization. If Skylar had accepted generalization as part of the process of proving a statement, she may have realized that demonstrating one case did not establish its validity in general form, and that it was impossible to numerically check if her conjecture and remembered formula worked for all natural numbers since there are infinitely many cases.Skylar was tasked with using a DGS to understand a dynamic visual representation of the geometric series and connect it to the proof. While Skylar initially found the DGS confusing, she eventually realized the role of each slider and connected it to the series. However, she tended to focus more on the symbolic representation at the bottom of the DGS than the visual representation, and her understanding of the DGS was only superficial. Despite this, she successfully answered the first two prompt questions for the task.



$$1 + r^1 + r^2 + r^3 + r^4 + r^5 = \frac{AD}{AE} = \frac{DG}{EF} = \frac{1 - r^6}{1 - r}$$

n = 6

r = 0.85

Figure 5: The proving GeoGebra task

She found it challenging to make sense of the DGS this time as it did not use area of shapes, which was different from previous DGS. Working with the DGS on her own did not help her find the similar triangles. However, after breaking down each part and studying them one by one in class discussions, she understood the proof and the similar triangles. Skylar mentioned that she was not a visual learner and did not trust visual proofs, but acknowledged the importance of visual representations and classroom discussions for learning:

> *They (teachers) are good at explaining how this applet works for what we're learning, and I think that's really cool, and I think it like, if you look at the bigger picture it's okay, these teachers are helping other teachers. A lot of your information doesn't come by working by yourself, it actually comes from learning and talking to other teachers as well. There are ways of proving we're talking about in an Algebra class, and you think algebra, you think numerical values… but you can take it more to a geometry standpoint. More of shapes and stuff like that, and I think that's the hard part, especially like I know in my curriculum it goes Algebra and then Geometry like Algebra one, Geometry, Algebra two... I know my… my Algebra one kids would never understand that.*

Skylar reported that the three assignments did not significantly alter her understanding of geometric series and proofs. She expressed frustration at what she perceived as repetitive problem-solving. However, she found exploring different visuals and connecting them to other representations a useful challenge. Skylar believed that using multiple representations helps in making sense of proofs and answering "why" questions. She thought technology could facilitate group discussions related to proving activities. Skylar's belief in using visual representations developed to the point where she expected her future students to make sense of DGS and visual representations as well as numerical and symbolic representations.

Conclusion

Skylar demonstrated a good understanding of mathematical proof as a means to support a claim through reasoning. Initially, Skylar equated proof with explanation and justification, but later she connected it to providing reasons when faced with "why" questions. However, she acknowledged the need to improve her understanding of different types of reasoning and their connection to mathematical proofs, such as inductive and deductive reasoning. Skylar's case illustrates how a student's experiences can shape their beliefs and perspectives towards mathematical concepts. Initially, Skylar valued numerical and symbolic representations more than visual representations, but through interventions, she came to recognize the importance of visual representations in understanding mathematical concepts. She even expressed interest in using visual representations in her future teaching but needed help in implementing them.Skylar struggled with recalling instances where she was required to construct conjectures as a K-12 student, which affected her confidence in coming up with conjectures. Nonetheless, she showed her ability to connect key ideas to visual representations, even though she had difficulty constructing the visuals herself. Skylar found dynamic geometry software (DGS) helpful in visualizing geometric series concepts, particularly the first two DGS. Although she found the third DGS less helpful, she still made sense of the proof of the sum of the geometric series by working with the DGS connected to the proof. Overall, Skylar believed that DGS played a positive role in visualizing dynamic concepts and enhancing understanding.

References

Abbaspour Tazehkand, S., "Supporting Secondary Teachers' Proof and Justification of Calculus Concepts Through the Intentional Use of Dynamic Technology" (2022). Electronic Theses and Dissertations, 2020-. 1166. https://stars.library.ucf.edu/etd2020/1166

Abbaspour, S., Safi, F. (2022) Enhancing Exploration and Conjecture Construction through Identifying Patterns by the Use of Technology. Proceedings of the National Council ofTeachers of Mathematics Research Conference. Los Angeles, CA.

Association of Mathematics Teacher Educators (AMTE). (2020). Standards for Preparing Teachers of Mathematics:(black+ White Version). Information Age Publishing, Incorporated.

Bramlet, D. C., & Drake, C. T. (2013). The role of proof in the curriculum. Journal of Mathematical Sciences and Mathematics Education, 8(1), 62-72.

Creswell, J. W. (2013). Research design: Qualitative, quantitative, and mixed methods approaches Sage publications.

Cuocci, Sophie, Padideh Fattahi Marnani, Iram Khan, and Shayla Roberts. "A Meta-Synthesis of Technology-Supported Peer Feedback in ESL/EFL Writing Classes Research: A Replication of Chen's Study." Languages 8, no. 2 (2023): 114.

National Center for Education Statistics., National Assessment of Educational Progress (Project), Educational Testing Service., & United States. (1992). NAEP ... reading report card for the nation and the states. Washington, D.C: National Center for Education Statistics, Office of Educational Research and Improvement, U.S. Dept. of Education.

National Council of Teachers of Mathematics. (2000). Principles and standards for school mathematics. Reston, VA: NCTM.

National Council of Teachers of Mathematics (NCTM). 2020. Catalyzing Change in Middle School Mathematics: Initiating Critical Conversations. Reston, VA: NCTM.

Raman, M. (2003). Key ideas: What are they, and how can they help us understand how people view proof?. Educational studies in mathematics, 52(3), 319-325.

Safi, F., Wenzel, T., & Trimble Spalding, L.A. (2020). Remote Learning Community: Supporting Teacher Educators During Unprecedented Times. Journal of Technology and Teacher Education, 28(2), 211-222

Steele, M. D., & Rogers, K. C. (2012). Relationships between mathematical knowledge for teaching and teaching practice: the case of proof. Journal of Mathematics Teacher Education, 15(2), 159-180.

Stylianides, G. J. (2005). Investigating students' opportunities to develop proficiency in reasoning and proving: A curricular perspective. University of Michigan.

Stylianides, G. J. (2008). An analytic framework of reasoning-and-proving. For the learning of mathematics, 28(1), 9-16.

Stylianides, G. J., & Stylianides, A. J. (2006). "Making proof central to pre-high school mathematics is an appropriate instructional goal": Provable, refutable, or undecidable proposition?. Proceedings of PME 30, 5, 209-216.

Zazkis, D., Weber, K., & Mejia-Ramos, J. P. (2014). Activities That Mathematics Majors Use to Bridge the Gap between Informal Arguments and Proofs. North American Chapter of the International Group for the Psychology of Mathematics Education.

USING CULTURAL IMMERSION EXPERIENCES IN TEACHER EDUCATION TO INFORM CULTURALLY-RESPONSIVE MATHEMATICS INSTRUCTION

Alesia Mickle Moldavan
Georgia Southern University
amoldavan@georgiasouthern.edu

Keywords: Preservice Teacher Education; Instructional Activities and Practices; Culturally Relevant Pedagogy

Teachers often struggle to bridge cultural connections between school mathematics and real-life applications to engage all learners (Brown et al., 2019). Cultural immersion experiences can be used to study mathematics as a cultural practice and develop mathematical identities to build cultural competence (Addleman et al., 2014; Smolcic & Katunich, 2017). This study reports on a cultural immersion field experience assigned to preservice teachers (PTs) in an elementary mathematics methods course to develop skills to recognize, interact with, and support the learning of mathematics as a cultural practice. The study examined the research question: How do PTs leverage cultural immersion field experiences to connect elementary mathematics content and inform culturally-responsive mathematics instruction?

Ten PTs were asked to learn about the local community surrounding their assigned internship schools within the context of their elementary mathematics methods course. More specifically, the PTs studied the local community paying particular attention to distinctive cultures different from their own. They spent 30-60 minutes participating alongside the community members in hands-on experiences and interactions (e.g., interviews) to learn how they used mathematics. In a reflection paper, they were asked to (a) describe the community visited, including a statement about why that particular community was selected and how its culture differed from their own; (b) how members of the community used mathematics; (c) how the observed mathematics could be connected to elementary mathematics content; and (d) how their understanding of mathematics as a cultural practice informs culturally-responsive mathematics instruction. I analyzed PTs' responses using a priori coding aligning with culturally relevant and sustaining practices (Ladson-Billings, 1995; Paris, 2012). Then, I looked for codes using in vivo and descriptive coding techniques (Saldaña, 2016). The codes informed themes capturing how the PTs leveraged the cultural immersion field experience as related to the research question. Analytic memos were also used to ensure transparency and researcher fidelity (Grbich, 2013).

The PTs had unique opportunities to learn about varying cultures in their local community while simultaneously connecting elementary mathematics content and culturally-responsive mathematics instruction. PTs visited lumber facilities, construction sites, and restaurants relevant to their students. They interviewed cooks, pastry chefs, electricians, estimators, pipe welders, and well drillers. Mathematics was discussed in terms of operations with whole numbers, decimals, and fractions to calculate orders with suppliers, cut materials, bend pipes, estimate materials, measure ingredients, and price inventory. PTs associated state standards with the concepts and suggested how to make mathematics relevant to students by recognizing mathematics from their communities. One PT noted, "The more places students can see mathematics used, the more students can make connections between mathematics and their lives, thereby keeping them motivated and building on their cultural strengths." Implications indicate that PTs must have learning opportunities to develop cultural awareness and relate it to their instruction. Cultural immersion experiences can be a resource for this work to prepare culturally competent teachers.

References

Addleman, R. A., Nava, R. C., Cevallos, T., Brazo, C. J., & Dixon, K. (2014). Preparing teacher candidates to serve students from diverse backgrounds: Triggering transformative learning through short-term cultural immersion. International Journal of Intercultural Relations, 43, 189–200. https://doi.org/10.1016/j.ijintrel.2014.08.005

Brown, B. A., Boda, P., Lemmi, C., & Monroe, X. (2019). Moving culturally relevant pedagogy from theory to practice: Exploring teachers' application of culturally relevant education in science and mathematics. Urban Education, 54(6), 775–803. https://doi.org/10.1177/0042085918794802

Grbich, C. (2013). Qualitative data analysis: An introduction (2nd ed.). SAGE.

Ladson-Billings, G. (1995). Toward a theory of culturally relevant pedagogy. American Educational Research Journal, 32(3), 465–491. https://doi.org/10.3102/00028312032003465

Paris, D. (2012). Culturally sustaining pedagogy: A needed change in stance, terminology, and practice. Educational Researcher, 41(3), 93–97. https://doi.org/10.3102/0013189X12441244

Saldaña, J. (2016). The coding manual for qualitative researchers (3rd ed.). SAGE.

Smolcic, E., & Katunich, J. (2017). Teachers crossing borders: A review of the research into cultural immersion field experience for teachers. Teaching and Teacher Education, 62, 47–59. https://doi.org/10.1016/j.tate.2016.11.002

USING SIMULATIONS TO PROVIDE SECONDARY MATHEMATICS TEACHER CANDIDATES WITH OPPORTUNITIES TO BUILD EQUITABLE TEACHING

Casedy Ann Thomas
University of Virginia
cat6hu@virginia.edu

Nationally, and at Crestmont University (pseudonym), secondary mathematics teacher candidates (TCs) often feel underprepared with classroom management, with consequences for career longevity and their learners. TCs at Crestmont University repetitively vocalized concerns with how to enact ambitious and responsive mathematics instruction while attending to classroom management. Mixed-reality simulations have the potential to provide TCs with a safe environment to build their skills. This study examined how simulations can be used with secondary mathematics TCs to engage in practice with classroom management while attending to lesson objectives, and to understand how such experiences impact TCs' self-efficacy with responsive, standards-based mathematics teaching. The findings focused on positive perceptions of learning despite anxiety, building discourse communities, and the actions of future educators.

Keywords: Preservice Teacher Education, Instructional Activities and Practices, Technology & Equity, Inclusion, and Diversity

Purpose & Conceptual Frameworks

Teacher education programs (TEPs) have a short time to prepare TCs for the complex work of teaching (Grossman et al., 2009). A disproportionate number of new teachers exit the profession in the first five years, overwhelmed by the many challenges that they face (Ingersoll, 2003; Sebastian et al., in press) including those associated with classroom management. Yet, the type of learning opportunities that TEPs provide to TCs has the potential to positively impact career longevity (Kennedy, 2016). When TCs have learning opportunities to practice that which they will be expected to do in the classroom, such practice experiences can improve their sense of readiness and efficacy (Grossman, et al., 2009; Ronfeldt et al., 2014), ultimately contributing to higher levels of teacher retention (Ball & Forzani, 2009; Sebastian & Krishnamachari, 2023).

In fields like medicine and aviation, novices often have opportunities to participate in simulated practice experiences that let them make mistakes, try new techniques, and grow in a safe environment (Brown, 1999; Cruickshank & Metcalf, 1993; Sebastian & Krishnamachari, 2023). Recently, more mathematics educators (Ayalon & Wilkie, 2020; Ghousseini & Herbst, 2016; Schack et al., 2013) are examining the use of simulations as approximations (Grossman et al., 2009) to better prepare TCs. *Approximations* provide TCs with opportunities to engage in practices proximal to teaching (Grossman, et al., 2009). As researchers (e.g., Amandor et al., 2021) continue to explore the influence of various approximations on TCs' practice, there is an emphasis on understanding how TCs perceive, "how the approximation process support[s] their own development and teaching" (p. 486). Mursion's mixed-reality platform features a live actor remotely controlling digital avatars of students in real time, allowing TCs to practice realistic interactions with simulated students. Unlike in field placements, which may vary by day and location, in simulations, instructors can standardize experiences, match the level of difficulty to TCs' needs, and offer immediate feedback (Issenberg et al., 2005; Sebastian et al., in press).

Teachers see classroom management as a source of stress, a cause of burnout, and an area of weakness in both their practice and preparation (Brouwers & Tomic, 2000; Dicke et al., 2015;

Veenman, 1984). Even in structured TEPs, TCs often receive little practice in classroom management, leaving them unaware of their own weaknesses and unprepared for classroom challenges (Freeman et al., 2014). In a classroom management simulation, the TC is placed in the role of the teacher and tasked with motivating student avatars and addressing off-task behaviors, while attending to the lesson objectives and can both practice and receive feedback. (Pankowski & Walker, 2016). Coaching in simulated environments has been shown to drive growth in TCs' skills and self-efficacy with classroom management (Cohen et al., 2020).

As a professor at Crestmont University, I heard concerns with classroom management from my secondary mathematics TCs, who did not take a course on classroom management, within our TEP. Thus, while they felt competent in their mathematics knowledge, when we discussed tenets of ambitious or standards-based instruction (Thomas, 2021; Walkowiak et al., 2018) such as building discourse communities and having learners actively engaged in investigation and discovery, it was evident that the TCs had concerns over how to implement such pedagogy while maintaining classroom management. Thus, if I wanted my TCs to build self-efficacy pertaining to their readiness to deliver ambitious and equitable mathematics instruction (Bartell, 2013), we had to first overcome some of their fears with classroom management and their reluctance to let students take the lead and have power. The purpose of this pilot, mixed-methods study was to examine how simulated technology can be used to help secondary mathematics TCs engage in repeated practice in classroom management simulations, offering a safe, standardized, and targeted complement to their field placement opportunities. Additionally, I wanted to understand how the TCs perceived these experiences and how such experiences impacted their self-efficacy with responsive (Thomas & Berry, 2019), standards-based mathematics teaching.

Research Questions

1. How do teacher candidates perceive and react to simulated classroom management practice opportunities?
2. How do teacher candidates respond to coaching on simulated practice?
3. How do teacher candidates perceive the usefulness of simulation and coaching experiences in supporting their self-efficacy in the delivery of ambitious or standards-based mathematics instruction?

Methods

Site and Sample

Crestmont University is a Research 1 (R1) institution in the Southeastern United States. The sample for this study included two cohorts of secondary mathematics TCs and four secondary mathematics doctoral students. The first undergraduate cohort consisted of five TCs, in their second semester in the TEP, who had a year remaining in the program. The second cohort of six, accelerated Master's program (AMP) students were in their final semester in the TEP. Doctoral students were included due to their enrollment in the final course with AMP students and to offer comparative insight into how previous, mathematics teachers respond to the simulations.

Data Collection

All participants completed individualized simulations. The participants were told in advance that within the simulation, they would be working with 6th grade learners in their classroom to establish classroom norms for their mathematics learning community (ideally in the first few days of school). They were encouraged to reflect upon how they would approach this opportunity. In each thirty-minute session, the students completed two simulations, receiving individualized coaching on strengths and ways to improve between the two rounds. All simulations and coaching sessions occurred remotely over Zoom and were video recorded.

During simulations, the coach took qualitative notes on the participant's reactions to minor incidences of off-task student behavior and noted a series of "hits" or "misses" that occurred throughout the simulations. The coach ranked each response/interaction in both simulation rounds on a scale of 1-4, such that: 1 indicated the need for a more *timely* response; 2 represented the need for a more *direct and specific* feedback with the learner; 3 represented the need for the teacher to be more *succinct* and cognizant of how much class time was used to redirect the behavior, leading to the lack of learning opportunities for other students; and, 4 represented the need for a more *calm* and composed interaction with the learner. Scores from the first round helped to guide the coaching conversation whereas scores in the second round provided evidence of participant improvement, especially in the ascent from timely to succinct. The coaching protocol progressed through the following stages:

- Asking TCs to identify how they felt about their performance.
- Coaching to provide affirmation for practices done well, supported with examples.
- Telling the TCs what to focus on in the next round accompanied by an example; stated as, "I noticed _____ in your last simulation. The next time the student does ____ I want for you to try _____."
- Modeling the practice by the coach.
- Asking the TCs why this approach might be more appropriate and responsive.
- Rehearsing such that the coach pretends to be a learner engaging in a behavior and the TCs practice implementing the coaching feedback.
- Telling the TCs any final improvements and providing positive encouragement.

Following simulations, all students participated in recorded focus group interviews of three to four, to gauge their perceived learning opportunities. Sample interview questions included: what aspects of the simulation did you find valuable for reflecting upon your own practice as a novice mathematics teacher; how did the coaching impact your instruction in the next simulated session; and, how does your self-efficacy with classroom management impact your mathematics teaching practices? Further, before and after each simulated experience, the TCs completed pre and post surveys on their self-efficacy with classroom management to monitor growth over time (Tschannen-Moran & Woolfolk, 2007).

Data Analysis

Coaching sessions, simulation sessions, and focus groups were recorded and transcribed. Fieldnotes from coaching and focus groups were also examined and analytic memos were written intermittently to document emerging themes. Dedoose, a qualitative data analysis software platform, was used for coding; the coding scheme focused on TCs' perceptions of the practice opportunities. Pre and post scores for self-efficacy were quantitatively analyzed for TCs' growth and triangulated with other data sources such as ranking for coaching (and growth from one simulation to the next) and clinical field observations collected through the TEP. Measures were taken to ensure credibility such as engaging in peer debriefs.

Findings

Positive Perceptions of Learning Despite Anxiety

TCs favorably viewed the simulations as influential learning opportunities despite the uncomfortableness that many of them experienced; these moments demonstrate important approximations in which TCs most often saw the need to be more specific in their expectations for learners. For example, a majority of the TCs initially acknowledged the students' behavior but made very vague comments about their expectations or they repetitively framed their statements as questions without being direct in what behaviors they would hope to see in their classroom community. This also uncovered some of their assumptions about cultural competency, as it relates to dominant ideals of cultural practices with forms of communication. When the TCs missed opportunities to have direct and specific interactions with the student avatars, the minor behaviors persisted, and the TCs generally became more anxious and frustrated; yet, many of them said later in interviews, that they were glad that they had these experiences in a simulation for the first time rather than with real children.

Building Mathematics Discourse Communities

In focus groups, the majority of the participants discussed how building their self-efficacy with classroom management positively impacted their perceptions of establishing discourse communities that give more ownership to their learners. Additionally, those students who approached the simulation as a collective experience in which teacher and students worked together to construct norms in their mathematics classrooms, generally had more favorable experiences. For instance, in these observations, TCs unpacked the importance of teacher and learner contracts and establishing norms collectively around what it means to contribute ideas, "respectfully" address the ideas of others, and engage in group work while working through mathematics tasks (to name a few).

Actions of Future Mathematics Teacher Educators

An unexpected finding is that, of the four doctoral students, two's first reaction was to immediately attempt to send the student avatar to the principal's office, while the other two became emotionally overwhelmed. When later questioned about these experiences, the doctoral students made comments about how they had not interacted with 6-12 learners in a long time or how they had taught "advanced" learners when in this setting. Some of them talked openly about questioning their own practices in hindsight, especially those whose first reaction was to attempt to send students out of the room. I believe that this shows that even our doctoral students who are typically entrusted with teaching courses for TCs need further professional development on what it means to be a culturally responsive mathematics teacher educator (Bartell, 2013).

Discussion and Significance

If we want TCs to engage in ambitious and equitable mathematics teaching practices then TEPs must provide TCs with a variety of opportunities to learn to teach mathematics in methods coursework (Grossman et al., 2009). While clinical experiences offer important learning opportunities, additional approximations, like simulations, can be incorporated and tailored to focus on aspects of teaching that TCs rarely get to practice in field-placements, like classroom management (Sebastian & Krishnamachari, 2023). In order for novice teachers to enact effective teaching practices, they need to feel prepared to navigate the social contexts of mathematics teaching and learning. This work, though on a small-scale, found that TCs positively perceived the simulation and coaching experiences and felt an improved sense of readiness to implement equitable instructional practices in mathematics like building mathematics discourse communities. More work is needed to better understand and develop the types of simulations and supports that can best assist TCs to develop their practice and build self-efficacy in secondary mathematics education while engaging in repeated simulated practice over time.

References

Amador, J. M., Kosko, K. W., Weston, T. & Weston, T. (2021). Prospective teachers' appraisals of technology platforms: Comparing perception and complexity. Technology Pedagogy and Education, 30(3), 473-489.

Ayalon, M., & Wilkie, K. J. (2020). Developing assessment literacy through approximations of practice: Exploring secondary mathematics pre-service teachers developing criteria for a rich quadratics task. Teaching and Teacher Education, 89.

Ball, D. L. & Forzani, F. (2009). The work of teaching and the challenge for teacher education. The Journal of Teacher Education, 60(5), 497-511.

Bartell, T. G. (2013). Learning to teach mathematics for social justice: Negotiating social justice and mathematical goals. Journal for Research in Mathematics Education, 44(1), 129-163.

Brouwers, A., & Tomic, W. (2000). A longitudinal study of teacher burnout and perceived self-efficacy in classroom management. Teaching and Teacher Education, 16, 239–253.

Brown, A. B. (1999). Simulated classrooms and artificial students: The potential effects of new technologies on teacher education. Journal of Research on Computing in Education, 32(2), 307–318.

Cohen, J., Wong, V., Krishnamachari, A., & Berlin, R. (2020). Teacher coaching in a simulated environment. Educational Evaluation and Policy Analysis, 42(2), 208-231.

Cruickshank, D. R., & Metcalf, K. M. (1993). Improving preservice teacher assessment through on-campus laboratory experiences. Theory Into Practice, 32, 86-92.

Dicke, T., Elling, J., Schmeck, A., & Leutner, D. (2015). Reducing reality shock: The effects of classroom management skills training on beginning teachers. Teaching and Teacher Education, 48, 1–12.

Freeman, J., Simonsen, B., Briere, D. E., & MacSuga-Gage, A. S. (2014). Pre-service teacher training in classroom management: A review of state accreditation policy and teacher preparation programs. Teacher Education and Special Education, 37, 106–120.

Ghousseini, H., & Herbst, P. (2016). Pedagogies of practice and opportunities to learn about classroom mathematics discussions. Journal of Mathematics Teacher Education, 19(1), 79–103.

Grossman, P., Hammerness, K., & McDonald, M. (2009). Redefining teaching, re-imagining teacher education. Teachers and Teaching: Theory and Practice, 15(2), 273-289.

Ingersoll, R. M. (2003). Is there really a teacher shortage? University of Washington, Center for the Study of Teaching and Policy.

Issenberg, S.B., McGaghie W.C., Petrusa E.R., Lee Gordon D., & Scalese R.J. (2005). Features and uses of high fidelity medical simulations that lead to effective learning: a BEME systematic review. Med Teach, 27(1), 10-28.

Kennedy, M. (2016). Parsing the practice of teaching. Journal of Teacher Education, 67(1), 6-17.

Pankowski, J., & Walker, J. T. (2016). Using simulation to support novice teachers' classroom management skills: Comparing traditional and alternative certification groups. Journal of the National Association for Alternative Certification, 11(1), 3-20.

Ronfeldt, M., Schwartz, N., & Jacob, B. (2014). Does pre-service preparation matter? Examining an old question in new ways. Teachers College Record, 116(10), 1–46.

Schack, E. O., Fisher, M. H., Thomas, J. N., Eisenhardt, S., Tassell, J., & Yoder, M. (2013). Prospective elementary school teachers' professional noticing of children's early numeracy. Journal of Mathematics Teacher Education, 16(5), 379–397.

Sebastian, R. & Krishnamachari, A. (2023). Unlocking the potential of introduction to teaching courses through simulations. Manuscript submitted for publication.

Sebastian, R., Krishnamachari, A., & McCartney, A. (In press). Understanding teacher candidates' strengths and areas of need in simulated parent-teacher conferences. Journal of Education for Teaching.

Thomas, C. A. & Berry, III, R. Q. (2019). Qualitative metasynthesis of culturally relevant pedagogy & culturally responsive teaching: Unpacking mathematics teaching practices. Journal of Mathematics Education at Teachers College, 10(1), 21-30.

Thomas, C. A. (2021). One university's story on teacher preparation in elementary mathematics: Examining opportunities to learn. Journal of Mathematics Teacher Education. https://doi.org/10.1007/s10857-021-09487-3

Tschannen-Moran, M., & Woolfolk Hoy, A. (2007). The differential antecedents of self-efficacy beliefs of novice and experienced teachers. Teaching and Teacher Education, 23, 944 –956.

Veenman, S. (1984). Perceived problems of beginning teachers. Review of Educational Research, 54(2), 143–178.

Walkowiak, T. A., Berry, R. Q., Pinter, H. H. & Jacobson, E. D. (2018). Utilizing the M-Scan to measure standards-based mathematics teaching practices: affordance and limitations. The International Journal on Mathematics Education, 50(3), 461-474.

UTILIZING ASYNCHRONOUS NUMBER TALKS AS A WAY TO ENGAGE ALL LEARNERS

Simon Byeonguk Han
Portland State University
byeonguk@pdx.edu

Eva Thanheiser
Portland State University
evat@pdx.edu

Due to the sudden and unexpected move to remote learning in 2020 influenced by COVID-19, both mathematics teacher educators (MTEs) and prospective elementary teachers (PTs) faced a new challenge in creating a productive remote teaching and learning environment. In this study, we used Parrish's (2014) addition strategies and Thanheiser's (2009) conception of multidigit numbers to analyze 41 prospective elementary teachers' responses in Number Talks (NTs) from two online asynchronous mathematics content courses for prospective elementary teachers. We found that (1) the order in which NTs are posed influences the strategies used, (2) some PTs identified the sameness of strategies differently than us (MTEs), and (3) PTs' conception of digits developed over time. In online asynchronous NTs, all PTs are asked to share their strategies before they see strategies provided by other students. This allowed all students to contribute and for instructors to observe and trace every PT's use of strategies over time. Therefore, we argue that asynchronous NTs can be a way to pursue the engagement of all learners in both face-to-face and online learning environments.

Keywords: Preservice Teacher Education, Online and Distance Education, Number Concepts and Operations. Equity, Inclusion, and Diversity.

Purpose of the Study

Discourse and participation are essential in mathematics learning (Boaler, 2002: Ernest, 1994; Lampert, 1990; Sfard, 1998; Staples, 2007; Wood, 1999) for two reasons: 1) To allow all students to share their own and engage with each other's mathematical thinking, and 2) to have a way for instructors to access their students' mathematical thinking (Carpenter et al., 1996; Franke et al., 2001; Kazemi et al., 2016). Focusing on students' mathematical thinking supports both students' learning of mathematics and instructors' learning of students' mathematical thinking (Durkin et al., 2017; Jacob & Spangler, 2017; Yackel & Cobb, 1996). Discourse-rich practices like Number Talks can support instructors in making sense of student thinking (Han & Thanheiser, 2021; Laustgarten & Matney, 2019; Stott & Graven, 2015). However, Number Talks are typically synchronous activities so implementing them asynchronously presents a challenge. Due to the sudden move to remote learning in 2020 influenced by COVID-19, both mathematics teacher educators (MTEs) and prospective elementary teachers (PTs) faced a new challenge in creating a productive remote teaching and learning environment.

To address this new challenge, we developed and implemented an online asynchronous version of Number Talks (NTs) with PTs in mathematics content courses for elementary teachers. We (Han & Thanheiser, 2021) demonstrated how NTs can be successfully enacted in online asynchronous learning. We argued that NTs can serve as a tool for more equitable learning and as a formative assessment in online asynchronous learning. In this study, we build on this prior work and argue that online asynchronous NTs can serve as a tool to engage all learners (in both face-to-face and online learning) by providing space for all learners to share their mathematical thinking. We also argue that the teachers can have a better understanding of their students' mathematical thinking since teachers can attend to all learners' mathematical thinking (rather than just a few students who share). In this study, we examined 1) the impact of purposeful sequencing of NT problems, 2) the strategies that sequencing elicited, 3) how PTs view their strategies as the same and different, and 4) PTs' conceptions of digits. Our research questions were:

1. Do the different sequences of NTs affect PTs' use of strategies?
2. How do PTs view each other's strategies as the same and different?
3. Can NTs support PT's development of conceptions of digits?

Literature Background / Theoretical Framework

Number Talks

Number Talks (NTs) are typically five to fifteen minutes of whole-class discussions on mental computations and/or mental problem-solving (Gerstenschlager, & Strayer, 2019; Han & Thanheiser, 2021; Humphreys & Parker, 2015; Johnson & Partlo, 2014; Okamoto, 2015; O'Nan, 2003; Parrish, 2011, 2014; Parrish & Dominick, 2016; Sun, et al., 2018; Woods, 2018, 2021, 2022). Parrish (2014) suggested five key components of NTs: a) classroom environment and community, b) classroom discussions, c) the teacher's role, d) the role of mental math, and e) purposeful computation problems. The teacher poses a purposeful computation problem or investigates students' strategies, elicits specific strategies, and/or makes connections between similar or different strategies. Students engage in private-think time to mentally solve the problem first and then the teacher facilitates a whole-class discussion. Parrish (2014) categorized addition strategies, which can be elicited during NTs, into eight categories (*counting all, counting up, breaking each number into its place value, making landmark or friendly number, doubles/near doubles, making tens, compensation, and adding up in chunks*).

NTs serve to engage students in mental math so that they strengthen their number sense and computation skills (O'Nan, 2003; Parrish, 2011; Johnson & Partlo, 2014; Parrish, 2014; Okamoto, 2015) by practicing mental math through purposefully selected computation problems followed by classroom discussions to debrief various strategies. NTs can play an essential role in developing accuracy, flexibility, and efficiency for computation and number sense (Han & Thanheiser, 2021; Okamoto, 2015; O'Nan, 2003). In NTs, students are asked to share their mental strategies and justify their thinking. Justification leads to better math understanding (Parris, 2011; Staples et al., 2012). NTs can also provide more equitable mathematics classrooms (Han & Thanheiser, 2021; Sun, Baldinger, & Humphreys, 2018) by allowing all students to participate and valuing all ideas. In online asynchronous NTs, every participant gets to share their thinking and see everyone else's thinking (Han & Thanheiser, 2021). NTs can help students to develop ownership of their mathematics learning (Parish, 2014) by helping them recognize what they can make sense of (Han & Thanheiser, 2021). Also, NTs serve to establish a classroom community in which all student thinking is valued and students are given time to complete their thinking (Parish, 2014; Woods, 2018, 2022). Students' private thinking time is essential to make sense of mathematical tasks (Anthony & Walshaw, 2009; Kelemanik et al., 2016; Staples, 2007). Establishing a classroom culture of mutual respect is essential for creating a safe environment for effective NTs (Parrish, 2014). This includes sharing incorrect answers, invalid strategies, unfinished strategies, etc. In face-to-face classrooms, making various strategies public on the board can elicit that there is more than one way to solve a problem (Lustrgarten & Matney, 2019). Also, NTs allow for comparing across strategies to determine their similarities and differences. This kind of reflection allows students to develop a deeper understanding of mathematics. (CCSS, 2010; Durkin et al., 2017; Jacobs & Spangler, 2017; NCTM, 2000; Yackel & Cobb, 1996). However, despite the wide use of NTs among teachers, schools, professional development, and teacher education, there is little rigorous research about if and how Number Talks support students' development of whole number conceptions and operations (Matney et al., 2020). Woods (2018, 2022) demonstrated that the Math Talk community (questioning, explaining mathematical thinking, the source of mathematical ideas, and responsibility for learning; Huffered-Ackles, 2004) was improved through NTs in third-grade classrooms. Woods (2021) studied PTs learning through the implementation of NTs through video simulations. PTs reported that they noticed students' different ways of thinking and strategies. They also learned the importance of public records, opportunities for students to participate, and support students to expand and revise their thinking.

Prospective Elementary Teachers' Conceptions of Whole Number and Operation

PTs typically enter mathematics content courses for teachers being able to perform addition and subtraction algorithms but unable to explain why they work (Browning et al., 2014; Thanheiser, Browning, et al., 2014; Thanheiser, 2009; 2010; 2018; Thanheiser, Whitacre, et al., 2014). This is because PTs are typically taught how to use the standard algorithms but do not connect the digits in the number to the values. For example, the number 321 would be interpreted as 3 next to a 2 next to a 1 (*concatenated digits only*) or 300 ones and 2 next to a 1 (*concatenated digits plus*) rather than as 3 hundreds, 30 tens, or 300 ones combined with 2 tens or 20 ones, combined with 1 one (*reference-units* or *groups-of-ones)* (Thanheiser, 2009; 2010; 2018). The concatenated digit conception restricts PTs' ability to explain regrouping conceptually and leaves them with a view of math that is not based on sense-making. Thus, NTs can be used as one way to allow PTs to rediscover the sense-making of digits and using numbers meaningfully.

Methods

We analyzed 41 PTs' responses in NTs from the two online asynchronous courses. Both courses were the first in a sequence of three mathematics content courses for elementary teachers in the United States. The courses were taught by the same instructor in Spring 2020 (23 PTs) and Fall 2020 (18 PTs). In Spring 2020, the course was suddenly moved to remote learning due to COVID-19. Since the school is on a quarter system this means that the whole course was moved from the first day. So, the PTs and the instructor participated in an online, asynchronous course despite never intentionally signing up for that format. The online asynchronous class was delivered through Google Slides and an Online Discussion Forum (in our case, D2L). The instructors did know in advance of Fall 2020 that the course would be online and asynchronous. Students were aware in Fall 2020 that they were registering for an online format of the course. No PTs in the study were exposed to different types of addition strategies (Parrish, 2014) or conceptions of whole number (Thanheiser, 2009; 2010; 2018) prior to the NTs.

We followed the part of the Online Asynchronous Collaboration (OAC) model of instruction, which supports teachers to develop their mathematical knowledge for teaching (MKT) (Clay et al., 2012; Silverman & Clay, 2009). The OAC model consists of six sequenced activities: "1) reviewing an expert model, 2) creating initial responses to the task, 3) listening to/viewing others' responses, 4) reviewing and commenting on others' responses, 5) discussing, and 6) revisit initial responses" (Clay, Silverman, & Fischer, 2012, p. 767). We did not use the first activity, reviewing an expert model, since NTs do not begin with sharing the instructor's strategies. OAC allows a slow pace of learning, makes all participants' ideas public and permanent, increases access to others' various thinking and understandings, and enables participants to return to their and others' previous thoughts to reinforce their understanding.

We analyzed the first two NTs from Spring 2020 (S1 and S2) and the first three NTs from Fall 2020 (F1, F2, and F3). The difference between the sequences is that in the Fall we started the sequence purposefully with F1, while the remaining NTs were the same, S1 = F2 and S2 = F3 (see Table 1). For each NT, we provided the problem through a Google Slide and asked PTs to solve the problem without writing anything down. Then we asked the PTs to share their strategies (written explanation on how they computed the NT problem) through D2L. To ensure every PT shared their own strategies, we utilized the function: *Users must start a thread before they can read and reply to other threads in each topic* in D2L. Once PTs posted their responses, they were able to see everyone else's responses. After their initial sharing, we asked PTs to look for other solutions. In this way, all PTs had to share their strategies to participate in the NTs. In both Spring 2020 and Fall 2020, we asked the PTs to look for strategies that were (a) the same as their own and (b) different from their own and to use the comment function in D2L to respond to the solution and to explain how their solutions were the same and/or different.

Table 1: Number Talks Problems

Term	Spring 2020		Fall 2020		
	S1	S2	F1	F2	F3
Problems	13+18	99+98	15+16	13+18	99+98
Timing	Week 4	Week 5	Week 1	Week 2	Week 4
Number of Participants	22	23	18	18	18
Number of Total Strategies	23	24	23	22	18

Analysis to answer the first research question and the third research question focused on each PT's use of strategies in each NT and across NTs. We had specific goals and anticipated strategies for each NT (see Table 2). Although Parrish (2014) did not include the standard algorithm as a category, we anticipated the algorithm as a strategy that PTs would use based on their experiences with school mathematics (Thanheiser, 2014). Analysis to answer the second research question focused on how PTs' use of strategies in F2 and F3 are similar and/or different compared to S1 and S2, respectively.

Table 2: Goal and Anticipated Strategies (Parrish, 2014) for Each NT

NT	Goal	Anticipated Strategies
S1 13+18	Introduce NT to the PTs. Observe PTs' initial strategy use.	*the standard algorithm*, *breaking each number into its place value, adding up in chunks*,
S2 99+98	Will PTs use the same strategies as S1 or different? Can 99+98 elicit *making landmark or friendly numbers* strategy?	the s*tandard algorithm, breaking each number into its place value, making landmark or friendly numbers, compensation,*
F1 15+16	Introduce NT to the PTs. Obverse PTs' initial strategy use. Can 15+16 elicit *doubles/near doubles* and *making landmark or friendly numbers* strategies, which were not used in S1?	*the standard algorithm, breaking each number into its place value, making landmark or friendly numbers,* *doubles/near doubles*,

F2 13+18	Will PTs use the same strategies as F1 or different? Will *doubles/near doubles* and *making landmark or friendly numbers* show up based on the inclusion of F1? Will *compensation* strategy emerge?	*the standard algorithm*, *breaking each number into its place value, adding up in chunks*, *making landmark or friendly numbers, doubles/near doubles*, *compensation,*
F3 99+98	Will PTs use the same strategies as F1 and F2 or different? Will *doubles/near doubles* and *making landmark or friendly numbers* show up based on the inclusion of F1?	*the standard algorithm, breaking each number into its place value, making landmark or friendly numbers, doubles/near doubles*, *compensation,*

We analyzed PTs' strategies through two different lenses: addition strategies by Parrish (2014) and conceptions of digits (Thanheiser, 2009). Through our initial analysis by using Parrish's (2014) categorization, we realized that some PTs used *concatenated concepts of digits* (Thanheiser, 2009). Thanhesier's (2009) original framework requires three-digit whole numbers to differentiate *reference units* and *groups of ones*, *concatenated digits plus* and *concatenated digits only*, respectively. Since Thanheiser (2009) found that the *concatenated digits plus* conception "arises only in numbers with three or more digits," (p. 279), that conception was left out of our analysis. Since we asked for two-digit addition problems, we simplified the categories in Thanheiser's (2009) framework into *sufficient* (reference units or groups of ones) and *limited* (concatenated) categories referring to the ability to explain regrouping.

Two researchers categorized PTs' strategies individually and reconciled them. For example, in F1, one PT responded, "*When I first look at this problem, my brain immediately added 15+15=30. Then, because one of the numbers is actually 16, instead of 15, I added a 1 to my answer to get 31.*" We highlighted 15+15=30, 16=15+1, 30+1=31, and matched the response to *doubles/near doubles* strategy since the student began with 15+15=30. We matched this student's response to *sufficient* conception of digits because the digits were clearly referred to as their place value. Responses were coded as the *standard algorithm* if the response described the standard algorithm without mention of place value, but as *breaking each number into its place value* if the response used the standard algorithm in conjunction with mention of place value to explain regrouping. For example, one response to F2 13+18 was *"I know 1+1= 2, then I added 3+8 = 11. I took the 1 from 11 adding it to 2 = 31."* Since there is no evidence that the student understood the "1 from 11" as one 10, this response was coded as the *standard algorithm* and *limited* conception of digits. Another response for F2 included *"My mind automatically places the 13 above the 18... 3+8 is equal to 11. Since we are working on the ones side, and your sum is greater than 10, you're going to subtract 10 from the 11, leaving you with 1. The 10 will be added in the tens place, making the problem 1+1+1 in the tens place."* This response includes a specific reference to place value to justify the regrouping in the *standard algorithm*. As such, this response was coded *breaking each number into its place value*. However, this response showed a *limited* conception of digits since the PT referred to the numbers in tens place as 1 and 3.

Results

Most of the PTs used the instructor's anticipated strategies (shaded in Table 3) in all NTs. Three exceptions emerged: we did not anticipate using the *compensation* strategy in S1 but one

PT used it, we did anticipate using the *compensation* strategy in F2 but no PT used it, and we did anticipate using the *standard algorithms* in F3 but no PTs used it. We think these exceptions were important. Unexpected use of strategies showed why instructors need to make sense of PTs' mathematical thinking and how NTs allowed PTs to safely bring their own way of sense-making to the class discussion. More specifically, the emergence of unexpected strategies showed the importance of carefully examining instructors' anticipated strategies. Instructors can reflect on the emergence of unexpected strategies and use this information in their future NT planning (as we included *compensation* strategy as an anticipated strategy in F2). When instructors do not observe their anticipated strategies, they can then think about how to elicit those strategies.

We did not expect drastic differences between S1 & S2 and F2 & F3. However, we anticipated some differences, including the use of *doubles/near doubles* and *making landmark or friendly number*s strategy in F2 and the use of *doubles/near doubles* in F3. We were able to observe some meaningful differences. First, some PTs successfully used the *doubles/near doubles* strategy for F2 (3 PTs) and F3 (4 PTs) and the *making landmark or friendly number*s strategy for F2 (1 PT), which were not the strategies used in S1 and S2 (shaded in dark gray in Table 3). Among 4 PTs who used the *doubles/near doubles* strategy in F3, two PTs used the *doubles/near doubles* strategy in F1 and the other two PTs did not. Second, no PT used *the standard algorithm* in F3 (shaded in dark gray in Table 3) while 1 PT used the *standard algorithm* in S2. Third, no PT used the *compensation* strategy in F2 whereas one PT used it in S1. As a result, the NTs S1 and F2 and the NTs S2 and F3 showed different distributions of strategy use. Our hypothesis is that the *doubles/near doubles* strategy in F1 influenced PTs to find a strategy other than the *standard algorithm, breaking each number into its place value, or compensation* strategies in F2 and F3. Some PTs explained their strategies in F2 by referring to F1, such as *"When I began to solve the problem, I thought about the fact that I could take 3 away from the 8, and add it to the 3, making it 16+15, like last week's problem,"* and *"I took 2 away from 18 and added it to the 13. Making it 15+16. I know that 15+15 is 30, so I added one more to arrive at 31."*

Table 3: Prospective Teachers' Use of Strategies for Each Number Talk

Number Talks Strategies	S1 13+18	S2 99+98	F1 15+16	F2 13+18	F3 99+98
Standard Algorithm for Addition	3	1	1	4	
Breaking Each Number Into Its Place Value	16	9	12	13	4
Making Landmark Or Friendly Numbers		9	1	1	9
Doubles/Near Doubles			9	3	4
Compensation	1	5			1
Adding Up In Chunks	3			1	
Total Strategies	23	24	23	22	18

Most PTs well identified the same and different strategies in both Spring 2020 and Fall 2020. However, we found a few examples that PTs identified as the same strategies, but we viewed them as not the same strategies (see Table 4). In S1, we categorized provided strategy as the *compensation* strategy and another strategy as *breaking each number into its place value* strategy. Although both PTs started by making 20, they used different strategies to make 20.

However, the second PT identified the two strategies as the same strategies. In F1, although we categorized both strategies as *breaking each number into its place value* strategy, one PT started with adding tens and another PT started with adding ones. We have more instances where the PTs identified as the same strategies regardless of the order of adding tens and adding ones. Since the PTs did not know about the strategy categories by Parrish (2014), their reasoning for the sameness did not rely on it. Our hypothesis is that some PTs captured some parts of the strategies and compared them and determined that those are the same strategies.

Table 4: Strategies Provided by a PT and Strategies Identified as the Same by another PT

NT	Strategy Provided by a PT	Identified as the Same by another PT
S1 (13+18)	18+2=20; 20+10=30; 30+1=31	10+10=20; 3+8=11; 20+11=31
F1 (15+16)	10+10=20; 20+6+5=31	10+6+10+5; 5+6=11; 10+10=20; 20+11=31

PTs' use of limited conceptions of digits decreased over time in both Spring 2020 and Fall 2020 (see Table 5). Also, the number of PTs who drew on limited conceptions of digits and their percentages in F2 and F3 were lower than S1 and S2, respectively. Again, our hypothesis is that this difference is due to the different sequences of NTs. It is possible that F3 had a lower number of PTs drawing on the limited conception of digits than S2 because F3 was the third NT of the term. So, PTs in F3 were more familiar with other strategies than *the standard algorithm* or *breaking each number into its place value* strategies compared to S2.

Table 5: Prospective Teachers' Conceptions of Digits Across the Number Talks

Conceptions of Digits	S1 (13+18)	S2 (99+98)	F1 (15+16)	F2 (13+18)	F3 (99+98)
Sufficient	15	19	16	16	16
Limited	8 (34.8%)	5 (20.8%)	7 (30.4%)	6 (27.2%)	2 (11%)
Total	23	24	23	22	18

In response to the first research question, "do the different sequences of NTs affect PTs' use of strategies?", we found that different sequences of NT do, indeed, affect PTs' use of strategies. As we discussed earlier, PTs' use of strategy in F2 and F3 showed slightly different distributions compared to S1 and S2. We observed *doubles/near doubles* and *making landmark or friendly numbers* strategy in F2 and *doubles/near doubles* strategy in F3. Moreover, in F3, no PTs used *the standard algorithm*, and the *making landmark or friendly numbers* strategy was used the most (9 out of 18 PTs). Throughout the five NTs, F3 was the only case where *breaking each number into its place value* was not the dominant strategy. We interpreted this difference as due to the fact that a) F1 elicited strategies that were not observed in S1 and S2 (*doubles/near doubles* and *making landmark or friendly numbers* strategy) b) PTs in F2 and F3 were more familiar with the strategies other than *the standard algorithm* or *breaking each number into its place value* strategies compared to S1 and S2 since PTs in Fall 2020 had one prior NT (F1) before they were asked to solve F2 and F3.

In response to our second research question, "how do PTs view each other's strategies as the same and different?", we found that most PTs successfully identified the sameness and differences between the strategies. However, as we explained earlier, there were some instances where the PTs identified their strategy as the same as other strategies, but we viewed them differently. We interpreted that these discrepancies occurred because we focused on the details of the strategies whereas some PTs captured and compared some parts of the strategies.

In response to the third research question, "can NTs support PT's development of conceptions of digits?", we found that the number of PTs who used limited conceptions of digits (*concatenated*) decreased in both Spring 2020 and Fall 2020. We acknowledge that we do not have enough evidence to argue that NTs were the main cause of PTs' improved conception of digits. However, participating in online asynchronous NTs required all PTs to explain their strategies. PTs had to share their strategies in a way that made sense not only to them but also to their peers and instructor. This process can allow PTs to reflect on their thinking and explanation so that they seek ways to improve their explanations. Therefore, we think NT can be one way to support PTs' development of the conception of digits.

Discussions/ Conclusions/ Implications

Online asynchronous NTs allowed instructors to observe every PT's use of different strategies over time, how different sequences of NTs can elicit different strategies, how PTs identified the same and different strategies, and how NTs can support the development of PTs' conception of digits. We found that in online asynchronous NTs, all PTs can share their strategies and compare and contrast their strategies with others, which is related to higher participatory equity (Reinholz & Shah, 2018). These aspects are hard to be examined in face-to-face NTs due to time constraints. As such, we were able to have a more comprehensive understanding of PTs' use of strategies and conception of whole numbers and digits. This result aligns with our previous study (Han & Thanheiser, 2021). Therefore, we argue that online asynchronous NTs can be a way to pursue the engagement of all students in both face-to-face and online learning environments. In a face-to-face or synchronous online classroom, the instructor can provide asynchronous NT through Google slides and Online Discussion Forum functions. Then, the instructor can attend to all students' strategies and their conception of digits. Also, as we mentioned earlier, examining whether some students used the strategies that the instructor did not anticipate could be crucial for making sense of students' strategies and planning future NTs.

We do not argue that NTs directly caused PTs' use of different strategies and the development of conceptions of digits. However, we think online asynchronous NTs did shed light on possible strong relationships because we could not assume these relationships in face-to-face or synchronous online NTs. In future research, we can have pre and post-tests on PTs' flexibility (use of different strategies) and conception of digits, and conduct interviews to elicit these relationships in detail.

References

Anthony, G., & Walshaw, M. (2009). Characteristics of effective teaching of mathematics: A view from the west. Journal of Mathematics Education, 2(2), 147-164.

Boaler, J. (2002). Learning from teaching: exploring the relationship between reform curriculum and equity. Journal for Research in Mathematics Education, 33(4), 239-358.

Browning, C., Thanheiser, E., Edson, A. J., Kimani, P., Olanoff, D., Tobias, J., & Whitacre, I. (2014). Prospective elementary teacher mathematics content knowledge: An introduction. The Mathematics Enthusiast, 11(2), Article 3.

Carpenter, T. P., Fennema, E., & Franke, M. L. (1996). Cognitively guided instruction: a knowledge base for reform in primary mathematics instruction. The Elementary School Journal, 97(1), 3-20.

Clay, E., Silverman, J., & Fischer, D. J. (2012). Unpacking online asynchronous collaboration in mathematics teacher education. ZDM Mathematics Education, 44, 761-773.

Durkin, K., Star, J., & Rittle-Johnson, B. (2017). Using comparison of multiple strategies in the mathematics classroom: lessons learned and next steps. ZDM Mathematics Education, 49(4), 585-597.
Ernest, P. (1994). Varieties of Constructivism: Their metaphors, epistemologies, and pedagogical implications. Hiroshima Journal of Mathematics Education, 1, 1-14.
Ertmer, P. A., Richardson, J. C., Belland, B., Camin, D., Connolly, P., Coulthard, G., Lei, K., Mong, C. J. (2007). Using peer feedback to enhance the quality of student online postings: an exploratory study. Journal of Computer-Mediated Communication, 12(2).
Franke, M. L., Carpenter, T. P., Levi, L., & Fennema, E. (2001). Capturing teachers' generative change: A follow-up study of professional development in mathematics. American Educational Research Journal, 38(3), 653-689.
Gerstenschlager, N., & Strayer, J. (2019). Number Talks for Statistics and Probability. Mathematics Teaching in the Middle School, 24(6), 362-367.
Huffered-Ackles, K., Fuson, K. C., & Shrin, M. G. (2004). Describing levels and components of math-talk learning community. Journal for Research in Mathematics Education, 35(2), 81-116.
Humphreys, C., & Parker, R. (2015). Making Number Talks: Developing mathematical practices and deepening understanding, grades 4-10. Stenhouse.
Jacobs, V. R., & Spangler, D. A. (2017). Research on core practices in k-12 mathematics teaching. In J. Cai (Ed.) The compendium for research in mathematics education. (pp. 766-792). National Council of Teachers of Mathematics.
Johnson, A., & Partlo, A. (2014). The impact of regular number talks on mental math computation abilities. Retrieved from Sophia, the St. Catherine University repository website: http://sophia.stkate.edu/maed/93.
Kazemi, E., Gibbons, H. K., Lomax, K., & Franke, M. L. (2016). Listening to and learning from student thinking. Teaching Children Mathematics, 23(3), 182-190.
Kelemanik, G., Lucenta, A., & Creighton, S. J. (2016). Routines for reasoning: fostering the mathematical practices in all students. Heinemann.
Lampert, M. (1990). When the problem is not the question and the solution is not the answer: mathematical knowing and teaching. American Educational Research Journal, 27(1), 29-63.
Lustgarten, A., & Matney, G. T. (2019). Examining number talks with secondary preservice teachers. Teaching and Learning Faculty Publications, 36.
Matney, G., Lustgarten, A., & Nicholson, T. (2020). Black Holes of Research on Instructional Practice: The Case of Number Talks. Investigations in Mathematics Learning, 1–15.
Murata, A., Siker, J., Kang, B., Baldinger, E. M., Kim, H. J., Scott, M., & Lanouette, K. (2017). Math talk and student strategy trajectories: The case of two first grade classrooms. Cognition and Instruction, 35(4), 290–316.
National Council of Teachers of Mathematics. (2000). Principles and standards for school mathematics. National Council of Teachers of Mathematics.
National Governors Association Center for Best Practices, Council of Chief State School Officers. (2010). Common core state standards (insert specific content area if you are using only one). Washington D.C. http://corestandards.org/
Okamoto, Y. (2015) The implementation process and impact of a six-week number talk intervention with sixth-grade middle school students in a large urban school district [Unpublished master's thesis, University of California Los Angeles]
O'Nan, M. (2003). Daily number talks and the development of computational strategies in fourth graders [Unpublished master's thesis, Johnson Bible College].
Parrish, S. D. (2011). Number talks build numerical reasoning. Teaching Children Mathematics, 18(3), 198-207.
Parrish, S. (2014). Number Talks; Whole number computation. Math Solutions.
Parrish, S., & Dominick, A. (2016). Number Talks; Fractions, decimals, and percentages. Math Solutions.
Reinholz, D. L., & Shah, N. (2018). Equity analytics: A methodological approach for quantifying participation patterns in mathematics classroom discourse. Journal for Research in Mathematics Education, 49(2), 140-177.
Sfard, A. (1998). On two metaphors for learning and the dangers of choosing just one. Educational Researcher, 27(2), 4-13.
Silverman, J., & Clay, E. L. (2009). Online asynchronous collaboration in mathematics teacher education and the development of mathematical knowledge for teaching. The Teacher Educator, 45(1), 54-73.
Staples, M. (2007). Supporting whole-class collaborative inquiry in a secondary mathematics classroom. Cognition and Instruction, 25(2), 161-217.
Staples, M., Bartlo, J., & Thanheiser, E. (2012). Justification as a teaching and learning practice: Its (potential) multifacted role in middle grades mathematics classrooms. Journal of Mathematical Behavior, 31(4), 447-462.

Stott, D., & Graven, M. (2015). Adapting number talks to foreground mathematical progression in South African classrooms. In J. Novotná & H. Moraová (Eds.), Developing mathematical language and reasoning - International Symposium Elementary Math Teaching (pp. 311–321). Prague: Charles University, Faculty of Education, Prague.
Sun, K. L., Baldinger, E. E., & Humphreys, C. (2018). Number talks: Gateway to Sense Making. Mathematics Teacher, 112(1), 48-54.
Thanheiser, E. (2009). Preservice Elementary School Teachers' Conceptions of Multidigit Whole Numbers. Journal of Research in Mathematics Education, 40(3). 251-281.
Thanheiser, E. (2010). Investigating further preservice teachers' conceptions of multidigit whole numbers: Refining a framework. Educational Studies in Mathematics, 75(3), 241- 251.
Thanheiser, E. (2014). Developing prospective teachers' conceptions with well-designed tasks: explaining successes and analyzing conceptual difficulties. Journal of Mathematics Teacher Education, 18(2), 141-172.
Thanheiser, E. (2018). The effects of preservice elementary school teachers' accurate self-assessments in the context of whole number. Journal for Research in Mathematics Education, 49(1), 39-56.
Thanheiser, E., Browning, C., Edson, A. J., Lo, J., Whitacre, I., Olanoff, D., & Morton, C. (2014). Prospective elementary mathematics teacher content knowledge: What do we know, what do we not know, and where do we go? The Mathematics Enthusiast, 11(2), 433-448.
Thanheiser, E., Whitacre, I., & Roy, G. (2014). Mathematical content knowledge for teaching elementary mathematics: A focus on whole-number concepts and operations. The Mathematics Enthusiast, 11(2), Article 4.
Wood, T. (1999). Creating a context for argument in mathematics class. Journal of Research in Mathematics Education, 30, 171-191.
Woods, D. M. (2018). Developing ambitious mathematics instruction through number talks. (Publication No. 10816956). [Doctoral dissertation, Southern Methodist University]. ProQuest Dissertations Publishing.
Woods, D. M. (2021). Enacting number talks in a simulated classroom environment: What do preservice teachers notice about students? International Journal of Technology in Education, 4(4), 772-795.
Woods, D. M. (2022). Building a math-talk learning community through number talks. Journal of Mathematical Behavior, 67, 1-21.
Xie, k., & Ke, F. (2011). The role of students' motivation in peer-moderated asynchronous online discussions. British Journal of Educational Technology, 42(6), 916-930.
Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. Journal for Research in Mathematics Education, 27(4), 458-477.

PROFESSIONAL DEVELOPMENT AND INSERVICE TEACHER EDUCATION

BUILDING ONLINE COMMUNITIES OF TEACHERS: AN INVESTIGATION OF INDICATORS OF SITUATIONAL INTEREST AND PERSISTENCE

Anthony Matranga
California State University San Marcos
amatranga@csusm.edu

Jason Silverman
Drexel University
Js657@drexel.edu

Amanda Reinsburrow
Drexel University
alr352@drexel.edu

Mikhail Miller
Drexel University
mm4879@drexel.edu

This paper reports on our investigation of online community building efforts with mathematics teachers. We aimed to better understand mathematics teacher interest and persistence in a sequence of online workshops focused on ambitious instructional practice. We theorize that situational interest is related to teachers' decisions to persist in online professional development and that an indicator of situational interest is teachers' descriptions of their local institutional context. We found that teachers who described a supportive institutional context were more likely to persist in our online workshops, while teachers who described tensions around instructional practice in their local schools were less likely to persist. Our findings suggest that situational interest may be an important construct to unpack to enhance community-building efforts and support teacher instructional improvement.

Keywords: Mathematics Teacher Education, Online Professional Development, Situational Interest

Objectives and Purpose

Studies of effective professional learning opportunities continue to demonstrate that sustained collaboration with colleagues is a key factor of teacher learning and instructional improvement (Darling-Hammond et al., 2017; Desimone, 2009; Sancar et al., 2021). Communities provide teachers with opportunities for sustained collaboration with colleagues, engagement in generative discourse, and mutual support (Horn & Garner, 2015; Ronfeldt et al., 2015; van Es, 2012). Communities emerge over time (Grossman et al., 2001; Dean, 2005) and the emergent process includes persistent interaction and collaboration (Lave & Wenger, 1991). Online environments offer flexibility around teacher interaction and collaboration with colleagues (Fletcher et al., 2007), enhancing potential for teacher persistent participation that is required for community building. We conjecture that it is important to sustain and ultimately cultivate teachers' *interest* in online professional learning opportunities to promote persistence and achieve community-building efforts. In this report, we argue that teachers' descriptions of their local institutional context can be an indicator of their interest in online professional learning programs and their persistence.

Theoretical Framework

Social life involves participation in multiple and potentially intersecting communities of practice (Wenger, 1998). Communities of practice are defined by norms and practices, a shared repertoire, and a joint enterprise. Teachers participate in a community of practice within their local school. This local community can be defined by instructional norms and practices, a shared curriculum, and goals to support student learning (Cobb et al., 2003). Teachers can also engage in a variety of online and optional professional development programs that allow them to pursue professional learning goals and build community (Lantz-Anderson et al., 2018). Relationships exist between the norms and practices that define teachers' local community and learning opportunities presented by professional development programs and these relationships may impact teachers' interest in the program – an essential component for building community.

We posit that teacher interest is related to their persistence in professional development programs. Interest is both "a psychological state and a predisposition to re-engage content over time" (Renninger, 2009, p. 106). Interest can develop over time and is often categorized along a continuum from situational interest to individual interest. Situational interest is triggered by a context or phenomenon and can fade as quickly as it is sparked. Individual interest is a more enduring form of interest that is not tied to a specific trigger and has been related to persistence (Ainly et al., 2002; Renninger & Hidi, 2020). While the development of interest can be non-linear (Rotgans & Schmidt, 2011), continued arousal of situational interest can support development of individual interest (Rotgans & Schmidt, 2017). Thus, there is a need to identify indicators of teachers' interest so that teacher educators can cultivate and sustain teachers' interest and increase their potential to persist in professional development programs.

Teachers' experiences within their local institutional context could influence their interest in a professional development program. Herbst et al. (2011) theorized the practical rationality, or the concept that teachers' instructional decisions are situated within the context of an institution that has a set of affordances and constraints for instruction. Similarly, Horn and Garner (2022) argue that teachers are likely to align their instructional practices with those that are normative within their school, otherwise they will be "adding the labor of trailblazing to the already considerable work of teaching" (p. 13). Horn and Garner (2022) argue further that teachers must see the utility of instructional strategies to their local context to apply them in their classes. Furthermore, recent work has suggested that teacher intentions to participate in professional development is associated with their recognition of the potential utility of the program to improve their classroom practice (Fütterer, 2023; Zhang, 2019). Thus, we argue that teachers' institutional context may shape their perception of the applicability of professional development learning opportunities to their classroom practice and ultimately their interest and persistence in the program.

Taken together, we argue that mathematics teachers' interest in online programs can be characterized along a continuum from situational to individual interest. While teachers with individual interest are more likely to persist in an online program, it is important to identify teachers with situational interest early in the program so teacher educators can cultivate and sustain their situational interest and ultimately support their persistence. We conjectured that teachers' descriptions of their local institutional context may be an indicator of their interest in a professional development program. Therefore, we investigated the following question: What relationships exist between teachers' descriptions of their local institutional context and their persistence in a sequence of online workshops?

Methodology

This research was conducted as part of a collaborative, online, and optional professional development program that features a sequence of three 6-week workshops. The workshop goals included supporting mathematics teachers in developing discourse-rich and evidence-based instructional practice by engaging in problem-solving, examining student math work, and connecting these experiences to canonical models of mathematics instruction such as the 5

Practices (Stein et al., 2008). Workshops activities focus on using student mathematical thinking as a starting point for rich conversations and the development of lessons that help students build on their current understandings. Across the workshops, the conversation foci evolve from, for example, what teachers attend to in student work to how to respond to students based on what they notice.

This study included 26 mathematics educators who participated in at least one of these workshops. Participants' classroom experience ranged from 3-25+ years. 75% of the participants identified as female and 25% identified as male. Participant racial breakdown is as follows: 70% White, 18% Black, 2% Latino, 8% Asian, 2% Native Pacific Islanders.

Data sources included participant persistence data from our workshops and transcripts of 1-hour semi-structured interviews of our participants following their completion of workshop one. The interview included questions intended to gather information about participant institutional context such as: how would you describe the culture of your school and district? We used open and axial coding procedures to analyze participants' responses to these questions and additional descriptions of their institutional context in the interviews. During research meetings we reviewed the transcripts and developed descriptive codes. We condensed codes into categories and developed themes that reflected commonalities in the categories. Once we developed common understandings of the data and coding scheme, we individually coded the rest of the data set. We brought uncertainties about codes to the research team and discussed the data and codes until consensus was reached. We generated the persistence data with a numerical coding system, where the digits one through three were used to denote the number of workshops completed by each participant. We looked across the themes in teachers' descriptions of their local institutional context and persistence in our workshops to identify relationships.

Results

This section introduces the themes from our analysis of how teachers talked about their local institutional culture and associations with their persistence in our online program. We identified two themes in how teachers described the institutional contexts of their schools and districts. We characterized these themes as *supportive* and *tensions.* When participants described a supportive culture, they discussed support and flexibility with curriculum and mentioned things like, "they leave it to our professional opinion as to how to conduct the class and teach the curriculum. I can choose which curriculum I want to use, the supplements I want to use, and there's no scripted teaching, so I can teach the students anyway I would like" (Vic). Participants also frequently discussed support around "best practices." For example, Gail noted that her math supervisor is "very supportive of best practices and not just best practices, but best practices for us." Participants also described opportunities for collaboration with colleagues during regular meeting times built into their schedule. For instance, Mia said, "We are very fortunate in my building to have common planning to talk about curriculum and student work and things like noticing and wondering."

When participants described tensions in their culture, they discussed a strict oversight over their instructional practice. For example, Clair mentioned, "But my director told me I should be doing, "I do, you do, we do." … and then I was observed doing a lesson that included doodle notes in my lesson. One of the comments in my observation was, "Coloring is not an appropriate activity for eighth grade students."" In addition, Pat mentioned, "it was so bad that we would have to have fake lesson plans for the days of the district walkthroughs. Because there was what we were really doing and then there was the stuff we did during walkthroughs." Participants also described an inconsistent or unclear vision for instructional practice in their schools. For instance, Ari mentioned, "my principal sees the value in teacher collaboration and instruction that builds on student thinking, but she also wants us to focus on standardized test prep, which the two don't really merge. So it's like, what do you value? Do you value conceptual understanding or that students get the right answer for a test?"

We noticed an association between how participants described their local institutional context and their persistence in our workshops. First, regarding persistence, 12 participants completed three workshops, eight participants completed two workshops, and six participants completed one workshop. Of the 15 participants who described a supportive context only, 10 completed three workshops and 5 completed one or two workshops. The nine participants who described tensions completed one or two workshops. The two participants who described tensions and a supportive context completed three workshops.

In summary, participants' descriptions of a supportive context suggest they teach in a context that is potentially conducive to ambitious instruction. For example, we shared evidence of participants discussing support with "best practices" and autonomy in their classes to "try things out" (Vic, Gail). Thus, the potential applicability of our workshop learning opportunities to these participants' classes may have sustained their situational interest in our program. On the other hand, participants who described tensions in their context suggest lack of potential to experiment with ambitious practice in their specific context. Clair's descriptions suggested that her school values "I do, you do, we do" - a common practice that emphasizes speed and accuracy with procedures - and is critical of efforts towards innovative practice. As Clair, for example, began engaging in our workshop that focused on discourse-rich instructional practice, she may have perceived these practices as not possible in her classroom on a consistent basis. This disconnect between learning opportunities and what is possible in teachers' classrooms may have led to fading situational interest and ultimately a lack in persistence in our workshops.

Discussion and Conclusions

We argue that online settings are an effective place for building communities of teachers because they can provide access to generative collaboration and interaction that might not be part of the norms and practices within teachers' local institutional context (e.g., Cobb et al., 2003). Despite being disconnected from teachers' schools and districts, our findings suggest a potential relationship between institutional context and persistence in online programs. These findings are similar to past work suggesting that the utility of professional development is related to teachers' intentions to participate in the program (e.g., Fütterer, 2023; Zhang, 2019). We extend this work and conjecture situational interest as one way of explaining the relationship between teachers' local context and persistence – teachers who describe tensions around pedagogical decisions in their schools are less likely to see the utility of the professional development foci for their local settings. In these cases, accordingly, we would expect to see situational interest fade.

Our findings suggest that one indicator of situational interest and potential to persist in an online program is teachers' descriptions of their local institutional context. Using this indicator to identify teachers with situational interest early in a professional development program may be important for engaging all learners and enhancing community-building efforts. While a portion of teacher attrition in our online workshops may be attributed to reasons such as family situations or changes in teaching assignments, it may be useful to tailor programs to teachers who are situated in contexts with tensions and have potential to lose situational interest in the program. This might include having discussions about how teachers can adapt instructional practices so that the practices, initially, represent small shifts beyond the norms for instruction in teachers' local context. Future research is needed to better understand teacher situational interest and its relationship to persistence in online programs and their institutional context.

References

Ainley, M., Hidi, S., & Berndorff, D. (2002). Interest, learning, and the psychological processes that mediate their relationship. *Journal of Educational Psychology*, *94*(3), 545–561.

Cobb, P., McClain, K., de Silva Lamberg, T., & Dean, C. (2003). Situating teachers' instructional practices in the institutional setting of the school and district. *Educational Researcher*, *32*(6), 13–24.

Darling-Hammond, L., Hyler, M. E., Gardner, M. (2017). *Effective teacher professional development*. Palo Alto, CA: Learning Policy Institute.

Dean, C. (2005). An analysis of the emergence and concurrent learning of a professional teaching community. In *Department of Teaching and Learning: Vol. Unpublished Doctoral Dissertation*. Department of Teaching and Learning, Vanderbilt University.

Desimone, L. M. (2009). Improving impact studies of teachers' professional development: Toward better conceptualizations and measures. *Educational Researcher*, *38*(3), 181–199.

Fletcher, J. D., Tobias, S., & Wisher, R. A. (2007). Learning anytime, anywhere: advanced distributed learning and the changing face of education. *Educational Researcher, 36*(2), 96–102.

Fütterer, T., Scherer, R., Scheiter, K., Stürmer, K., & Lachner, A. (2023). Will, skills, or conscientiousness: What predicts teachers' intentions to participate in technology-related PD? *Computers & Education, 198*, 104756.

Grossman, P., Wineburg, S., & Woolworth, S. (2001). Toward a theory of teacher community. *The Teachers College Record*, *103*(6), 942–1012.

Herbst, P., Nachlieli, T., & Chazan, D. (2011). Studying the practical rationality of mathematics teaching: What goes into "installing" a theorem in geometry? *Cognition and Instruction*, *29*(2), 218–255.

Horn, I. S., & Kane, B. D. (2015). Opportunities for professional learning in mathematics teacher workgroup conversations: Relationships to instructional expertise. *Journal of the Learning Sciences*, *24*(3), 373–418.

Horn, I. S., & Garner, B. (2022). *Teacher Learning of Ambitious and Equitable Mathematics Instruction: A Sociocultural Approach*. Routledge & CRC Press.

Lantz-Andersson, A., Lundin, M., & Selwyn, N. (2018). Twenty years of online teacher communities: A systematic review of formally-organized and informally-developed professional learning groups. *Teaching and Teacher Education*, *75*, 302–315.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge University Press.

Ronfeldt, M., Farmer, S. O., McQueen, K., & Grissom, J. A. (2015). Teacher collaboration in instructional teams and student achievement. *American Educational Research Journal, 52*(3), 475–514.

Renninger, K. A. (2009). Interest and identity development in instruction: An inductive model. *Educational Psychologist, 44*(2), 105-118.

Renninger, K. A., & Hidi, H. S. (2020). To level the playing field, Develop Interest. *Policy Insights from the Behavioral and Brain Sciences*, 7(1), 10-18.

Rotgans, J. I., & Schmidt, H. G. (2011). Situational interest and academic achievement in the active-learning classroom. *Learning and Instruction*, *21*(1), 58–67.

Rotgans, J. I., & Schmidt, H. G. (2017). Interest development: Arousing situational interest affects the growth trajectory of individual interest. *Contemporary Educational Psychology*, *49*, 175–184.

Sancar, R., Atal, D., & Deryakulu, D. (2021). A new framework for teachers' professional development. *Teaching and Teacher Education, 101*, 103305.

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning*, *10*(4), 313–340.

van Es, E. (2012). Examining the development of a teacher learning community: The case of a video club. *Teaching and Teacher Education, 28*(2), 182–192.

Wenger, E. (1998). *Communities of practice: Learning, meaning, and identity*. Cambridge university press.

Zhang, S., & Liu, Q. (2019). Investigating the relationships among teachers' motivational beliefs, motivational regulation, and their learning engagement in online professional learning communities. *Computers & Education, 134*, 145–155.

A COMBINED FOCUS ON TEACHING MATHEMATICS FOR SOCIAL JUSTICE AND MATHEMATICAL MODELING ON TEACHER AGENCY

Manjula Peter Joseph
Fresno Pacific University
manjula.joseph@fresno.edu

Christopher Brownell
Fresno Pacific University
chris.brownell@fresno.edu

This study grew out of the findings of an earlier one in the context of a semester-long graduate course on teaching mathematics for social justice, which revealed that participant teachers were unconvinced of their agency to address frustrations with administrative expectations and job-related limitations. To address these concerns an intentional pairing of two courses was created. Teaching Math for Social Justice (TMfSJ) and Mathematical Modeling (MM) in the Classroom were paired with the purpose of increasing teacher agency. Data came from teachers' reflective writings, forum discussions, and a final collaborative project. Preliminary findings suggest that teachers were assured of the impact of social justice math lessons on their students, found strength in collaboration, and were willing to start small to bring about change within their classrooms and school systems. The study has implications for teacher preparation.

Keywords: teacher agency, teaching mathematics for social justice, mathematical modeling.

An earlier study (Joseph & Reimer, 2022) examined teacher beliefs in a 16-week graduate course in Teaching Mathematics for Social Justice (TMfSJ) in the MA in Mathematics Education program. Data came from the "Teacher beliefs survey" (Enterline, S., Cochran-Smith, M., Ludlow, L., & Mitescu, E., 2008), and reflective writing assignments. Results showed that teachers were convinced that they could bring about positive change in the lives of their students and their communities but were challenged by systems working to inhibit them. They were frustrated by their inability to freely design lessons to meet the needs of their students and work alongside them in co-creating curricula. Furthermore, even though they were convinced that math education needs to be refreshed, they reported feeling helpless in the face of the policies that govern them. They also seemed reluctant to include social justice in their lessons since it might mean bringing politics into the classroom.

The current study followed with an intentional pairing of two courses: Teaching Mathematics for Social Justice (TMfSJ), and Mathematical Modeling in the Classroom (MM). As the techniques of mathematical modeling lend themselves to "reading and writing the world with mathematics" (Gutstein, 2006), we intended to enhance the teachers' perception of their agency to bring about change in their classroom/school/community in this pairing.

We sought to address the following questions through this study: a) *To what extent does the combination of TMfSJ and MM support the development of teacher agency? and b) How is teacher agency evidenced within their designing and implementing lessons that focus on mathematical modeling within a social justice context?*

Theoretical Framework

Priestley, Biesta, & Robinson (2015) provide a conceptual model of agency as something that is both temporal and relational, that can be developed through responses to experiences in one's environment. Drawing on the seminal work of Emirbayer and Mische (1998), Priestley et al. (2015) theorized that teacher agency can also be both temporal and ecological. Hence, agency is not so much an individual's ability but rather something that can be achieved based on the context. Our goal through the courses was to provide this context.

If education reform is a goal, one must begin by empowering teachers, who are the agents of change in institutional instructional practice (Bridwell-Mitchell, 2015). Teachers as agents is a concept that has been researched extensively in education (Villegas and Lucas, 2002; Buchanan, 2015; Biesta, Priestley, & Robinson, 2015; etc.) particularly in the context of preparing culturally responsive teachers (McGee Banks & Banks, 1995, Darling-Hammond, 2002). Villegas and Lucas (2002) offer a vision of culturally responsive teachers who, in response to perceived inequities in schools, construct their classroom activities based on what their students already know with the intention of extending into new avenues of knowledge. In order to do this, teachers are making decisions every day in their classrooms that impact their students, their colleagues, and themselves. But many mathematics teachers, while actively engaging their agency, do not view their teaching as 'political acts' except when it involves social justice tasks (Gutiérrez, 2016). The goal of the courses was to build on their perceptions of themselves as change agents and highlight all aspects of this work within and outside the classroom.

The Earlier Study

This study explored in-service teachers' beliefs about teaching mathematics for social justice throughout a sixteen-week online graduate course. Twelve teachers (9 women, 3 men) in the Master of Arts program in Mathematics Education (secondary focus) at a mid-sized private university were enrolled in the sixteen-week course, "Teaching Mathematics for Social Justice". The group consisted of two Latinx, one Middle Eastern, one Indian, and eight White teachers. Their ages ranged from 27 to 57, and their teaching experience from 3 to 17 years. Data were derived from pre-post surveys (Enterline et al., 2008), written reflections, online forum discussions, and a final reflection. Analysis was done using a grounded theory approach.

Findings

In responding to whether an important part of learning to teach for social justice is examining one's own identity and role, even though teachers were convinced of their roles as 'facilitator', they were challenged by what it meant to give up control. One participant wrote: "I wonder about the dynamics in a co-constructed classroom, because in reality, the teacher is still in a role of power, so there will never be true co-construction." Secondly, they questioned their own sense of agency, evidenced by responses like "I struggle to see how we can start to make a difference in the whole school system." In reflecting on their beliefs about students, while teachers expressed the importance of knowing their students, this was complicated by the demands of curriculum and large class sizes. Lastly, they hesitated with discussing topics that may be divisive in the current political scene. A typical response was, "I am fearful that we would bring politics into the classrooms and sway students to a certain side depending on the teacher's own political beliefs and I do not think that is fair."

As we reflected on these findings, we chose to focus on the one related specifically to teacher agency. We sought to engage in-service teachers in opportunities to interrogate their own agency within the systems that govern them in order to initiate mathematics education reform. With this intention, we created the pairing of two courses described below.

Course(s) Re-Design

In a graduate program in Mathematics Education we, the instructors, were intentional in pairing the two courses: Teaching Mathematics for Social Justice and Mathematical Modeling in the Classroom during the same semester.

As before, in the TMfSJ course, we drew on 'patterns of practice' (Cochran-Smith, Grudnoff, Haigh, Hill, & Ludlow, 2016) for supporting teacher development in teaching for social justice: ongoing critical self-reflection, experiences in naming and challenging systems of inequity, a collaborative inquiry stance to learning, opportunities for dialogue and sense-making, connection to student cultures and communities, and integrating theory and practice.

In the Mathematical Modeling course, we began with a close reading of the Common Core Standard of Mathematical Practice # 4 Mathematical Modeling (MM) (California. Department of Education. 2013) with a primary focus on expanding the definition of modeling into the broader perspective held within the professional mathematics community; wherein mathematics is an applied science, targeted at modeling data, relationships, phenomena etc. in order to solve authentic problems.

The Study

Twenty-one teachers (no participants were repeated from the earlier study) enrolled in the program and twenty of them consented to participate in the study. The 11 female and 9 male teachers represent diverse ethnicities, ages, and teaching experiences. The planned methodology for the study consisted of a phenomenological analysis of the teachers' reflections on a signature assignment (described below) across both courses.

Signature assignment: Working in pairs, teachers had to plan and create a modeling lesson in collaboration with their students. The lesson should be based on a social justice issue their students cared about. One teacher taught the lesson in their classroom, while the other teacher observed. The teachers then debriefed, made improvements to the lesson, following which, the second teacher taught the same lesson to their students. This iterative process speaks to the inherent nature of a mathematical modeling task. The teachers were then asked to reflect on the entire process that included planning strategies for future teaching. Their submissions included lesson plans, samples of student work, and individual reflections from both teachers.

Data Analysis

Final reflections of the participants were coded and analyzed for emergent themes. A first read to identify positive and negative feelings of agency revealed that the positive statements outweighed the negative. We drew on grounded theory for the next stage of analysis to make sense of the teachers' willingness to take steps towards the reform they envisioned for their classrooms. The questions for the study guided the thematic analysis and coding was applied to identify and organize descriptive categories.

Results

The lesson reflections came from a variety of lessons based on topics like minimum wage, human trafficking, and fair trade. Three main themes emerged: a) student engagement and excitement; b) collaboration with colleagues and connections with students; and c) future planning.

Student engagement and excitement: "They showed true maturity!" Eighteen (of twenty) of the participants reported their students being excited about the lesson and being fully engaged in the activity. Responses included: "The students surprised and encouraged me with their high level of engagement…[t]hey really became interested in increasing awareness about the issue and in how they could do something to help. They showed true maturity and a willingness to get involved and to try to bring change." and "I have never seen the students so attentive, creative, respectful and involved!"

One teacher wrote about giving up control and said she was pleasantly surprised when she "stepped back, giving them the reins to take ownership of their learning."

Collaboration with colleagues and connections with students: "We all learned and grew together." Teachers expressed excitement at the opportunity to collaborate with a fellow teacher, which, as one said, "is something I need to do more often. It also required me to think critically about what happened and how I could improve this lesson in the future.

Teachers also reported connecting with their students in a deeper and more meaningful way. We saw statements like "this lesson allowed me to interact more with the students and have organic conversations about the work," and "I was able to connect with my toughest class as most were able to participate in this lesson and we all learned and grew together".

Future planning: "Start somewhere!" All the participants expressed feeling motivated to continue making changes to their classroom practice to include social justice themes. They reported feeling more confident by "finding the places to "start somewhere" with mathematical modeling combined with teaching mathematics for social justice." After having tried and seeing the positive results, they were willing to continue the practice. As one participant put it, "I keep coming back to this lesson series as one that could continue to grow and evolve over time." While teachers felt challenged about balancing both the mathematical and social justice components, they felt confident "it will enhance an already exceptional lesson sequence for the benefit of all.

Participants made action plans to spread the message in their schools. For example, one participant wrote: "It would be beneficial to discuss such lessons in our professional learning community (PLC) to have everyone onboard and be inclusive of everyone else who is following the same pacing calendar."

Conclusion

Even though teachers are generally convinced that teaching mathematics for social justice is a lofty ideal, many feel overwhelmed by what it would entail to make it a reality (Bartell, 2013). The challenges may come from within, like giving up control of their carefully crafted lesson plans to empower students, or from outside influences like parents who are uncomfortable with social issues being discussed in the math classroom or from administration that is focused on a set curriculum that needs covering. The results of this study give teacher educators a possible direction to help teachers overcome their perceived lack of agency. By designing courses that not only focus on aspects of pedagogy and lessons focusing on social justice mathematics education but including mathematical modeling as an avenue for this work has the potential of providing teachers strategies for this work.

References

Bartell, T. G. (2013). Learning to teach mathematics for social justice: Negotiating social justice and mathematical goals. Journal for Research in Mathematics Education, 44(1), 129-163.

Biesta, G., Priestley, M., & Robinson, S. (2015). The role of beliefs in teacher agency. Teachers and Teaching: Theory and Practice, 21(6), 624–640.

Bridwell-Mitchell, E. N. (2015). Theorizing teacher agency and reform: How institutionalized instructional practices change and persist. Sociology of education, 88(2), 140-159.

Buchanan, R. (2015). Teacher identity and agency in an era of accountability. Teachers and teaching, 21(6), 700-719.

California. Department of Education. (2013). California common core state standards: Mathematics. California Department of Education.

Cochran-Smith, M., Ell, F., Grudnoff, L., Haigh, M., Hill, M., & Ludlow, L. (2016). Initial teacher education: What does it take to put equity at the center? Teaching and Teacher Education, 57, 67-78.
Darling-Hammond, L. (2002). Learning to teach for social justice. In L. Darling-Hammond, J. French, & S. P. Garcia-Lopez (Eds.), Learning to Teach for Social Justice, Multicultural Education Series. New York, NY: Teachers College Press.
Emirbayer, M., & Mische, A. (1998). What is agency?. American journal of sociology, 103(4), 962-1023.
Enterline, S., Cochran-Smith, M., Ludlow, L., & Mitescu, E. (2008). Learning to teach for social justice: Measuring change in the beliefs of teacher candidates. The New Educator, 4(4), 267-290. doi: 10.1080/15476880802430361
Gutiérrez, R. (2016). Strategies for creative insubordination in mathematics teaching. Special Issue Mathematics Education: Through the Lens of Social Justice, 7(1), 52-62.
Gutstein, E. (2006). Reading and writing the world with mathematics: Toward a pedagogy for social justice. New York, NY: Routledge.
Joseph, M. P. & Reimer, C. (2022, Feb. 10-12). Engaging Teacher Beliefs in a Graduate Course on Teaching Mathematics for Social Justice. 26th Annual Conference of the Association of Mathematics Teacher Educators, Henderson, NV, USA. https://amte.net/sites/amte.net/files/2022AMTEConf_Program_FINAL%20PRINT%20PROGRAM%2002032022.pdf
McGee Banks, C. A., & Banks, J. A. (1995). Equity pedagogy: An essential component of multicultural education. Theory into practice, 34(3), 152-158.
Priestley, M., Biesta, G., & Robinson, S. (2015). Teacher agency: What is it and why does it matter?. In Flip the system (pp. 134-148). Routledge.
Villegas, A. M., & Lucas, T. (2002). Preparing Culturally Responsive Teachers: Rethinking the Curriculum. Journal of Teacher Education, 53(1), 20–32.

CREATING MONSTERS ISN'T FUN: CHALLENGES IN CREATING PLAYFUL MATH PROFESSIONAL DEVELOPMENT

Candice Love	Amy Parks	Brittany Caldwell
Vanderbilt University	Michigan State University	Vanderbilt University
candice.love@vanderbilt.edu	parksamy@msu.edu	brittany.caldwell@vanderbilt.edu

This paper examines two activities in a weeklong Professional Development (PD) focused on implementing mathematical play in kindergarten classrooms. Though the weekly activities all had different goals, the two focal activities (Pattern Blocks and Create-a-Monster) were designed specifically to be both playful and mathematically engaging for the kindergarten teachers taking part in the PD. This analysis looks at why Create-a-Monster fell short of those goals and compares it to the more successful Pattern Blocks to better understand how to design activities that are playful and mathematically interesting activity for early elementary educators. We finish with implications and future directions on playful PD design.

Keywords: Professional Development; Early Childhood Education; Elementary School Education.

Over the last few decades, a consensus has developed about the importance of play in early mathematics learning (Seo & Ginsburg, 2004; Wolfgang, Stannard & Jones, 2003). Research has documented that young children learn a variety of mathematics through play, such as developing spatial reasoning through block building (Casey et al, 2008) and making sense of magnitude through linear board games (Siegler & Ramani, 2009). More broadly, mathematicians have argued that mathematical play provides opportunities for children to engage in mathematical ways of knowing, such as exploration and argumentation, in ways similar to the work of mathematicians (e.g., Oughton et al, 2022).

However, the routines and practices of everyday school mathematics can make playful experiences challenging for teachers to realize (Putnam & Borko, 2000). Pressures around mandated curricula, classroom management, pacing guides, and even knowledge of productive mathematical play contexts can make it difficult for elementary teachers to take up mathematics in playful ways in their classrooms. Research has also demonstrated that primary grade teachers find it challenging to recognize mathematics in children's free play and to leverage those interactions in formal lessons (Ginsburg & Ertle, 2008). To make the context even more challenging, children in the primary grades are generally receiving fewer and fewer opportunities to play in school, even as researchers recognize the academic, social, and emotional benefits of play (Burson & Castelli, 2022).

Given the benefits of mathematical play as well as the challenges of putting it into practice, professional development around math and play seems like a productive path to support primary grade teachers; however, few models for professional development aimed specifically at promoting playful learning exist. In an effort to figure out what transformational professional development around mathematics and play might look like, our research team has spent a year working with kindergarten educators at a public charter school in a southern city. This work included a weeklong workshop in the summer to introduce the idea of mathematical play, four daylong planning workshops throughout the year, the video recording of four weeks of play-based mathematics lessons taught throughout the year, and video clubs based on those play-based lessons three times over the year.

This paper focuses on the week of summer PD and our efforts to build teachers' understandings of what playful mathematics might look like in their classrooms. We explore what playful mathematics learning looked like for teachers, and how our design of activities seemed to support—or thwart—their engagement in an effort to describe the characteristics of playful PD in mathematics.

Literature Review

In order to identify whether or not the activities in our PD promoted playful engagements in mathematics, we turned toward the broader literature on play. Scholars have studied play across the human lifespan, in a variety of social contexts, and even in non-human species. Burghardt (2011) provides general characteristics of play, intended to be applicable across mammal species. He argues that it is pleasurable, functional (doing something for those engaged in the play), different in some way from more serious activities, often repeated, and typically initiated in the absence of stress. Lifter and Bloom, who focused on humans, defined play as consisting of "spontaneous, naturally occurring activities with objects that engage attention and interest" (1998, p. 164). Brown (2009, p. 17) include many of the same characteristics of play but add that play inspires a "diminished consciousness of self" and an "improvisational potential."

Although adult play (in both humans and animals) has been studied less than child play, Elkind (2007) has argued that tinkering with materials offers adults opportunities to develop both creativity and perseverance and Brown and Vaughn (2009) suggest that including play in adults' work lives makes people happier and more productive.

In relation to mathematical play, Oughton and colleagues (2022) have noted similarities between their own play with mathematics as research mathematicians and the engagements of preschool children engaged in play-based mathematics, such as free exploration of materials and discussion with peers to explore mathematical ideas. To guide our analysis, we considered a task to be *playful* in our PD if it contained multiple features of play identified by scholars (e.g., pleasurable, functional, or engaging, non-serious, improvisational). We considered a task *mathematically* playful if it also invited teachers to think more deeply about some aspect of mathematics (e.g., magnitude, comparison, cardinality, similarity, rotations, geometric vocabulary, proportional reasoning, etc.).

Context

The data for this paper comes from a week-long professional development with kindergarten teachers at Strong! Elementary, a public charter school in a mid-sized Southern United States city and situated in a racially and economically diverse neighborhood. The larger research team, which all three authors are part of, is engaged in a multi-year partnership with Strong!, with the larger goal of implementing more mathematical play into teachers' classrooms. This data is from the very beginning of the project, in the summer of 2022.

Four kindergarten teachers participated in this professional development: Ms. Conway and Ms. Lane, two lead kindergarten teachers, and Ms. Nelson and Ms. Clarkson, their respective assistant teachers. The PD was led primarily by the second author and occasionally by other members of the team.

Over the four days of PD, teachers engaged in a variety of activities, including analyzing videos of playful mathematics lessons in early childhood classrooms and videos of children's mathematical thinking, reading, and discussing articles related to mathematical play, analyzing standards, assessments and curricula, and engaging in mathematical tasks. Our analysis focuses on the mathematical tasks teachers engaged in.

At multiple times throughout the week, teachers engaged in free play with materials designed to support mathematical thinking, such as magna-tiles, art supplies, wooden blocks, and Cuisenaire rods. During two different 30-minute blocks, teachers also engaged in guided play with these same materials, where the facilitator gave a specific goal for engagement with the toys, such as making connections between the toys and the kindergarten content standards or thinking about how children in their classrooms might play with particular toys.

Methods and Data

This study draws its data from a larger interpretive project (Erickson, 1986) that seeks to understand both how to support teachers in designing playful mathematics experiences as well as how children engage in mathematics through play. For the current study, we used case study methods (Flyvbjerg; 2006; Stake, 1995) to create contrasting critical cases of enacted mathematical tasks to examine both the characteristics of the tasks in relation to each other as well as differences in the quality of the teachers' engagement in the tasks.

During the weeklong PD, multiple cameras were used to film interactions. A swivel camera was used to record whole-group sessions. During small group activities, cameras were typically mounted at the small tables where partners worked.

Data for this project included the videos taken during the week, field notes taken by graduate student observers, video recorded interviews with teachers at the end of the PD, and written documents used in the PD.

We began analysis by drawing on the fieldnotes and daily PD agendas to identify mathematical tasks that were designed to support teachers in engaging in mathematics (in contrast to activities such as analyzing videos or standards). We found eight distinct mathematical tasks, each lasting 20 to 30 minutes, across the four days of PD. These ranged from episodes of free play with mathematical games and toys to tasks more closely directed by the facilitator.

We then used the video to code each of these tasks for the features of play identified in the theoretical framework and for the mathematical content present. We had expected two activities in particular to be noticeably more productive for supporting the teachers in engaging playfully with mathematics; however, we found that only one of these two activities had significant features of both play and mathematical engagement in the teachers' participation when it was enacted.

As a result of this finding, we engaged in a closer analysis of these two contrasting critical cases that included repeated viewings of the videos, written memos, and conversations among researchers. The analysis below briefly describes the landscape of mathematical tasks from the PD and then focuses on the two activities we had expected to be most successful in promoting both play and mathematical engagements: Pattern Blocks (which we did find to have many features of play and to promote mathematical engagements) and Create-a-Monster (which we found to have many fewer characteristics of play and mathematical engagements, despite our expectations to the contrary).

Findings

Two of the eight mathematical tasks during PD involved free play with carefully selected toys and games. Analysis showed that free play tasks tended to be playful but not mathematical (as represented in the table below). During free play, mathematical ideas were rarely discussed. Conversations often veered toward personal lives or to broader challenges of teaching. However, free play tasks did have characteristics of play. Teachers laughed, chose to work with the materials most interesting to them, and seemed to find genuine pleasure in the activities, such as painting a watercolor landscape, as evidenced by their desires to finish the tasks they had chosen and to go back to them during breaks.

In some guided play activities, there were fewer conversations as teachers took notes or worked on their own, or conversations were muted, such as when two teachers discussed the opportunities for practicing one-to-one correspondence and cardinality as they played a game of Hi Ho Cherry-O. In this case, teachers discussed mathematical ideas relevant to kindergarteners, agreed the game would be playful for their students, but did not seem (unsurprisingly) to find the game playful themselves. The table below locates four sample activities from the PD on axes representing the depth of the mathematics and playfulness that we identified in each activity based on our analysis. These four activities were chosen as exemplars. The following section goes into more depth about the features and enactment of the Pattern Block and Create-a-Monster activity, in order to make explicit characteristics of mathematically playful PD.

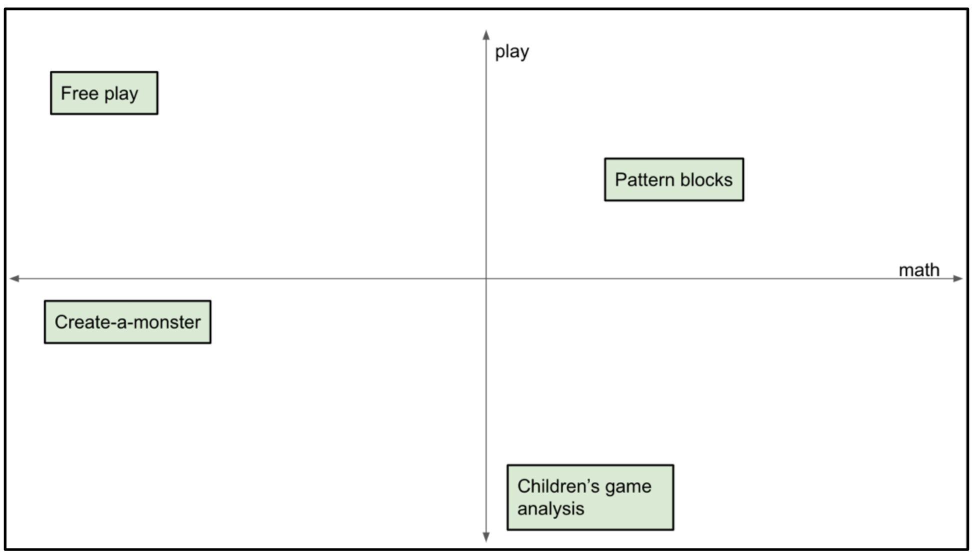


Figure 1: Tasks During PD Organized by Playfulness and Mathematical Learning for Teachers

While the free play activities did not reveal high levels of mathematical engagement and the game analysis did not reveal high levels of playfulness, these findings were expected. The free play activities were intended to provide some of the social and emotional benefits of play and the children's game analysis was intended to support teachers in thinking about how to encourage their children's playful engagement with mathematics. So, while they were not coded high on both scales, they did achieve the intended purpose of the facilitator. However, the other two activities, Pattern Blocks and Create-a-Monster, were intended to provide teachers with the opportunity to experience playful mathematics as learners.

The "Critical Case" Activities

In the "Create-a-Monster," activity, teachers, working in pairs, were invited to use any measuring device they chose (links, ribbons, rulers, etc.) to measure their arms, legs, torso, and head to create a monster twice as large as themselves. They could also choose whether to make the representation 2D or 3D. The mathematical goal was to explore measurement, including units and estimation, as well as proportional reasoning. We conjectured that the creativity involved in the design of the monster and in the variety of approaches to solving the measurement task would support mathematical play. In addition, we felt that the choice of making a monster twice as big would help the teachers see the task as potentially possible with their kindergarteners (perhaps with some modifications), which would also increase their engagement.

The second activity, Pattern Blocks, involved working in pairs. One person used pattern blocks to create a design that was hidden behind a divider, and then explained to a partner how to recreate the design without showing the design itself. After the partner completed their replica, the divider was lifted, and the images compared. The mathematical goals of the activity were to explore geometric vocabulary and attributes, language about position, and spatial reasoning. No constraints were placed on the number or type of blocks that could be used except that each block had to be touching another. We conjectured that this lack of constraint would make the task both playful and cognitively demanding for the teachers, while also allowing them to imagine how imposing additional constraints (e.g., limiting the number of blocks) would make the same task manageable for kindergarteners.

In our analysis of the two mathematical activities, we found that the Pattern Blocks activity became highly playful for the teachers and invited multiple mathematical conversations. By comparison, Create-A-Monster did not become as playful as we hoped, despite opportunities for creative expression, social engagement, and invitations to silliness and that mathematical conversations were relatively limited.

Create-a-Monster. Teachers seemed to find the Create-a-Monster task both mathematically and creatively uninteresting. While Ms. Conway initially seemed to attempt making the monster activity playful by asking her partner teacher, Ms. Nelson, about creating the arms of the "monster," Ms. Nelson did not take up the invitation.

Ms. Conway: Are they gonna be, like, person-like arms or are they gonna be, like, monster-like arms? How should I cut them?

Ms. Nelson: I think however you want.

This response from Ms. Nelson could be taken to signal a lack of engagement in the activity, either as a result of the relative simplicity of the math being used or a lack of interest in the ultimate goal of creating a monster. Ms. Conway seemed to take the feedback as meaning this activity was not a time for silliness, and decided to create human arms and would also go on to create "person-like" features for the legs, head, and torso.

There was some novelty seen at the beginning of the activity, when Ms. Conway initially cut out the first arm and seemed amused at how long it was:

Ms. Conway: (Holding up the length of the paper arms) Look at these arms. (chuckles) That's how long the arm is.

Ms. Nelson: (Laughs, begins measuring her own head independently)

However, the "twice as long" novelty seemingly wore off fairly quickly, and they went on to complete the task without even taping the "monster" together.

The other set of teacher partners, Ms. Lane and Ms. Clarkson, worked mostly independently on separate body parts, partially in silence, and partially engaged in casual, non-mathematical conversations. The independent nature of their experience with this activity and the lack of mathematical conversations suggests a lack of mathematical interest around this task. Additionally, the task of making a monster did not seem to inspire a playful interest. Just like the other pair, they created a large person, who did not have any "monster-like" characteristics.

Pattern Blocks. In contrast, the Pattern Block activity, seemed to create many playful engagements as well as mathematical conversations. In her group, Ms. Lane was the first to build and describe what she made. Yet, when Ms. Conway was trying to interpret her directions, she ran into trouble almost immediately: Ms. Lane's directions called for a parallelogram, but in the set of pattern blocks, there were three different parallelograms to choose from. In the clip, Ms. Conway paused for a moment, looking between the choices she had to recreate the image being described to her, before deciding to use one parallelogram over the others.

Another piece of evidence that this was a mathematically demanding task for the teachers was that both sets of teachers were not able to successfully recreate their partners' images on the first try. In both partner groups, teachers squealed and laughed when they raised their folders to see the differences in their designs, showing pleasure in the process even after a failed attempt. In addition, both groups were eager to play again, demonstrating a genuine engagement. Additionally, after their reveals, the teachers discussed new vocabulary or strategies that they needed to draw on when playing this game in order to succeed.

During the pair's initial reveal, Ms. Conway brought up the presence of multiple parallelograms.

Ms. Conway:	That's the right one though! Because there were also these (Picks up an alternative parallelogram) and I was like oh, no which… which one?
Ms. Lane:	(smiling) Ohhh yeahh… okay

She then brought up the importance of how to orient a trapezoid within the picture to connect with a hexagon in the second round:

Ms. Conway:	I should have specified, like this (*pointing to one edge of the trapezoid she used to connect to the hexagon*) instead of this (*pointing to the different edge of the trapezoid that Ms. Lane used to connect to the hexagon.*

In the next round of play, teachers used, more mathematically precise language and were more successful in their recreations.

A simple game of putting geometric shapes together while following directions from your partner became a site both for mathematical engagement and play. This was reiterated by Ms. Lane during a reflection, where she cited the reveal as being joyful, even when they got the shapes wrong. Adding another layer of their engagement, the activity also invited teachers to consider ways in which the game could be modified so that their students could play, such as starting with fewer shapes.

Discussions and Conclusions

In our analysis, we found that for adults, the appropriate level of mathematical challenge was an important feature in creating a playful math environment. Interestingly, there was no trade-off between "math" or "fun." Instead, we found that the activity that was more mathematically challenging and interesting was also more playful, suggesting that future PD activities should be planned with a focus on creating mathematically interesting tasks for teachers, rather than on demonstrating that the tasks could also be used with kindergarteners. Also, we recognized that challenging mathematical content included new mathematical practices. Naming shapes is not what makes the pattern tiles mathematically interesting for teachers, but learning how to communicate about shapes to a partner who cannot see becomes a complex venture with many parts. Finally, we recognize that tasks alone do not carry activity. For brevity, we do not explore the norms that allowed for exploration and a task with a high failure rate to be experienced as playful and enjoyable, but further work will explore these conditions.

In conclusion, we found our task that promoted both more playfulness and more mathematical engagement was more challenging mathematically, invited laughter and surprise and was seen as interesting to adults. Both tasks used interesting materials, allowed for social engagement, made room for improvisation, and had explicit connections between the task done in PD and a potentially similar task that could be done in kindergarten. However, only the Pattern Block task seemed to promote actual meaningful mathematical engagement and playful interactions among the teachers.

The Pattern Block activity required teachers to engage in discussions about shape orientation, relationships between shapes, and connections to the real world. This task was not just about naming shapes but rather about relationships, composing, and then relaying that information without any visuals. The surprise teachers felt in seeing that their designs did not match provided them with pleasure and created a functional desire to repeat the activity to get it right. The Create-A-Monster activity, while in many ways had the potential to be playful, did not inspire visible pleasure or desire for deep engagement or improvisation in the teachers. This seemed to be in part because the teachers did not find creating a monster to be serving a meaningful function in their lives and because the mathematical task of doubling a person did not provide a mathematically interesting challenge. The teachers' engagement in the activity was evidently less playful, and the teachers did not show a desire to continue with the task.

This work has implications for PD as it provides beginning guidance for designing playful mathematical tasks for PD. In addition, it suggests the need for further research on what it takes to create playful learning environments in PD and preservice classrooms, particularly focused on differences between child and adult learners.

Acknowledgements

This material is based upon work supported by NSF Grant No. 2101356. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation. We would like to thank Melissa Gresalfi, Anita Wager, Madison Knowe, and Jamie Vescio for their feedback and support of this work.

References

Brown, S. L. (2009). Play: How it shapes the brain, opens the imagination, and invigorates the soul. Penguin.

Burghardt, G.M. (2011). Defining and recognizing play. In A. Pellegrini (Ed.), Oxford handbook of the development of play (pp. 9–18). New York, NY: Oxford University Press.

Burson, S. L., & Castelli, D. M. (2022). How Elementary In-School Play Opportunities Relate to Academic Achievement and Social-Emotional Well-Being: Systematic Review. Journal of School Health, 92(10), 945-958.

Casey, B. M., Andrews, N., Schindler, H., Kersh, J. E., Samper, A., & Copley, J. (2008). The development of spatial skills through interventions involving block building activities. Cognition and Instruction, 26(3), 269-309.

Elkind, D. (2007). The power of play: How spontaneous, imaginative activities lead to happier, healthier children. Da Capo Lifelong Books.

Erickson, F. (1986). Qualitative methods in research on teaching. In M. Wittrock (Ed.), Handbook of research on teaching (3rd ed., pp. 119-161). New York: Macmillan.
Flyvbjerg, B. (2006). Five misunderstandings about case-study research. Qualitative inquiry, 12(2), 219-245.
Ginsburg, H. P., & Ertle, B. (2008). Knowing the mathematics in early childhood mathematics. Contemporary perspectives on mathematics in early childhood education, 1(3), 45-66.
Henningsen, M., & Stein, M. K. (1997). Mathematical tasks and student cognition: Classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. Journal for Research in Mathematics Education, 28(5), 524-549.
Lifter, K., & Bloom, L. (1998). Intentionality and the role of play in the transition to language. In A.M. Wetherby, S.F. Warren, & J. Reichle (Eds.), Transitions in prelinguistic communication (pp. 161-195). Baltimore, MD: Paul H. Brookes Publishing Company.
Oughton, R., Nichols, K., Bolden, D. S., Dixon-Jones, S., Fearn, S., Darwin, S., ... & Townsend, A. (2022). Developing 'deep mathematical thinking' in geometry with 3-and 4-year-olds: a collaborative study between early years teachers and university-based mathematicians. Mathematical Thinking and Learning, 1-20.
Putnam, R. T., & Borko, H. (2000). What do new views of knowledge and thinking have to say about research on teacher learning? Educational Researcher 29, 1(4-15).
Seo, K. H., & Ginsburg, H. P. (2004). What is developmentally appropriate in early childhood mathematics education? Lessons from new research. Engaging young children in mathematics: Standards for early childhood mathematics education, 91-104.
Siegler, R. S., & Ramani, G. B. (2009). Playing linear number board games—but not circular ones—improves low-income preschoolers' numerical understanding. Journal of educational psychology, 101(3), 545.
Stake, R. E. (1995). The art of case study research. Sage.
Stein, M. K., Smith, M. S., Henningsen, M. A., & Silver, E. A. (2000). Implementing standards-based mathematics instruction: A casebook for professional development. New York: Teacher College Press.
Wolfgang, C., Stannard, L., & Jones, I. (2003). Advanced constructional play with LEGOs among preschoolers as a predictor of later school achievement in mathematics. Early Child Development and Care, 173(5), 467-475.

CRITIQUE AS SUPPORT IN JAPANESE INSTRUCITONAL CIRCLES

Matthew Melville
Purdue University Fort Wayne
mdmelvil@pfw.edu

Douglas Corey
Brigham Young University
corey@mathed.byu.edu

Bryant Jensen
Brigham Young University
Bryant_jensen@byu.edu

Keywords: Professional Development, Teacher Educators, Teacher Knowledge

Introduction

Japanese instructional circles are a form of teacher community designed to address problems of practice proposed by active teachers (Author, 2022). Teacher communities are one form to help teachers provide a higher quality and more student-centered instructional practices (Horn & Little; 2010). Productive communities of practice share three main characteristics: 1) mutual engagement, 2) a joint vison of teaching, and 3) common language around instructional practices (Gallimore et al., 2009; Horn, 2007; Wenger, 1998). To enable these three features, the use of critique is often avoided to maintain a cordial relationship amongst Western teachers (Borko et al., 2008). They often are hesitant to focus on other teachers' practices or comment on their own even when they are unsatisfied with the outcomes that occurred as a result of their instructional practices (Borko et al., 2008). Teachers from the United States often avoid giving or accepting critique though the privatization their own practices (Author, 2022).

Japanese instructional circles utilize a focus on improving teaching rather than teachers to allow for teachers to begin to de-privatize their practice. This de-privatization of practice allows for teachers to engage in more productive discourse around the improvement of instructional practices and improve on their own problems of practices (Lefstein et al., 2020). Because of this method of helping teachers de-privatize their practice, we were curious as to how teachers in this Eastern professional learning environment both gave and received critiques about their instructional practices. We were guided by the following research question.

1) In what ways do teachers participating in Japanese instructional circles navigate giving and receiving critique?
2) How do the practices of critique in Japanese instructional circles compare to other professional development situations, i.e., Lesson study?

Conclusion

For this study, we engage in a discourse analysis of all the meetings of Japanese instructional circle located in Saitama prefecture in Japan for one year. We triangulated our findings with knowledgeable others in Japan who are familiar with Japanese instructional circles and the Japanese educational culture.

We found that teachers in Japan use giving and receiving critique as support. Where normally in the United States teachers need to receive support along with critiques to maintain relationships and positive dispositions. The idea of using critique as support stems from the focus and belief of the teachers that the critique is designed to improve the instructional practices instead of improving the "teacher". There is no need "to combine support with critique"; rather, in our data, support *was* the critique. When teachers are open to making their practices public, they encourage others to provide that critique by selecting their own problem of practice to improve during the sessions. These findings are not unique to instructional circles, but similar instances can be found in the reflect phase of lesson study. Through these findings we hope to utilize these ideas with how to incorporate critique into teacher communities in teacher learning settings in Western schools as well.

References

Author, 2022

Borko, H., Jacobs, J., Eiteljorg, E., & Pittman, M. E. (2008). Video as a tool for fostering productive discussions in mathematics professional development. Teaching and teacher education, 24(2), 417-436.

Gallimore, R., Ermeling, B. A., Saunders, W. M., & Goldenberg, C. (2009). Moving the learning of teaching closer to practice: Teacher education implications of school-based inquiry teams. The elementary school journal, 109(5), 537-553

Horn, I. S. (2007). Fast kids, slow kids, lazy kids: Framing the mismatch problem in mathematics teachers' conversations. The Journal of the Learning Sciences, 16(1), 37-79.

Horn, I. S., & Little, J. W. (2010). Attending to problems of practice: Routines and resources for professional learning in teachers’ workplace interactions. American educational research journal, 47(1), 181-217.

Lefstein, A., Vedder-Weiss, D., & Segal, A. (2020). Relocating research on teacher learning: Toward pedagogically productive talk. Educational researcher, 49(5), 360-368.

Wenger, E. (1998). Communities of practice: Learning, meaning, and Identity. Cambridge University Press, New York.

DEVELOPING POLITICAL CONOCIMIENTO: TRAJECTORIES OF MATHEMATICS TEACHERS

Brenda Aguirre-Ortega
University of Colorado Boulder
brenda.aguirre@colorado.edu

Victoria Hand
University of Colorado Boulder
victoria.hand@colorado.edu

Tarah Donoghue
University of Colorado Boulder
tarah.donoghue@colorado.edu

Victor Leos
University of Colorado Boulder
victor.leos@colorado.edu

This paper focuses on the trajectories of two mathematics teachers in developing Political Conocimiento through one year of Professional Development (PD) on culturally responsive mathematics teaching. The PD was organized around teacher and student noticing, positionality, community partnerships, and action research. The study found that the teachers' discourse practices shifted from whiteness pedagogies towards politicized notions of schooling, caring, and mathematics learning. The paper discusses the dominant ideologies that teachers reproduced in their discourses around mathematics education and interactions with students. It also illustrated the teachers' trajectories of Political Conocimiento through the deconstruction of the role that race plays in their positionalities, their classrooms, and school.

Keywords: professional development, Political Conocimiento, dominant ideologies, teacher education

Introduction and Framing

Mathematics teachers are increasingly being asked to engage in critical analyses to question and subvert dominant practices in mathematics education that re-minoritize groups of youth and their communities. While some teachers have the opportunity to scrutinize mathematics education in preparation or professional development programs (Leonard et al., 2010), most teachers only have access to content and pedagogical knowledge (Kumashiro, 2013). Researchers like Gutiérrez et al. (2021) position this lack of opportunity for developing what they call *Political Conocimiento in Teaching Mathematics (PCTM)*, a problem because schools and the system of mathematics education are necessarily political institutions. This paper reports on the trajectories of two mathematics teachers in developing Political Conocimiento through a one-year Professional Development (PD) on culturally responsive mathematics teaching and antiracist practices. The PD was organized around teacher and student noticing, positionality, community partnerships, and action research. Our findings suggest that teachers' discourse practices shifted from whiteness pedagogies towards politicized notions of schooling, caring, and mathematics learning.

Political Conocimiento is "knowledge that allows you to see how politics permeates everything we do" (Gutierrez, 2017, p. 20). This knowledge entails understanding that education is necessarily a political institution (Bishop, 1990), and that educational reform often serves neoliberal ends instead of dismantling systems of oppression (Philip et al., 2019). This is particularly true in mathematics education, which tends to reinforce whiteness (Martin, 2013) and deficit perspectives of minoritized youth (Berry et al., 2014). It also involves close scrutiny of the self in relation to systems of power. Teachers' positionalities tend to shape the degree to which teachers are aware of themselves as cultural and racial beings (Rubel, 2017; Gutiérrez et al., 2021) who organize the mathematics classroom based on their implicit perspectives and practices (Battey & Leyva, 2018). Without knowledge of self in relation to others, mathematics teachers may inadvertently reinforce systems of power they may be seeking to subvert. Mathematics teachers who are developing Political Conocimiento are more attentive to dominant ideologies and (discourse) practices, notions of goodness, fairness, and competence organized around white norms (Louie, 2018), and their role in cultural and racial processes in mathematics activity (Gutiérrez, 2017).

Literature Review

Programs aimed at supporting cultural and political awareness have met with mixed results. Mathematics teachers from dominant backgrounds tended to maintain colorblind ideologies and discourses of whiteness (de Freitas, 2008, Gutiérrez et al., 2021), held the Black, Indigenous, and people of color (BIPOC) students in their classrooms at a distance (Rubel, 2017), treated mathematics as a neutral subject (Sleeter, 2017), and engaged in benign notions of caring (Bartell, 2011; Daniels & Varghese, 2020; Matias & Zembylas, 2014). Critical forms of care and compassion in mathematics education must confront deficit perspectives, challenge whiteness, and white fragility, and focus on political solidarity with minoritized students, families, and communities (Bartell, 2011).

Our research questions included: (1) How did dominant ideologies around mathematics education manifest in teachers' discourses?, and (2) In what ways did these discourse practices shift over the course of the PD towards Political Conocimiento?

Methods

This qualitative research study took place in the context of a Professional Development (PD) on culturally responsive and antiracist mathematics instruction for secondary mathematics teachers in one school district. The PD comprised bimonthly meetings over Zoom and took place during the 2021-2022 school year. Teachers engaged with various activities and artifacts to support their development of political knowledge. Data collected for the study included video recordings of the PDs on Zoom, video recordings of interviews with teachers, and artifacts from the PD and teacher work.

Design of the Professional Development

The PD was thoughtfully designed to embrace multiple dimensions of noticing, antiracist practices, and culturally responsive teaching. Its goal was to disrupt traditional notions of teaching and learning mathematics by engaging in critical explorations of teachers' identities and interconnectedness, fostering a collective consciousness rooted in shared experiences. Through intentional acts of noticing, teachers were encouraged to challenge the dominant discourse and critically reflect on their pedagogical praxis. Immersive reading discussions on subjects like anti-deficit noticing, antiracist education, and historical racism were employed to deepen their comprehension of equity and social justice in education. Student noticing surveys were employed as a means for teachers to gather insights from their students, notice students' perspectives and identify areas of improvement. Moreover, teachers contributed diverse artifacts—ranging from videos and personal reflections—that allowed them to share their observations with facilitators and teachers. Ultimately, the PD culminated in teachers constructing action research plans tailored to address specific problems of practice, empowering them to challenge their own biases and actively confront systemic racism.

Subjects

Carrie and Charlotte were two white, female, mathematics teachers who work at the same high school and have more than ten years of experience teaching mathematics. The demographics of the high school at the time was: 64% white, 14 % Hispanic, 11% Black, 8% two or more races, 2% Asian or Pacific Islander, and 1% Native American. Both taught three mathematics classes that followed a tracking system: low, average, and high. Teachers reported that in their 'high track' class, most students were white, in the 'average track' class, the students were white and BIPOC, and in the 'low track' class, the majority were BIPOC students.

Positionality

The PD was led by the authors of this paper. The first author, Brenda Aguirre-Ortega, identifies as an international Latina researcher. who has been working as a facilitator and researcher of learning spaces that promote Political Conocimiento for mathematics teachers. The second and third authors are white, female mathematics education researchers who have experience teaching culturally responsive mathematics teaching to students and teachers. The fourth author identifies as a first-generation Latino, who engages math and science teachers in equitable teaching practices and inclusive pedagogies.

Data Collection

Teachers' discourse practices were captured through initial interviews, noticing interviews, transcripts of PD Zoom meetings, final interviews, and teacher PD artifacts such as reflection journals, action research plans, and collaborative jamboards.

Analysis

To analyze the data, the first author transcribed the PD sessions and interviews, read observational notes, and examined the teachers' PD artifacts. In this process, she developed ideas and categories related to the learning opportunities teachers had around antiracist practices. The data was coded using Dedoose, with categories including power relations, whiteness, reform math pedagogy, tensions, teachers' perceptions, critical discourses, emotions, and caring. The second author recoded a subset of the data and the coders sought interrater consensus around the codes. We went back to the coding process and ensured we followed the agreed-upon guidelines and criteria for coding. We discussed individual interpretations of the data, compared notes, shared thought processes, and identified any areas where our understanding diverged. We worked together to reach a consensus on the analysis.

Later, the data was recoded based on Political Conocimiento and dominant ideologies. These concepts were used as analytical frameworks to analyze the data collected from various sources. The initial interviews provided a baseline understanding of teachers' perspectives on political issues in mathematics education and allowed the authors to identify any initial awareness. The noticing interviews enabled the authors to gain insights into how teachers recognized and responded to power dynamics, dominant ideologies, and antiracist practices within videos of mathematics classrooms. The noticing interviews captured teachers' awareness and engagement with political conocimiento. The transcripts of PD Zoom meetings provided a rich source of data for analyzing discussions around articles read, shared experiences, and students noticing surveys.

The authors could observe how teachers addressed power relations, critically examined dominant ideologies, and discussed antiracist pedagogies. The final interviews provided an opportunity for teachers to reflect on their learning and level of awareness of political issues in education. Finally, the authors analyzed various artifacts that provided concrete evidence of how teachers applied and integrated political conocimiento and confronted dominant ideologies in their teaching practice. For example, the action research plans demonstrated their intention to address power dynamics and challenge dominant ideologies, while collaborative jamboards served as visual representations of teachers' collective engagement with political conocimiento.

By using the concepts of political conocimiento and dominant ideologies as analytical frameworks across the different data sources, the authors were able to examine and showcase the ways in which teachers' initial understandings changed, how they recognized and addressed power dynamics, and the impact of the PD on their pedagogical practices. This analysis highlighted the teachers' engagement with political conocimiento and their efforts to challenge and transform dominant ideologies within their own teaching contexts.

Findings

The analysis revealed that after the second half of the school year, the two mathematics teachers experienced notable shifts along a trajectory of Political Conocimiento. Their learning trajectories are represented in three main themes: colorblind distancing, discourses of whiteness, and development of critical discourses. The first finding illustrates how the teachers described their "low-track" math classes in ways that ignored the hypersegregation of African American students and held them at arm's length. The second finding focuses on the values and ideologies that the teachers manifest in their discourse that reinforce the superiority of white people. The third theme traces the shifts from discourses of whiteness to critical discourses, where teachers start to notice or question their biases.

Colorblind distancing

At the beginning of the PD, Carrie, and Charlotte identified their respective "low track" classes as "challenging". Carrie described her "challenging" class as a "hard class with a tough culture", and Charlotte described hers as "combative and yucky". The teachers acknowledged that coming back from the Covid year was difficult for students in general, but both positioned their "low track" classes (in this case hypersegregated African American students) as particularly challenging.

Carrie, for example, shared at the PD how it felt like she and her students were on different sides, and that nothing she tried changed that dynamic:

> …that one class that's pretty challenging this year …it feels very much like the kids against the teacher. It feels like it's all of them against me, and that is a brand new feeling. I've been teaching for a very long time and I've never felt like that before. I've been trying all kinds of things to improve the culture of the classroom and the behaviors of the classroom.

In this quote, Carrie is portraying an entire class of students as against her. We view this move as a way of *distancing* herself from them and treating them as a singular body with shared motivation to oppose her teaching. Yet, classrooms are necessarily heterogeneous spaces, where youth as individuals have varied orientations, goals, and values. Ignoring this heterogeneity and treating the class as of one mind is a way of "othering". Additionally, to Carrie, changing the "tough" classroom culture is about changing the behaviors of the students, who she views are the ones creating the opposition. Locating the blame for an unhealthy classroom culture with students is a deficit move that ignores interactional dynamics. We note also that Carrie's description of the dynamic in her classroom does not attend to issues of race and culture, but instead comes from a colorblind discourse.

Like Carrie, Charlotte also described her classroom as oppositional. In this case, she mentioned race, but not in terms of a racialized learning environment, and employed deficit discourses:

> I do have several students of color that just put their heads down. I have a lot of kids that have opted out. I have twelve that would do absolutely nothing, or they pretty much do absolutely nothing. Like, I just can't get them to pick up a pencil. And, we had a few really big personalities of kids too. At the beginning, I think [it was] just kind of a toxic start.

Charlotte used the term "opting out" to describe the majority of students, in this case all of whom were African American, whom she viewed were not behaving appropriately (e.g., putting heads down, not picking up a pencil, having a big personality). She made a judgment call based on students' behavior that they were choosing not to learn mathematics in her classroom. In this case, again, we argue that the way Charlotte is positioning the African American students is both distancing and deficit. She is distancing herself from them by not accounting for the way that her class and the broader system of education may be harming them, and by treating their behavior as self-explanatory.

In both cases, we view the teachers' discourse to lack criticality and self-reflection about the ways their classrooms operate as racialized learning environments for African American students. The distancing that both teachers engage in is bound up in their whiteness (Rubel, 2017) and colorblind discourses of teaching. We examine other characteristics of their discourses of whiteness in the next section.

Discourses of Whiteness in mathematics teaching

A second finding is that Carrie and Charlotte's discourses gravitated around white people being the norm. While Carrie centered her white students and their mathematics learning, Charlotte centered herself on the analysis of her classroom.

The PD involved teachers in interrogating dominant ideologies around race, culture, and mathematics. Teachers read an article about how a mathematics teacher concerned with equity challenged dominant ideas around smartness in her classroom. When discussing dominant ideologies around mathematics education, we started populating a jamboard with examples. Some common themes from the teachers were: meritocratic ideology, good scores as a proxy for intelligence, math as pure and objective, and math as a space that privileges certain types of knowledge. After the jamboard activity, we asked if any teachers wanted to elaborate on one of the themes on our jamboard. Carrie started describing math as a space that privileges certain types of knowledge. In doing so, she used an example from her high track class:

> I teach an advanced level class. They're eighth graders doing 10th grade math and so I've got a lot of really, really bright kids in there .. And, you know, there's still kids who are very smart in that class but don't get it as fast…I think it's interesting. The entire class is full of high achieving kids and yet they still are subscribing the same sort of ideologies that smart looks one way.

In this excerpt, Carrie elaborates on how the dominant ideology that smartness in mathematics looks one way affects "bright students" in her "advanced" level class. Importantly, the students in her "advanced" class were white. Carrie notes how some of the students in that class do not get mathematics quickly because they accept dominant ideologies around smartness. We note here how Carrie's focus is placed on her white "advanced" class, versus her "low track" class, which is predominantly African American. The fact that students in the "advanced" class were white could have been an opportunity to reflect on the overrepresentation of white students in advanced level mathematics courses. Instead, Carrie's discourse centered white students who are "bright" and targets of dominant ideologies of smartness.

Similar to Carrie, Charlotte participated in discourses that centered whiteness. In Charlotte's case, she centered her experiences as a white person and teacher. For example, Charlotte wanted African American students to like her and to see how she was like them in some ways:

> I think they [students] just assume that I've always had everything I ever needed which I mean I haven't struggled like a ton, but my parents were on food stamps when I was a kid. We didn't get to go on family vacations. Obviously it's not the same, … and it's like it's hard to say, like, "I kind of can relate", but it's difficult sometimes to let your students know that you might be similar more similar than you think.

In this excerpt, Charlotte is attempting to explain the commonality between her experiences growing up with socioeconomic challenges and the experiences of African American students in her classroom. In naming her experiences with class oppression growing up but not contrasting them with the privileges she has had as a white woman, Charlotte is potentially manifesting white fragility. Here, she is hoping that the students in her class will relate to her experiences of poverty and find her more appealing as a teacher.

We see the discourses of white fragility manifest again in another PD session. In it, Charlotte commented on how the "low track" class challenged her authority and made it difficult for her to teach. She argued, "When I try to express vulnerability, they [students] come on to attack me. [I tell them], "It's really difficult for me to do my job!" And then they tell me I am not a very good teacher. [I tell them] "I'm on your side!"" Charlotte's discourse in these utterances again reflects her white fragility around her BIPOC students. Charlotte became defensive when the students did not respond to what she viewed as acts of vulnerability with them, and she centered her pain and frustration. Similar to the excerpt above, Charlotte appears to have expected students to feel a connection with her and to support her as a teacher. In both sets of utterances, Charlotte's discourse brings the attention to her, as a white woman, focusing away from the experiences of BIPOC students.

Developing critical discourses in mathematics teaching

Major themes that arose from the last sessions of the PD and the final interviews indicated that the teachers were starting to take up critical discourses around positionality and race. Both Carrie and Charlotte's discourses showed awareness of themselves as white women, the impact their positionalities had on their students, and engagement in less colorblind analysis when referring to their "low track" classes.

In her last interview, Carrie shared a deep reflection on her journey in the PD, she pointed out her growth, her desire to continue learning, and how she enjoyed her "challenging" class, in her words:

> I've grown a lot and I still have a lot to grow, but just understanding that my position in this world as a white woman and how I position myself in the classroom can impact my students. My hardest class, they've become my favorite class, they were the easiest to build relationships with, I enjoy them as people in a way that I didn't expect.

Similarly, Charlotte also demonstrated awareness of her whiteness. She recounted an interaction she had with her "low track" class:

> I was like, "Tell me about your experience here." [I said], "I know I'm white but I want to hear anything you have to say." I think that's the big part for our students. Most of their teachers are white. Most of their teachers don't look like them.

In these utterances, Charlotte and Carrie demonstrated an awareness of their racial identities as white women and the potential impact it may have on their students. While Carrie emphasized in how this awareness has brought her to establish closer relationships with her students, Charlotte has noticed the lack of racial diversity among teachers and the need for more representation for students of color in her school. Both teachers have started a process of deconstruction of narratives around their "challenging" classes and their students of color. We argue that both discourse practices indicated movement along a trajectory of Political Conocimiento. According to Gutierrez (2017), teachers who have developed Political Conocimiento are able to deconstruct deficit narratives around students.

As Charlotte and Carrie began to acknowledge their whiteness, they engaged in deeper reflections on the role that race plays in their classrooms. Carrie, for example, noted that her black students consistently showed up late or in some cases didn't show up. In the following quote, she starts to wonder about the meaning of what she sees from a point of curiosity:

> I have all of my [white] students show up on time, and then 5, 10, 20 minutes later I get a sea of black students that show up late every single day, and I think that is one of the most obvious differences in my [challenging] class. I don't know what I think about that, but it's interesting to me. And then, I have a few perpetual students who just don't come, and they are all students of color the ones that I never see.

Carrie engages in a discourse that challenges colorblindness. For the first time, she was able to articulate the racialized experiences and behaviors of the students in her "challenging" classroom. By acknowledging that there is a difference in behavior between her white and black students, she is recognizing that race does play a role in her classroom. This utterance shows a departure from a colorblind approach that would ignore any racial disparities.

Charlotte, also notices the racialized experiences of students, but from another point of view:

> I think at our school our white kids have opportunities such as tutoring. I don't think a lot of my other kids ..it's not something that they have. I think they [white students] just have more access to tools to feel more confident and to feel better at math. But they're not actually better at math.

We note several points in the utterance. First, Charlotte continues to name white students explicitly, while using "other" to refer to students of color. In this way, Charlotte was still engaging in colorblind discourses. At the same time, she showed awareness of the privileges of her white students that BIPOC students did not have. Importantly, for Charlotte, this privilege is reflected in opportunities and resources that enable white students to "feel more confident and to feel better at math." The word "feel" here is critical, because she is not arguing that the resources do produce these differences. The final utterance, "But they're not actually better at math", reinforces this interpretation, and signals that Charlotte is disrupting the discourse around mathematical identity, mathematical performance, and ability.

We argue that the critical discourses of both teachers around positionality and race show Political Conocimiento. For Gutierrez (2017) signs of Political Conocimiento development include: “the ability to deconstruct competing messages about concepts like equity, mathematics, and learning that circulate in society; consideration of not just one-to-one interactions but historical, systemic, and institutional aspects of schooling that affect particular students..” (p. 29)

Carrie’s discourse demonstrates an understanding of how her positionality as a white woman can impact her students, she acknowledges that there is a role her whiteness plays in her classroom. This awareness brings her to deconstruct her mathematics classroom from a neutral space to one mediated by race. She is also aware of the racial differences in her classroom, such as the different patterns of lateness among her students. This may indicate that she is aware of systemic and institutional factors that can impact student learning. Similarly, Charlotte’s discourse shows an awareness of the historical and systemic lack of representation of students of color in her school. She starts to question institutional aspects of her school such as the overrepresentation of white teachers as something that affects her students of color. She also recognizes the unequal opportunities and resources available to different groups of students, indicating an understanding of the institutional factors that can impact student learning.

Both Charlotte and Carrie demonstrated signs of Political Conocimiento development, including the ability to deconstruct competing messages about equity, mathematics, and learning, and an awareness of systemic, and institutional aspects of schooling that affect students of color disproportionally.

Discussion and Conclusion

The aims of this paper were to illustrate dominant (discourse) practices around mathematics teaching represented in two white female teachers’ narratives, and the teachers’ trajectories towards politicized forms of mathematics knowing. The paper drew upon the concepts of Political Conocimiento for Teaching Mathematics (Gutierrez, 2013, 2017, 2021), dominant ideologies in mathematics education (Louie, 2018), and whiteness (Sleeter, 2017) to examine the shifts in teachers’ discourse through participation in a culturally responsive math professional development.

Our findings regarding the teachers’ discourses validate extant research that mathematics teachers tend to engage in dominant ideologies that center whiteness, sustain colorblindness, and equate whiteness with smartness (Gutierrez, 2017; Louie, 2018). Mathematics teachers from dominant backgrounds, such as Carrie and Charlotte, tend to reproduce dominant, deficit ideologies in their discourses and teaching practices centering white students as the norm and positioning BIPOC students as the “other”. The study also identifies *particular* deficit discourses that the teachers took up, such as colorblind distancing (Rubel, 2017) and white fragility, and the ways these discourses manifest in everyday talk about classroom life. Finally, the study also reinforces findings from Gutierrez et al. (2021) that teachers tend to be race-evasive about their own identities, and by confronting whiteness and questioning the role of race in the classrooms and school they can gain political knowledge. This research also informs policy and decision-making around professional development opportunities for mathematics teachers that focus on culturally responsive teaching and antiracist practices by highlighting distinctions between dominant discourses and ones that are aligned with antiracist ideologies.

References

Aguirre, J. M., & del Rosario Zavala, M. (2013). Making culturally responsive mathematics teaching explicit: A lesson analysis tool. Pedagogies: An international journal, 8(2), 163-190.

Bartell, T. (2011). Caring, race, culture, and power: A research synthesis toward supporting mathematics teachers in caring with awareness. Journal of Urban Mathematics Education, 4(1), 50-74.

Battey, D., & Leyva, L. A. (2018). Making the implicit explicit: Building a case for implicit racial attitudes to inform mathematics education research. Toward equity and social justice in mathematics education, 21-41.

Berry III, R. Q., Ellis, M., & Hughes, S. (2014). Examining a history of failed reforms and recent stories of success: Mathematics education and Black learners of mathematics in the United States. *Race Ethnicity and Education*, *17*(4), 540-568.

Bishop, A. J. (1990). Western mathematics: The secret weapon of cultural imperialism. Race & Class, 32(2), 51-65.

Daniels, J. R., & Varghese, M. (2020). Troubling practice: Exploring the relationship between Whiteness and practice-based teacher education in considering a raciolinguicized teacher subjectivity. *Educational Researcher*, *49*(1), 56-63.

de Freitas, E. (2008). Troubling teacher identity: Preparing mathematics teachers to teach for diversity. *Teaching education*, *19*(1), 43-55.

Freire, P. (1996). Pedagogy of the oppressed (revised). *New York: Continuum*, *356*, 357-358.

Gutiérrez, R. (2013). Why (urban) mathematics teachers need political knowledge. *Journal of Urban Mathematics Education*, *6*(2).

Gutierrez, R. (2017). Political Conocimiento for teaching mathematics: Why teachers need it and how to develop it. S. E. Kastberg, A.M. Tyminski, A. E. Lischka, & W. B. Sanchez (Eds.). (2017). Building support for scholarly practices in mathematics methods. (11-37).

Gutiérrez, R., Myers, M., & Kokka, K. (2021). Political Conocimiento in teaching mathematics: Intersectional identities as springboards and roadblocks. *Exploring new ways to connect*, 507.

Kumashiro, K. K. (2013). Against common sense: Teaching and learning toward social justice. Routledge.

Leonard, J., Brooks, W., Barnes-Johnson, J., & Berry III, R. Q. (2010). The nuances and complexities of teaching mathematics for cultural relevance and social justice. Journal of Teacher Education, 61(3), 261-270.

Louie, N. L. (2017). The culture of exclusion in mathematics education and its persistence in equity-oriented teaching. Journal for Research in Mathematics Education, 48(5), 488-519.

Louie, N. L. (2018). Culture and ideology in mathematics teacher noticing. *Educational Studies in Mathematics*, *97*, 55-69.

Martin, D. B. (2013). Race, racial projects, and mathematics education. *Journal for Research in Mathematics Education*, *44*(1), 316-333.

Matias, C. E., & Zembylas, M. (2014). 'When saying you care is not really caring': Emotions of disgust, whiteness ideology, and teacher education. *Critical Studies in Education*, *55*(3), 319-337.

Philip, T. M., Souto-Manning, M., Anderson, L., Horn, I., J. Carter Andrews, D., Stillman, J., & Varghese, M. (2019). Making justice peripheral by constructing practice as "core": How the increasing prominence of core practices challenges teacher education. Journal of Teacher Education, 70(3), 251-264.

Rubel, L. H. (2017). Equity-directed instructional practices: Beyond the dominant perspective. *Journal of Urban Mathematics Education*, *10*(2).

Shah, N., & Coles, J. A. (2020). Preparing teachers to notice race in classrooms: Contextualizing the competencies of preservice teachers with antiracist inclinations. *Journal of Teacher Education*, *71*(5), 584-599.

Sleeter, C. E. (2017). Critical race theory and the whiteness of teacher education. *Urban Education*, *52*(2), 155-169.

Yeh, C. (2018). Democratic accountability in the neoliberal era: The politics of teaching and teacher education in mathematics classrooms. *Policy Futures in Education*, *16*(6), 764-780.

EXHAUSTED, DRAINED, AND APATHETIC: AN EMERGING COACH'S EMOTIONAL TRAJECTORY DURING HER FIRST YEAR

Pavneet Kaur Bharaj
California State University Bakersfield
pavneet.k.bharaj87@gmail.com

Dionne Cross Francis
University of North Carolina at Chapel Hill
dicross@unc.edu

Kathryn Habib
University of North Carolina at Chapel Hill
kaldrich@unc.edu

Anna Hinden
University of North Carolina at Chapel Hill
annarh@unc.edu

Anna Gustaveson
University of North Carolina at Chapel Hill
annagus@unc.edu

While instructional coaching can support teacher improvement and student learning, their effectiveness and longevity in the role may be influenced by teachers' emotions, the quality of the teacher-coach relationship, as well as the cognitive and emotional climate of the school. In this study, we analyze the emotional experiences of a novice mathematics instructional coach at multiple time points over one school year, through 14 reflective interviews. We categorize the coach's descriptions of her emotions as positive, negative, and neutral and describe themes that explain why she felt the way she did in her experiences. Our analysis points to three major reasons underlying her emotional trajectory which ultimately led to her decision to discontinue her role as a coach: (a) unmet expectations (b) student learning, and (c) teacher engagement.

Keywords: Professional Development, Affect, Emotions, Beliefs, and Attitudes

Introduction

Instructional coaching is a promising way to support teachers in improving their instruction and content knowledge (Campbell & Griffin, 2017; Kraft & Hill, 2020), and has been shown to support student learning (Campbell & Malkus, 2011; Kraft et al., 2018). Thus, the needs of instructional coaches are important to consider. The work of instructional coaches has been framed in a range of ways because these roles are often designed to serve the needs of the school or district, in addition to teachers. Coaches often function in leadership roles but are distinct from other school leaders because they are also closely connected to teachers' work (Wenner & Campbell, 2017). In many cases, teacher-coach interactions significantly advance teachers' thinking and practices as they provide support in a safe, non-evaluative way (Costa & Garmston, 2016). However, how the coach is positioned within the school can influence the quality of the relationships they develop with teachers and the extent to which they can establish trust and rapport (Comstock & Margolis, 2021). Straddling the boundary between being affiliated with leadership and building trust with teachers is a potentially emotionally challenging aspect of coaching, which may ultimately contribute to decreased effectiveness or attrition. Given the dearth of research on coaches' emotional experiences, in this study, we explored the experiences of an emerging coach during her first year to understand how they influenced her emotional well-being and decisions about continuing in her role as coach. We focused on answering the following research questions: (i) What is the trajectory of emotions of an emerging math coach during her first year? (ii) What factors influence this emotional trajectory? (iii) In what ways do these emotions contribute to her career decision-making?

Teachers' and Coaches' Emotions

When considering the complexities of coaching in a school setting, it's important to consider the multiple layers of emotional interactions that can ultimately influence teaching and learning. We frame this study by examining the impact of teacher emotions, teacher engagement, and the overall emotional climate of the school on a coach's emotional well-being. Our definition of emotions is guided by prior work as "socially constructed, personally enacted ways of being that emerge from conscious and/or unconscious judgments regarding perceived successes at attaining goals or maintaining standards or beliefs during transactions as part of social-historical contexts" (Schutz et al., 2006, p. 344). Emotions are relational, meaning they are elicited through person-environment interactions based on how the person perceives progress towards achieving desired goals.

Teachers' emotions are elicited based on their perceptions of how they are progressing towards achieving their instructional or other teacher-related goals (Schutz & Zembylas, 2009). They are influenced by various factors, including interactions with students (Chaves, 2009; Hargreaves, 2000; Sutton, 2003), student learning (Reyna and Weiner, 2001; Russo et al., 2020, Trigwell, 2012), level of autonomy in their teaching (Lee, 2005; Russo et al., 2020), and contextual factors such as lack of support (Van Veen et al., 2005; Chaves, 2009) or time (Van Veen et al., 2005). Emotions have also been found to play a significant role in the interactions between teachers and instructional coaches (Cross Francis et al., 2021). Similar to teachers, goal setting is a key aspect of coaches' work. They too, set goals that are teacher-, student- and/or school-focused and experience emotions as the judge how well events unfolding daily are helping them make progress towards those goals. To be effective, it is important that coaches support teachers in regulating their emotions in ways that bode well for student learning, while also regulating their own emotions as they manage the complexities of their own work. By acknowledging the emotional aspect of their work and engaging in discussions about emotions, coaches and teachers can understand and support each other's experiences, as well as develop a shared vision for teaching and learning (Hunt, 2016). When difficult emotions arise in the context of coaching interactions, coaches' engagement with these emotions can support teachers' change toward effective practices (McGugan et al., 2023). By acknowledging, addressing, and leveraging their emotions, both coaches and teachers can contribute to a more supportive and productive educational environment.

The multi-faceted nature of coaches' roles may also play a part in their emotional experience. Coaches often work with multiple teachers within a school or across schools. Although not an explicit requirement of their role, coaches are integral in maintaining both a positive academic and emotional climate within schools by supporting teachers' efficacy and engagement (Lobaugh, 2022). Additionally, coaches must navigate their relationships with administrators and communicate expectations from school leadership, while also building trust and rapport with teachers. Efforts to meet these implicit expectations can be a physically and emotionally burdensome endeavor.

Methods

Participant

Arabella was a first-year instructional math coach in an urban K-5 Title I elementary school. She was supported by a colleague who was an experienced coach and had been the instructional literacy coach for six years. She was concurrently working on her doctoral degree at a large public university in the United States. Prior to being the math coach, she worked as a math interventionist for Grades 4-5. As a coach, she was required to support core curriculum implementation, including 1:1 coaching cycles, facilitating professional learning teams, co-planning, analyzing data, and providing professional development.

Data Collection

Semi-structured interviews were the main data source. There were two anchor interviews, a pre-interview with Arabella before she began coaching and a post-interview after 14 reflection sessions at the end of her coaching role. During the coaching experience, the research team conducted bi-weekly reflection sessions with her to hear about her experiences, reflections, and the events that occurred between sessions. For this paper, we focused on two questions from these reflections that targeted her emotions: What were your primary emotions during these past couple days while working as a math coach? What were the reasons for these emotions?

Analysis

We identified the statements that Arabella shared in relation to her emotions. We organized the data in a table with the stated emotions in one column and the accompanying emotion reason in an adjoining column. The research team read each stated emotion and coded it in terms of valence – positive, negative, and neutral – referring to the pleasant and unpleasant multi-component responses to external stimuli respectively. This is displayed visually in Figure 1. We identified the responses provided in the pre- and post-interviews to capture the emotions at distinct time points – prior to the start of her coaching role and at the end of the year – and the reasons underlying the emotions. Then we focused on understanding the trends of emotions across the reflective interviews to document how events unfolding within the school, and with teachers and students impacted the coach's emotional experiences. We read and coded each emotion reason using a code that captured the essence of what Arabella was trying to communicate. Then, we organize the codes grouping those with similar meanings into themes (Braun & Clarke, 2006). In what follows, we describe the emotions and reasons that she experienced these emotions at the two distinct time points. We also describe the themes that summarize the reasons underlying the emotions she experienced over the year.

Findings

We describe Arabella's emotional experience at the beginning and end of her coaching role. Then we discuss how her emotional experiences changed across the 14 reflective interviews, and her emotions after deciding to leave her position as a coach.

Emotions at the beginning and end of the coaching role

Before starting her work with teachers, Arabella was eager, excited, and grateful to start her role as a coach because she had worked hard to get this position. She stated her feelings and described why she wanted to become a coach.

> I feel like I've worked hard for this position as well. Like this is something that I've wanted for a couple years now. And I knew I had to go to grad school for it and even after I went to grad school for it, I still like there wasn't an opportunity and I didn't necessarily like really

seek it out … so it's this is something that I've wanted to do, and now that I'm in it, I'm just like really loving it. And so, I think just like eagerness and fun and exciting like every day.

I enjoy working with the teachers, I enjoy being in their classrooms. I enjoy when they come to me and ask me questions. I just enjoy seeing their classes...be successful.

Arabella wanted to be a coach. She worked hard to achieve the qualifications for the position and had exercised patience. Feelings of *enjoyment* seemed to come from the fact that she loved working with teachers, providing them with effective support, and positively influencing students through that support. Arabella also mentioned that she was anticipating feeling some *disappointment* in cases where she might not be able to help teachers. She was looking forward to functioning as a "good support" for teachers to help ensure that they "feel successful".

At the end of the year, when asked about her emotions related to the coaching experience in its entirety, Arabella described being *exhausted, frustrated, challenged,* and *confused.* She explained the reasons for these emotions,

I think that was just really frustrating not feeling impactful. The lack of accountability. All that was frustrating day to day. I'm exhausted in multiple ways, like physically just like getting up every morning at 6 am being here all day. And then going home and like just feeling totally burnt out. than being surrounded by people who were feeling burnt out. Just all very exhausting. [I am also] I'm hopeful because there were times where like, I did get to do coaching cycles with the teachers and there were teachers who, you know, were working with me and and you found the value in talking to me and working with me.

Though there were multiple negative emotions that she shared, she still sounded *hopeful* and *enjoyed* it when she did have opportunities to engage in the work of coaching, particularly when she was able to directly work with teachers and build relationships.

To understand how Arabella's emotions changed over time we explored the nature of the transitionary emotions she experienced from the reflective interviews and organized the reasons for these emotions in themes.

Trajectory of Emotions

Figure 1 shows Arabella's emotions, categorized as positive, negative, or neutral, over the first year in the role. It shows that over the course of the first year Arabella experienced a range of positive, negative, and neutral emotions; however, she consistently experienced negative emotions following reflective interview 6.

Positive	X		X	X	X		X	X	X	X		X			X	X
Neutral		X	X						X							
Negative			X	X	X	X		X	X	X	X	X	X	X	X	X
	Pre	RI 1	RI 2	RI 3	RI 4	RI 5	RI 6	RI 7	RI 8	RI 9	RI 10	RI 11	RI 12	RI 13	RI 14	Post

Figure 1: Trajectory of Emotions Across Reflective Interviews (RIs)

Unmet Expectations

During the initial days of her role as a coach, Arabella experienced ***neutral*** emotions. She noticed that teachers in the school appeared to have a depressed mood (as they were transitioning back to in-person learning after COVID) and were feeling overwhelmed by the increased expectations they were experiencing. Though these negative emotions (e.g., burn-out) were not Arabella's emotions, they dampened her excitement about her new role. Though she had started her position on a positive note with a zeal to help teachers, during her second through fifth reflective interviews, Arabella expressed some **negative emotions**, which she described as a *feeling of being lost.* This reflected her struggle to understand the purpose of her role as a coach. She mentioned during the second week's reflection,

> I was feeling like, I'm just kind of like lost, what is my purpose? When I started this, I was so excited. But now I feel it's been hard for teachers coming back from COVID. And they've been feeling so drained. I think drained and burnt out would be the words they would use...so I think I've been like, well, how do I help? Basically, it's like, what is my purpose here? Like, because I want to coach, I want to help but like, that's not where we're at. Right? What I can do to help at that moment is like, okay, we can, you know, work on this math problem, We can have conversations, like, it's not the typical coaching cycles. That's not, that's not how it works.

Arabella's emotions started to mirror those of the teachers within the first few months in her role. She described experiencing *stress, exhaustion,* and *irritation* due to *stress on teachers* and recognizing that her school's administration considered improving students' standardized test scores to be a major part of her role. Specifically, during her fifth reflection, she described feeling irritated because her role required her to wear multiple hats and perform at distinct levels, (i.e., ensuring that students' scores improved) which took away time she wanted to spend on her actual job role - which she saw as coaching and working with teachers.

Student Learning

Arabella reported **positive emotions** throughout the sixth to eighth reflective interviews. She had introduced EOG Fridays with 3rd-5th grade teachers. Every other Friday, students would take a short standards-based assessment to help teachers gather data to analyze student learning to help them prepare for their upcoming standardized tests. EOG Fridays was received well by some teachers while other teachers pushed back. They intended to be encouraging and motivating and students earned rewards for growth and proficiency. Students enjoyed participating so they continued to the end of the year. During this time, she expressed emotions like – excitement, hopeful, feeling good, and contentment. Arabella proposed the idea for EOG Fridays based on her prior experiences with students. The leadership accepted the idea and following its implementation they observed an improvement in test scores. Although implementing this plan seemed tangential to what she perceived was her coach role, she felt acknowledged within the leadership team, and was pleased to see that her hard work seemed to improve students' scores. Soon after, during her ninth reflection, Arabella experienced **a mix of positive and negative emotions.** On one hand, she was excited as she found that students had scored well after implementing the new practice of EOG Friday in the school. She shared,

> I'm excited I think that the math data the trajectory from this past benchmark was like pretty good. We saw a lot of growth. I think overall the math teachers felt pretty good about it. That was exciting. The students felt good about it so definitely excited.

Contrarily, she reported feeling frustrated because of the behavior of some teachers in the school. She was feeling some tension with some of the teachers who remained distant and not open to engaging in coaching cycles.

Teacher Engagement

The residue of these **negative emotions** carried forward in Arabella's experience for the next four reflection interviews (tenth to thirteenth interviews). During this time, she reported emotions like tired, frustrated, drained, exhausted, and apathetic. The primary reason she associated with these emotions was the lack of teacher engagement she was encountering. She started feeling animosity with one teacher in particular, whose behavior she interpreted as resistant to engage in coaching aimed at improving her instruction. This created a toxic environment which she felt made it difficult for her to function effectively as a coach. She explained her feelings as follows:

> I think tired, exhausted, drained, I don't know if drained is an emotion, but I think the negativity and the toxicity of some of the attitudes are really hard to like deal with and so I think just like tired. I was just like...exhausting to deal with.

Although Arabella focused on her interactions with one teacher in this statement, she noted that the experience with this teacher was reflective of her broader feelings of disconnection in her interactions with teachers at the school in general. The negative emotions that Arabella began to express in the second interview persisted for most of the school year, with only sporadic positive emotions, and this ultimately seemed to inform Arabella's decision to actively seek other job opportunities at the end of the school year. After making this decision (by the fourteenth interview), she re-focused on the positive aspects of her role. She reported feeling hopeful and excited about two teachers who were willing to work with her, and about the continued improvements in students' test scores, ending the school year on a positive note.

Discussion and Significance of this Study

This study's findings highlight the transitory nature of the emotions of an emerging math coach during her first year and point to teachers' emotions, coach-teacher interactions, and the emotional climate of the school as key factors that influenced a novice coach's emotional trajectory and her ultimate decision to leave her role as coach. After beginning her new role with optimism and excitement, Arabella experienced increasingly negative emotions related to the challenges of coaching, followed by a brief improvement in her emotional well-being toward the end of the school year, which can be attributed to her sense of relief at having made the decision to leave her position. As the school year progressed and challenging learning conditions persisted because of the COVID 19 pandemic, Arabella described the way in which teachers' feelings of overwhelm negatively influenced her own emotional state, noting the uncertainty and anxiety that resulted from not knowing how she could support teachers. The nature of her interactions with teachers also contributed to Arabella's emotional decline, as she described feeling exhausted by teachers' resistance to improving their instruction through coaching. The overall emotional climate of the school was a third factor to which Arabella attributed her negative emotional trajectory as the year wore on. The expectations placed on Arabella to increase test scores while trying to build relationships with overwhelmed teachers resulted in increased levels of stress. Arabella was bolstered throughout the year by continued improvements in students' standardized test scores, which she felt was an indication that despite the challenges she faced in working with teachers, she was contributing positively to student learning. Ultimately, however, this was not enough to counteract the heightened negative emotional experience resulting from teachers' emotions, challenging teacher-coach interactions and the overall school climate, finally leading to her decision to leave her coaching role.

The trajectory of negative emotions we observed in this study extends the literature which documents the relationship between teachers' and students' emotions (e.g., Chaves 2009; Hargreaves, 2000; Sutton, 2003), and between student learning and teachers' emotions (e.g., Reyna and Weiner, 2001; Russo et al., 2020), by demonstrating existing commonalities in the relationship between teachers' and coaches' emotions. Arabella's experience illustrates that coaches' emotions are influenced by factors including teachers' emotions, teacher (dis) engagement, student learning outcomes, and the overall cognitive and emotional climate of the school. Similarly, Arabella reported feelings of excitement and hope in connection with teachers' engagement and growth, suggesting that like teachers, coaches experience positive emotions when they see the learners in their charge developing positively (i.e., teachers growing professionally in ways that support student learning). By illustrating the influence of teachers' emotions and engagement on coaches' emotions, these findings add to the existing literature (e.g., Hunt, 2016) on the role of emotions in instructional coaching. Additionally, this study provides insight into the emotional challenges that novice instructional coaches may face and extends what we know about the emotionality of coach-teacher relationships (McGugan et al., 2023). We also noted that many of the words used by the coach were not emotion words as described in the literature; instead, many described the physical state of their bodies. This aligns with existing work (Cross Francis et al., 2019, Cross Francis et al., 2020) but also emphasizes that coaching can take both an emotional and physical toll.

Hiring and retaining effective coaches requires a significant investment of resources and time on the part of schools and districts, and our results highlight the importance of supporting coaches as they manage the emotions they may experience while engaging with teachers and supporting schools. Based on the work coaches do, engaging with human beings in contexts that are challenging and complex, it is inevitable they have to contend with the emotions of others and their own. Thus, it is essential that coaches acknowledge the emotional and affective terrain of teachers and learn to understand and support individuals in regulating their emotions (Cross Francis et al., 2019). To prevent burnout and support the emotional well-being of instructional coaches, it's important for schools to prioritize building a positive emotional culture, provide ongoing support and professional development, and ensure that coaches have clear expectations and boundaries. By creating a supportive environment, instructional coaches are more likely to thrive in their roles and make a positive impact on student learning.

References

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. Qualitative Research in Psychology, 3(2), 77–101. https://doi.org/10.1191/1478088706qp063oa

Campbell, P. F., & Griffin, M. J. (2017). Reflections on the promise and complexity of mathematics coaching. The Journal of Mathematical Behavior, 46, 163–176. https://doi.org/10.1016/j.jmathb.2016.12.007

Campbell, P. F., & Malkus, N. N. (2011). The impact of elementary mathematics coaches on student achievement. The Elementary School Journal, 111(3), 430–454. https://doi.org/10.1086/657654

Chaves (2009). "Teachers' feelings towards indiscipline of adolescent students in elementary education," in Affectivity & Learning: Contributions from Henri Wallon, L. R. Almeida and A. A. Mahoney (São Paulo: Loyola), 151–173.

Comstock, M., & Margolis, J. (2021). "Tearing Down the Wall": Making Sense of Teacher Leaders as Instructional Coaches and Evaluators. Journal of school leadership, 31(4), 297-317.

Costa, A. L., & Garmston, R. J. (2015). Cognitive coaching: Developing self-directed leaders and learners. Rowman & Littlefield.

Cross Francis, D., Liu, J., Bharaj, P. K., & Eker, A. (2019). Integrating Social-Emotional and Academic Development in Teachers' Approaches to Educating Students. Policy Insights from the Behavioral and Brain Sciences, 6(2), 138–146. https://doi-org.libproxy.lib.unc.edu/10.1177/2372732219864375

Cross Francis, D. I., Hong, J., Liu, J., Eker, A., Lloyd, K., Bharaj, P. K., & Jeon, M. (2020). The Dominance of Blended Emotions: A Qualitative Study of Elementary Teachers' Emotions Related to Mathematics Teaching. Frontiers in Psychology, 11, 1865. https://doi.org/10.3389/fpsyg.2020.01865

Cross Francis, D., Hong, J., Liu. J., Eker, A., Bharaj, P. K., & Jeon, M. (2021 July). Examining teachers' emotional experiences through the process of mathematical instructional change. In Proceedings of the Fourteenth International Conference on Mathematics Education. Shanghai, China.

Hargreaves, A. (2000). Mixed emotions: Teachers' perceptions of their interactions with students. Teaching and Teacher Education, 16(8), 811–826. https://doi.org/10.1016/S0742- 051X(00)00028-7

Hunt, C. S. (2016). Getting to the heart of the matter: Discursive negotiations of emotions within literacy coaching interactions. Teaching and Teacher Education, 60, 331-343.

Kraft, M. A., Blazar, D., & Hogan, D. (2018). The effect of teacher coaching on instruction and achievement: A meta-analysis of the causal evidence. Review of Educational Research, 88(4), 547–588. https://doi.org/10.3102/0034654318759268

Kraft, M. A., & Hill, H. C. (2020). Developing ambitious mathematics instruction through web-based coaching: A randomized field trial. American Educational Research Journal, 57(6), 2378–2414. https://doi.org/10.3102/0002831220916840

Lee, H. J. (2005). Understanding and assessing preservice teachers' reflective thinking. Teaching and teacher education, 21(6), 699-715.

Lobaugh, S. S. (2022). Instructional coaching: The perceived relationship between instructional coaching, efficacy, and school climate (Order No. 29061204). Available from ProQuest Dissertations & Theses Global. (2656808283). Retrieved from http://libproxy.lib.unc.edu/login?url=https://www.proquest. com/dissertations-theses/instructional-coaching-perceived-relationship/docview/2656808283/se-2

McGugan, K. S., Horn, I. S., Garner, B., & Marshall, S. A. (2023). "Even when it was hard, you pushed us to improve": Emotions and teacher learning in coaching conversations. Teaching and Teacher Education, 121, 103934.

Reyna, C., & Weiner, B. (2001). Justice and utility in the classroom: An attributional analysis of the goals of teachers' punishment and intervention strategies. Journal of Educational Psychology, 93(2), 309-319. https://doi.org/10.1037/0022-0663.93.2.309

Russo, J., Bobis, J., Sullivan, P., Downton, A., Livy, S., McCormick, M., & Hughes, S. (2020). Exploring the relationship between teacher enjoyment of mathematics, their attitudes towards student struggle and instructional time amongst early years primary teachers. Teaching and Teacher Education, 88, 102983.

Schutz, P. A., Hong, J. Y., Cross, D. I., & Osbon, J. N. (2006). Reflections on investigating emotion in educational activity settings. Educational psychology review, 18, 343-360.

Schutz, P. A., & Zembylas, M. (2009). Advances in Teacher Emotion Research: The Impact on Teachers' Lives. NY: Springer. https://doi.org/10.1007/978-1-4419-0564-2

Sutton, R. E., and Wheatley, K. F. (2003). Teachers' emotions and teaching: a review of the literature and directions for future research. Educ. Psychol. Rev. 15, 327–358.

Trigwell, K. (2012). Relations between teachers' emotions in teaching and their approaches to teaching in higher education. Instructional Science, 40(3), 607– 621. https://doi.org/10.1007/s11251-011-9192-3

Van Veen, K., Sleegers, P., & Van de Ven, P. H. (2005). One teacher's identity, emotions, and commitment to change: A case study into the cognitive–affective processes of a secondary school teacher in the context of reforms. Teaching and teacher education, 21(8), 917-934

Wenner, J. A., & Campbell, T. (2017). The theoretical and empirical basis of teacher leadership. Review of Educational Research, 87(1), 134–171. https://doi.org/10.3102/0034654316653478

EXPLORING CHANGES IN MATHEMATICS TEACHER PRACTICE FROM PROFESSIONAL DEVELOPMENT ROOTED IN THE TRU FRAMEWORK

Victoria Bonaccorso
Montclair State University
bonaccorsov@montclair.edu

Helene Leonard
Montclair State University
leonardh2@montclair.edu

Amy Daniel
Montclair State University
daniela4@montclair.edu

Youngjun Kim
Montclair State University
kimy8@montclair.edu

Joseph DiNapoli
Montclair State University
dinapolij@montclair.edu

Equitable and accessible classrooms should engage all learners with mathematics content in meaningful ways. However, practicing teachers need support from professional development (PD) to learn to teach with this ambitious vision. Informed by sociocultural theory, we employed an evaluative case study methodology to describe, explain, and assess the experiences of one middle school mathematics teacher's longitudinal participation in a continuous PD model focused on the Teaching for Robust Understanding (TRU) framework. Based on classroom observations and interview data, our findings show evidence of TRU-aligned changes in teaching practice as a result of years of participation in the PD model. These findings strengthen the call for PD programs focused on equity and access and suggest design elements of such PD to support effectiveness.

Keywords: Professional Development, Instructional Vision, Instructional Activities and Practices

Recent reform movements have framed engaging all learners as an issue of equity and access to mathematics content (Larson, 2017). Teachers must create a classroom culture that empowers students to actively participate in mathematics lessons through productive struggle, collaboration, and explaining their reasoning, in contrast to lecture-based pedagogies (Baldinger & Louie, 2014; Porter et al., 2011). Thus, there is a strong need for professional development (PD) to support instructional shifts that help teachers develop practices aligned to this ambitious vision (Gallagher, 2016; Rosli & Aliwee, 2021; Sztajn et al., 2017). In this paper, we report on the changes in one middle school mathematics teacher's classroom practice based on her participation in a professional learning community (PLC) within a PD model (AIM-TRU) focused on creating engaging and powerful mathematics classrooms. This case study addresses the research question guiding our work: How does participation in a community of practice centered on the collective investigation of video cases grounded in high-quality instructional materials impact teachers' use of these materials and practice?

Theoretical Perspective and Review of Literature

Our work is framed by sociocultural theory in the ways we have studied the engagement of a PLC of mathematics teachers in PD and the impacts of those experiences on one middle school mathematics teacher's classroom practice. The following sections will describe socioculturalism and communities of practice (CoPs), tenets and examples of effective professional development, the Teaching for Robust Understanding (TRU) framework, and Formative Assessment Lessons (FALs). Literature in these areas helps preview the AIM-TRU PD model and its impacts on a middle school mathematics teacher's practice that we will report in this paper.

Sociocultural Theory and Communities of Practice

Sociocultural theory claims that learning and the activities, contexts, and cultures in which it takes place are inseparable (Brown et al., 1989; Collins et al., 1988; Lave & Wenger, 1991). Within socioculturalism, CoPs are groups of people who mutually engage in an activity, are connected by a joint enterprise, and engage with a shared repertoire of resources (Wenger, 1998, 1999). Within a CoP, evidence of learning can occur through changes in participation and reification. In teaching, the changes in the practice of one teacher in a PLC can demonstrate that individual's learning while engaging with a CoP through participation in the PD program.

Effective Professional Development

To support teachers' instructional shifts toward creating engaging and powerful mathematics classrooms, effective PD should be centered on coherent mathematics content, have sustained duration, and involve teachers in collective and active participation (Garet et al., 2001). Also, effective PD should be explicitly connected to classroom lessons to help facilitate changes in teaching practice (Desimone & Garet, 2015). When a PD program incorporates these features of effectiveness, research has shown that teachers can alter their instructional practice toward ambitious standards, develop their content and pedagogical knowledge, form productive beliefs about engaging all learners, and better support learning in their classrooms (Desimone, 2011).

PD research has also shown that PLCs focused on video case study can help teachers productively shift classroom practices. Borko et al. (2008) found that when a PLC of teachers was actively engaged with familiar mathematics lessons and collaborated to understand students' solution strategies in videos, they developed specialized content knowledge and pedagogical content knowledge (see Ball et al., 2008). This PD was also effective in helping teachers change their formative assessment practices in their classrooms. Santagata and Bray (2016) found that video-based error analysis helped teachers in a PLC develop new lesson-planning practices. Teachers started incorporating anticipations and responses to common student misconceptions about big mathematical ideas into lessons, providing more opportunities for student discourse during class. van Es and Sherin (2008) found that a PLC video club focused on professional noticing in teachers' classrooms helped them interpret and examine classroom interactions in new ways, which informed the teachers' implementation of a reform mathematics curriculum. In our own work, changes in participation and reification of TRU concepts were found to be associated with video case analysis in the AIM-TRU PD model (Leonard et al., 2022).

Teaching for Robust Understanding via Formative Assessment Lessons

Imperative to a CoP engaged in PD is the development of a shared repertoire built around best practices. In the context of the AIM-TRU PD model, this shared repertoire consists of the TRU framework and FALs. The TRU framework outlines an ambitious vision of mathematics classrooms that create engaging and equitable learning environments to support all students in becoming independent mathematical thinkers (Schoenfeld, 2015). The TRU framework details five interrelated dimensions: The Mathematics; Cognitive Demand (CD); Equitable Access (EA); Agency, Ownership, and Identity (AOI); and Formative Assessment (FA).

The Mathematics dimension refers to the rich, coherent mathematical content that forms the foundation of powerful mathematics classrooms (National Council of Teachers of Mathematics, 2000; National Governors Association, 2010; Schmidt et al., 2005). Engaging students with such content through mathematical tasks requiring high levels of *CD* with appropriate scaffolds creates the opportunity for the productive struggle necessary for developing conceptual understanding (Hiebert & Grouws, 2007). Such tasks are associated with improved opportunities to learn (Jackson et al., 2013; Stein et al., 1996), higher student achievement (Boaler & Staples, 2008; Stigler & Hiebert, 2004), and the development of sophisticated solution strategies (Downton & Sullivan, 2017).

The *EA* dimension highlights instructional practices that can meaningfully engage all students with rich mathematical content. For example, selecting tasks with multiple entry points and solution strategies provides students with different ways of connecting their prior knowledge to new content, thus positioning all students as capable doers of mathematics (Boaler, 2016; Hodge & Cobb, 2019). *AOI* refers to the extent to which students are positioned with agency as creators of mathematical knowledge in the classroom, rather than passive recipients (Engle & Conant, 2002). When teachers establish classroom norms wherein students are responsible for making mathematical arguments and for evaluating the validity of those made by their peers, students are more likely to identify themselves as mathematically competent (Cobb et al., 2009). The use of *FA* in the classroom to elicit student thinking in order to inform instruction and provide feedback has been connected with improved student learning outcomes (Andersson & Palm, 2017; Black & Wiliam, 1998). In contrast to performance-based summative assessments (e.g., tests and quizzes), FA practices can foster intrinsic motivation for learning (Shepard, 2000), and can encourage students' development of a growth mindset and metacognitive habits (Darling-Hammond et al., 2020; Granberg et al., 2021).

Related to the TRU framework are FALs, which are high-quality, research-based lessons developed by Schoenfeld and his team to support teachers in creating TRU-aligned classroom experiences. Designed to be incorporated into existing curricula, each FAL includes structures to support teachers in formatively assessing student thinking (e.g., pre-assessment tasks) and in providing access to the mathematics for all students (e.g., a whole-class introduction wherein teachers can preview the context of the lesson). FALs also include high-CD small-group tasks designed to support students' AOI as they collaboratively construct mathematical knowledge.

Methodology

We used an evaluative case study methodology (Merriam, 1998) to describe, explain, and assess the experiences of one middle school mathematics teacher's longitudinal participation in a continuous PD model focused on creating engaging and powerful mathematics classrooms. This case study is part of a larger, multi-year PD research project involving over 150 teachers spanning multiple regions in the United States. To gain insight into learning within this CoP, we chose Ms. Chaves (pseudonym), a middle school teacher, to be the focus of our case study because of her active involvement in the AIM-TRU PD model as both a participant and facilitator throughout the four years of this study.

Ms. Chaves

At the beginning of her participation in the AIM-TRU PD model, Ms. Chaves had nine years of middle school mathematics teaching experience in a suburban district in the northeast United States, teaching primarily Math 8 and Algebra 1 classes. For the first year of her participation in this PD model, Ms. Chaves engaged as a teacher-participant in a PLC. Since then, she has led a PLC as a facilitator-participant. Prior to her involvement in this PD model, she had engaged in various PDs about classroom practices.

The AIM-TRU PD Model

The AIM-TRU PD model engages middle and high school mathematics teachers in a collaborative investigation of FALs and their enactment to deepen instructional knowledge and support shifts in practice aligned to the TRU framework. Grounded in tenets of effective PD (e.g., Desimone & Garet, 2015; Garet et al., 2001), this PD model provides opportunities for teachers and facilitators to collectively generate professional knowledge for teaching and learning mathematics using the dimensions of ambitious instruction that are necessary and sufficient to produce equitable environments supporting deep learning opportunities for all students (Schoenfeld & the TRU Project, 2016). This PD model focuses on the following components: (a) unpacking the big mathematical ideas in a TRU-aligned FAL, (b) making observations about video cases demonstrating students' mathematical thinking while engaging in FALs, and (c) sets of video case reflective discussion questions based on the TRU framework (see Figure 1). During the reflective discussion of the video analysis, teachers often co-construct understandings about TRU-aligned teaching practices through dialogue (Leonard et al., 2022).

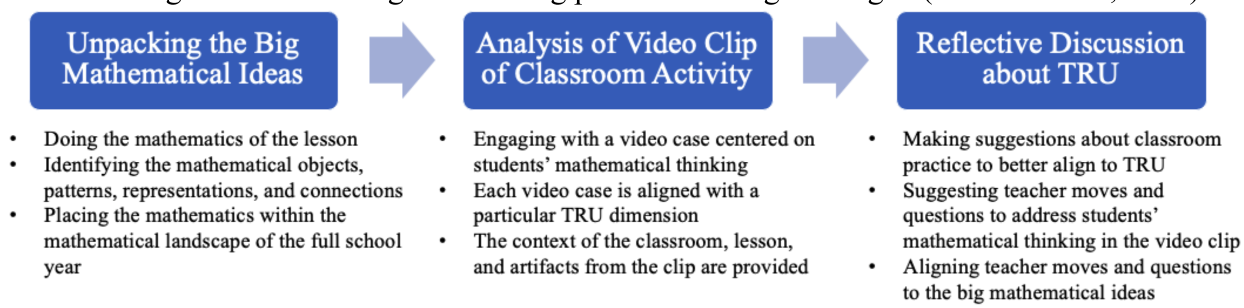


Figure 1. The AIM-TRU PD Model

Data Sources and Analysis

Data collected and analyzed for this case study included one classroom observation video from each Year 1 and Year 4 and a follow-up interview. We watched the classroom video data, segmented it by class activity structure (whole class, set-up, small group, etc.), and used thematic analysis to describe Ms. Chaves' teaching practices. We used an observation rubric (see Schoenfeld et al., 2014) to assess her classroom practices in Year 1 and Year 4 relative to the pedagogical TRU dimensions to assess the alignment of teaching practices with the TRU framework. On a 1-3 half-point scale, with 3 being the highest alignment, we assessed pedagogical alignment with the following guiding questions:

- CD: How are students supported in productive struggle?
- EA: How is access to the content supported for all students?
- AOI: How are students the source of ideas and discussion?
- FA: How is students' thinking surfaced and built upon?

We also conducted a semi-structured follow-up interview with Ms. Chaves to gather details about the practices captured in her classroom observation video data and to learn from her perspective on the impact that the AIM-TRU PD model had on changes to her practice.

Findings

One teacher's journey with the AIM-TRU PD model provides a window into the ways that teacher learning manifests in a CoP through participation and how that learning can be reified with changes in a teacher's practice. In Year 1, Ms. Chaves had little prior knowledge of the TRU framework, relied on lecture-based practices, was influenced by institutional expectations, and was hesitant to implement FALs with fidelity. After prolonged participation in this PD model, we found evidence of shifts in her classroom practices and implementation of FALs. Her exposure to and immersion in the TRU framework and our analysis of her teaching practices can help explain these shifts.

Year 1: Ms. Chaves' Classroom at the Start of the AIM-TRU PD Model

In this section, we describe Ms. Chaves' teaching practices and her implementation strategies for FALs during Year 1 of her involvement in the AIM-TRU PD model.

Classroom practices. In the video of Ms. Chaves' classroom from Year 1, she was observed facilitating a whole-class homework review. Students were expected to self-check solutions by comparing their answers to a posted key. Following the self-check, Ms. Chaves asked, "Does anyone have any questions?" Seeing none, she moved on to the next part of the lesson. During her interview, Ms. Chaves reflected on her motivations for homework:

> But it's because I was supposed to, and everyone in our district, starting in sixth grade does this sort of homework, and this is how much it is. And all of the teams give the same. And it's due on this day. And this is how we grade it.

This indicates that Ms. Chaves was conforming to the institutional expectations for assigning, reviewing, and grading homework. This excerpt was coded using the observation rubric as CD: 1.5, EA: 1, AOI: 1, and FA: 1. Within the CD dimension, her homework assignments provided students an opportunity to productively struggle through problem solving, but this opportunity was not fostered or built upon through classroom practices. Her standardized homework practices created differential access for students because some students may not have had the background knowledge needed to enter the tasks (EA). In addition, students were limited to individually accessing solutions and could not engage in student-to-student discussion (AOI). This practice also limited assessment to purely corrective feedback on student solutions (FA).

Another classroom practice we noted through our analysis was the use of accountability talk. Ms. Chaves prompted discussions in small groups by reminding the students that they had structures of accountability talk. She indicated during her interview that she supported student accountability talk by hanging a poster of prompts on the wall for students to reference. This practice supported student engagement and discourse, but the placement of the resource may have limited students' access to these supports.

Implementation of FALs. Observations from Year 1 also provided a baseline for how Ms. Chaves implemented FALs. In one observation, she made significant changes to the format of the FAL, the tasks involved, and the questions suggested. For instance, Ms. Chaves chose not to use the FAL's pre-assessment task, eliminating a critical opportunity for her to formatively assess students' prior knowledge. She also eliminated the whole-class introduction, which would have increased students' access to the mathematics by providing them opportunities to engage with the content prior to small group work. When implementing the FAL's card sort activity, Ms. Chaves removed a pair of matches from the card sort and provided students with information about the number of matches, lowering the CD of the activity. Using the observation rubric, this episode was coded as CD: 2, EA: 2, and AOI: 2. These scores show that Ms. Chaves attended to CD, EA, and AOI, yet productive struggle was scaffolded away, access was inhibited, and means of fostering student agency were not promoted to the fullest extent.

Ms. Chaves shared in her interview that in Year 1, she was unfamiliar with FALs and used them to piece together her existing classroom practices and new practices related to FALs: "[I was] looking at those [FAL activities] like, oh, this would be good. And like I would just pull it and plop it in and like, trying to figure it out as I went." Ms. Chaves also expressed that she felt compelled to implement only the main task from the FAL because she was under institutional pressure to cover content. She shared that she needed to "keep pace" with other teachers, even if her students needed more time with particular lessons.

Year 4: The Impact of the AIM-TRU PD Model on Ms. Chaves' Teaching

In this section, we describe Ms. Chaves' practices and her implementation strategies for FALs demonstrated after four years of involvement with the AIM-TRU PD model.

Classroom practices. In the Year 4 observation video, we observed changes to Ms. Chaves' classroom practices around homework. In her interview, she shared that she thought her Year 1 homework practices were inequitable because "whatever these kids are going home to may or may not be conducive to them doing [home]work" and to "then penalize a child [for not doing homework] … seems like a one-two punch." She explained that she has adjusted her intentions regarding homework since Year 1, assigning less homework, but increasing the emphasis on making connections between mathematical ideas. She viewed this as a more equitable practice to promote students' retention of mathematical understandings. Furthermore, if students report struggles with homework assignments, she finds time during class for students to collaborate to explore mathematical ideas rather than posting an answer key, as in Year 1. This practice fosters discussion among students around concepts and connections and encourages students to evaluate their own mathematical thinking. She described this in her interview by stating, "I'm letting them come to those conclusions by themselves now." This practice shows her alignment with the TRU framework: she attended to EA by recognizing that conditions at home may not be favorable to completion, raised CD by having students productively struggle to form conclusions, and provided opportunity for students to develop their AOI by having the students work on the problems together to construct mathematical truths, and introduced more opportunities for FA by eliciting student thinking.

Supporting students to come to their own mathematical conclusions was also observed in the classroom observation from Year 4. During her launch of the FAL, Ms. Chaves asked students to determine if 0.123 was a terminating, non-terminating repeating, or a non-terminating non-repeating number. After a student shared their choice, Ms. Chaves asked the student, "Why did you choose that one?" The student gave their justification, and Ms. Chaves then asked another student, "Is that what you are thinking, would you like to add on?" She then asked if students had "any argument" for the other two choices. Ms. Chaves proceeded to ask if the number was rational or irrational. After giving time for students to think individually, she solicited student responses. Students responded with various ideas, including "both," and when the students did not agree on a choice, she did not disclose the correct answer. She instead told the class, "we will be figuring this out in our task." We coded this setup and exchange as FA: 3 and CD: 3 because she used students' emerging understandings to build on student thinking and engaged and supported students in productive struggle by not scaffolding away challenges, respectively.

In her interview, Ms. Chaves contended that her years of participating in this PD model helped her reify the dimensions of the TRU framework. When confronted with outside curricular materials or resources, she now critically analyzes them with a TRU framework lens:

> [How can I] make sure all kids have access to the lesson, but also make it cognitively demanding, and also give the kids agency? If someone comes up to me and says, I want you to teach like this now, I'm going to naturally throw that up against TRU in my mind.

This shows that Ms. Chaves changed her evaluation of classroom experiences, considering whether they raise CD, provide EA for all students, promote student AOI, and allow her the opportunity to formatively assess students' thinking effectively (FA).

Classroom observation of Ms. Chaves from Year 4 also revealed classroom practices that differed from those observed in Year 1. She reminded students to use the "accountability-talk stems" on their desks. When interviewed, she explained that these stems contained prompts for responding to peers, asking peers for clarifications, and sharing new ideas with peers. By using these prompts, students were supported in engaging in conversations by challenging others and justifying their own mathematical thinking. Unlike Year 1, these accountable-talk stems were placed on student desks instead of the wall, supporting student engagement with these practices. This teaching move attended to EA by helping more students to engage in mathematical conversations and to AOI because students were supported in sharing their ideas and building on others' understandings.

Implementation of FALs. In our classroom observation from Year 4, we observed Ms. Chaves implementing an FAL with fidelity and more closely aligned with TRU. In Year 4, she used the pre-assessment on definitions of decimals as recommended in the FAL rather than omitting it as in Year 1. Then, she used multiple approaches to formatively assess students' understandings. She prompted students to work on whiteboards and display them so she could assess their thinking. Next, she facilitated a class discussion based on some perceived misunderstandings, prompting students to justify their thinking and reasoning (FA: 3). Each portion of the FAL was implemented with fidelity, which was a stark difference compared to her Year 1 observation. In her interview, Ms. Chaves attributed this change, in part, to the work done within this PD model:

> There were always discussions … my kids can't do this, but if I edit it, maybe they can get it and then we would talk about, what does that do to the lesson if you edit it? If you make this easier, or if you scaffold this up, because you want to increase your access. But are you simultaneously lowering your cognitive demand? How do you do both? The more we would talk about editing the FALs, the more you question if you should be editing the FAL at all.

In her interview, Ms. Chaves referenced discussions from previous PD sessions in which teachers debated the impact of altering the format and structure of an FAL. Through these learning experiences in the PLC, which were marked by changes in participation and reification in the discussion, she was able to make shifts in her implementation of FALs in her classroom. She also noted that student engagement with this FAL has shifted over multiple years:

> I've done one lesson … three or four years, I finally feel like I let it breathe enough. And all of a sudden, these kids figured out things throughout the lesson that they had never done in previous years. It was like, oh, my gosh, what just happened? The answer is I gave them more time. I didn't try to rush.

Ms. Chaves now gives students time, space, and structures to make connections and persevere and has seen students making better mathematical connections than in previous years. She stated in her interview that this PD "is the only one that I've done that's been long term, sustained." The sustained duration of her involvement in this PD afforded her the opportunity to enact lessons multiple times, reflect on them with others, and improve her practice.

Discussion

Ms. Chaves' participation in the AIM-TRU PD model motivated changes in her teaching practices and the fidelity with which she now implements FALs in her classroom. Through her learning experiences shared across the CoP, she was able to demonstrate clear changes in teaching practices aligned with the TRU framework. Also, the increased fidelity to FALs positioned Ms. Chaves to attain a closer alignment with the TRU framework. These changes in practice help answer recent calls to engage all learners with mathematical content in an equitable way (Larson, 2017). Our analysis of Ms. Chaves' changing classroom practices and her own reflections suggest that it was her continued participation in this PD model and the design of the PD itself (Desimone & Garet, 2015; Garet et al., 2001) that provided her the opportunity to reify ideas about powerful mathematics teaching and the implementation of high-quality materials.

We noted specific shifts in classroom practice regarding homework expectations and student discussion strategies. Ms. Chaves' altered her homework practices due to recognizing inequities in her prior practices. Through sustained duration and collaborative interrogation of teaching moves related to equity, Ms. Chaves chose to alter her practice to create a more equitable space for students. Related to Borko et al. (2008), Ms. Chaves leveraged her PD experiences to allow for more FA opportunities in her classroom as students discussed homework and sought to make their own connections between mathematical ideas to overcome any challenges. Additionally, while Ms. Chaves' change in structures for accountable-talk stems may seem small, shifting from a whole-class anchor chart to individual small-group reference sheets provided additional support for more students to engage in mathematical discourse. The teaching move of providing students with individual prompts was present throughout Ms. Chaves' participation in the PD. Drawing from Cobb et al. (2009), this practice situated students to view themselves as more mathematically competent. There were tangible and available resources for small groups to access and enter mathematical discourse and build on each other's thinking. The FAL instructions for student small-group work are also intentionally designed to promote student discourse. Akin to Santagata and Bray (2016), Ms. Chaves planned for more student discourse by drawing on discussions from PD sessions about the importance of setting and maintaining the FAL expectations for student talk structures; she fostered these interactions among students by making the structures clear and providing reminders.

The ways in which Ms. Chaves implemented FALs changed dramatically, due in part to her involvement in this PD model. Desimone and Garet (2015) stressed the importance of sustained PD with active learning experiences that can connect to teacher practice. Ms. Chaves' participation in the PD model provided her with sustained time reading, analyzing, and reflecting on the implementation of FALs within a CoP. Through discussions in the PLCs, she shifted her perception of how she can use FALs in her classroom and her opinions of the impact of altering the resource. Aligned with these tenets of effective PD, Ms. Chaves also shifted the way she adopts other materials for her classroom by analyzing the alignment of the materials with the TRU framework. This provided her the opportunity to push past institutional norms to adopt TRU aligned teaching practices. It was through her prolonged investigation of teaching practices using the TRU framework that she was able to take this resource and use it to select materials and moves for implementation that leads to ambitious teaching practices.

Conclusion

In this paper, we used sociocultural theory and an evaluative case study methodology to describe, explain, and assess Ms. Chaves' longitudinal experiences in the AIM-TRU PD model. Our findings show that a well-designed PD program focused on the TRU framework can inform shifts in classroom practice toward engaging all learners in ambitious learning opportunities with mathematics. A next step in our research is to broaden the scope of our methodology to study the impact of the AIM-TRU PD model on the collective learning of the entire PLC.

Acknowledgements

This material is based upon work supported by the National Science Foundation under grant number 1908319.

References

Andersson, C., & Palm, T. (2017). The impact of formative assessment on student achievement: A study of the effects of changes to classroom practice after a comprehensive professional development programme. *Learning and Instruction*, *49*, 92-102.

Baldinger, E., & Louie, N. (2014, March 25). *TRU Math conversation guide: A tool for teacher learning and growth*. Graduate School of Education, University of California, Berkeley & College of Education, Michigan State University. https://www.shastacoe.org/uploaded/SCMP2/2015_Spring_Content_Day/trumath_conversation_guide_alpha.pdf

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389-407.

Black, P., & Wiliam, D. (1998). Assessment and classroom learning. *Assessment in Education: Principles, Policy & Practice*, *5*(1), 7-74.

Boaler, J. (2016). Designing mathematics classes to promote equity and engagement. *The Journal of Mathematical Behavior*, *100*(41), 172-178.

Boaler, J., & Staples, M. (2008). Creating mathematical futures through an equitable teaching approach: The case of Railside School. *Teachers College Record*, *110*(3), 608-645.

Borko, H., Jacobs, J., Eiteljorg, E., & Pittman, M. E. (2008). Video as a tool for fostering productive discussions in mathematics professional development. *Teaching and Teacher Education*, *24*, 417–436.

Brown, J. S., Collins, A., & Duguid, P. (1989). Situated cognition and the culture of learning. *Educational Researcher*, *18*(1), 32-42.

Cobb, P., Gresalfi, M., & Hodge, L. L. (2009). An interpretive scheme for analyzing the identities that students develop in mathematics classrooms. *Journal for Research in Mathematics Education*, *40*(1), 40-68.

Collins, A., Brown, J. S., & Newman, S. E. (1988). Cognitive apprenticeship: Teaching the craft of reading, writing and mathematics. *Thinking: The Journal of Philosophy for Children*, *8*(1), 2-10.

Darling-Hammond, L., Flook, L., Cook-Harvey, C., Barron, B., & Osher, D. (2020). Implications for educational practice of the science of learning and development. *Applied Developmental Science*, *24*(2), 97-140.

Desimone, L. M. (2011). A primer on effective professional development. *Phi Delta Kappan*, *92*(6), 68-71.

Desimone, L. M., & Garet, M. S. (2015). Best practices in teacher's professional development in the United States. *Psychology, Society, & Education*, *7*(3), 252-263.

Downton, A., & Sullivan, P. (2017). Posing complex problems requiring multiplicative thinking prompts students to use sophisticated strategies and build mathematical connections. *Educational Studies in Mathematics*, *95*(3), 303-328.

Engle, R. A., & Conant, F. R. (2002). Guiding principles for fostering productive disciplinary engagement: Explaining an emergent argument in a community of learners classroom. *Cognition and Instruction*, *20*(4), 399-483.

Gallagher, H. A. (2016). *Professional development to support instructional improvement: Lessons from research*. SRI International. https://www.sri.com/wp-content/uploads/2021/12/professional_development_to_support_instructional_improvement.pdf

Garet, M. S., Porter, A. C., Desimone, L., Birman, B. F., & Yoon, K. S. (2001). What makes professional development effective? Results from a national sample of teachers. *American Educational Research Journal*, *38*(4), 915-945.

Granberg, C., Palm, T., & Palmberg, B. (2021). A case study of a formative assessment practice and the effects on students' self-regulated learning. *Studies in Educational Evaluation*, *68*, Article 100955. https://doi.org/10.1016/j.stueduc.2020.100955

Hiebert, J., & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. *Second Handbook of Research on Mathematics Teaching and Learning*, *1*(1), 371-404.

Hodge, L. L., & Cobb, P. (2019). Two views of culture and their implications for mathematics teaching and learning. *Urban Education*, *54*(6), 860-884.

Jackson, K., Garrison, A., Wilson, J., Gibbons, L., & Shahan, E. (2013). Exploring relationships between setting up complex tasks and opportunities to learn in concluding whole-class discussions in middle-grades mathematics instruction. *Journal for Research in Mathematics Education*, *44*(4), 646-682.
Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge University Press.
Larson, M. (2017, July 19). Positioning NCTM for a second century. *National Council of Teachers of Mathematics*. https://my.nctm.org/blogs/veronique-nguyen/2017/07/19/positioning-nctm-for-a-second-century
Leonard, H. S., DiNapoli, J., Murray, E., & Bonaccorso, V. D. (2022). Collegial frame processes supporting mathematics teacher learning in a community of practice. In the *American Education Research Association Online Paper Repository.* American Educational Research Association. https://doi.org/10.3102/1891262
Mathematics Assessment Resource Service. (2015). *Classifying Rational and Irrational Numbers*. Mathematics Assessment Project. https://www.map.mathshell.org/download.php?fileid=1710
Merriam, S. B. (1998). *Qualitative research and case study applications in education: Revised and expanded from case study research in education* (2nd ed.). Jossey-Bass Publishers.
National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*.
National Governors Association. (2010). *Common core state standards for mathematics*. http://corestandards.org/
Porter, A., McMaken, J., Hwang, J., & Yang, R. (2011). Common core standards: The new US intended curriculum. *Educational Researcher*, *40*(3), 103-116.
Rosli, R., & Aliwee, M. F. (2021). Professional development of mathematics teacher: A systematic literature review. *Contemporary Educational Research Journal*, *11*(2), 81-92. https://doi.org/10.18844/cerj.v11i2.5415
Santagata, R., & Bray, W. (2016). Professional development processes that promote teacher change: The case of a video-based program focused on leveraging students' mathematical errors. *Professional Development in Education*, *42*(4), 547-568.
Schmidt, W. H., Wang, H. C., & McKnight, C. C. (2005). Curriculum coherence: An examination of US mathematics and science content standards from an international perspective. *Journal of Curriculum Studies*, *37*(5), 525-559.
Schoenfeld, A. H. (2015). Thoughts on scale. *ZDM Mathematics Education*, *47*(1), 161-169. https://doi.org/10.1007/s11858-014-0662-3
Schoenfeld, A. H., Floden, R. E., & the Algebra Teaching Study and Mathematics Assessment Project. (2014, July 31). *The TRU Math scoring rubric*. Graduate School of Education, University of California, Berkeley & College of Education, Michigan State University. https://www.map.mathshell.org/trumath/tru_math_rubric_alpha_20140731.pdf
Schoenfeld, A. H., & the Teaching for Robust Understanding Project. (2016). *An introduction to the Teaching for Robust Understanding (TRU) framework*. Berkeley, CA: Graduate School of Education. http://map.mathshell.org/trumath.php
Shepard, L. A. (2000). *The role of classroom assessment in teaching and learning* (Report No. CSE-TR-517)*.* Office of Educational Research and Improvement. https://cresst.org/wp-content/uploads/TECH517.pdf
Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal*, 33(2), 455-488.
Stigler, J. W., & Hiebert, J. (2004). Improving mathematics teaching. *Educational Leadership*, 61(5), 12-17.
Sztajn, P., Borko, H., & Smith, T. (2017). Research on mathematics professional development. In J. Cai (Ed.), *Compendium for Research in Mathematics Education* (pp. 793-823). National Council of Teachers of Mathematics.
van Es, E. A., & Sherin, M. G. (2008). Mathematics teachers' "learning to notice" in the context of a video club. *Teaching and Teacher Education*, *24*(2), 244-276.
Wenger, E. (1998). Communities of practice: Learning as a social system. *Systems Thinker*, *9*(5), 2-3.
Wenger, E. (1999). *Communities of practice: Learning, meaning, and identity*. Cambridge University Press.

EXPLORING DYNAMIC ASPECTS OF TEACHING IN EARLY-CAREER TEACHERS' CLASSROOMS

Annika Perlander
University of South-Eastern Norway
annika.perlander@usn.no

Annica Andersson
University of South-Eastern Norway
annica.andersson@usn.no

Keywords: Communication, Classroom discourse, High School Education

This classroom study takes a dialogic research approach to investigate how early-career mathematics teachers engage students in conversations about mathematics. Investigating teaching is not only about what toolbox of actions or activities a teacher uses, but how these tools are used to create opportunities to engage with mathematics and with other students (Franke et al., 2007). The meaning, or learning, develops through the ways objects are talked about and used together with others. The teacher thus needs to link different ideas in meaning-making processes, guiding students' learning (Scott et al., 2011). The guiding also has the function to ensure that the learning follows mathematics conventions and goals in the curriculum (Walshaw & Anthony, 2008). Through teachers' approach to communication (communicative approach), the frames for meaning-making processes are created when teaching (Mortimer & Scott, 2003). Balancing purposes, students' understanding, learning goals, and the mathematical content, the teacher may plan for activities with different levels of interaction or space to explore the meaning of students' diverse perspectives. This creates a range of participation for students to engage in (Otten et al., 2015). Depending on students' responses, activity interactions affect what happens in the next lesson or several lessons to come, hence creating dynamic aspects of teaching and learning. The guiding research questions are: What communicative approaches are used by the teacher? What shifts in those approaches can be found within and between lessons?

Dialogical theories emphasize the importance of contributions from others to understand communication, cognition, and human actions (Linell, 2009). In interactions, people negotiate the meaning of what is said or done. From a dialogical perspective, an individual's actions are both responses to previous interactions and initiatives for new actions influencing what will happen next. In this study, this is relevant when looking at how early-career teachers approach communication, not just within a lesson but also between lessons.

The empirical material consists of classroom observations with two early-career teachers in Scandinavia. For each teacher, a sequence of consecutive lessons (eight for one teacher, ten for the other) has been video and audio recorded. The teachers were interviewed at the end of the observation period. The material is analysed using a meaning-making framework by Mortimer and Scott (2003). The teachers' communicative approaches in whole class settings are identified using two dimensions: dialogic/authoritative and interactive/non-interactive approach. The analysis focuses on how the teacher shifts between communicative approaches, and in what ways the students' ideas of the mathematical content are included in the teaching depending on the type of activities. Questions for future analysis (and for discussion with the PME-NA community) include how these shifts in communicative approaches might support the students' meaning-making processes both within and between lessons.

References

Franke, M. L., Kazemi, E., & Battey, D. (2007). Mathematics teaching and classroom practice. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning: A project of the National council of teachers of mathematics* (pp. 225–256). IAP.

Linell, P. (2009). *Rethinking language, mind, and world dialogically*: *Interactional and contextual theories of human sense-making*. Information Age Pub.

Mortimer, E., & Scott, P. (2003). *Meaning making in secondary science classrooms*. Open University Press.

Otten, S., Engledowl, C., & Spain, V. (2015). Univocal and dialogic discourse in secondary mathematics classrooms: the case of attending to precision. *ZDM, 47*(7), 1285–1298. https://doi.org/10.1007/s11858-015-0725-0

Scott, P., Mortimer, E., & Ametller, J. (2011). Pedagogical link-making: a fundamental aspect of teaching and learning scientific conceptual knowledge. *Studies in Science Education, 47*(1), 3–36. https://doi.org/10.1080/03057267.2011.549619

Walshaw, M., & Anthony, G. (2008). The teacher's role in classroom discourse: a review of recent research into mathematics classrooms. *Review of Educational Research, 78*(3), 516–551. https://doi.org/10.3102/0034654308320292

EXPLORING THE ENACTMENT OF A COACHING STANCE: A CASE OF DISSONANCE FROM ONE COACH-TEACHER DYAD

Evthokia Stephanie Saclarides
University of Cincinnati
saclares@ucmail.uc.edu

Ryan Gillespie
University of Idaho
rgillespie@uidaho.edu

Drawing on data collected from one coaching cycle during mathematics instruction for one coach-teacher dyad, this study explores one instructional coach's discursive enactment of their coaching stance. Qualitative analyses indicate that there was dissonance between the coach's stance for coaching and their discursive enactment of coaching, and that the coach's disciplinary expertise seemed to influence the enactment of her coaching stance. Implications for research and practice are discussed.

Keywords: instructional leadership, classroom discourse, professional development

Coaching is complex as coaches must navigate multiple, and often competing, tensions in their roles and responsibilities as they support teachers. In the context of a three-part coaching cycle (Bengo, 2016), a coach has simultaneous obligations, including: working with a teacher to design, prepare for, and implement a high-quality lesson; supporting the teacher to learn content and pedagogy; establishing a trusting and productive relationship with the teacher; and developing a teacher's capacity to plan for and reflect upon lessons without the presence of a coach. This complexity is further amplified as a coach must balance the roles of an expert (in content, pedagogy, or both) and collegial thinking partner. Coaches typically operate from a position of authority, given their formal title and comparative experience to the teachers they support (Mosley Wetzel et al., 2017). From this position of power, coaches can choose to leverage their expertise as they work towards the aforementioned obligations. However, the conception of coaching (e.g., Showers & Joyce, 1996) and more current definitions of coaching (e.g., Baker et al., 2021) call for a coach to act as a non-evaluative colleague available for job-embedded collaboration. Thus, coaches are tasked with making continual choices during coaching interactions regarding how to balance these competing roles and responsibilities, making a coach's capacity to creatively manage these tensions a central feature of intentional coaching (West & Cameron, 2013).

Within mathematics education, there is a dearth of literature on how coaches enact their stance, defined here as a coach's behaviors when managing the competing roles of expert and colleague (e.g., Deussen et al., 2007; Ippolito, 2010), in ways that best support teacher learning. Furthermore, little is known about the specific discursive practices coaches use when enacting different stances during interactions with teachers. Our study seeks to fill this gap through understanding how one instructional coach discursively enacts their coaching stance throughout planning, teaching, and debriefing interactions in a coaching cycle with one elementary teacher during mathematics instruction. Specifically, three research questions are explored: (1) What is one coach's stance about their coaching practice, (2) How does the coach discursively enact their coaching stance, and (2) How might the coach's disciplinary expertise influence the enactment of their coaching stance?

Related Literature

We frame our study with the argument that coaching interactions are situated in a system of negotiation in which the language choices and roles of interlocutors continuously shape and are shaped by the context of the interaction (Halliday & Matthiessen, 2004). This system of negotiation involves one speaker assuming a role of a primary "knower" relegating the other participant to a role of secondary "knower" during a linguistic exchange (González & DeJarnette, 2012). Researchers studying literacy coaching have provided language describing the two competing stances for how coaches leverage their role and potential position of power when talking with teachers: *reflective* or *directive* (Deussen et al., 2007; Ippolito, 2010; Sailors & Price, 2015). Coaches using a *reflective* stance use discourse moves to position the teacher as the primary "knower" when examining the effectiveness of their practice and student outcomes (Ippolito, 2010). Discourse moves associated with a reflective coaching stance include invitational moves (e.g., questions) and paraphrases as these forms of discourse invite teacher cognition and do not involve the coach sharing their own thinking or opinions (Costa & Garmston, 2016; Deussen et al., 2007). In contrast, a *directive* stance involves the coach using language to position themselves as the "primary knower". Discourse moves associated with a directive stance include suggestions, explanations, and evaluative feedback which all involve the coach sharing ideas generated from their opinions and beliefs (Ippolito, 2010). We also reference facilitation literature within mathematics education (e.g., van Es et al., 2014) to argue the existence of a third coaching stance: *facilitative.* Coaches enacting a facilitative stance use language to guide the direction, focus, and clarity of the conversation without positioning either the coach or teacher as the "primary knower". Discourse moves associated with a facilitative stance include sharing low-inference, non-evaluative observations, clarifying ideas being discussed to ensure a shared understanding, and framing conversations by reminding the teacher about goals or larger purposes established during prior discussion.

Researchers, in both literacy and mathematics education, have examined the discursive tendencies of coaches. Heineke (2013) noted that in one particular context, literacy coaches "dominated the discourse" through the use of many suggestions (p. 424). In these cases, teacher participation and learning opportunities in the conversations were limited. In a study directly exploring how literacy coaches balance differing stances, Ippolito (2010) found that effective coaches shifted between reflective, and directive moves within a single coaching session. Similarly, when studying three mathematics coaches, Gillespie and colleagues (2019) found all three mathematics coaches frequently shifted between using reflective and directive moves during planning and debriefing conversations with teachers. However, despite working in a similar context, the coaches had contrasting tendencies with respect to the duration, intensity, and frequency in their use of directive moves. Furthermore, these tendencies remained stable across multiple interactions with the same teacher. Witherspoon et al. (2021) found that nearly all coaches held directive stances during planning conversations but, in more effective coaching sessions, coaches "remained engaged in longer discussions about each pedagogical decision before telling the teacher their own interpretation" (p. 892). In other words, effective coaches balanced different coaching stances by holding reflective and facilitative stances prior to shifting into a directive stance. In sum, these studies highlight that both reflective and directive moves are common forms of discourse within coaching interactions, yet coaches differ in how they employ such moves to support their teachers.

To our knowledge, only one research study has examined how coaches enact a set of beliefs comprising a theory of action for effectively supporting teacher learning. Russell et al. (2020) examined how coaches deviated from the inquiry-based principles of a coaching model by providing teachers with explicit directions or suggestions. Russell et al. claimed some project coaches frequently used directive coaching moves with teachers, despite the model's theory that reflective coaching moves that support teacher inquiry most effectively support improvements to teaching. However, Russell et al. deemed that directive coaching moves, even in contradiction of the beliefs underpinning the coaching model, were efficacious in supporting teacher development under appropriate conditions. These studies point to the importance of understanding coaches' beliefs about balancing competing stances when supporting teachers and how coaches make decisions about shifting between stances.

Methods

Setting and Participants

This study was situated in Midtown District, which enrolled about 10,000 students in 18 schools. We partnered with one full-time, school-based elementary instructional coach (Jade) and one fourth grade teacher (Jennifer). Both individuals were white females. At the time of the study, Coach Jade has been an educator for 12 years across grades K-5. She had worked in her current school district as an instructional coach for four years. Teacher Jennifer was entering her third year as a fourth-grade elementary teacher in Midtown. Coach Jade and Teacher Jennifer appeared to have a positive professional relationship that was marked by co-respect and trust.

The data for this study come from a larger study exploring teachers' learning opportunities during one-on-one coaching (Saclarides, 2022; Saclarides & Lubienski, 2021; Saclarides & Munson, 2021). Coach Jade was specifically selected as the focal coach for the current analysis given that she explicitly discussed her desire to develop and sustain a reflective coaching stance when engaged with teachers as her own personal coaching goal. Hence, we perceived that Coach Jade's coaching cycle with Teacher Jennifer could enable us to better understand our broad research focus, which is how coaches enact their coaching stance when engaged with teachers as well as the mediating factors that may influence that stance.

Data Source and Analytic Technique

Data sources encompassed observations and field notes of Coach Jade's and Teacher Jennifer's coaching cycle, which included two planning meetings (range of 21-46 minutes), one modeled lesson (75 minutes), and one reflection meeting (5 minutes). Furthermore, five semi-structured interviews (Lareau, 2021) were conducted separately with Coach Jade and Teacher Jennifer to establish context for the analysis (range of 10-44 minutes). All observations and interviews were audio recorded and transcribed.

We first read through all interview transcripts to better understand Coach Jade's coaching stance (RQ1). We used the literature-driven codes for reflective and directive stances (e.g., Ippolito, 2010), and facilitative stance (e.g., van Es et al., 2014) when coding for coaching stance during this phase. Next, we explored Coach Jade's discursive enactment of her coaching stance (RQ2). To do this, we organized the planning, modeling, and reflection transcripts for analysis. First, we created excel files where each alternating vertical cell contained either the coach's or teacher's talk turn. Next, we focused our attention on only the cells containing the coach's talk turn given our interest in exploring the coach's discursive enactment of her coaching stance. We further parsed each coaching talk turn at the sentence-level, which was our unit of analysis. Next, the second author used a comprehensive coding scheme developed through prior work (Gillespie et al., 2019), and informed by literature (Ippolito, 2010; van Es et al., 2014), to code at the sentence-level for the coach's discursive stance. One of the following three broad codes was applied: Directive, Reflective, Facilitative. Last, to better understand how the coach's disciplinary expertise influenced the enactment of her coaching stance (RQ3), we engaged in one last coding round. The first author coded for the substance of each coach-spoken sentence using one of the following codes: pedagogy, content, logistics, other. All codes and code definitions can be found in Table 1 below. Last, matrices were created, and counts were completed to better understand the prevalence and intersection of the applied codes.

Table 1: Codes, Definitions, and Examples

Codes		Definition	Example
Stance	Directive	Coach moves in which the coach shares their thinking and opinions with the teacher.	*Coach Jade*: That's something we can work on, that's something they can work on independently, but I think we have to put that in place for kids we know don't have single digit multiplication in place.
	Facilitative	Moves in which the coach establishing the focus, direction, and/or clarity of the conversation	*Coach Jade*: I heard you today having them repeat directions.
	Reflective	Coach moves in which the coach invites the teacher to share their thinking and opinions**.**	*Coach Jade*: So what did you think of yesterday? What did you take away from that?
Substance	Content	The coach's interaction is focused on the mathematics content, including unpacking student mathematics learning goals and mathematics standards, as well as the district-provided mathematics curriculum, including seeking to understand precisely what the curriculum requires of students.	*Coach Jade:* Just so I know what we're talking about. Like, 2 times 2, and 4 times 4. [Are those] like square arrays?
	Logistics	The coach's interaction is focused on logistics, including discussing the timing for the lesson, lesson materials, and classroom management.	*Coach Jade*: I looked at your schedule. Is this pretty accurate as far as, 12:45-2 [for mathematics instruction] across the day?
	Pedagogy	The coach's interaction is focused on pedagogy, which encompassed conversations about student discourse, differentiating instruction, monitoring student learning, using data to inform next instructional steps, and planning and/or creating original activities that extend beyond the district-provided curriculum.	*Coach Jade:* So, maybe for those lower ones, it's talking about, it's three plus three plus three. Or three, three times. 'Cuz I'm guessing if we gave them three times…I'd be curious even three times three to see how they said they solved it.

Findings

Coach Jade's Coaching Stance

Overall, Coach Jade articulated a goal of enacting a reflective coaching stance when engaged with teachers. Instead of using a directive approach that would position her as an authority or primary knower, Coach Jade sought to position her teachers as primary knowers as she centered their ideas, goals, and questions. Coach Jade shared, “I ask them what they want. I ask them what they want to grow in.” As previously mentioned, this was even Coach Jade’s personal goal for her coaching cycle with Teacher Jennifer: to grow in her ability to enact a reflective coaching stance. Coach Jade shared, “I think that every coaching opportunity provides me with the opportunity to listen and try to not tell as much. I should do more questioning instead of telling. That will be my coaching focus [with Teacher Jennifer]”.

Coach Jade’s Discursive Enactment of her Coaching Stance

When coding Coach Jade’s spoken sentences for her discursive coaching stance, the most common code applied was *directive* (137/288) as she engaged with Teacher Jennifer during their coaching cycle (see Table 2). Although it was Coach Jade’s personal coaching goal to focus on enacting a reflective stance with Teacher Jennifer, we applied the reflective code to Jade’s spoken sentences the least frequently (40/288) (See Table 2).

Table 2: Discursive Enactment of Coaching Stance

	Planning Meeting 1	Planning Meeting 2	Modeled Lesson	Reflection Meeting	Total
Directive	28	88	10	11	137
Facilitative	56	50	4	1	111
Reflective	13	21	0	6	40

The following excerpt illustrates what it sounded like as Coach Jade enacted the most prevalent coaching stance - that of being directive - with Teacher Jennifer. During planning meeting one, when discussing how students’ prior experiences with ten-frames might support students during the lesson, coach Jade shared the following (numbers added for explanatory purpose here and below):

> (1) I moved that dot over here to a ten and there's one left over, so I know that's ten, one, eleven. (2) So that really helps them. (3) Kindergarten has done phenomenally with that, so that might be a place we play with for some of that.

In this talk-turn, coach Jade spoke three sentences, each coded as directive because each sentence involved her sharing her thoughts or opinions with Jennifer, temporarily positioning herself as the primary knower within this moment. Jade begins by explaining how she used the ten-frame to conceptualize the number 11 (sentence 1). Next, Jade shares her opinion that ten-frames help students (sentence 2). Last, Jade suggests they consider incorporating the ten-frames as they collaboratively plan their lesson (sentence 3).

To illustrate discursive moves in which Coach Jade shifted between a facilitative and reflective stance, we provide the following example also from planning meeting one. In this moment, Jade and Jennifer are examining a single question within the lesson they are planning as it is printed in the curriculum resource. Coach Jade says: “(1) It just says prime and composite numbers. (2) What does that look like?” In this conversational moment, Jade tells Jennifer that the question, as printed in the curriculum resource, involves prime and composite numbers (sentence 1). Because Jade is highlighting the language contained in the printed question without sharing her own interpretation or thinking, we coded this sentence as facilitative. Coach Jade then asks Jennifer to share her thinking about what student thinking might look like as they attempt to answer the question (sentence 2). In doing so, Jade initiates an opportunity for

Jennifer to share her thinking about how students would approach this question. Thus, we coded this sentence as reflective.

The Influence of Disciplinary Expertise on Coaching Stance

As previously discussed, we were interested in better understanding the mediating influence of Coach Jade's disciplinary expertise on the enactment of her coaching stance. Although Coach Jade was an instructional coach and was expected to coach across all content areas, she did not feel as confident coaching teachers in mathematics as opposed to English Language Arts. In her interview, Coach Jade stated, "I think that I'm definitely stronger for [coaching] literacy [as compared to mathematics]". However, Coach Jade was eager to deepen her own mathematics content knowledge in the context of her coaching cycle with Teacher Jennifer. She shared, "I'm going to try my best for math! It will really push me".

We highlight several noteworthy trends in the relationship between the substance and stance of Coach Jade's discourse to understand how disciplinary expertise may mediate the enactment of Jade's coaching stance (see Table 3). First, when Coach Jade enacted a directive stance, positioning herself as a primary knower, most of the time (85/137) it was about pedagogy. In contrast, there were fewer instances in which she discussed content (31/137) when enacting a directive stance. Second, there was more of an even split between pedagogy (16/40) and content (14/40) when enacting a reflective stance, where she positioned the teacher as the primary knower.

Table 3: Influence of Disciplinary Expertise on Coaching Stance

	Pedagogy	Content	Logistics	Other	Overall
Directive	85	31	18	3	137
Facilitative	19	36	56	0	111
Reflective	16	14	10	0	40

The following excerpt illustrates what it sounded like as Coach Jade enacted a directive stance about pedagogy - the most commonly noted trend in Table 3 above. During the reflection conversation, Teacher Jennifer shared her observations of Coach Jade's use of wait time, to which Coach Jade responded:

> (1) Yeah, I wasn't consciously thinking about wait time. (2) Maybe I just, I don't know. (3) I knew when I came in, I wanted them to do more talking than me. (4) And part of that's from our district-sponsored literacy professional development. (5) Which is crazy because it's a literacy based training.

This conversational moment was coded as directive because Coach Jade momentarily positioned herself as a primary knower as she shared her thinking about the pedagogical move of giving students wait time to spur student discourse. Although Coach Jade acknowledges that she was not intentionally thinking about wait time while she modeled instruction (sentences 1-2), she implemented this particular structure as a way to create a rich discursive environment for students in which students talked more than the teacher (sentence 3). Coach Jade ends by referencing the literacy-based training sponsored by the district office (sentences 4-5), which presumably is where she learned about this connection between wait time and enhanced student discourse.

As a second example, the following excerpt illustrates Coach Jade enacting a reflective stance about mathematics content during the second planning meeting. At the beginning of their meeting, Coach Jade initiates the following interaction with Teacher Jennifer to clarify the mathematical content embedded in the district-provided curriculum, "(1) Just so I know what (content) we're talking about. (2) Like, 2 times 2? (3) 4 times 4? (4) (Is that) like square arrays?" In this discursive moment, Coach Jade enacted a reflective coaching stance as she momentarily positioned Teacher Jennifer as the primary knower about 4th grade mathematics content. Specifically, Coach Jade wanted to ensure that she understood what the mathematics term "square array" means (sentence 1). She proceeded to provide two examples of what she perceived a square array was (sentences 2-3) and ended by asking Teacher Jennifer if this was correct (sentence 4).

Discussion and Implications

All coaches have a preference for the coaching stance they seek to enact when working with teachers. This coaching stance may be informed by beliefs about what it means to be an effective coach, as well as coaches' perceptions about the needs of individual teachers. However, when managing simultaneous obligations and balancing roles of expert and colleague based on the needs of individual teachers, our findings suggest it may be difficult for coaches to enact their preferred stance for coaching. Like prior studies on literacy and mathematics coaches (e.g., Gillespie et al., 2019; Heinke, 2013; Ippolito, 2010; Witherspoon et al., 2021) we agree that coaching inherently involves coaches using, and productively balancing, differing stances and discourse moves. We highlight the challenge and complexity of enacting a set of coaching beliefs to extend this prior research and make visible potential assumptions about coaching.

In our study, we found dissonance between Coach Jade's stance for coaching and the discursive enactment of her stance. Although Coach Jade desired to enact a reflective coaching stance with Teacher Jennifer and explicitly set a coaching goal to enact this belief, in practice Coach Jade primarily used directive coaching moves (137/288). In other words, although Coach Jade had the goal of primarily positioning Teacher Jennifer as the primary knower, just the opposite happened in the context of their coaching cycle: most of the time Coach Jade positioned herself as the primary knower. We consider this case of dissonance connected to the findings from Russell et al. (2020). Recall that Russell et al. found that coaches operating in an inquiry-based model coaching made responsive adaptations by using directive coach moves. In this context, the dissonance existed between the beliefs inscribed in an established coaching model and the discursive actions of coaches learning to use the model. Our findings highlight a similar case of dissonance but between a coaches' own articulation of her stance for coaching and her discursive actions, further highlighting the challenge of efficaciously enacting a stance for coaching.

The literature points to various factors that can influence a coach's ability to enact their coaching stance, including relational and organizational contexts (Russell et al., 2020) as well as the alignment between the goals, instructional vision, and evaluation mechanisms of an administrator, coach and teacher (Ippolito, 2010). In this study, we highlight one such factor, disciplinary expertise, that appeared to mediate Coach Jade's enactment of her discursive coaching stance. That is, when Coach Jade enacted a directive stance and positioned herself as the primary knower, most of the time it was about pedagogy (85/137) instead of content (31/127). This can be understood in light of the fact that Coach Jade reportedly felt less confident about mathematics content, so it is conceivable she would be more likely to leverage her expertise and use directive moves when discussing pedagogy. Surprisingly, when Coach Jade enacted a reflective stance and positioned Teacher Jennifer as the primary knower, there was a near even split between content (14/40) and pedagogy (16/40). Given Coach Jade's self-reported lack of confidence with 4th grade mathematics content, we hypothesized that there would be many more instances in which Coach Jade enacted a reflective stance about mathematics content than was noted in the data.

This study has implications for both research and practice. We raise methodological questions for the research community regarding difficulties with adequately characterizing or "measuring" coaches' discursive enactment of their coaching stance. For one, we acknowledge that we are seeking to understand a complex phenomenon – the enactment of a coach's stance for coaching – by assigning codes at the sentence-level and then aggregating counts into broad categories. We posit studying a coach's discursive enactment of their stance for coaching is more nuanced and complex, and we do not mean to oversimplify this complexity with our analysis. However, given that the research community has not yet developed tools to adequately capture and study our phenomenon of interest, we invite other researchers to continue to grapple with us about this important dilemma. Relatedly, we raise methodological questions associated with talking about prevalence of codes when there may be a natural imbalance with the codes themselves. That is, when coaches enact a directive stance with teachers and are positioning themselves as the primary knower in an effort to explain or suggest something, it may take more sentences to offer clear and direct assistance in that discursive moment. Conversely, when coaches enact a reflective stance with teachers and pose a question, it takes many fewer sentences to enact that stance in that discursive moment. For example, a single question from a coach may create lengthy verbal contributions from the teacher. Hence, we wonder how future research on coaching can better attend to and examine such discursive moves that may carry a natural imbalance of counts. Last, we call upon future research to invite coaches to reflect on their own discursive enactment of their coaching stance by, for example, member checking. This would help elevate coaches' voices in the research process and importantly tap into their insider's, or emic, perspective about the discursive enactment of their coaching stance.

For school districts with coaching programs, district- and school-level administrators must clearly articulate their vision regarding the coaching stance that they expect their coaches to enact when engaged with teachers. What is important here is that there is a shared understanding among district- and school-level administrators and coaches about the particular coaching stance that is to be enacted in that particular school district. Once this is established, it should not be assumed that coaches instinctively understand how to enact a particular coaching stance or set of stances. Rather, coaches should be provided with ongoing, job-embedded professional development that helps them understand the coaching stance they should enact, as well as the accompanying discursive moves that will enable coaches to enact their coaching stance. Furthermore, and in the context of professional development, coaches must be supported to reflect about the variety of mediating factors that may influence the enactment of their coaching stance, and think about how they can still enact their preferred coaching stance amid such mediating factors.

References

Baker, C., Saclarides, E. S., Harbour, K. E., Hjalmarson, M., & Livers, S. (2021). Trends in mathematics specialist literature: Analyzing research spanning four decades. *School Science and Mathematics*. https://doi.org/10.111.ssm.12507

Costa, A. L., & Garmston, R. J. (2016). *Cognitive coaching: Developing self-directed leaders and learners*. Rowan & Littlefield.

Deussen, T., Coskie, T., Robinson, L., & Autio, E. (2007). *"Coach" can mean many things: five categories of literacy coaches in Reading First* (Issues & Answers Report, REL 2007–No. 005). U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance, Regional Educational Laboratory Northwest. Retrieved from http://ies.ed.gov/ncee/edlabs
Gillespie, R., Amador, J., & Choppin, J. (2019). *Exploring the discursive differences of mathematics coaches within online coaching cycle conversations.* In S. Otten, A. G. Candela, Z. de Araujo, C. Haines, & C. Munter (Eds.), Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (pp. 442-450). St. Louis, MO: University of Missouri.
González, G., & DeJarnette, A. F. (2012). Agency in a geometry review lesson: A linguistic view on teacher and student division of labor. *Linguistics and Education, 23*, 182-199. https://doi.org/10.1016/j.linged.2012.02.001
Halliday, M.A.K., & Matthiessen, C. (2004). *An introduction to functional grammar.* Arnold. https://doi.org/10.4324/9780203783771
Heineke, S. F. (2013). Coaching discourse: Supporting teachers' professional learning. *Elementary School Journal, 113*(3), 409-433. https://doi.org/10.1086/668767
Ippolito, J. (2010). Three ways that literacy coaches balance responsive and directive relationships with teachers. *Elementary School Journal, 111*(1), 164-190. https://doi.org/10.1086/653474
Lareau, A. (2021). Listening to People: A Practical Guide to Interviewing, Participant Observation, Data Analysis, and Writing it All Up. University of Chicago Press.
Mosley Wetzel, M., Taylor, L. A., & Vlach, S. K. (2017). Dialogue in the support of learning to teach: A case study of a mentor/mentee pair in a teacher education programme. Teaching Education, 28(4), 406-420. https://doi.org/10.1080/10476210.2017.1309016
Russell, J. L., Correnti, R., Stein, M. K., Bill, V., Hannan, M., Schwartz, N., Booker, L. N., Pratt, N. R., & Matthis, C. (2020). Learning From Adaptation to Support Instructional Improvement at Scale: Understanding Coach Adaptation in the TN Mathematics Coaching Project. *American Educational Research Journal, 57*(1), 148–187. https://doi.org/10.3102/0002831219854050
Saclarides, E. S. (2022). Studying coach-teacher interactions during co-teaching. *Investigations in Mathematics Learning*, 1-17. https://doi.org/10.1080/19477503.2022.2052664
Saclarides, E. S., & Lubienski, S. T. (2021). Teachers' mathematics learning opportunities during one-on-one coaching conversations. *Journal for Research in Mathematics Education, 52*(3), 257-300. https://doi.org/10.5951/jresematheduc-2020-0092
Saclarides, E. S., & Munson, J. (2021). Exploring the foci and depth of coach-teacher interactions during modeled lessons. *Teaching and Teacher Education, 105.* https://doi.org/10.1016/j.tate.2021.103418
Sailors, M., & Price, L. (2015). Support for the improvement of practices through intensive coaching (SIPIC): A model of coaching for improving reading instruction and reading achievement. *Teaching and Teacher Education, 45*, 115-127. https://doi.org/10.1016/j.tate.2014.09.008
Showers, B., & Joyce, B. (1996). The evolution of peer coaching. Educational Leadership, 53(6), 12-16.
West, L., & Cameron, A. (2013). *Agents of change: How content coaching transforms teaching and learning*. Heinemann.
van Es, E. A., Tunney, J., Goldsmith, L. T., & Seago, N. (2014). A framework for the facilitation of teachers' analysis of video. *Journal of Teacher Education, 65*(4), 340-356. https://doi.org/10.1177/0022487114534266
Witherspoon, E. B., Ferrer, N. B., Correnti, R. R., Stein, M. K., & Schunn, C. D. (2021). Coaching that supports teachers' learning to enact ambitious instruction. *Instructional Science, 49*(6), 877–898. https://doi.org/10.1007/s11251-021-09536-7

HOLISTIC INDIVIDUALIZED COACHING: FOREGROUNDING TEACHERS' PSYCHOLOGICAL AND AFFECTIVE ATTRIBUTES TO SUPPORT TEACHER LEARNING

Dionne Cross Francis
University of North Carolina at Chapel Hill
dicross@unc.edu

Kathryn Habib
University of North Carolina at Chapel Hill
kaldrich@unc.edu

Anna Gustaveson
University of North Carolina at Chapel Hill
annagus@unc.edu

Pavneet Kaur Bharaj
University of Massachusetts Boston
pavneetkaur.bharaj@umb.edu

Anna Hinden
University of North Carolina at Chapel Hill
annarh@unc.edu

Coaching has been shown to be an effective professional development approach to improving teachers' mathematics instruction. However, research on existing coaching models foregrounds teachers' knowledge and instructional practices and tends to overlook the psychological and affective aspects of teaching. In this study, we explored if and how two elementary teachers' participation in Holistic Individualized Coaching (HIC), designed to attend to the psychological and affective aspects of teaching, supported instructional change and teachers' well-being. Results show that while both teachers mathematics instruction improved, changes to the beliefs, teaching efficacy, and emotions were different. Findings suggest that teachers navigate different psychological and affective pathways towards change and the importance of attending to these attributes during coaching interactions.

Keywords: Professional Development, Affect, Emotions, Beliefs, and Attitudes

Supporting teachers to engage in high-quality mathematics instruction is considered essential for developing students' problem-solving abilities (Cobb et al., 2018). This support often takes the form of professional development (PD) programs which align in whole, or in part, with what are regarded as the central characteristics of PD [see Garet et al., (2001) for a full description of these features]. However, their design and focus are not always responsive to the individual needs of teachers (Cross Francis et al., 2019). To address this drawback, coaching has emerged as an effective job-embedded approach to support teachers in improving their instruction (Cobb & Jackson, 2012; Kraft et al., 2018). Despite evidence of coaching's effectiveness at promoting teacher development, there are two significant gaps in coaching research that this study aims to fill. First, researchers (e.g., Gibbons and Cobb, 2016) argue that the mathematics education community has yet to document what coaches actually do to support teachers' learning. Without this information, it will be difficult to replicate effective coaching practices. Second, research on existing coaching models foregrounds teachers' knowledge and instructional practices and tends to overlook the psychological and affective aspects of teaching, which have been shown to influence instructional quality and student learning (Cross Francis et al., 2021). In this study, we respond to a call by Gibbons and Cobb (2016) to document how accomplished coaches work with teachers to support their learning. We focus on coaching as embedded within a learner-centered, holistic coaching model. Specifically, we investigated the following question: *what* *changes occur in two elementary teacher's attributes and instructional practices through participation in an innovative coaching model, referred to as Holistic, Individualized Coaching (HIC)?*

Theoretical Framework

Teaching is a psychologically and emotionally demanding endeavor (Schutz et al., 2020). Thus, attending to teachers' psychological and affective attributes is essential both for supporting their learning and enhancing their well-being. We conceptualize teacher learning holistically to include enhancement of their instructional practices as well as their psycho-social-emotional development, which includes attending to their beliefs, emotions, efficacy, and identity. We consider deep and flexible knowledge (Willingham & Riener, 2019), productive beliefs (Cross Francis, 2015), a robust identity (Hong, 2010), and positive emotions (Cross Francis et al., 2020) central aspects of teachers' wellbeing and essential attributes for effective mathematics teaching. This robust conceptualization of teacher learning underlies a coaching approach referred to as Holistic Individualized Coaching (HIC).

Holistic Individualized Coaching (HIC)

The Holistic Individualized Coaching (HIC) model arose in response to the call (e.g., Brackett & Cipriano, 2020) for PD initiatives to attend to the multi-dimensional aspects of teaching. HIC attends to teachers' psycho-socio-emotional needs. HIC takes into account the teacher as an individual and utilizes information about teachers' emotions, beliefs, identity, knowledge and current instructional practices that the teachers bring with them to the coaching experience (Cross Francis et al., 2019). [Full descriptions and empirical support for the inclusion of these attributes can be found at: beliefs (Fives & Gill, 2015); identity (Hong, 2010); emotions (Schutz et al., 2020)]. HIC attempts to meaningfully enhance teachers' work by attending to them comprehensively; thus, HIC is designed to be *holistic*. What a teacher believes, feels, or considers themselves to be in relation to math is personal; teachers' emotions, beliefs and identity are unique to the teacher. Thus, to encourage productive shifts in these attributes, HIC is *individualized* and focused on attending to the specific needs of the teacher. HIC involves six steps articulated in Figure 1.

Step 1. Multi-dimensional Teacher profile.
- Prepare a multi-dimensional teacher profile about the teacher's instructional quality, MKT, beliefs, identity, and emotions.
- Coach uses information to shape the nature of the interaction with the teacher and the type of support to provide.

Step 2. Pre-coaching conversation of a lesson.
- A pre-coaching conversation of a lesson to be coached to gain an insight into the teacher's history with the content, including associated feelings and current understanding of the content.
- Coach and teacher agree on nature of support needed before and during lesson

Step 3. Content-specific mini-profile
- Developing a content-specific mini teacher profile that considers all the constructs described above (MKT, beliefs, identity, emotions in relation to teaching the content of the lesson), and information yielded from the pre-coaching conversation

Step 4. Pre-lesson support (if needed).
- Dependent on information from the content-specific mini profile (coach and teacher collaboratively determine this)

Step 5. In-class coaching.
- Coached lessons are video recorded

Step 6. Post-coaching conversation
- Teacher and coach engage in a discussion organized around video clips from the coached lesson.

Figure 1. Holistic, Individualize Coaching Model Steps

Distinct Features of HIC

1) Data on teachers' beliefs, emotions, efficacy and identity are used to determine development over time but more centrally to inform the coach about *how* to engage with the teacher.
2) The coach collects and uses data on teachers' generalized beliefs (trait), emotions, efficacy related to math, as well their beliefs, emotions, efficacy related to specific mathematics content (state).
3) In Steps 1 and 6 (pre and post-coaching conversations), the coach directly gathers content-informed data related to beliefs, emotions, efficacy and identity, then uses these data as relevant in subsequent HIC cycles.

Figure 1: Holistic, Individualize Coaching Model Steps

Methods

This study is embedded within a larger study including six teachers. We focus on two Ghanaian teachers, Rehema and Peace, who wanted to incorporate more student-centered approaches in their teaching following Ghana's nationwide mathematics education reform.

Participants. Rehema was a fourth-grade teacher who struggled in math throughout most of her schooling. Rehema initially refused to participate in coaching as she was not comfortable with someone else seeing her teach or with being videorecorded. However, she participated in four HIC cycles over the course of the study. Peace was a second-grade teacher who enjoyed math in her early years but struggled later on due to poor instruction. She desired to teach in ways that were impactful but consistently felt that her students were not grasping math concepts deeply and welcomed support.

Data Sources and Analyses

Five data sources were used in this study. *First,* the Mathematical Quality of Instruction (MQI) instrument (Hill et al, 2008) measures mathematical instruction quality along four dimensions (see Table 2). Two scorers watched 8-minute clips of the teachers' instructional videos and scored indicators on each dimension as either *NP-not present; L-low; M-mid; H-high.* Then, scorers met to discuss and resolve any discrepancies. *Second*, adapted Teachers Emotions Scale (TS; Frenzel et al., 2016) is a 5-point Likert scale used to determine teachers' levels of anxiety, anger and enjoyment related to teaching mathematics. Numeric values [1 – strongly disagree; 5- strongly agree] were assigned to the items on the adapted survey, then an average score for items aligned with each emotion was calculated for each teacher. *Third*, teachers completed comprehensive, semi-structured interviews included questions related to all constructs. We focus here on the math-related beliefs questions, in particular, their response to the question "What would you say to a new teacher to support her in engaging in student-centered instruction?". The interviews were transcribed and teachers' responses to the beliefs questions were summarized. Teachers' responses to abovementioned question captured their stated beliefs about math teaching and learning. *Fourth,* teachers completed the Single Item Measure (adapted from McDonald et al., 2019) which includes a series of overlapping circles with increasing degrees of overlap. The overlap represents the degree to which their identity as a teacher overlaps with their perception of a math teacher. Circle 7 has the greatest overlap representing a robust identity as a teacher. *Fifth*, the participants completed the *adapted* Self-Efficacy Teaching and Knowledge Instrument for Science Teachers (SETAKIST) survey (Roberts & Henson, 2000) which measures teaching efficacy (confidence in math teaching) and knowledge efficacy (confidence about their math knowledge). Each scale includes a five-point Likert scale. Scores from each subscale were averaged indicating the level of knowledge and personal efficacy of each teacher.

Findings

Table 1 includes findings from the analyses of the data related to beliefs, efficacy, emotions and identity, which was administered in a pre-post format. Table 2 shows the participants' scores on the four dimensions of the MQI.

Table 1: Findings related to teachers' beliefs, efficacy, identity and emotions

	Rehema		Peace	
Construct	**Pre**	**Post**	**Pre**	**Post**

Beliefs about teaching and learning	So if the teacher's method was a method that the student didn't get it, I would teach mine for the teacher to understand.	So first of all, you just have to prepare, get to know what you are about doing. Get a starter, let the kids think for themselves. Get some manipulatives attached to your storytelling, or any other game that you want the kids to do.	Okay. So I would advise the teacher to have the children in watching a lot of video and then involving them in a lot of activities by themselves	So I'll just tell the person to calm down and have extra time for those students. If in the first lesson, the teacher didn't use a lot of manipulatives, I think this time around, they can use it and see if there will be a change.
	Shift from an approach to learning as telling to one where learner is engage in sense-making		***Shift from an approach to learning with focus on watching others accompanied by independent practice to one where students use tools to engage in sense-making***	
Knowledge Efficacy	2.88	4.38	4.25	3.43
Personal Efficacy	2.13	3.63	4.38	3.38
Identity	4	6	No response	6
Emotions	Anxiety: 2.5 Enjoyment:3.0 Anger: 2.25	Anxiety: 1.75 Enjoyment: 3.5 Anger: 1.5	Anxiety: 1.0 Enjoyment:3.75 Anger: 1.25	Anxiety: 2.0 Enjoyment: 3.0 Anger: 2.25

Table 2. Participants' scores on Mathematical Quality of Instruction (MQI) Instrument

Construct	Participant	Richness of the Mathematics	Working with Students and Mathematics	Errors and Precision	Common Core Aligned Practices
Math Instruction	Peace (Pre)	NP (not present)	L (low)	NP	NP
	Peace (Post)	L	M (Mid)	NP	L
	Rehema (Pre)	NP	L	L	L
	Rehema (Post)	M	M	NP	M

Discussion

Coach's work with Rehema. As Table 1 indicates, Rehema made notable shifts in all attributes related to her teaching. Rehema entered the program with low efficacy (see pre-scores) based on her prior experiences with math which manifested in designing lessons with low cognitive demand, avoiding rigorous content and meaningful student engagement. Due her low efficacy she was not initially open to trying new teaching methods, so the coach focused on improving her efficacy while ensuring that it was calibrated with her knowledge and skill. The coach focused on enhancing her content knowledge, knowledge of students' thinking, and strategies to address related misconceptions around the mathematical ideas. The goal was to intentionally craft mastery experiences. Drawing on the work around self-efficacy, mastery experiences have been shown to improve efficacy beliefs (Bandura, 1997), as we observed with Rehema. To centralize the mastery experience for Rehema, the coach to not model practices or co-teach; instead, they supported Rehema in practicing teaching ahead of time, so she was able to attribute the successful aspects to the lesson to her own teaching. In this regard, the HIC approach deviates from recommendations in the coaching literature that encourages modeling and co-teaching. Instead, we caution that this approach may not be effective for all teachers based on their profiles. As described in the literature (Keltchermans, 2009), we subsequently saw how her changing efficacy enhanced her positive emotions and a stronger conception of herself as a math teacher. In alignment with her increased efficacy, we observed shifts in her instruction towards working more collaboratively with students and allowing for more student ideas to be shared.

Coach's Work with Peace. Peace entered the program with high efficacy in both her knowledge of math and ability to teach math well. However, her high efficacy was not aligned with her teaching quality, and she believed it was the teacher's job to show students what to do. In this regard, the coach focused on creating better alignment between her efficacy beliefs and instruction. So instead of providing more problematic tasks for Peace, the coach allowed her to lead in task design just offering suggestions about how to tweak textbook tasks and incorporate manipulatives. The coach was intentional about helping her identify the ways in which the minor changes Peace made in her lesson supported students' thinking. Although her instruction improved overall, Peace struggled in shifting her role from being the one to show students how to think about and do math to students having greater voice and agency in doing math. Although she developed stronger teaching skills and having a robust math teacher identity, we observed declines in her efficacy and overall enjoyment in teaching math, and slightly increased anxiety. Researchers (Favre & Knight, 2016) have noted that declines in teachers' efficacy beliefs may occur when reforms require changes to teaching performance. Teachers experience a period of "pedagogical discontentment" (Southerland et al., 2011) and are inclined to reevaluate their efficacy beliefs.

Our experiences with Peace and Rehema show that framing interactions with teachers based on who they are as professionals and learners increases their receptiveness to changing their instructional practices. However, as teachers are individual adult learners in the context of professional development, it is essential that we acknowledge their psychological and affective needs and ensure that coaching attends to these individual differences in ways that bode well for teacher and student outcomes.

References

Bandura, A. (1997). The anatomy of stages of change. *American journal of health promotion: AJHP*, *12*(1), 8-10.

Brackett, M., & Cipriano, C. (2020) Retrieved at https://www.edsurge.com/news/2020-04-07-teachers-are-anxious-and-overwhelmed-they-need-sel-now-more-than-ever

Cobb, P. & Jackson, K. (2012): Analyzing Educational Policies: A Learning Design Perspective, *Journal of the Learning Sciences*, 21:4, 487-521

Cobb, P., Jackson, K., Henrick, E. C., & Smith, T. M. (2018). *Systems for instructional improvement: Creating coherence from the classroom to the District Office*. Harvard Education Press.

Cross Francis, D. (2015). Dispelling the notion of inconsistencies in teachers' mathematics beliefs and practices: A three-year case study. *Journal of Mathematics Teacher Education, 18*(2), 173 - 201.

Cross Francis, D., Eker, A., Liu, J., Lloyd, K., & Bharaj, P.K. (2021). Mis (alignment) between noticing and instructional quality: The role of psychological and cognitive constructs. *Journal of Mathematics Teacher Education.* https://doi.org/10.1007/s10857-021-09509-0

Cross Francis, D., Liu, J., Bharaj, P. K., & Eker, A. (2019). Integrating social-emotional and academic development in teachers' approaches to educating students. Policy Insights from the Behavioral and Brain Sciences, 6(2), 138–146. https://doi-org.libproxy.lib.unc.edu/10.1177/2372732219864375

Cross Francis, D. I., Hong, J., Liu, J., Eker, A., Lloyd, K., Bharaj, P. K., & Jeon, M. (2020). The Dominance of Blended Emotions: A Qualitative Study of Elementary Teachers' Emotions Related to Mathematics Teaching. Frontiers in Psychology, 11, 1865. https://doi.org/10.3389/fpsyg.2020.01865

Favre, D. E., & Knight, S. L. (2016). Teacher Efficacy Calibration in Education Reform: When Highly Efficacious Teachers Don't Spell "Implement." *International Journal of Educational Reform*, *25*(4), 361–383. https://doi.org/10.1177/105678791602500402

Fives, H., & Gill, M. G. (Eds.). (2015). *International handbook of research on teachers' beliefs.* Routledge.

Frenzel, A. C. (2014). Teacher emotions. In R. Pekrun & L. Linnenbrink-Garcia (Eds.), International handbook of emotions in education (pp. 494–519). New York: Routledge.

Frenzel, A. C., Pekrun, R., Goetz, T., Daniels, L. M., Durksen, T. L., Becker-Kurz, B., & Klassen, R. M. (2016). Measuring teachers' enjoyment, anger, and anxiety: The Teacher Emotions Scales (TES). *Contemporary Educational Psychology*, *46*, 148-163.

Garet, M. S., Porter, A. C., Desimone, L., Birman, B. F., & Yoon, K. S. (2001). What Makes Professional Development Effective? Results From a National Sample of Teachers. *American Educational Research Journal*, *38*(4), 915–945. https://doi.org/10.3102/00028312038004915

Gibbons, L., & Cobb, P. (2016). Content-Focused Coaching: Five Key Practices. *The Elementary School Journal, 117*, 237 - 260.

Hill, H. C., Blunk, M. L., Charalambous, C. Y., Lewis, J. M., Phelps, G. C., Sleep , L., & Ball, D. L. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. Cognition and Instruction, 26 (4), 430 – 511.

Hong, J.Y. (2010) Pre-Service and Beginning Teachers' Professional Identity and Its Relation to Dropping out of the Profession. Teaching and Teacher Education, 26, 1530-1543. https://doi.org/10.1016/j.tate.2010.06.003

Kelchtermans, G. (2009) Who I am in how I teach is the message: self-understanding, vulnerability and reflection, Teachers and Teaching, 15:2, 257-272, DOI: 10.1080/13540600902875332

Kraft MA, Blazar D, Hogan D. The Effect of Teacher Coaching on Instruction and Achievement: A Meta-Analysis of the Causal Evidence. Review of Educational Research. 2018;88 (4) :547-588.

McDonald M.M., Zeigler-Hill, V., Vrabel J,K., & Escobar, M. (2019). A Single-Item Measure for Assessing STEM Identity. Frontier in Education 4:78. doi: 10.3389/feduc.2019.00078

Roberts, J. K., & Henson, R. K. (2000). *Self-Efficacy Teaching and Knowledge Instrument for Science Teachers* (SETAKIST): A Proposal for a New Efficacy Instrument.

Schutz, P. A., Hong, J. & Cross Francis, D. I. (2020). Teachers' Goals, Beliefs, Emotions, and Identity Development. Routledge Publishing: New York, NY.

Southerland S. A., Sowell S., Blanchard M., & Granger E. M. (2011). Exploring the construct of pedagogical discontentment: A tool to understand science teachers' openness to reform. *Research in Science Education,* 41(3), 299–317.

Willingham, D., & Riener, C. (2019). Cognition: The Thinking Animal (4th ed.). Cambridge: Cambridge University Press. doi:10.1017/9781316271988

INCREASING STUDENT ENGAGEMENT IN MATHEMATICAL MODELING THROUGH TEACHER PROFESSIONAL DEVELOPMENT

Geena Taite
Montclair State University
taiteg1@montclair.edu

Amanda Provost
Montclair State University
provosta1@montclair.edu

Frank Forte
Montclair State University
fortef@montclair.edu

Andre Perkoski
Montclair State University
perkoskia1@montclair.edu

Youngjun Kim
Montclair State University
kimy8@montclair.edu

Research shows the benefits of mathematical modeling but provides little guidance on how to implement it successfully. This action research study documents and responds to the challenges perceived by five in-service high school mathematics teachers while implementing mathematical modeling at a STEM school in the U.S.. The team of researchers (led by a teacher at the school) aligned modeling tasks to the school's curricula, created and delivered professional development (PD) sessions, conducted surveys and interviews, and made classroom observations. The constant comparative method was used to analyze data and to design PD sessions that provided meaningful supports informed by teachers' challenges. The final stage of analysis revealed that some of the student engagement challenges teachers faced at the beginning of the school year were reported as benefits by the end of the school year.

Keywords: Modeling, Professional Development, High School Education, Research Methods

Review of the Literature

International and national mathematics achievement tests show a drop in U.S. students' mathematics performance over the last several years (e.g., Mullis et al. 2020). Additionally, "by the time many students enter high school, disengagement from course work and serious study is common" (National Research Council, 2004, p. ix). Due to this trend, it is imperative to capture student interest and increase their engagement in productive mathematics learning. This can be accomplished through mathematical modeling. Garfunkel and Montgomery (2019) define mathematical modeling (modeling) as "a process that uses mathematics to represent, analyze, make predictions or otherwise provide insight into real-world phenomena" (p. 8). Modeling engages students in relevant, accessible, and real-world problems while providing opportunities to connect to students' lives (Akapame, 2022) and to integrate interdisciplinary content that garners students' interest (Maiorca & Stohlmann, 2016).

Ingram (2013) defines engagement as "students' involvement in the mathematical activity of the classroom and their commitment to learning mathematical content" (p. 1). Modeling engages students in collaboration, critical thinking, and creativity (Suh et al., 2017) as well as the negotiation and justification of mathematical reasoning (Scott-Wilson et al., 2017). By using culturally relevant tasks, modeling also supports students pursuing STEM careers which are driven by students' interests (Caspi et al., 2019) while preparing them for the critical thinking required in STEM professions (Green & Emerson, 2010). The Common Core State Standards (National Governors Association, 2010) include Mathematical Practice 4: Model with mathematics, that requires K-12 teachers to implement modeling activities within their classrooms. Ferri (2013) argues that teacher educators and researchers have a duty to prepare pre-service and in-service teachers for modeling. Garfunkel and Montgomery (2019) lay out numerous best practices and guiding principles for teachers as they enact modeling tasks with their students. However, little guidance is provided on how teachers should implement these practices, especially when these principles conflict with teacher beliefs (Zbiek, 2016). Research has focused on the actions teachers should take when enacting modeling tasks (Gerhardt, 2017; Mason, 2001), rather than the challenges teachers face. In response to these issues, PD can provide teachers with support to successfully implement modeling.

For in-service teachers to adequately implement modeling in the classroom, they need support through effective PD. Desimone (2009) lays out five components of effective professional development: content focus, active learning, coherence, sustained duration, and collective participation. Additionally, Desimone and Garet (2015) noted that “PD is more successful when it is explicitly linked to classroom lessons” (p. 254). These features of effective PD have strong implications for the way that effective modeling PD is designed, including the development of curricula aligned tasks and teacher collaboration to scaffold and adapt tasks to their classroom. Jung and Brady (2016) posit that teacher-researcher partnerships could be an avenue for effective modeling PD but note that more institutional support for such collaboration is needed due to numerous challenges that can arise as teachers implement modeling.

Theoretical Framework

Collaborative teacher participation in PD can be facilitated through the theoretical perspective of a community of practice (CoP). Via sociocultural theory, a CoP is a group of people mutually engaged in an activity for a joint enterprise developing a shared repertoire of resources (Lave & Wenger, 1991; Wenger, 1998a, 1998b). In this study, in-service high school mathematics teachers were engaged in PD to develop pedagogies for teaching modeling. Thus, our participants formed a CoP through the common goal of engaging their students in modeling.

This study implements PD as a CoP through action research. Smit (2013) identifies action research as an avenue for meaningful change within the school structure and for teacher PD. Frequent feedback from the community of teachers was used to responsively design relevant, effective PD to address teacher challenges with implementing modeling, a shared goal within the CoP. While this work is part of a larger study, this paper addresses the research question: How do in-service high school mathematics teachers’ perceptions of student engagement evolve as teachers participate in PD and enact mathematical modeling tasks with their students?

Methods

This study took place over the 2021-2022 school year at a public STEM high school in the northeast U.S.. The school offers Algebra 1 and Algebra 2, each as a two-part course, one part focused on content standards and the other part focused on applications. Ellie, Savannah, and Aimee taught Algebra 1 Applications and Zach taught Algebra 2 Applications. Zach left the school in November and was replaced by another participant, Jack. Aimee, Zach, and Jack have eight to ten years of experience while Ellie and Savannah have fewer than two years’ experience.

This action research study (Herr & Anderson, 2014) was led by an insider teacher-researcher (first author) who taught both Applications courses. The other research team members are four fellow doctoral students and a student volunteer from the school. The research team designed and implemented PD as a CoP to address the evolving challenges in-service mathematics teachers face while implementing mathematical modeling. Teachers participated in modeling tasks at PD before enacting them with their students. For example, the September PD engaged them in the Having Kittens task which posed the question, is it realistic for a female cat to have

2000 descendants in just 18 months? (see Mathematics Assessment Resource Service [MARS], 2015). Teachers also had access to a library of modeling tasks organized by the research team and aligned to their curriculum like the Team Ranking task (Carmona & Greenstein, 2013). Given a unit-less axes graph with losses on the x-axis and wins on the y-axis, modelers had to decide a fair way to rank the top five soccer teams.

Data was collected through surveys, interviews, check-ins, observations and the PD sessions to triangulate data from multiple sources. The five PD sessions were developed by the research team based on the analysis of the data from the teachers throughout the year by the insider and one or more research team members. Surveys were distributed at the end of each PD session (to gather feedback on PD and supports desired for future sessions) and a few weeks before each interview (to check in with teachers about their modeling challenges and supports needed). In addition, surveys with questions about teacher beliefs were also conducted. Three rounds of interviews were conducted and recorded. Member checking for clarification and confirmation of findings was conducted during interviews. Finally, the insider researcher observed participants engaging their students in modeling when invited by the participating teachers.

This study used the constant comparative approach (Merriam & Tisdell, 2016) to identify changes in the teachers' evolving challenges related to engagement while implementing modeling. First, open coding was done in teams of two or more researchers to ensure inter-rater reliability. Some examples of our initial coding categories included challenges, supports needed, teacher beliefs, positives, and perceptions. Within each category, we organized similar responses. For example, student engagement and perseverance were initial subcategories in challenges, and engagement was a sub-category in positives. Within those categories, we then looked for trends. All five researchers coded during the axial stage of coding that was used to inform each PD session. Final selective coding by all five researchers took place after the school year ended.

Findings

Early on, participants indicated challenges related to student engagement. As a research team, we addressed these challenges at the December 2021 PD. We asked our participants to brainstorm ways to increase engagement and to build perseverance (Figure 1). Then we shared some additional ways to build engagement specific to modeling and based on research (Cirillo et al., 2016; DiNapoli & Miller, 2022; Vorholter & Kaiser, 2016).

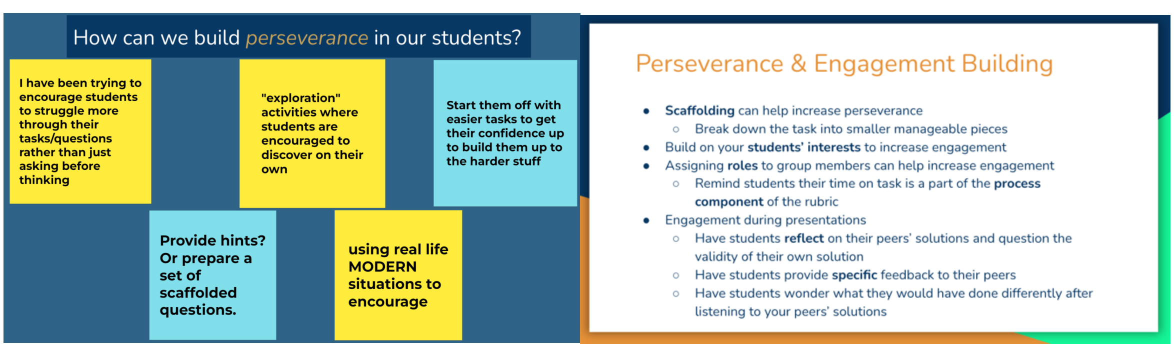


Figure 1: Perseverance and Engagement - December 2021 PD

Once the school year concluded, the final stage of analysis revealed some interesting findings related to teachers' perceptions of student engagement. Participants indicated engagement as a challenge in the first half of the year, but noted more times that it was a positive of modeling toward the end of the year. For example, Savannah indicated in her October 2021 survey that her students tend to "groan a lot more when I tell them we're working on a project." In her June 2022 interview, Savannah shared that her students "loved that they could make a video out of it" for the TikToker Car Purchase task. This was a task developed by our student volunteer in which students created a video explaining which of the three deals would be the best choice and why.

Another example of a shift in student engagement can be seen through Ellie's experience. In Ellie's October 2021 interview, she indicated that while students were presenting their final solutions to the class, "they got very bored which I was not expecting." During a classroom observation in November, five out of six groups of students were fully engaged in the Team Ranking task. In Ellie's June 2022 interview, she indicated, "they get pretty into it and interested in it…in the beginning of the year it was definitely more of a problem though than it is now."

Aimee identified in her January 2022 interview that student perseverance was an issue and that perseverance was important to keep students engaged. In her June 2022 interview, she cited the scaffolds and hints given to students as ways she kept students engaged by increasing student perseverance in the modeling tasks. Ellie also commented on students' perseverance as it related to their engagement and said "…most of them, they get pretty into it and interested in it…In the beginning of the year, it was definitely more of a problem than it is now" (June 2022 Interview).

Finally, students' view of modeling in mathematics appeared as a greater challenge in the beginning of the year than the end. This played a role in their engagement. In her October 2021 interview, Ellie mentioned that because some of her students were too focused on a "right versus wrong answer" on the Having Kittens task, they struggled, a challenge that was echoed by all other participants in the study during the October 2021 interview. In contrast, in Ellie's January 2022 interview, she mentioned that her students were feeling "more comfortable" with mathematical modeling. She also mentioned the tasks became "less overwhelming to them."

Despite challenges, participants indicated student engagement to be a positive feature of mathematical modeling a total of 19 times throughout the year, citing that modeling tasks made it "a much livelier classroom" (Aimee, June 2022 interview) and "instead of memorizing the patterns, they'll remember the activity they did, and how much meaning that provides them through the space" (Ellie, June 2022 interview).

Discussion & Conclusion

Through our constant comparative analysis, we noted engagement challenges teachers faced implementing modeling in their classrooms, necessitating ongoing reflexiveness in the design of modeling PD. Particularly, student engagement in modeling shifted over the year. At the beginning of the school year, teachers noted that students were intimidated or frustrated with modeling tasks because of their complexity and open-ended nature. By the end of the year, teachers said that students were more comfortable and exhibited more engagement with and perseverance through modeling problems.

The literature recognizes that implementing modeling can be a challenge, yet there is little guidance on how to successfully support teachers to engage students (Zbiek, 2016). Our research highlights the benefit of PD as a CoP for supporting teachers of modeling. We provide an example of a "multiplier" teacher-researcher (Ferri, 2013) who shared knowledge on modeling and an example of PD as part of action research that addressed teachers' challenges related to student engagement. Adopting these strategies will allow students the opportunity to engage in authentic modeling tasks which build problem solving skills and allow them to make the connection between learning in the classroom and how mathematics is applied in the real world.

Acknowledgments

We would like to thank Dr. Kathryn Herr and Cameron Farid for their contributions.

References

Akapame, R. (2022). Connecting the “real-world” to the math classroom: Implementing professional development for mathematical modeling. Northwest Journal of Teacher Education, 17(1). https://doi.org/10.15760/nwjte.2022.17.1.2

Carmona, G., & Greenstein, S. (2013). Investigating the relationship between the problem and the solver: who decides what math gets used?. Modeling Students' Mathematical Modeling Competencies: ICTMA 13, 245-254.

Caspi, A., Gorsky, P., Nitzani-Hendel, R., Zacharia, Z., Rosenfeld, S., Berman, S., & Shildhouse, B. (2019). Ninth-grade students' perceptions of the factors that led them to major in high school science, technology, engineering, and mathematics disciplines. Science Education, 103(5), 1176-1205. https://doi.org/10.1002/sce.21524

Cirillo, M., Bartell, T. G., & Wager, A. A. (2016). Teaching mathematics for social justice through mathematical modeling. In C. R. Hirsch & A. R. McDuffie (Eds.), Mathematical modeling and modeling mathematics (pp. 87-96). National Council of Teachers of Mathematics.

Desimone, L. M. (2009). Improving impact studies of teachers’ professional development: Toward better conceptualizations and measures. Educational Researcher, 38(3), 181-199.

Desimone, L. M., & Garet, M. S. (2015). Best practices in teachers’ professional development in the United States. Psychology, Society and Education, 7(3), 252-263.

DiNapoli, J., & Miller. E. K. (2022). Recognizing, supporting, and improving student perseverance in mathematical problem-solving: The role of conceptual thinking scaffolds. Journal of Mathematical Behavior, 66, 1-20. https://doi.org/10.1016/j.jmathb.2022.100965

Ferri, R. B. (2013). Mathematical modeling-The teacher’s responsibility. Journal of Mathematics Education at Teachers College.

Garfunkel, S. & Montgomery, M. (2019). GAIMME: Guidelines for assessment and instruction in mathematical modeling education. COMAP and SIAM.

Gerhardt, I. (2017). How to make a math modeling class from scratch in six (not-so) easy steps. PRIMUS, 27(6), 618-640.

Green, K. H. & Emerson, A. (2010). Mathematical reasoning in service courses: Why students need mathematical modeling problems. The Mathematics Enthusiast, 7(1), 113-140. https://doi.org/10.54870/1551-3440.1178

Herr, K., & Anderson, G. L. (2014). The action research dissertation: A guide for students and faculty. Sage Publications.

Ingram, N. (2013). Mathematical engagement skills. In V. Steinle, L. Ball, & C. Bardini (Eds.), Proceedings of the 36th Annual Meeting of the Mathematics Education Research Group of Australasia (MERGA), 402-409. https://eric.ed.gov/?id=ED572907

Jung, H., & Brady, C. (2016). Roles of a teacher and researcher during in situ professional development around the implementation of mathematical modeling tasks. Journal of Mathematics Teacher Education, 19(2-3), 277-295. https://doi.org/10.1007/s10857-015-9335-6

Lave, J., & Wenger, E. (1991). Situated learning: Legitimate peripheral participation. Cambridge University Press.

Maiorca, C., & Stohlmann, M. S. (2016). Inspiring students in integrated STEM education through modeling activities. In C. R. Hirsch & A. R. McDuffie (Eds.), Mathematical modeling and modeling mathematics (pp. 153-161). National Council of Teachers of Mathematics.

Mason, J. (2001). Modelling modelling: Where is the centre of gravity of-for-when teaching modelling? In J. F. Matos, W. Blum, K. Houston, S. P. Carreira (Ed.), Modelling and Mathematics Education (pp. 39-61). Woodhead Publishing.

Mathematics Assessment Resource Service. (2015). Modeling population growth: Having kittens. Mathematics Assessment Project. https://www.map.mathshell.org/download.php?fileid=1708

Merriam, S. B., & Tisdell, E. J. (2016). Qualitative research: a guide to design and implementation. Fourth edition. John Wiley & Sons.

Mullis, I. V. S., Martin, M. O., Foy, P., Kelly, D. L., & Fishbein, B. (2020). TIMSS 2019 International Results in Mathematics and Science. Boston College, TIMSS & PIRLS International Study Center. https://timssandpirls.bc.edu/timss2019/international-results/

National Governors Association. (2010). Common core state standards for mathematics. https://learning.ccsso.org/wp-content/uploads/2022/11/Math_Standards1.pdf

National Research Council. (2004). Engaging schools: Fostering high school students' motivation to learn. National Academies Press.
Scott-Wilson, R., Wessels, D. C. J., Wessels, H. M., Swart, E. (2017). The hidden benefits of mathematical modelling for students with disabilities. In G. Stillman, W. Blum, & G. Kaiser (Eds.), Mathematical modelling and applications: International perspectives on the teaching and learning of mathematical modelling (pp. 455-465). Springer, Cham. https://doi.org/10.1007/978-3-319-62968-1_38
Smit, B. H. (2013). Young people as co-researchers: Enabling student participation in educational practice. Professional Development in Education, 39(4), 550-573.
Suh, J. M., Matson, K., & Seshaiyer, P. (2017). Engaging Elementary Students in the Creative Process of Mathematizing Their World through Mathematical Modeling. Education Sciences, 7(2), 62. https://doi.org/10.3390/educsci7020062
Vorholter, K., & Kaiser, G. (2016). Theoretical and pedagogical considerations in promoting students' metacognitive modeling competencies. In C. R. Hirsch & A. R. McDuffie (Eds.), Mathematical modeling and modeling mathematics (pp. 273-280). National Council of Teachers of Mathematics.
Wenger, E. (1998a). Communities of practice: Learning as a social system. Systems Thinker, 9(5), 2-3.
Wenger, E. (1998b). Communities of practice: Learning, meaning, and identity. Cambridge University Press.
Zbiek, R.M. (2016). Supporting teachers' development as modelers and teachers of modelers. In C. R. Hirsch & A. R. McDuffie (Eds.), Mathematical modeling and modeling mathematics (pp. 263-272). National Council of Teachers of Mathematics.

INTERPRETACIONES SOBRE LA MODELACIÓN, CONECTANDO EL DESARROLLO PROFESIONAL CON EL APRENDIZAJE EN EL AULA

INTERPRETATIONS ON MODELING, CONNECTING PROFESSIONAL DEVELOPMENT WITH CLASSROOM LEARNING

Verónica Vargas-Alejo	Luis E. Montero-Moguel	Luis A. Muñoz-Gonzalez
Universidad de Guadalajara	The University of Texas at San Antonio	Universidad de Guadalajara
veronica.vargas@academicos.udg.mx	luis.monteromoguel@utsa.edu	luis_87mg@hotmail.com

En este artículo se presentan resultados de una investigación cuyo objetivo fue estudiar la evolución de interpretaciones de un docente al participar en un proceso de desarrollo profesional centrado en la modelación matemática. La investigación fue cualitativa, tipo estudio de caso. El marco teórico fue la Perspectiva de modelos y modelación. El análisis de los datos recolectados permitió identificar las interpretaciones del docente y su evolución. Como resultado, se identificó que el docente reconoció que la modelación puede ser un vehículo para la enseñanza y aprendizaje de las matemáticas más allá de la aplicación de conocimiento matemático aprendido.

Palabras clave, Desarrollo profesional, Modelado, Formación de maestros en servicio.

Es importante enfocar esfuerzos para respaldar el desarrollo profesional docente con el fin de lograr el aprendizaje exitoso de los estudiantes en todos los niveles educativos (OCDE, 2022; UNICEF, 2022). Sin embargo, se ha identificado que faltan investigaciones orientadas a analizar cómo influye lo que aprenden los profesores en lo que sus estudiantes requieren o deben aprender (Desimone, 2023). Una forma de apoyar a los profesores en servicio o en formación para que desarrollen conocimiento y habilidades en modelación y su uso en el salón de clases es mediante la incorporación de actividades de modelación en los procesos de desarrollo profesional docente en los que participen como estudiantes y futuros profesores (Brady, 2018; Garofalo y Corum, 2019). Estos ambientes son importantes porque ofrecen la oportunidad a los profesores de experimentar la modelación en el aula como modeladores, lo cual hace posible que se reflexione sobre los procesos que podrían seguir los propios alumnos al desarrollar experiencias similares.

En este reporte se muestra la evolución de las interpretaciones de un profesor sobre la modelación mientras participaba en un proceso de desarrollo profesional, que implicaba desarrollar modelos al resolver MEAs y promover la modelación en el aula. La pregunta de investigación que se aborda es ¿cómo un profesor modifica y amplía sus interpretaciones acerca de la modelación durante este proceso de desarrollo profesional?

Marco teórico

Varios investigadores (English, 2021; Lesh, 2010; Lesh y Doerr, 2003; Makar et al., 2020) señalan que los estudiantes deberían aprender en el aula aspectos clave de la práctica de la modelación. "La gente usa modelos todos los días sin siquiera darse cuenta. Los modelos crean una estructura para predicciones (inferencias) que se pueden usar o adaptar a medida que cambian las situaciones" (Makar et al., 2020, p. 147). Lesh (2010), Ärlebäck y Doerr (2018) entre otros investigadores mencionan que los profesores deberían ofrecer a los estudiantes la oportunidad de experimentar la modelación en el aula tanto para aprender a modelar, como para profundizar en sus conocimientos matemáticos.

Investigadores como Brady (2018), Doerr y Lesh (2003), Lesh (2010) y Makar et al. (2020), promueven la incorporación de la modelación en el salón de clase bajo el enfoque de la Perspectiva de Modelos y Modelación [MMP, por sus siglas en inglés]. Esta perspectiva reconoce a la modelación como un objetivo de aprendizaje en sí mismo y un vehículo para la enseñanza y aprendizaje de las matemáticas (Ärlebäck y Doerr, 2018). Pone énfasis en fomentar procesos de desarrollo profesional docente [DPD] que involucren a administradores, investigadores, profesores y estudiantes para apoyar el desarrollo de procesos de modelación en el aula. Se fomenta que los investigadores construyan, revisen y refinen sus formas de pensar sobre los procesos de aprendizaje seguidos por los profesores y sus estudiantes. Se busca, también, la construcción, revisión y refinamiento de las formas de pensar de los docentes sobre los procesos de aprendizaje de sus estudiantes al realizar actividades en el aula (Schorr y Lesh, 2003).

La MMP sugiere utilizar actividades cercanas a la vida real (Lesh, 2010) denominadas MEAs (por sus siglas en inglés: Model Eliciting Activities), las cuales se diseñan con base en seis principios de diseño (Lesh et al., 2000). Brady (2018) señala que involucrar a los profesores en la resolución de estas los apoya para reconocer la importancia de la modelación, debido al potencial de estas actividades para promover el aprendizaje de las matemáticas. Esto coincide también con lo mencionado por Garofalo y Corum (2017)

> Al incorporar MEAs en los cursos de pedagogía, los formadores de profesores de matemáticas pueden apoyar el desarrollo de la propia comprensión conceptual de los PSMT [profesores de matemáticas de preservicio], acerca de la modelación matemática y el conocimiento necesario para incorporar tales tareas en su futura enseñanza. (p. 78)

Este investigador realizó una investigación para identificar el compromiso que podían desarrollar futuros maestros con la modelación matemática como estudiantes al resolver una MEA, así como su compromiso como futuros docentes para promover la modelación en el aula. Entre otros hallazgos encontró que a los futuros docentes les pareció que la MEA resuelta sería una actividad emocionante y atractiva para los estudiantes “que podían ayudarlos a ver las matemáticas como auténticas y útiles” (p. 84).

Lo anterior resalta una de las aportaciones de Doerr y Lesh (2003) en relación con la necesidad de dar la oportunidad al profesor de interpretar y reflexionar sobre su propia práctica docente, pues las interpretaciones de cualquier situación influyen en lo que decide o no hacer, cuándo hacerlo y por qué. Al igual que los estudiantes, el profesor aprende en función de la interpretación de sus propias experiencias. Por lo tanto, una característica distintiva de la excelencia en la enseñanza, de acuerdo con estos investigadores de la MMP, se refleja no solo en las acciones, sino en la riqueza y diversidad de las formas como el profesor interpreta, describe y reflexiona su propia práctica. Esto hace importante poner atención en el desarrollo de formas eficaces para analizar e interpretar las experiencias propias.

Metodología

La investigación fue cualitativa. La investigación cualitativa implica una descripción detallada del entorno o de las personas, seguida de un análisis de los datos por temas o problemas (Creswell y Creswell, 2017). Se utilizó Estudio de caso.

El profesor cuyos resultados se describen en este artículo estaba cursando tercer semestre de un posgrado en enseñanza de las matemáticas y, simultáneamente, impartía clases de estadística en el nivel universitario. El profesor estaba interesado en mejorar su práctica docente y tomó este curso por interés propio, siempre asistió a las sesiones y realizó todas las actividades solicitadas. Durante el curso interactuó con otros cinco profesores en servicio, quienes también participaban en el proceso de DPD.

El proceso de DPD fue virtual, se llevó a cabo durante 10 sesiones programadas en Moodle equivalentes a 30 horas en total. Cada sesión contenía alguna lectura, tareas, y un foro de reflexión o discusión grupal donde los profesores respondían alguna pregunta y comentaban entre sí las aportaciones de sus compañeros. Tres de las 10 sesiones fueron síncronas y se utilizaron para compartir reflexiones, retroalimentar avances, dudas y resultados de las implementaciones. El resto de las sesiones fueron asíncronas. El proceso de DPD consistió en las siguientes fases:

- Resolución individual y posteriormente en binas de una MEA1 "El Gigante Bondadoso" (Adaptación de la MEA Pie Grande de Lesh y Doerr, 2003)
- Reflexión sobre los modelos construidos y artículos de la MMP en sesión síncrona
- Resolución individual y posteriormente en binas de la MEA2 Reforestación (Vargas-Alejo y Cristóbal-Escalante, 2017)
- Reflexión sobre los modelos construidos en foros y tareas
- Implementación de una MEA con alumnos
- Reflexión sobre resultados en sesión síncrona
- Diseño de una MEA.

Los conceptos matemáticos subyacentes comunes en ambas MEAs fueron: proporcionalidad y función lineal. La MEA1 solicita que los profesores construyan modelos para estimar la altura de una persona con relación a la longitud de su pie. La MEA2 solicita que se construya un modelo para proyectar la reforestación de cierta cantidad de hectáreas de bosque a partir de cierto monto inicial. Los instrumentos de recolección de datos fueron: reportes de tareas, foros, diario, audio de una de las sesiones síncronas y la bitácora del investigador.

Se codificaron los datos en dos direcciones, relacionadas con las interpretaciones del docente sobre: la naturaleza de la modelación y la importancia de utilizarla en el aula. Estos códigos permitieron clasificar las interpretaciones en las categorías siguientes: la modelación como un objetivo de aprendizaje en sí mismo, y la modelación como un vehículo para la enseñanza y aprendizaje de las matemáticas. Los instrumentos de recolección de datos permitieron triangular la información y profundizar en ella para corroborar los hallazgos.

Resultados y Discusión

El análisis de los datos recopilados reveló el desarrollo, modificación y ampliación de las interpretaciones del profesor. Estas modificaciones y ampliaciones se manifestaron en sus reflexiones por escrito y verbalmente, así como en sus acciones al implementar una de las MEAs en el aula y en su informe sobre la implementación. En relación con la naturaleza de la modelación, el docente validó el uso de problemas que requerían la construcción de un modelo como respuesta, en lugar de simplemente obtener un resultado numérico. En cuanto a la importancia de utilizar la modelación en el aula, el docente destacó que las MEAs fomentan los procesos de matematización y otorgan significado a los conceptos matemáticos.

El docente consideró que era relevante proponer MEAs en el aula sin explicar previamente un tema matemático. A continuación, se describe en detalle el desarrollo y ampliación de las interpretaciones observadas durante el estudio.

Primera interpretación

Al preguntarle al docente sobre el tipo de actividades, situaciones o problemas que utiliza en el salón de clases para el aprendizaje de las matemáticas, se obtuvo la siguiente respuesta:

> Docente: En mi caso planteo problemas enfocados en el área de ciencias de la salud, pues mis alumnos son de la carrera de medicina. Se utilizan datos que pueden ser obtenidos en pacientes con alguna enfermedad.

Esta respuesta demuestra la importancia que el docente otorga al uso de problemas relacionados con la futura profesión de los estudiantes. A continuación, se presenta un ejemplo de uno de los problemas utilizados en clase:

> Docente: Se hizo un estudio sobre los pesos en onzas de tumores malignos del abdomen de 57 personas, los cuales se indican a continuación: 68, 63, 42, 27, 30, 36, 28, 32, 79, 27, 22, 23, 24, 25, 44, 65, 43, 25, 74, 51, 36, 42, 28, 31, 28, 25, 45, 12, 57, 51, 12, 32, 49, 38, 42, 27, 31, 50, 38, 21, 16, 24, 69, 47, 23, 22, 43, 27, 49, 28, 23, 19, 46, 30, 43, 49, 12. Realizar la tabla de distribución de frecuencias, la gráfica correspondiente y describir qué indican las medidas de tendencia central y dispersión.

En este problema se pueden destacar aspectos interesantes, como el uso de un contexto relacionado con la carrera de los estudiantes y la promoción del uso de diferentes representaciones (tablas y gráficas) con el objetivo de fomentar la comprensión de las medidas de tendencia central y dispersión.

En cuanto al proceso de resolución de problemas, el docente mencionó que sus estudiantes abordaban los problemas después de la explicación de ciertos temas matemáticos vistos en clase. La intención al incorporar problemas en el aula era: “ayudar al alumno a entender la utilidad de las matemáticas”.

> Docente: Propongo problemas con datos de pacientes, que pueden ser ficticios, creados por mí, obtenidos de un libro o de un artículo, pero que brindan al estudiante un panorama más concreto de la utilidad de esta área en su trabajo.

El docente también destacó que ponía énfasis en la argumentación, el análisis de los datos y la toma de decisiones cuando planteaba a los estudiantesproblemas y proyectos en el aula.

Segunda interpretación

La experiencia de resolver MEAs en el aula permitió al docente modificar y ampliar su concepción de los problemas. A continuación, se describen algunas de las acciones que tomó el docente al resolver la MEA1, así como las correspondientes interpretaciones que muestran cómo se diferenció un problema tradicional de una MEA.

El docente se enfrentó inicialmente a la dificultad de no contar con datos directos en la MEA, a diferencia de un problema tradicional. Resolver las MEAs resultó controvertido para el profesor debido a que se le pedía construir modelos en lugar de obtener resultados precisos.

En el siguiente extracto se observa cómo el profesor abordó la MEA 1 de forma individual. En primer lugar, señaló que surgieron dudas acerca de si la MEA1 se podía resolver. A continuación, reflexionó sobre el tipo de solución que podría encontrar: exacta o aproximada.

Docente: Me puse a investigar [en internet], porque dije ¿si se podrá? Luego dije: bueno… pues no se va a poder encontrar un [resultado] exacto; pero, le vamos a llegar con una aproximación.

El docente buscó datos en internet para ofrecer una respuesta a la situación-problema planteada, pero mencionó que no tuvo en cuenta el contexto de la situación y utilizó datos que no estaban relacionados. Esto es común según Lesh y Doerr (2003), ya que en un entorno de clase tradicional, el contexto de un problema generalmente no es considerado importante. Propuso un modelo que permitiera al cliente encontrar una solución y encontró una altura aproximada como respuesta.

En la resolución de la MEA2, el docente aprovechó su experiencia previa al resolver la MEA1. Esto concuerda con los hallazgos de Lesh y Yoon (2004), quienes señalan que las experiencias al resolver una MEA influyen en las acciones y el pensamiento de los estudiantes cuando se enfrentan a una segunda MEA. El docente intentó que el modelo propuesto fuera reutilizable, es decir, siguiendo los principios de la MMP (Brady, 2018; Lesh et al., 2000), buscó generalizar su solución para que permitiera al cliente responder a una variedad de situaciones similares.

Docente: Asimismo, podrías evitarte todos los cálculos anteriores con el archivo que puedes descargar del siguiente enlace […] En el cual encontrarás las tablas programadas con fórmulas en Excel (considerando los cálculos descritos con anterioridad). Las celdas en gris puede modificarlas para calcular en caso de tener otro presupuesto, querer reforestar un área con porcentajes de cada tipo de árbol distinta o simplemente que los árboles tengan un costo diferente.

La experiencia de resolver dos MEAs y discutirlas en sesiones grupales llevó al docente a reflexionar sobre la diferencia entre los problemas de los libros de texto y las MEAs. A continuación, se presenta su reflexión.

Docente: Me pareció importante identificar la diferencia entre los problemas tradicionales y las actividades provocadoras de modelos [MEAs]. En un problema tradicional el alumno debe sólo seguir pasos ya definidos de manera específica, pero en una actividad provocadora de modelos [MEA] el estudiante analiza, crea, modifica, para poder generar “el modelo” que le ayudará a resolver ciertas problemáticas, lo cual tuvimos que hacer en la actividad de Pie grande.

En este extracto se destaca que el análisis del problema y la construcción de “el modelo” fueron aspectos importantes del proceso de resolución de una MEA, según lo identificado por el docente. Al señalar “el modelo que le ayudará a resolver ciertas problemáticas”, el docente se refiere a la construcción de un modelo reutilizable permita resolver situaciones similares, pero con datos diferentes.

La interpretación de las MEAs como actividades donde no se siguen instrucciones precisas permite observar cómo el docente validó la existencia de problemas distintos al problema propuesto, el cual tenía instrucciones precisas, como: “realizar la tabla de distribución de frecuencias, la gráfica correspondiente y describir qué indican las medidas de tendencia central y dispersión”. Además, el docente reconoció el potencial de las MEAs al mencionar que “el estudiante analiza, crea, modifica”. Esto evidencia cómo se amplió la interpretación sobre la naturaleza de los problemas que se pueden incorporar en el aula y la importancia de utilizar

MEAs en el aula.

Tercera interpretación

La interpretación que el docente había construido hasta ese momento sobre la naturaleza de la modelación y la importancia de su uso en el aula influyó en la forma en que implementó una MEA con sus alumnos. Las acciones que el docente llevó a cabo durante la implementación de la MEA1 fueron las siguientes. El profesor decidió no explicar previamente algún tema matemático relacionado y no proporcionó instrucciones específicas para abordar la MEA1. En cambio, permitió a los estudiantes expresar sus ideas sobre la situación planteada en la MEA, utilizar sus propios recursos para resolver el problema, matematizarlo y encontrar una solución. Esto le permitió describir lo siguiente.

Docente: Aprender matemáticas bajo la perspectiva de modelos y modelación implica entonces matematizar problemas y situaciones cotidianas a las que se enfrentan los alumnos o personas en general.

El docente reflexionó sobre la importancia de fomentar el uso de MEAs en el aula para que los estudiantes puedan comprender la utilidad de las matemáticas

Docente: Resolver un problema sin una receta de pasos a seguir es una tarea que deberíamos fomentar en nuestros estudiantes, pues al generar un modelo para resolver una problemática de la vida cotidiana puede ayudar a entender al alumno la utilidad de las matemáticas.

El docente describió la evolución del conocimiento de los estudiantes al resolver la MEA1 utilizando ciclos de entendimiento "cualitativos y cuantitativos", usando los mencionados por Vargas-Alejo, Reyes-Rodríguez y Cristóbal-Escalante, 2016). Explicó que hubo una transición de formas de pensamiento cualitativas a formas de pensamiento cuantitativas a medida que los estudiantes avanzaban en la resolución del problema. Además, observó que, de acuerdo con la MMP, todos los estudiantes intentaron construir algún modelo, incluso aquellos con dificultades en matemáticas. Todos lograron dar sentido a varios conceptos de manera más accesible y comprensible.

Docente: Los modelos generados y procesos realizados ayudaron a mis estudiantes a apropiarse de sus conocimientos de una manera más sencilla.

El docente amplió su comprensión sobre la importancia de proponer problemas en el aula en el sentido de Ärlebäck y Doerr (2018). Reconoció a la modelación como un vehículo fundamental para la enseñanza y el aprendizaje de las matemáticas.

Docente: En conclusión, en este curso conocí las APM [MEA] en las cuales los estudiantes crean modelos propios, lo cual permite que se apropien de la situación y genere en ellos un aprendizaje significativo.

En uno de los últimos fragmentos escritos por el docente, después de diseñar una MEA, se puede confirmar una vez más su interés en motivar el aprendizaje de las matemáticas al mostrar su utilidad en futuras profesiones.

Docente: El objetivo de aprendizaje va más con la comprensión de la utilidad de las matemáticas en la medicina y la importancia de esta utilidad, tratar de eliminar o disminuir la idea en los alumnos de que no necesitan matemáticas en su área, otro objetivo va encaminado al uso de la tecnología en estos procedimientos y pienso concluir, después de esta actividad, orientada al uso de fórmulas en Excel, pues será de gran ayuda como introducción en las materias que imparto.

En conclusión, la evolución de las interpretaciones puede describirse de la siguiente manera (Tabla 1).

Tabla 1: Evolución de las interpretaciones

	1ª Interpretación	2ª Interpretación	3ª Interpretación
Naturaleza de la modelación	No se percibe, pero el docente utilizaba problemas con caracteristicas similares a los incluidos en los libros de texto	El docente reconoció que en la modelación no hay pasos específicos a seguir; el estudiante debe crear un modelo para resolver problemas	El docente reconoció que aprender matemáticas bajo la MMP es matematizar problemas y situaciones similares a las que se pueden enfrentar los individuos en su vida real
Importancia de utilizar la modelación	No se percibe la importancia. El docente usaba los problemas para denotar la utilidad de las matemáticas en la carrera profesional	El docente reconoció que en la modelación el estudiante analiza, crea, modifica, para poder generar "el modelo"	El docente reconoció a la modelación como un vehículo para la enseñanza y aprendizaje de las matemáticas. Reforzó que las matemáticas se pueden utilizar en la profesión futura

Conclusiones

En relación con la pregunta de investigación, el análisis nos permite concluir que el docente modificó sus interpretaciones iniciales respecto a la naturaleza de los problemas que puede usar en su clase. Esta modificación surgió de la reflexión sobre las MEAs, las acciones que llevó a cabo al resolverlas y los resultados obtenidos al implementar una MEA en el aula. El docente evaluó positivamente el potencial de la MEA para apoyar el aprendizaje de las matemáticas, ya que sus estudiantes pudieron utilizar sus propios recursos para resolverla y profundizaron en algunos conceptos matemáticos durante el proceso de construcción de modelos. Esto se reflejó en su papel durante la implementación de la MEA en el aula y en el ambiente de trabajo que creó.

Coincidimos con Doerr y Lesh (2003) en que la interpretación de las acciones realizadas por el profesor al resolver la MEA influyó en las decisiones tomadas durante su implementación en el aula, incluyendo cuándo y por qué llevarlas a cabo. Por ejemplo, el docente permitió que los estudiantes tuvieran el tiempo necesario para resolver la MEA utilizando sus propios recursos, sin darles instrucciones precisas a lo largo del proceso. Observó, analizó y brindó apoyo en la construcción de modelos.

Aunque se logró ampliar las interpretaciones sobre la naturaleza de la modelación y la importancia de utilizarla en el aula durante el proceso de DPD basado en la modelación, consideramos que estas interpretaciones aún pueden ser refinadas. Según los resultados de investigaciones previas (Montero-Moguel et al., 2020), sabemos que el profesor puede continuar ampliando, modificando y refinando estas interpretaciones a medida que participe en procesos de reflexión sobre su propia práctica docente. De esta manera, se puede fomentar un mayor desarrollo en las dos direcciones mencionadas en el marco teórico: promover el aprendizaje de la modelación en el aula como un objetivo en sí mismo, y al mismo tiempo, utilizarla como un vehículo para la enseñanza y el aprendizaje de las matemáticas.

Los resultados de esta investigación concuerdan con los hallazgos de estudios previos realizados por investigadores como Brady (2018), Clark y Lesh (2003) y Lesh (2010), quienes han mencionado que a través de enfoques multinivel (investigador, profesor y estudiante) y la resolución de MEAs, los docentes logran replantear su práctica docente. Estos enfoques les permiten reflexionar sobre cómo fomentar la modelación en el aula y por qué es importante hacerlo.

Además, es importante destacar que, aunque no se profundizó en ello en este artículo, se ha identificado que estos hallazgos se amplían al reconocer que el apoyo a la evolución de las interpretaciones de los docentes, a través del uso de MEAs, no se limita únicamente a entornos presenciales. También es posible promover esta evolución en entornos virtuales.

En resumen, esta investigación se suma a los hallazgos existentes y destaca la relevancia de utilizar MEAs como herramientas efectivas para promover cambios en la práctica docente y apoyar la evolución de las interpretaciones de los docentes tanto en entornos presenciales como virtuales.

Reconocimiento

Agradecimiento a CONACYT por el apoyo brindado a través de las becas a estudiantes, a la Universidad de Guadalajara y al proyecto Campus Viviente.

Referencias

Ärlebäck, J. B., y Doerr, H. (2018). Students' interpretations and reasoning about phenomena with negative rates of change throughout a model development sequence. *ZDM Mathematics Education*, *50*(1-2), 187–200. https://doi.org/10.1007/s11858-017-0881-5

Brady, C. (2018). Modelling and the representational imagination. *ZDM Mathematics Education*, *50*(1-2), 45–59.

Clark, K. K., y Lesh, R. (2003). A modeling approach to describe teacher knowledge. En R. Lesh y H. M. Doerr (Eds.), Beyond Constructivism. Models and modeling perspectives on mathematics problem solving, learning, and teaching (pp. 159-174). Mahwah, NJ: Lawrence Erlbaum Associates.

Creswell, J., y Creswell, D. (2017). *Research design: Qualitative, quantitative, and mixed methods approaches.* SAGE Publications.

Desimone L. M. (2023) Rethinking teacher PD: a focus on how to improve student learning. *Professional Development in Education*, *49*(1), 1–3, doi:10.1080/19415257.2023.2162746

Doerr, H. M., y Lesh, R. (2003). A modeling perspective on teacher development. En R. Lesh y H. M. Doerr (Eds), *Beyond Constructivism. Models and modeling perspectives on mathematics problem solving, learning, and teaching* (pp. 125–140). Mahwah, NJ: Lawrence Erlbaum Associates.

English L. D. (2021). Mathematical and Interdisciplinary Modeling in Optimizing Young Children's Learning. En J. M. Suh, M. H. Wickstrom y L. D. English (Eds.), *Exploring Mathematical Modeling with Young Learners*. (pp. 3-23). Springer. https://doi.org/10.1007/978-3-030-63900-6_1

Garofalo, J., y Corum, K. (2019). Engaging Preservice Secondary Mathematics Teachers in Authentic Mathematical Modeling: Deriving Ampere's Law. *Mathematics Teacher Educator*, *8*(1), 76-91.

Lesh, R. (2010). Tools, researchable issues and conjectures for investigating what it means to understand statistics (or other topics) meaningfully. *Journal of Mathematical Modeling and Application*, *1*(2), 16–48.

Lesh, R., Hoover, M., Hole, B., Kelly, A., y Post, T. (2000). Principles for developing thought revealing activities for students and teachers. En M. Hoover y R. Lesh (Eds.), Handbook of research design in mathematics and science education (pp. 591–645). Routledge.

Lesh, R., y Doerr, H. M. (2003). Foundations of a Models and modeling perspective on Mathematics teaching, learning, and problem solving. En R. Lesh y H. M. Doerr (Eds.), *Beyond constructivism: A models and modelling perspective on mathematics problem solving, learning and teaching* (pp. 3-34). Mahwah, NJ: Lawrence Erlbaum Associates.

Makar, K., Doerr, H., y DelMas, R. (2020). Developing statistical modeling with paper helicopters. *Mathematics Teacher: Learning and Teaching Pk–12*, *113*(2), 147–151.

Montero-Moguel, L., Vargas-Alejo, V. y Rodríguez-González, I. (2020). Conceptual system when implementing model-eliciting activities. En A.I. Sacristán, J.C. Cortés-Zavala y P.M. Ruiz-Arias (Eds.). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico (pp. 1653-1661). Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

OCDE. (2022). *Education GPS The world of education at your fingertips.* https://gpseducation.oecd.org/revieweducationpolicies/#!node=41732&filter=all

Schorr, R. Y., y Lesh, R. (2003). A modeling approach for providing teacher development. En R. Lesh y H. M. Doerr (Eds.), *Beyond constructivism: A models y modelling perspective on mathematics problem solving, learning y teaching* (pp. 141–157). Mahwah, NJ: Lawrence Erlbaum Associates.

UNICEF. (2021). *Educación. Cada niño tiene derecho a aprender*. https://www.unicef.org/es/educacion

Vargas-Alejo, V. y Cristóbal-Escalante, C. (2017). *MEA Propuesta de Reforestación*. Recuperada el 10 de diciembre de 2017 de https://plus.google.com/u/0/communities/100320204402382732939

Vargas-Alejo, V., Reyes-Rodríguez, A. V. y Cristóbal-Escalante, C. (2016). Ciclos de entendimiento de los conceptos de función y variación. *Educación Matemática, 28*(2), 59-84.

LEARNING TO FACILITATE REFLECTIVE CONVERSATIONS: EXPLORING CHANGES IN THE PRACTICES OF MATHEMATICS COACHES

Ryan Gillespie
University of Idaho
rgillespie@uidaho.edu

Jennifer Kruger
University of Rochester
jkruger@warner.rochester.edu

Adam Hanan
University of Idaho
hana3274@vandals.uidaho.edu

Julie M. Amador
University of Idaho
jamador@uidaho.edu

We examined changes in how mathematics coaches facilitated debriefing conversations after learning about a debrief conversational structure we created based on the principles of our content-focused coaching model. We compared participant coaches' first debriefing conversation, which occurred prior to learning about the debrief conversational structure to their second debriefing conversation, which occurred after learning about the debrief conversational structure. Findings indicate that in the second debriefing conversation, participant coaches participated more, prioritized different discursive moves, developed unique data collection systems to share observations from the co-taught lesson, and more frequently structured the conversation around content ideas related to the guiding principles of our content-focused coaching model.

Keywords: Teacher Educators, Professional Development

Debriefing conversations with a coach after teaching a lesson can support mathematics teacher growth through reflection in at least three ways: (a) identifying practical next steps to improve their practice (Gibbons & Cobb, 2016), (b) practicing using evidence of student thinking to consider responsive instructional decisions (West & Cameron, 2013), and (c) developing productive cognitive habits of reflecting and projective thinking (Costa & Garmston, 2016). However, these benefits depend on a coach's ability to facilitate debriefing conversations with clear goals and intentional decision-making—a process not clearly outlined in existing literature. To fill this gap, we developed a debrief conversational structure for content-focused coaches and designed a coherent set of learning experiences for our project participants to make sense of and implement the conversational structure with mathematics teachers. Participants, henceforth called participant coaches, were mathematics specialists from diverse contexts (Baker et al., 2022) who wanted to learn to facilitate content-focused coaching cycles (Callard et al., 2022) with mathematics teachers. We examined changes in the facilitation practices of participant coaches across two debriefing conversations; conversation one occurred prior to participant coaches learning about the debrief conversational structure and conversation two occurred after. Specifically, we answered two research questions: (1) How did the coaches' participation in debriefing conversations change after learning about a conversational structure? (2) How did the content of the coaches' verbal contributions in the debriefing conversations change after learning about the conversational structure? We strove to understand changes in both *how* the coaches talked to teachers and participated in the conversations and *what* content ideas the coaches prioritized after learning about the conversational structure.

Theoretical Perspective and Related Literature

We frame our study with the notion that content-focused coaching is complex work (e.g., Carlson et al. 2017; Saclarides & Kane, 2021). We conceptualize this complexity by viewing content-focused coaching as a form of disciplined improvisation, in which coaches' actions must be responsive to the contingent needs of a mathematics teacher but are also bound by the guiding principles and structures of a content-focus coaching model (e.g., Callard et al., 2022). Sawyer (2004) explained that frameworks play a critical role in disciplined improvisation as they support the interplay of structure and agency. Frameworks and structures can serve as productive common artifacts which help organize fluid and responsive interactions. In other words, collaborative discussions involve structure to be productive, but the structure must not be deterministic, allowing space for interlocutors to co-construct knowledge in novel and unpredictable ways.

Mathematics coaches are charged with fostering these types of fluid conversations to effectively support teachers; however, the field lacks knowledge on effective strategies for coach professional learning around how to facilitate productive coaching conversations as part of coaching cycles (Carlson et al., 2017; Kane & Saclarides, 2022; Mangin & Stoelinga, 2010; Stein et al., 2022). Two studies have examined how the inclusion of conversational structures and protocol can support coaches to grow and refine their practice. Baker and Knapp (2019) created the Decision-Making Protocol for Mathematics Coaching (DMPMC) and trained coaches to use the protocol. Baker and Knapp found that the DMPMC effectively supported coaches to grow in their ability to enact research-based coaching practices. Similarly, Russell et al. (2020) created and trained coaches in an inquiry-based coaching model. Russel et al. claimed coaches used the key practices outlined in the model in coaching cycles with teachers, resulting in conversations with greater depth and detail. Russell et al. also noted that coaches adapted their behaviors in unique ways that appeared contradictory to the principles of the model but were deemed productive in supporting teacher learning after more thorough analysis. Both studies highlight the potential of comprehensive models and protocols to support mathematics coaches to effectively act in ways that are both responsive and structured (i.e., disciplined improvisation). However, little is known about how coaches learn to use a comparatively smaller conversational structure within the specific context of facilitating debriefing conversations.

The Debrief Conversational Structure

We constructed our debrief conversational structure for mathematics coaches by considering existing debrief conversational structures from other coaching models (e.g., Costa & Garmston, 2016; Knight, 2007) and our primary content-focused coaching goals: (a) increase the teacher's content knowledge in a specific subject area and (b) build the teacher's knowledge of effective instructional practices related to that subject area, referred to as pedagogical content knowledge (Ball et al., 2008). Additionally, we considered recommendations from existing practitioner resources for content-focused coaches (e.g., West & Cameron, 2013) about the importance of examining evidence of student learning in relation to established mathematical and instructional goals. Thus, our debrief conversational structure contained four phases to guide coaches: (1) Reviewing goals established in the planning session; (2) Examining evidence of student learning related to the mathematical and instructional goals; (3) Considering contributing factors that may have supported or limited success toward the mathematical and instructional goals of the lesson; and (4) Reflecting on implications for the teacher's future practice (see Callard et al., 2022 for a fuller explanation of the debrief conversational structure).

Methods

Participants

Participant coaches for the study included four mathematics specialists who held various roles in their contexts (Baker et al., 2022). Table 1 shows years of experience (prior to beginning the project) and the positioning of each participant coach in their context outside of our project.

Table 1. Participant Coach Demographics

Participant Coach	Teaching Experience	Specialist Experience	Mathematics Specialist (MS) Positioning (Baker et al., 2022)
Briggs	4	0	MS as coach, Organization
Delgado	27	0	MS as coach, Mathematics Coach: District Level
Kennedy	6	0	MS as teacher, Teacher Leader
Lee	11	1	MS as coach, Mathematics Coach: District Level

Project Context

This study was situated within a larger professional development project in which we designed, implemented, and researched a fully online learning model for practicing or aspiring mathematics coaches. We engaged participants in three professional development components that included (a) five course sessions on content-focused coaching, (b) four video coaching clubs, and (c) two supported coaching cycles (Amador et al., 2020).

This study focused on the two supported coaching cycles within the project. For each coaching cycle, a participant coach facilitated a planning conversation, lesson implementation, and debriefing conversation with a teacher in their context. A mentor coach supported a participant coach to prepare for each coaching cycle. The first coaching cycle occurred immediately after participant coaches learned about facilitating planning conversations, through the course and video clubs, and their preparation with their mentor coach focused only on preparing for their upcoming planning conversation. Thus, participant coaches facilitated their first debriefing conversation without guidance or project support. The second cycle occurred after participant coaches learned about our debrief conversational structure and their preparation with their mentor coach focused on preparing for all three parts of the coaching cycle.

Data Collected

For this study, we analyzed the debriefing conversations for the first and second coaching cycles of the four participant coaches. This resulted in the analysis of four, cycle-one debriefing conversations (occurring prior to participants learning about the debrief conversational structure) and four, cycle-two debriefing conversations (occurring after participants learned about our debrief conversational structure). All eight coaching conversations were transcribed verbatim.

Data Analysis

To analyze conversation transcripts, individual talk turns for the coach and teacher were copied into separate rows in a spreadsheet. The teacher talk-turns were not analyzed but used for context when coding the coach talk-turns. Longer coach talk-turns (i.e., those containing substantially different ideas) were separated into different rows and considered distinct talk-turns. A single coach-talk turn was the unit of analysis.

Our research questions focused on understanding changes in *how* the coaches participated in the conversations and *what* content ideas the coaches prioritized. Given these two foci, we developed two codebooks. The first codebook detected content ideas in the coaches' talk-turns related to our debrief conversational structure. This codebook contained four dimensions, each connected to one part of our debrief conversational structure, with specific codes nested within each dimension. Figure 1 shows the details of this codebook.

Dimension	**Code**	**Definition** *Coach shares or invites the teacher to share a/an:*
Goals	Instructional Practice	instructional practice goals established by the teacher in the planning conversation.
	Math Learning Goal	mathematical concepts the teacher/coach wanted the students to learn.
Evidence of Student Learning	Student Thinking	observation about students' mathematical thinking during the lesson.
	Student Action	observation about students' actions or behaviors during the lesson.
Contributing Factors	Teacher Actions	observation about the teachers' actions/behaviors during the lesson.
	Lesson Design	observations about the design of the lesson/task.
Implications for Future Practice	Content	new idea involving the teachers understanding of mathematics that can be drawn from the lesson or prior discussion.
	Content-specific pedagogy	new idea about how students learn mathematics or how to teach mathematics that can be drawn from the lesson or prior discussion.
	Pedagogy	new idea about instruction (not specific to teaching and learning mathematics) that can be drawn from the lesson or prior discussion.

Figure 1. Codebook Focused on Content of Coaches' Talk

The second codebook characterized the coaches' discursive moves in each talk-turn and contained three dimensions: *directive*, *reflective*, and *facilitative*. Each of these dimensions contained three codes, created from existing literature on coach and facilitator discourse, to depict specific coach discourse moves (see Figure 2). *Directive* moves involved the coach sharing their thinking and opinions with the teacher which in turn positions the coach as the thinking authority within that instance of conversation (Ippolito, 2010; Sailors & Price, 2015). *Reflective* moves involved the coach inviting the teacher to share thinking and opinions which in turn positions the teacher as the thinking authority within that instance of conversation (Ippolito, 2010; Sailors & Price, 2015). *Facilitative* moves involved the coach establishing the (van Es et al., 2014). Facilitative moves did not contain the coach's thinking, nor did they invite substantial thinking from the teacher.

To analyze the data, three researchers independently coded the coach talk-turns in all eight transcripts and met to reconcile disagreements. Then, for each coach, the three researchers drafted analytic memos describing specific coach actions indicative of change between the coaches' cycle one and cycle two facilitation. The three researchers again met to reconcile individual memos into a single memo for each coach. Descriptive statistics related to the coaches' participation and facilitation were also recorded, including length of conversations, number of coded coach talk-turns, average words in a coded coach talk-turn, and the percentage of conversation words spoken by the coach.

First-Level Code	Definition	Second-Level Code	Definition
Directive	moves in which the coach shares their thinking and opinions with the teacher	Explaining	provide an interpretation of an event, interaction, or mathematical idea
		Suggesting	recommends an action
		Validating	Confirm and support teacher's contributions or actions
Reflective	moves in which the coach invites the teacher to share their thinking and opinions	Launching	pose general prompts to elicit teacher ideas
		Pressing	prompt teacher to explain their reasoning and/or elaborate on their ideas
		Paraphrasing	restating the teacher ideas in a manner that prompts the teacher to elaborate on their ideas
Facilitative	Moves in which the coach establishing the focus, direction, and/or clarity of the conversation	Describing	Direct attention to noteworthy ideas or events by describing the idea or event without inference, evaluation, and interpretation
		Clarifying	Prompts teacher to verify an idea to ensure common understanding
		Framing	Directing attention to the larger purpose or goal that was previously discussed

Figure 2. Codebook Focused on Coaches' Discursive Moves

Changes in Coach Participation

We found three patterns with respect to changes in how coaches participated in the conversations. First, all four coaches showed increased participation in the second debriefing conversation based on conversation lengths and how much and how often coaches spoke (see Table 2). For example, Briggs had roughly double the number of talk-turns in the second conversation (55) when compared to the first conversation (27) and conversation two (33:37) was approximately 13 minutes longer than conversation one (20:01). We found more dramatic changes in Lee's participation. In conversation one, Lee had 20 talk-turns that contained, on average, 6.5 words. In conversation two, Lee had 73 talk-turns containing an average of 36 words. This suggests that Lee spoke nearly 3.5 times more often in conversation two and each of these talk-turns was approximately 6 times longer than talk-turns in conversation one. We highlight that this increase in participation did not mean coaches dominated the second conversation (e.g., Heineke, 2013) as coaches contributed approximately 50% or less of the conversation words in conversation two. Instead, this pattern suggests that coaches became more equal contributors in their second conversation.

Second, we found changes in coaches' discursive patterns with respect to four discourse moves. All four coaches increased their use of *reflective: press* moves. In their first debriefing conversation, the four coaches collectively used 3 press moves. In conversation two, the coaches collectively used 28 press moves with each coach increasing their use of this move from conversation one to conversation two. This suggests that in the second conversation, coaches more frequently asked teachers to explain their reasoning and/or elaborate on or make connections between ideas currently being discussed. All four coaches also increased their use of *facilitative*: *describe* moves collectively increasing from 13 to 37 from conversation one to conversation two. To illustrate both these discourse moves, we share the following example from Lee in conversation two:

> When you think of the learning goal—so "understanding inverse operations and property equality and using them, applying that understanding to solve one-step equations", what evidence do you have then for the students in terms of what you saw that day or what you've seen today, the day after, that it worked?

In this example, Lee first described the learning goal established in the planning conversation and then asked the teacher to elaborate on their prior claim that the lesson had felt successful using specific evidence of student thinking. Thus, we coded the discursive moves in the talk-turn as *facilitative: describe* and *reflective: press*. We consider this noteworthy as Lee used zero *press* moves and only three *describe* moves in her first conversation, and this distinctive combination of describing and pressing connects directly to our debrief conversational structure. All four coaches also increased their use of *directive: validate* and *directive: explain* moves, further highlighting shifts in the ways coaches participated in the second debriefing conversation.

Table 2. Data Related to Coach Participation in Debriefing Conversations

Participant Coach	Conversation	Total Conversation Time	Coded Talk Turns	Words Per Talk Turn	Percent of Conversation Words Spoken by Coach
Briggs	Cycle 1	20:01	27	28.8	30.7%
	Cycle 2	33:37	55	29.4	29.4%
Delgado	Cycle 1	16:43	36	20.5	40.6%
	Cycle 2	20:16	53	31.7	49.8%
Kennedy	Cycle 1	5:42	19	9.6	52.1%
	Cycle 2	39:19	55	45.3	49.5%
Lee	Cycle 1	7:14	20	6.5	25.2%
	Cycle 2	25:34	73	36.0	45.5%

Third, from our thematic analysis of analytic memos, all four coaches created and used unique data collection systems to record teacher actions and student thinking in conversation two that were not found in conversation one. Furthermore, in conversation two, three of the four coaches structured the conversations in ways that allowed both the coach and teacher to collaboratively examine collected data. For example, coach Kennedy recorded student quotes throughout the lesson. During the debriefing conversation, Kennedy recommended taking 10 minutes of private think time for the coach and teacher to individually examine the quotes, select quotes that felt important to discuss further, and determine rationale for why the quotes felt salient. We find this noteworthy as the learning experiences we provided for coaches about using the debrief conversational structure never recommended or shared ways of creating data collection systems nor ways to publicly share data with teachers.

Changes in the Content of the Coaches' Verbal Contributions

We identified two patterns with respect to changes in the content of coaches' verbal contributions from debriefing conversation one to two. First, all four coaches talked more about *evidence of student learning, contributing factors*, and *implications* in their second conversation (see Table 3).

Table 3. Content of Coach Contributions

Participant Coach	Conversation	Goals	Evidence of Student Learning	Contributing Factors	Implications
Briggs	Cycle 1	3	2	0	8
	Cycle 2	7	6	11	13
Delgado	Cycle 1	4	5	8	3
	Cycle 2	3	13	20	5
Kennedy	Cycle 1	3	0	2	4
	Cycle 2	0	10	5	12
Lee	Cycle 1	2	3	2	2
	Cycle 2	7	6	18	18

This finding highlights that coaches more frequently incorporated three of the focal topics articulated in the debrief conversational structure into their second debriefing conversation. As a second finding, we found each coach shifted the content of their verbal contributions from conversation one to two in unique ways. For example, in debriefing conversation one, Briggs never shared or elicited thinking from the teacher about *contributing factors* such as teacher moves or the lesson design. In the second debriefing conversation, eleven of Brigg's talk-turns focused on the *contributing factors*. As a second example, Delgado's talk-turns in conversation two referenced *evidence of student learning* (13) and *contributing factors* (20) more frequently than talk-turns in conversation one (five and eight respectively). This change in the content of Delgado's talk-turns connects to changes in her participation and discourse patterns. In conversation one, six of Delgado's talk-turns were coded as *facilitative: describe*. In conversation two, the number of Delgado talk-turns coded as *facilitative: describe* increased to 18. This suggests that Delgado more frequently directed attention to student thinking and actions as well as teacher moves and lesson design in conversation two when compared to conversation one. To illustrate this claim, we share the following Delgado talk-turn from her second debriefing conversation:

> One group that you were with, the question was, "What fraction card should you use to make eight sixths?" You said, "you have wholes, thirds, fifths—which should you use?" Then they answered. Then you said, a whole—you questioned a whole. And you said, "how are you going to split that into sixths?" So what's you're thought on that? Listening to what you said?

In this example, Delgado references her data collection system to recall specific teacher questions and language. This low-inference observation then became the basis of a question to elicit thinking from the teacher. Thus, we coded the content of this talk-turn *contributing factor: teacher actions* and the discourse moves *facilitative: describe* and *reflective: launch*.

Discussion and Implications

The findings from both research questions provide new insight into how a conversational structure may have supported coaches in the act of disciplined improvisation (Sawyer, 2004) when facilitating a debriefing conversation with a mathematics teacher. From research question one, we found three patterns in changes to coaches' participation in the debriefing conversations after learning about our debrief conversational structure. First, coaches participated more fully in the debriefing conversations by speaking more often and at greater length, without dominating the conversation. Second, in conversation two, coaches more frequently pressed teachers to explain or elaborate on their thinking, shared low-inference descriptions of classroom events for public consideration, and provided explanations involving their own interpretations. Third, coaches created and used unique data collection systems in the second conversation to record and publicly share observed classroom events. From research question two, we found that the content of coaches' verbal contributions related to *evidence of student thinking*, *contributing factors*, and *implications* for future practice, increased after learning about our debrief conversational structure.

Similar to Baker and Knapp (2019) and Russell et al. (2020), we consider the changes documented in our findings to be productive in relation to the guiding principles and goals of our content-focused coaching model. Furthermore, we argue that providing a debrief conversational structure played a role in supporting participant coaches to act with enhanced disciplined improvisation. For example, in the first debriefing conversation, Lee made or invited only five total references to *evidence of student learning* or *contributing factors* and only two references to *implications*. In the second debriefing conversation, Lee made or invited 24 total references to *evidence of student learning* or *contributing factors* and 18 references to *implications*. The change in the content of Lee's verbal contributions suggests she increased the structure of her second conversation to focus more on topics central to the guiding principles of content-focused coaching (Callard et al., 2022; West & Cameron, 2013). Lee also used more press moves in the second conversation (13) than in the first conversation (0). This suggests Lee also acted more responsively, or with greater improvisation since press moves involved the coach asking a teacher to elaborate on or further explain shared thinking. We argue that Lee, and the other three participant coaches, were relatively inexperienced in coaching mathematics teachers and facilitating debriefing conversations. Thus, they lacked any disciplinary structure to support productive improvisational decisions that are a natural part of facilitating coaching conversations. Once equipped with a basic debrief conversational structure and a modest set of learning experiences, participant coaches made substantial changes to the structure and responsiveness of their coaching in ways that aligned with the guiding principles of our content-focused coaching model.

Coaches or mathematics specialists are often promoted into these new roles based on their expertise as classroom teachers, yet being an effective teacher is not a sufficient condition to becoming a successful coach (Carlson et al., 2017; Chval et al., 2010). Based on our findings and these claims from prior researchers, we offer implications for those supporting the development of coaches and conducting future research on coaching. Given the changes our four participant coaches demonstrated, we urge those working with coaches to consider adopting, adapting, or creating structures and protocols that operationalize guiding principles and goals of a coaching model in a particular context. Such structures should explicitly name desired coaching behaviors while also providing coaches space to operate responsively based on the needs of teachers. We also encourage future researchers to more carefully examine how changes in coach actions, driven by protocols and structures, influence teachers' opportunities to learn. Similar to Baker and Knapp (2019) and Russell et al. (2020), we argue that the changes in our participants' coaching behaviors were productive given the theories of teacher learning underpinning our coach model. However, future research should directly examine the relationship between changes in coaching behaviors and teacher learning opportunities to validate these inferences.

Acknowledgments

This material is based upon work supported by the National Science Foundation under Grant #2006263.

References

Amador, J., Choppin, J., & Callard, C. (2020). Synchronous online video-based development for rural mathematics coaches. Grant submitted to the National Science Foundation, DRK-12.

Baker, C. & Knapp, M. (2019). The decision-making protocol for mathematics coaching: Addressing the complexity of coaching with intentionality and reflection. *Mathematics Teacher Educator*, *7*(2), 27-43. https://doi.org/10.5951/mathteaceduc.7.2.0027

Baker, C. K., Saclarides, E. S., Harbour, K. E., Hjalmarson, M. A., Livers, S. D., & Edwards, K. C. (2022). Trends in mathematics specialist literature: Analyzing research spanning four decades. *School Science and Mathematics*, *122*(1), 24-35. https://doi.org/10.1111/ssm.12507

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special?. *Journal of Teacher Education, 59*(5), 389-407. https://doi.org/10.1177/0022487108324554

Callard, C., Kruger, J., Gillespie, R., & Foster, G. (2022). *Coaching mathematics teachers in-person and online: A content-focused coaching model* [White paper]**.** Center for Professional Development and Education Reform, University of Rochester. https://www.rochester.edu/warner/center/wp-content/uploads/2022/10/online-video-coaching241.pdf

Carlson, M. A., Heaton, R., & Williams, M. (2017, September). Translating professional development for teachers into professional development for instructional leaders. *Mathematics Teacher Educator*, 6(1), 27 - 39. https://doi.org/10.5951/mathteaceduc.6.1.0027

Chval, K. B., Arbaugh, F., Lannin, J. K., van Garderen, D., Cummings, L., Estapa, A. T., & Huey, M. E. (2010, September). The transition from experienced teacher to mathematics coach: Establishing a new identity. *The Elementary School Journal,* 111(1), 191-216. https://doi.org/10.1086/653475

Costa, A. L., & Garmston, R. J. (2016). *Cognitive coaching: Developing self-directed leaders and learners*. Rowan & Littlefield.

Gibbons, L. K., & Cobb, P. (2016). Content-focused coaching: Five key practices*. Elementary School Journal, 117*(2), 237-260. https://doi.org/10.1086/688906

Heineke, S. F. (2013). Coaching discourse: Supporting teachers' professional learning*. Elementary School Journal, 113*(3), 409-433. https://doi.org/10.1086/668767

Ippolito, J. (2010). Three ways that literacy coaches balance responsive and directive relationships with teachers. *Elementary School Journal, 111*(1), 164-190. https://doi.org/10.1086/653474

Kane, B. D., & Saclarides, E. S. (2022). Doing the math together: coaches' professional learning through engagement in mathematics. *Journal of Mathematics Teacher Education*, 1-30. https://doi.org/ 10.1007/s10857-021-09527-y

Knight, J. (2007). *Instructional coaching: A partnership approach to improving instruction*. Corwin Press.

Mangin, M. M., & Stoelinga, S. R. (2010). The future of instructional teacher leader roles. *The Educational Forum*, 74(1), 49-62. https://doi.org/10.1080/00131720903389208

Russell, J. L., Correnti, R., Stein, M. K., Thomas, A., Bill, V., & Speranzo, L. (2020). Mathematics coaching for conceptual understanding: Promising evidence regarding the Tennessee math coaching model. *Educational Evaluation and Policy Analysis*, 42(3), 439-466. https://doi.org/10.3102/0162373720940699

Saclarides, E. S., & Kane, B. D. (2021). Understanding mathematics coaches' development: Coaches' attributions of their professional learning in school districts. *International Journal of Educational Research,* 109, 1-12. https://doi.org/10.1016/j.ijer.2021.101815.

Sailors, M., & Price, L. (2015). Support for the improvement of practices through intensive coaching (SIPIC): A model of coaching for improving reading instruction and reading achievement. *Teaching and Teacher Education, 45*, 115-127. https://doi.org/10.1016/j.tate.2014.09.008

Sawyer, R. K. (2004). Collaborative discussion as disciplined improvisation. *Educational Researcher, 33*(2), 12-20.

Stein, M. K., Russell, J. L., Bill, V., Correnti, R., & Speranzo, L. (2022). Coach learning to help teachers learn to enact conceptually rich, student-focused mathematics lessons. *Journal of Mathematics Teacher Education*, 25(3), 321-346. https://doi.org/10.1007/s10857-021-09492-6

van Es, E. A., Tunney, J., Goldsmith, L. T., & Seago, N. (2014). A framework for the facilitation of teachers' analysis of video. *Journal of Teacher Education, 65*(4), 340-356. https://doi.org/10.1177/0022487114534266

West, L., & Cameron, A. (2013). *Agents of change: How content coaching transforms teaching and learning.* Heinemann.

MATHEMATICAL MEANING MAKING WITH STUDENTS THROUGH A PROCESS OF DOUBLE REFLECTION

Wesley Shumar
Drexel University
shumarw@drexel.edu

Amanda Reinsburrow
Drexel University
alr352@drexel.edu

Sienna Medina
Drexel University
sm4688@drexel.edu

Jason Silverman
Drexel University
js657@drexel.edu

This poster reports on a project that produced a series of online teacher professional development workshops. The workshop series was designed to support teachers in thinking about their own mathematical work, to share that work with other teachers and listen to how other teachers think about the same mathematical work, and finally to transfer that process to students, listening to what students are saying to better understand how students are thinking about the math. As one of the main goals of the workshops was to support teachers as reflective practitioners, the noticing and wondering framework was used to help the teacher stay focused on the specific work or ideas that the student has written in the problem-solving activity.

As mathematics researchers began to see math as a meaning making process (Boaler & Greeno, 2000; Bruner, 1996; Horn & Kane, 2015; Ma, 2016; Ray 2013; Schoenfeld, 1992), this led to the importance of understanding how one thinks about what they are doing and how others think about what they are doing. The workshops supported a process of “double reflection.” Similar to the work of other researchers (Gehlbach & Brinkworth, 2012; Teuscher, Moore, & Carlson, 2016), double reflection is about a relationship with an “other” and learning to think deeply about the other. The poster is based on preliminary research looking at three types of teachers, those who have a developed sense of reflection, those who have a developing sense of reflection and those who are just starting to reflect on their own and student work.

We utilized three data sources for this study, semi-structured interviews, discussion board conversations, and journal reflections. We completed three cycles of coding and triangulated across data sources (Merriam & Tisdell, 2015; Stake, 2010; Yin, 2009). The final stage of coding looked at how the forms of reflection lined up with each teacher in the workshop.

While the reflection comments are separate from the individuals who make them, it was clear from the data that most teachers tended toward one type of reflective comment which indicated their actual ability to engage in reflection. The teachers who fit mostly into the Developing Reflection category were the one exception where they occasionally made more Developed comments or also more Starting comments. The findings of this poster contribute to the understanding of how math teachers come to reflect on and engage with student’s mathematical thinking. As mathematics educators and researchers seek to create professional learning opportunities that not only develop mathematical knowledge, but also encourage deeper connections with student thinking, this study contributes to the growing knowledge regarding beneficial professional development for math teachers.

Acknowledgments

This work was supported by NSF Discovery Research K-12 Grant # 2010306. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

References

Boaler, J., & Greeno, J.G. (2000). Identity, agency, and knowing in mathematical worlds. In J. Boaler (Ed.), Multiple Perspectives on Mathematics Teaching and Learning (pp. 45-82). Stamford, CT: Ablex.

Bourdieu, P. (1990). The Logic of Practice. Stanford, CA: Stanford University Press.

Bruner, J. (1996). The Culture of Education. Cambridge, MA: Harvard University Press.

Dewey, J. (1938). Experience and Education. New York: Macmillan.

Gehlbach, H. & Brinkworth, M. E. (2012). The social perspective taking process: Strategies and sources of evidence in taking another's perspective. Teachers College Record, 114, 010308.

Heath, S. B. (1983). Ways with Words: Language, Life, and Work in Communities and Classrooms. Cambridge: Cambridge University Press.

Horn, I. S. & Kane, B. D. (2015). Opportunities for professional learning in mathematics teacher workgroup conversations: relationships to instructional expertise, Journal of the Learning Sciences, 24:3, 373-418, DOI:10.1080/10508406.2015.1034865

Ma, J. Y. (2016). Designing disruptions for productive hybridity: the case of walking scale geometry, Journal of the Learning Sciences, 25:3, 335-371, DOI: 10.1080/10508406.2016.1180297

Merriam, S. B., & Tisdell, E. J. (2015). Qualitative research: A guide to design and implementation. John Wiley & Sons.

Ray, M. (2013). Powerful Problem Solving: Activities for Sense Making with the Mathematical Practices. Portsmouth, NH: Heinemann.

Renninger, K. A. (2009). Interest and identity development in instruction: An inductive model. Educational Psychologist 44(2), pp. 105-118.

Schoenfeld, A. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In Grouws, D.A. (Eds.), Handbook of Research on Mathematics Teaching and Learning (pp. 334-370). New York: Macmillan.

Sfard, A. (2008). Thinking as communicating: Human development, the growth of discourses, and mathematizing. Cambridge University Press. https://doi.org/10.1017/CBO9780511499944

Stake, R. E. (2006). Multiple case study analysis. Guilford.

Teuscher, D., Moore, K. C., Carlson, M. P. (2016). Decentering: A construct to analyze and explain teacher actions as they relate to student thinking. Journal of Math Teacher Education 19, 433–456. DOI 10.1007/s10857-015-9304-0

Vygotsky, L. (1978). Mind in Society, Cambridge: MA: Harvard University, (Originally published in 1934).

Vygotsky, L. (1986). Thought and Language, Cambridge, MA: Massachusetts Institute of Technology, (Original work published in 1934).

Wittgenstein, L. (1953). Philosophical Investigations, New York: Macmillan.

Yin, R. K. (2008). Case Study Research: Design and Methods (4th edition). SAGE Publications, Inc.

MATHEMATICS COACHES' SUGGESTIONS: FOUCS ON TOPIC, LESSON PHASE, AND CLARITY TO SUPPORT TEACHERS' INSTRUCTIONAL PRACTICE

Julie M. Amador
University of Idaho
jamador@uidaho.edu

Ryan Gillespie
University of Idaho
rgillespie@uidaho.edu

Cynthia Carson
University of Rochester
ccarson@warner.rochester.edu

We examined the suggestions mathematics coaches provided to teachers as part of one-on-one coaching cycles. The purpose was to understand the object (content) of the suggestions, the lesson phase in which the suggestion would occur, and the clarity of the suggestion (how actionable the suggestion would be if the teacher followed the suggestion). Twenty-three coaching planning meetings were recorded, transcribed, and analyzed. Findings indicated that suggestions commonly focused on lesson design (how the lesson plan should be completed) or teacher questions (actual questions the teacher should ask). Almost half of the suggestions were about the explore phase (or middle) of the lesson and a majority of the suggestions were coded as medium or high clarity, meaning the coach clearly articulated what the suggestion would look like in the classroom. Implications for coaching and future coaching research are provided.

Keywords: Professional Development, Coaching, Online Professional Development, Directive, Mathematics Teacher Educator

Coaching is a professional development process used to support teachers to improve their instruction (West & Staub, 2003). Within mathematics education, content-focused coaching (e.g., West & Cameron, 2013) is a common model. Content-focused coaching involves iterative cycles in which a coach works one-on-one with a teacher with a focus on students' mathematical learning goals. Each coaching cycle contains three sequential components: a pre-conference discussion to plan a lesson; a collaboratively taught lesson; and a post-conference discussion to debrief the lesson (Bengo, 2016; West & Staub, 2003). Coaching is a dialogic endeavor in which the coach responds to multiple simultaneous obligations, such as designing a high-quality lesson, supporting the teacher to learn mathematics content and pedagogy, and establishing a trusting and productive relationship with the teacher. Given the complexity of coaching, it is important to explore the levers by which coaches manage these obligations (Gibbons & Cobb, 2016). A primary mechanism coaches employ is to suggest a course of action to the teacher. Doing so serves several purposes in relation to the obligations listed above: first, suggestions influence the design of the lesson; second, suggestions offer new or alternative courses of action for the teacher, thus creating opportunities to encounter new content and pedagogy; and third, suggestions push the boundary of the relationship between coach and teacher. From a research perspective, suggestions provide insights into the coach's perspectives on mathematics teaching and learning, and the coach's priorities with respect to a given coaching cycle. In short, studying the suggestions of coaches is a way to delve into the complexities of coaching.

There is a need for more research on how mathematics coaches using a content-focused coaching model interact with teachers (Gibbons & Cobb, 2016); especially around coaches' suggestions to teachers. Our study is situated within an innovative online mathematics video coaching experience that we adapted from an in-person modality to an online modality. We describe our three-part coaching model, with a specific focus on the video-assisted interactions between the coach and teacher. We studied the suggestions coaches provided to mathematics teachers by looking at the *Object* (content) of the suggestion, the *Lesson Phase* to which the suggestions applied, and the *Clarity* of the suggestion. Specifically, we answer the following research question: What is the nature of the suggestions coaches offered to teachers when using a content-focused coaching model?

Theoretical Framing and Related Literature

We theoretically frame this paper from a perspective of discourse, with a focus on how conversational turns occur, with attention to the particular stances of actors in a conversation (Ippolito, 2010). Research on coaching has highlighted two competing stances for how coaches talk with teachers: reflective or directive (Deussen et al., 2007; Ippolito, 2010; Sailors & Price, 2015). Coaches using a *reflective* stance emphasize collaborative inquiry in which the coach elicits ideas from the teacher; these ideas become the basis of the coach-teacher discussion (Ippolito, 2010). Coaching moves associated with a reflective stance include probing questions and low-inference non-evaluative observations, as means to catalyze teacher thinking (Costa & Garmston, 2016). In contrast, a *directive* coaching stance involves the use of suggestions and evaluative feedback (Ippolito, 2010). The challenge in content-focused coaching is to know how and when to provide a teacher with direct assistance, such as a suggestion, and when to employ an inquiry stance (West & Staub, 2003). It is crucial to understand how coaching moves, particularly suggestions, impact teacher learning and the uptake of new practices.

Despite the importance of mathematics coaches' strategic decision making when choosing appropriate actions when working with teachers, little is known about how mathematics coaches using a content-focused coaching model interact with teachers in ways they envision supporting teachers (Gibbons & Cobb, 2016). Furthermore, researchers such as Witherspoon et al. (2021) specifically call for new research on how direct assistance from a coach during a coaching cycle supports a teacher to implement new instructional practices. We focused on the suggestions coaches gave teachers in a coaching cycle (which included a planning meeting, lesson implementation, and a debriefing meeting) to characterize the focal topics for suggestions. The way coaches use suggestions and what they actually suggest can impact opportunities for teacher growth (Costa & Garmston, 2016; Heineke, 2013, Witherspoon et al., 2021); yet discerning moments and techniques for sharing suggestions with teachers is a primary challenge for coaches (West & Cameron, 2013). Researchers have identified that suggestions occur in the process of coaching (Gillespie et al., 2020), as noted with directive coaching stances (Deussen, et al., 2007; Ippolito, 2010; Sailors & Price, 2015), but researchers have yet to characterize coaches' suggestions in ways that would support professional developers, coaches, and researchers to consider the efficacy of suggestions in influencing the professional growth of teachers.

Methods

In our video-assisted online coaching model, coaches engaged teachers in typical content-focused coaching cycles that included a planning meeting, lesson implementation, and a debriefing meeting (see Figure 1). The planning and debriefing meetings were conducted via Zoom and videorecorded. The lessons were recorded using Swivl robot technology to capture audio and video data from middle grades mathematics classrooms. These data were automatically uploaded to the online Swivl platform. The coach and teacher then independently watched and annotated the video prior to the debriefing meeting. Figure 1 details the online video coaching process, showing both the collaborative and independent aspects of the cycle.

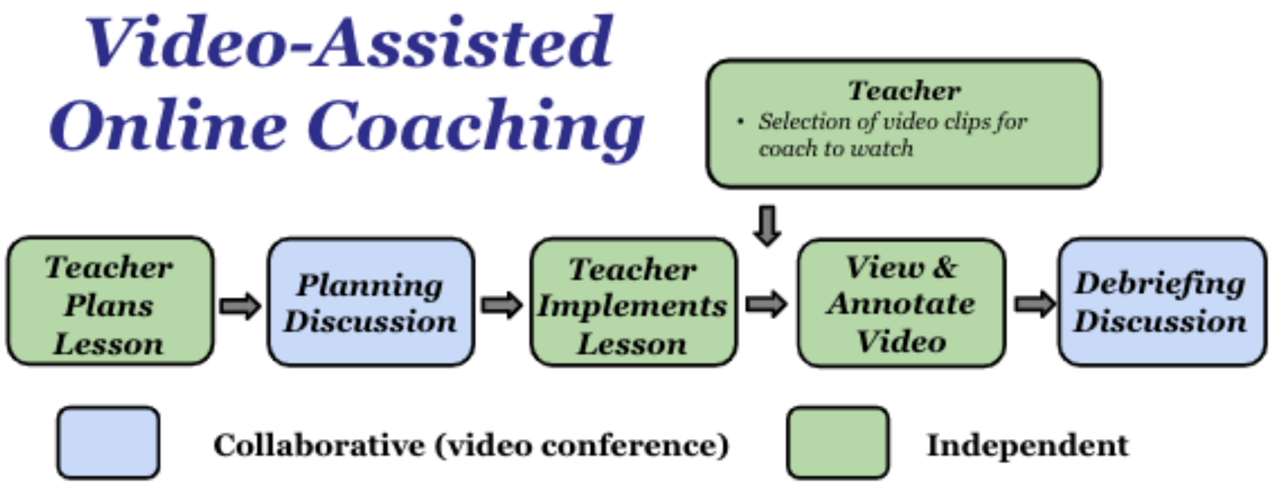


Figure 1. Online Video Coaching with content-focused Coaching

Data Collected

In total, there were three cohorts of teachers in the project, staggered over a four-year span, with each cohort participating for two years. Data for this study are transcripts from the Cohort One coach-teacher planning meetings as part of the one-on-one coaching cycles. Cohort One had eight teachers and four coaches, for a total of 23 transcripts of their online planning meetings.

Data Analysis

To identify the suggestions the coaches made during the collaborative planning conversations, we used results from a broader analysis of the coaching conversations. In that broader analysis, we parsed the transcripts of the coach-teacher planning and debriefing conversations into stanzas, which included a coach's statement and the teacher's response, as well as text needed for context (Saldaña, 2013). This broader data set included the analysis of 1719 stanzas from coaching conversation transcripts. We developed a codebook to analyze the discursive moves of the coaches and teachers as well as the content of the conversations within stanzas. We coded stanzas in pairwise teams after a lengthy calibration process that involved five researchers. We met via video conferencing software, Zoom, to reconcile disagreements. Kappas ranged from 0.39 to 0.65, which is considered moderate to strong reliability (Landis & Koch, 1977). Analysis for this study focused on the discursive moves of the coach. The section in the codebook on coaching discursive moves was comprised of five categories, including suggestions (see Figure 2). We defined a suggestion as a statement from the coach recommending an action for the teacher. We identified 273 suggestions in the 23 transcripts.

Coaching Stance	Discursive Move	Description	Example
Reflective	Invitation	Statement or question that invites the teacher to reflect or respond	"What might be some strategies we could use to increase student participation?"
	Description	Statements that share a direct observation and do not contain inference, interpretation, judgement or opinion	"I noticed that during the turn-and-talk, Alex did not say anything to his partner."
Directive	Suggestion	Statement that recommends an action	"I think we should use a turn-and-talk prior to the whole class discussion."
	Explanation	Statements that provide an interpretation or rationale of an event, interaction, or mathematical idea	"Turn-and-talk is a powerful strategy to use prior to a whole class discussion because it increases student participation."
	Evaluation	Statements that offer praise or critique	"I think it was a great idea to use a turn-and-talk."

Figure 2: Excerpt from the larger codebook focusing on coaching discursive moves

The following example is a coach's comments that was coded as a *suggestion*:

> One of the really nice moves you can do if the group shares a thought about something, and it's somewhat ambiguous, is you can turn to the class and say, "Can someone else use their own words to explain what Dave is saying?"

In this comment, the coach recommended that the teacher prompt students to paraphrase a peer's explanation as a means to increase student participation in classroom discussions.

Following the aforementioned coding to identify suggestions, two researchers wrote low inference paraphrases for each suggestion and later reconciled these to ensure the main action or idea of the suggestion was consistent across researchers. Table 1 shows two examples of excerpts from a planning meeting on a lesson on congruence, and the assigned low inference paraphrase.

Table 1. Suggestion examples from a coach-teacher planning meeting

	Transcript of Suggestion	Low Inference Paraphrase
Example 1:	"Have a discussion, I hope—- about which ones they put where, here and the fact there's nothing here. How are they going to justify that they're congruent?"	Have a discussion and encourage students to justify congruence
Example 2:	You could say, "Well, why don't you think you have anything over there? What have you seen?"	Probe students to explain their thinking if they struggle

We then analyzed the suggestions based on the actual transcript and the low inference paraphrase for the object (content), lesson phase, and clarity. To arrive at the Object codes, four researchers open coded (Corbin & Strauss, 2008) a subset of the larger data set and met to discuss and agree on the Object codes. Once we had a small set of Object codes, we went back to the data and coded another subset of the suggestions using the Object codes and met to reconcile and refine the definitions again. This resulted in the final list of Object codes, which referred to the topic of the suggestion. Codes for Object included Lesson Design (a suggestion about how the lesson plan or design should be completed and what should be included), Facilitating Discourse (a suggestion about promoting and facilitating classroom conversations), Teacher Questions (a suggestion of actual question(s) that should be asked during the lesson), and Teacher Action (a suggestion of what the teacher should do physically). Given the frequency and diverse nature of suggestions about Lesson Design, we divided the Lesson Design category into four subcategories to provide increased clarity about the aspect of Lesson Design being addressed. Subcodes for Lesson Design included: Represent (a suggestion about how representations of student work would materialize in the lesson), Resource (a suggestion about a particular resource or manipulative to use), Task (a suggestion about a particular task or modification of a task), and Participation Structure (a suggestion about how students should be grouped, whether whole-class, small groups, independent). To assign the Object codes, we used the low inference paraphrase from the initial analysis in addition to the original text for each suggestion. Lesson Phase codes included Launch, Explore/Investigate, and Summarize (e.g., Van de Walle et al., 2019), and identified the phase of the lesson in which the action proposed by the coach would occur. Clarity codes included Low, Medium, and High. Low clarity implied there were many ways for the teacher to enact the suggestion. Medium clarity implied there was more than one way to follow the suggestion, and High clarity implied there was only one way to follow a particular suggestion. Table 2 provides an example of the coding process for Object, Lesson Phase, and Clarity.

Table 2. Suggestion examples from a coach-teacher planning meeting

	Transcript of Suggestion	Object	Lesson Phase	Clarity
Example 1:	I think that would be great to try the warm-up of that. Then have a discussion about what they really found. You can call even the airplane distance between two points.	Facilitating Discourse	Launch	Low
Example 2:	That gives you the opportunity to ask, “How is this, when you get 4 S plus 4, how is that connected to S plus S plus 4?”	Teacher Questions	Summarize	High

In example one, the Object code is “Facilitating Discourse” because the suggestion focused on the teacher hosting a discussion after students attempted the warm-up activity. In example two, the Object code is “Teacher Questions” because the suggestion focused on a question the teacher should ask. The Lesson Phase of example one was coded as Launch because the suggestion pertained to the beginning phase of the lesson. Based on the context of the suggestion (in the transcript, not included here), the Lesson Phase of example two was coded as Summarize because the suggested question was to be used during the summary discussion. Example one was coded as Low clarity because there were many ways for the teacher to facilitate the discussion. In example two, the coach told the teacher exactly what to ask, which was considered a High

Clarity suggestion because the question was provided word for word. Following this additional round of coding, we then calculated frequencies for each of these codes for each coach across the coaching cycles.

Findings

We focus first on the Object of the suggestions, followed by Lesson Phase and Clarity. Table 3 shows the percent of the suggestions for each code within the three categories.

Table 3. Content of suggestions

Code	Percent of Suggestions
Object	
Lesson Design	30.0%
Teacher Questions	27.6%
Teacher Actions	25.4%
Facilitating Discourse	17.0%
Lesson Phase	
Launch	19.3%
Explore/Investigate	46.6%
Summary	34.2%
Clarity	
Low	8.5%
Medium	51.8%
High	39.7%

Lesson Design was the most frequently occurring Object code as 30.0% of suggestions focused on the design of the lesson. The focus of Lesson Design is unique from the other three Object codes as it related to planning decisions made prior to the lesson versus what the teacher should do during the lesson. These suggestions specifically targeted how students should represent their work (Lesson Design- Represent, 3.9%), how teachers could use resources (Lesson Design- Resource, 4.2%), particulars of tasks (Lesson Design- Task, 14.5%), and participation structure (Lesson Design- Participation Structure, 7.4%). Almost half of the Lesson Design suggestions focused explicitly on the mathematical task that would be used during the lesson. These included modifications to the task, suggestions about a particular task to be used, or ways to implement the task.

More than half of the suggestions focused on Teacher Questions or Teacher Actions, meaning the coach provided questions to ask or described movements or actions to make during the lesson. As evidenced in the table, more than a quarter of all suggestions focused on Teacher Questions. These were suggestions where the coach suggested that the teacher should ask a question. The following is an example that was coded a suggestion characterized as Teacher Questions:

> [ask] if we know what interior is and we know what alternate is. “Can you identify a pair of alternate interior angles?” Then, “What is the relationship between those angles based on the work we did before? Are they congruent or are they supplementary?”

In this example, the coach provided the teacher with the exact questions the coach thought should be asked. About one-quarter of the suggestions were coded as Teacher Actions. In the following suggestion, the coach encouraged the teacher to share student work on a document camera, "If you see some kids with some interesting work, good work, you could have them put it on there. The other option is you take a picture." The coach specifically provided direction to the teacher about what to do (an action) if a particular situation occurred (students with interesting work).

With respect to Lesson Phase, the majority of the suggestions focused on the exploration of the lesson (Explore/Investigate, 46.6%). The suggestions related to the Explore/Investigate phase often referenced the participation structure or other aspects of the lesson design that related specifically to what students would do after the lesson was launched, but prior to a summary or discussion. Summarize, as a lesson phase, was the focus on more than a third of the suggestions. A majority of the suggestions that focused on the Summarize phase of the lesson addressed how to structure a summarizing discussion or questions that should be asked during this discussion.

Regarding Clarity, most of the suggestions were coded as Medium (51.8%) or High (39.7%) Clarity, meaning there was more than one way (Medium) or only one way (High) that the suggestion could be accomplished. As an example of a High Clarity suggestion, meaning there was only one way to implement the suggestion, one coach said, "Have them in partners," which offered only one course of action. In contrast, an example of a Low Clarity suggestion (8.5%) was "Give students the opportunity to arrive at their own solution." This was coded as Low Clarity because there would be many different ways that a teacher could provide an opportunity for students to come to their own solution.

Discussion and Conclusion

The suggestions coaches provided to teachers occurred through *directive* comments that occurred as the coach and teacher interacted in the coaching conversation (Ippolito, 2010). In this negotiated space of learning, the coaches often used dialogue to tell the teacher specifically how the lesson should be designed, including what task should be used or how a task should be modified. Coaches commonly gave direct suggestions of exact questions teachers should ask students while teaching. They also provided input on what the teacher should do physically during the lesson, such as how to show student work. The directive comments of the coach focused mostly on the Explore/Investigate and Summarize parts of a lesson and were Medium to High clarity. We consider these suggestions to be the coaches' way of conveying direct instruction in the context of content-focused coaching (West & Staub, 2003).

We recognize the need for more extensive research on coaches' suggestions and how those suggestions relate to the relational dynamics of a coaching context, but we believe the findings of this study provide a typology of suggestions made by coaches engaging in coaching cycles with teachers. We see this way of characterizing suggestions as the first step in gaining more insight of how coaches' suggestions can be effective for supporting teachers. Knowing the content of the suggestions not only provided insight on what is being talked about, but it raises questions about why those topics are being discussed and how those topics could be intentionally selected to support teachers in particular aspects of practice. Research has shown that teachers respond to feedback (e.g., Cherasaro et al., 2016), which indicates that coaches' suggestions can be purposeful in what they talk about with teachers.

The findings about the Object, Lesson Phase, and Clarity of suggestions raise questions for future research. First, the present study focused on the coaching conversations and the suggestions that were provided. Knowing whether or not the teachers actually implemented these suggestions would provide insight on the extent to which teachers follow what the coaches recommend. Further research on whether or not teachers follow the suggestions, and perhaps why or why not, would add to the existing literature on coach-teacher dynamics. Second, it would be interesting to know if the coach and teacher discussed the suggestions when they met after the lesson was taught during the debriefing meeting. Analysis of these interactions would also provide details on the process of providing suggestions.

Acknowledgments

This material is based upon work supported by the National Science Foundation under Grant #1620911. We acknowledge the contributions of Jeffrey Choppin and Cynthia Callard.

References

Bengo, P. (2016). Secondary mathematics coaching: The components of effective mathematics coaching and implications. Teaching and Teacher Education, 60, 88-96.

Cherasaro, T.L., Broderson, R.M., Reale, M.L., & Yanoski, D.C. (2016). Teachers' response to feedback from evaluators: What feedback characteristics matter? U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance, Regional Educational Laboratory Center, Washington, DC. https://files.eric.ed.gov/fulltext/ED570162.pdf

Corbin, J., & Strauss, A. (2008). Basics of qualitative research: Techniques and procedures for developing grounded theory (3rd ed.), SAGE Publications. https://doi.org/10.4135/9781452230153

Costa, A. L., & Garmston, R. J. (2016). Cognitive coaching: Developing self-directed leaders and learners. Rowan & Littlefield.

Deussen, T., Coskie, T., Robinson, L., & Autio, E. (2007). "Coach" can mean many things: five categories of literacy coaches in Reading First (Issues & Answers Report, REL 2007–No. 005). Washington, DC: U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance, Regional Educational Laboratory Northwest. http://ies.ed.gov/ncee/edlabs

Heineke, S. F. (2013). Coaching discourse: Supporting teachers' professional learning. The Elementary School Journal, 113(3), 409-433. https://doi.org/10.1086/668767

Gillespie, R., Amador, J., & Choppin, J. (2020). Examining how teachers enact the suggestions of a coach: Critique of a methodology. In A.I. Sacristán, J.C. Cortés-Zavala & P.M. Ruiz-Arias, (Eds.). Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Mexico (pp. 1809-1817). Cinvestav /AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

Gibbons, L. K., & Cobb, P. (2016). Content-focused coaching: Five key practices. The Elementary School Journal, 117(2), 237-260. https://doi.org/10.1086/688906

Ippolito, J. (2010). Three ways that literacy coaches balance responsive and directive relationships with teachers. The Elementary School Journal, 111(1), 164-190. https://doi.org/10.1086/653474

Landis, J. R., & Koch, G. G. (1977). The measurement of observer agreement for categorical data. Biometrics, 33, 159–174.

Sailors, M., & Price, L. (2015). Support for the improvement of practices through intensive coaching (SIPIC): A model of coaching for improving reading instruction and reading achievement. Teaching and Teacher Education, 45, 115-127. https://doi.org/10.1016/j.tate.2014.09.008

Saldaña, J. (2013). The coding manual for qualitative researchers. SAGE Publications.

Smith, M. S., & Stein, M. K. (2011). 5 practices for orchestrating productive mathematical discussions. National Council of Teachers of Mathematics.

Van de Walle, J.A., Karp, K.S., Bay-Williams, & J. M. (2019). Elementary and middle school mathematics: Teaching Developmentally (2nd ed.), Pearson.

West, L., & Cameron, A. (2013). Agents of Change. Heinemann.

West, L., & Staub, F. C. (2003). Content-focused coaching: Transforming mathematics lessons.

Heinemann. Witherspoon, E. B., Ferrer, N. B., Correnti, R. R., Stein, M. K., & Schunn, C. D. (2021). Coaching that supports teachers' learning to enact ambitious instruction. Instructional Science, 49(6), 877–898. https://doi.org/10.1007/s11251-021-09536-7

MATHEMATICS TEACHER EDUCATION IN CHANGING TIMES: MASTER STUDENTS STORYLINES

Annica Andersson
University of South Eastern Norway
Annica.Andersson@usn.no

Trine Foyn
University of South Eastern Norway
trine.m.foyn@usn.no

Anita Movik Simensen
The Arctic University of Norway
anita.m.simensen@uit.no

Keywords: Diversity, teacher education, minoritised students

This research is part of the Norwegian Research Council's FINNUT-granted project MIM: Mathematics Education in Indigenous and Migrational contexts: Storylines, Cultures and Strength- Based Pedagogies, (see https://www.usn.no/mim), a research collaboration with researchers in the US and Canada. We draw on participatory approaches to investigate educational possibilities and desires, in times of societal changes and movements. Although we focus on the Norwegian context, we recognize that societal movements impact many countries throughout the world. With these changes and movements of people, language diversity may be the most obvious challenge, but this reality also connects to cultural differences and conventional characteristics of the discipline (Andersson et al, 2022). Indigenous communities have experienced linguistic and other challenges for decades as a result of colonization.

As Bascia and Jacka (2001) show, the way institutions respond to rapid changes in social societies, with an increasingly diverse student population, are both slow, piecemeal, and fragmented. It is of huge importance that teacher students are well prepared through their education when it comes to mathematical knowledge as well as being prepared for meeting all students regardless of their social, linguistically, and cultural backgrounds. In this research, we draw on interviews with master students who wrote their thesis within mathematics education. Teacher students' voices about mathematics teacher education is unique and have potential to raise awareness on aspects less visible in dominant conversations about mathematics teacher education. In this poster session, we want to learn from the North American contexts and listen to how colleagues talk about the preparation of becoming mathematics teachers for classrooms with a high number of minoritised students. Hence, we wish to encourage critical conversations about mathematics teacher education. The research question guiding this poster presentation is: *What storylines emerge in interviews with master students in mathematics teacher education?*

The data comes from in depth semi structured interviews with seven master students who completed their five-year mathematics teacher education with a master thesis connected to the MIM-project. Our analysis is informed by positioning theories (Wagner & Herbel-Eisenmann, 2009). We independently read each interview multiple times and took notes on the range of what the master students talked about from their teacher education and what they (not) learnt or critically reflected on and discussed during their teacher education. This allowed us to reflect on the emerging storylines in the students' narratives individually, before we met and discussed these as a group. Our analysis has made us aware of strong storylines connected to a) language b) method rigidity in schools and in mathematics teacher education c) cultural invisibility among students in mathematics classrooms and mathematics as a culturally neutral subject.

In a time where the world has gone through a rapid demographic change with increasing diversity which is mirrored in classrooms, we want to explore and discuss these storylines to raise a discussion connected to the social, ethical, and political dimensions of mathematics education.

References

Andersson, A., Ryan, U., Herbel-Eisenmann, B., Huru, H. L., & Wagner, D. (2022). Storylines in public news media about mathematics education and minoritized students. Educational Studies in Mathematics, 111(2), 323–343.

Bascia, N., & Jacka, N. (2001). Falling in and filling in: ESL teaching careers in changing times. Journal of Educational Change, 2(4), 325-346. doi:1014668925172

Wagner, D., & Herbel-Eisenmann, B. (2009). Re-mythologizing mathematics through attention to classroom positioning. Educational Studies in Mathematics, 72(1), 1–15.

PRE-SERVICE TEACHERS' INTERROGATION OF MATHEMATICS EDUCATION FOLLOWING A DIGITAL CLINICAL SIMULATION

Erin Barno
Boston University
ebarno@bu.edu

Greg Benoit
Boston University
gbenoit1@bu.edu

This preliminary study explores how digital clinical simulations can reveal pre-service teachers' understanding of equity and mathematical instructional decisions. Pre-service teachers engaged in a digital clinical simulation where their responses to scripted mathematical small group scenarios led them to different simulated outcomes. The analysis of the full class debrief conversation following the simulation experience surfaced how pre-service teachers interrogated the boundaries of mathematics education. This revealed how they felt tension about certain conceptualizations of mathematics teaching and equitable pedagogies. This study suggests that digital clinical simulations could support pre-service teachers' sensemaking of the inherent relationship between mathematical pedagogy and equitable orientations through moments of tension within the simulation, and interrogation of that norm during the debrief.

Keywords: Preservice Teacher Education, Teacher Beliefs, Technology

Teacher choices, including those of pre-service teachers, operate within contexts where power shapes classroom interactions (Chen et al., 2021; Chen et al., 2022; Yeh et al., 2020). Power is connected to teachers' instructional choices by seeing how those smaller interactions perpetuate, challenge, or reimagine hierarchical systems as they exist within mathematics classrooms (Philip et al., 2019). If a decision falls outside of the *normative boundaries of mathematics education* (i.e., how mathematics education is understood by hierarchical systems of power), teachers can perpetuate a "culture of exclusion" and whiteness within mathematics thinking, doing, and teaching (Battey, 2013; Battey and Leyva, 2016; Louie, 2017b; Martin, 2009; 2012). In other words, there is a potential for pre-service teachers to position students at fault when an instructional decision "fails" when power is not seen as part of every interaction (Battey et al., 2018; Mendoza et al., 2021; Philip et al., 2019; Shah, 2017). To understand the interweaving of instructional decisions and the boundaries of mathematics education, pre-service teachers may need a place to reflect on their moment-to-moment choices and think of how they potentially expand or enclose opportunities towards more inclusive learning experiences.

This study aims to explore *digital clinical simulations* as form of practice for pre-service teachers to reflect on how they understand equity within designed mathematics classroom interactions. Digital clinical simulations are a form of practice with prompts to respond in-the-moment to scripted characters (Sullivan et al., 2020); improvisational decisions made within the simulation cause a scripted linear path to split at specific moments within the simulation. Because this form of practice is not on real students, digital clinical simulations can be used with pre-service teachers to engage in practice and reflect on how decisions they make within the simulation are shaped by a culture of exclusion surrounding mathematics teaching and learning. In this exploratory paper, we will be analyzing how pre-service teachers engaged with a digital clinical simulation and made sense of equity following this experience. In this paper, we begin to answer this question: *How are the normative boundaries of mathematics education interrogated by pre-service teachers within a debrief following a digital clinical simulation experience?*

Conceptual Framework

We believe digital clinical simulations can be used as a conduit to surface the ways pre-service teachers *interrogate* the *boundaries of mathematics education.* A key part of pre-service teacher practice includes recognizing, questioning, and reflecting on the ways some mathematics content, thinking, and doing are positioned as more valid than others within what we are calling the *normative boundaries of mathematics education* (Louie, 2017a; Warren et al., 2020). Building from an Anzaldúan philosophy of mathematics education (Anzaldúa, 1987; Leyva, 2022), we believe the process of *interrogation* can be used to examine how pre-service teachers question normative boundaries of mathematics education. To optimize the way interrogation of boundaries can take place, digital clinical simulations can be designed with a queering lens to create moments of *tension* towards complicating pre-service teachers' understandings of normative ways of doing and learning mathematics (McWilliams and Penuel, 2016; Yeh & Rubel, 2020). Digital clinical simulations that pair moments of tension within practice positions the subsequent debrief to focus on these moments of tension. This, in turn, positions their reflections to show an expansion or reification of boundaries at these moments of tension in relation to more racial, queer, and/or intersectional form(s) of inclusion or exclusion (Kavanagh et al., 2020a; 2020b).

Methodology

This study took place in the fall of 2022 within a pre-service teachers' mathematics methodology course at a large private research university in an urban city in the northeastern United States. The eight students in this methodology class were members of a fellowship program that trained teachers to work in urban schools within the same city. For this study, a pair of mathematics teacher educators facilitating the methods course designed a digital clinical simulation using a platform called *Teacher Moments*. Within this simulation, three small groups were designed to model experiences that positioned instructional decisions within group dynamics in relation to equitable mathematics teaching (Bartell et al., 2019; Hand, 2012). Within the simulation, pre-service teachers were asked to either select a response from a set of responses or record an auditory response. Although all prompts for the auditory responses led down the sane linear path in the simulation, prompts for a selection response meant that pre-service teachers went down different paths depending on their selection. The pre-service teachers gathered as a class to debrief their experience with the mathematics teacher educators after engaging with the simulation. In the first phase of the debrief, pre-service teachers were asked to discuss their experience working with the first simulated small group.

The first small group was designed with three simulated students (Aliyah, Sandro, and Willow). This small group was designed to elicit moments of tension about *student status.* When considering student status, a culture of exclusion influences the decision making of teachers to not perceive students to be valuable thinkers unless engaging in particular ways of showing their mathematics or regurgitating knowledge (Battey & Leyva, 2016; Langer-Osuna, 2017; 2018; Louie, 2017b). In the design, Aliyah dominated the discussion and was the only student explicitly explaining their mathematical thinking. Along with this, other moments of tension were designed, such as Sandro prompting the pre-service teachers that their group fine (e.g., "We are good over here"), or Willow continually asking for specific support (e.g., "Are we heading down the right path?" and "I'm about to give, us, can you please just help us?).

The audio recordings of the pre-service teachers debrief with the math teacher educators were transcribed and analyzed for moments of interrogation. Luis Leyva's conceptualization of

Anzaldúan philosophy of mathematics education was adopted as a lens to identify moments of *interrogation* in relation to the *boundaries of mathematics education* (2022). The set of codes that emerged during this preliminary analysis identified the modes of interrogation that were used by the pre-service teachers. The emergent modes of interrogation were in relation to the way in which pre-service teachers questioned boundaries of mathematics education. Examples of these include a *singular interrogation*, where there is evidence within the discourse of a pre-service teacher reifying the boundaries of mathematics education by providing reifying the boundaries of mathematics education by providing a singular explanation or reasoning with one or more pieces of evidence; a *multifaceted interrogation,* where there is evidence within the discourse of a pre-service teacher questioning the boundaries of mathematics education by providing multiple explanations or reasoning with one or more pieces of evidence. This framework was adopted to highlight the role of fluidity within recognizing, questioning, and reflecting on pre-service teacher learning about equity in mathematics classrooms; we are using the multiplicity implicit within Anzaldúan and broader queer perspectives to highlight interrogation as a way towards expansive learning and understanding (Anzaldúa, 1987; Leyva, 2022; McWilliams and Penuel, 2016; Yeh & Rubel, 2020). Below, we will focus on the pre-service teachers' debriefing their experience with the first small group.

Preliminary Findings

Analyzing the debrief showed how the lens of interrogating boundaries highlighted pre-service teachers' values in terms of students' mathematical thinking and doing. In this phase of the discussion, pre-service teachers used both singular and multifaced forms of interrogation. Whereas almost all the singular forms of interrogation reified normative boundaries of mathematics education, the multifaceted interrogation examples were a mix of enclosing or expanding boundaries. We will present two pre-service teachers' reflections from the same part of debrief conversation that showcase multifaceted interrogation and illustrate how these moments of interrogation helped surface the types of beliefs and actions pre-service teachers valued.

Example of Multifaceted Interrogation towards Expanding Boundaries

Some pre-service teachers used multifaceted interrogation in a way that ideologically expanded their understanding of the boundaries of mathematics education. Alice, a white female pre-service teacher in her early twenties, for instance, said the following:

> [1] Sandro's was like, "We are good over here." I almost kind of thought that, like, he was just one of the kids that, like, didn't want the teacher over at his group….[2]and he didn't want to admit that he needed help… [3] like he just kind of wanted to be with his groupmates and maybe like, get off, get sidetracked and talk about something else. [4] So, I kind of said, like, "I'm glad you're doing okay. [5] I'm glad you think you understand it, but do you mind walking me through what you did and explain it to me, so I have a better understanding?".

Here, Alice is questioning the boundaries of mathematics education by providing multiple explanations or reasoning. First, Alice names tensions between what type of status Sandro is showing within the simulation. She refers to [1] Sandro not wanting the teacher to come over because of either [2] his uncomfortableness of asking for help or [3] because he did not want to do work. To address this with instruction decisions, Alice shared that, within the simulation, she [4] responded the way she did to echo that she knows he is okay, [5] to affirm he knows the contents, and [6] to ask him to clarify so she as the teacher has more details on where he is mathematically. With an interrogation lens, Alice can be seen valuing that Sandro may have had multiple reasons for wanting the teacher to leave the small group. Although [3] part of that interpretation could be understood as a deficit view about Sandro, Alice [4-5] tried to hold Sandro in a way where she valued him as a person and as a doer of mathematics who (potentially) had mixed feelings about sharing his thinking.

Example of Multifaceted Interrogation towards Enclosing Boundaries

Other instances of pre-service teachers using multifaceted interrogation did so in a way where their questioning reinforced existing boundaries of mathematics education. Dylan, a white male pre-teacher in his mid-twenties, said the following about the small group later within the debrief:

> [6] I kind of picked up on that… [7] Sandro hadn't necessarily really done the problem or really engaged with it… [8] I thought maybe Aliyah knew what was going on and it was relatively easy for her. [9] And then Willow and Sandro maybe weren't totally on the same page. [10] So, I engaged in a similar strategy…by calling Sandro out…like, "Are you done?" [11] Which I didn't say that way, but the idea was, "Sandro can you explain what Aliyah did here?" [12] And just, as like, getting everybody on the same page.

Here, Dylan is reifying the boundaries of mathematics education because his multiple explanations why Sandro did not have status here related to how he views value within the mathematics. First, Dylan seems to identify valuable mathematics thinking and doing if [7] the problem is done and [8] there is evidence that the problem is easy to do. Dylan's description of wanting to [12] make sure everyone understands what is going on mathematically is bookended by [10-11] Dylan jokingly saying he will "Call Sandro out." Although this could be understood as jest, there is a question in terms of Dylan valuing accountability and compliance by the students, along with an assumption solely from the simulation that Sandro had none no valuable work with the task.

Discussion

Within the debrief conversation, we found the lens of *interrogation* surfaced pre-service teachers' beliefs and actions about the boundaries of mathematics education. Mathematical boundaries were enclosed or expanded in terms of how pre-service teachers questioned what was deemed as valuable mathematical thinking and doing within simulated moments of tensions. Although some tensions and interrogations could be seen as ideologically moving toward expansive boundaries, almost the same amount moved towards leaving those boundaries as is. Not connecting the boundaries of mathematics education as it functions as teacher decision making within approximations of practice, therefore, can perpetuate deficit perspectives about students.

The analysis of the full class debrief conversation illustrates how pre-service teachers feel tension around certain conceptualizations of mathematics teaching and equitable pedagogies within digital clinical simulations. Revealing their process of interrogating, in turn, shows that there is a continued need to create practice spaces where pre-service teachers values and beliefs can be seen and addressed if needed. If practice spaces are not designed to surface tensions for pre-service teachers to question these beliefs, mathematics teacher educators may be further fueling settled perspectives of mathematics teaching and learning to new teachers. If this happens during practice spaces, there is a potential for pre-service teachers to continue making the same decisions in the classroom with actual students. This study suggests that digital clinical simulations could support pre-service teachers' understanding of the inherent relationship between mathematical pedagogy and equitable orientations if moments of tension are designed within the simulation, and opportunities to interrogate that norm are given during the debrief.

References

Anzaldúa, G. (1987). Borderlands/La frontera: The new mestiza. San Francisco, CA: Aunt Lute Books.

Bartell, T. G., Drake, C., McDuffie, A.-R., Aguirre, J. M., Turner, E. E., & Foote, M. Q. (Eds.). (2019). Transforming mathematics teacher education: An equity-based approach. Springer International Publishing. https://doi.org/10.1007/978-3-030-21017-5

Battey, D. (2013). Access to mathematics: A possessive investment in whiteness. Curriculum Inquiry, 43(3), 332–359.

Battey, D., & Leyva, L. A. (2016). A framework for understanding whiteness in mathematics education. Journal of Urban Mathematics Education, 9(2), 49–80.

Chen, G. A., & Horn, I. S. (2022). A Call for Critical Bifocality: Research on Marginalization in Mathematics Education. *Review of Educational Research*, 003465432110700. https://doi.org/10.3102/00346543211070050

Chen, G.A., Marshall, S.A., & Horn, I. S. (2021).'How do I choose?': Mathematics teachers' sensemaking about pedagogical responsibility. Pedagogy, Culture & Society, *29*(3), 379–396.

Hand,V. (2012). Seeing culture and power in mathematical learning:Toward a model of equitable instruction. *Educational Studies in Mathematics*, *80*(1), 233–247.

Hillaire, G., Waldron, R., Littenberg-Tobias, J., Thompson, M., O'Brien, S., Marvez, G. R., & Reich, J. (2022). Digital Clinical Simulation Suite: Specifications and Architecture for Simulation-Based Pedagogy at Scale.

Kavanagh, S. S., Conrad, J., & Dagogo-Jack, S. (2020a). From rote to reasoned: Examining the role of pedagogical reasoning in practice-based teacher education. *Teaching and Teacher Education*, *89*.

Kavanagh, S. S., & Danielson, K. A. (2020b). Practicing justice, justifying practice: Toward critical practice teacher education. *American Educational Research Journal*, *57*(1), 69-105.

Langer-Osuna, J. M. (2017). Authority, identity, and collaborative mathematics. Journal for Research in Mathematics Education, 48(3), 237-247.

Langer-Osuna, J. M. (2018). Exploring the central role of student authority relations in collaborative mathematics. ZDM, 50(6), 1077-1087.

Leyva, L.A. (2022). Deconstructing Dissonance and Resonance Across Historically Marginalized Learners' Mathematics Experiences to Advance Intersectional Justice. Proceedings of the forty-fourth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (PME-NA 44). Middle Tennessee State University.

Louie, N. L. (2017a). Supporting Teachers' Equity-Oriented Learning and Identities: A Resource-Centered Perspective. *Teachers College Record: The Voice of Scholarship in Education*, *119*(2), 1–42. https://doi.org/10.1177/016146811711900203

Louie, N. L. (2017b). The culture of exclusion in mathematics education and its persistence in equity-oriented teaching. Journal for Research in Mathematics Education, 48(5), 488–519. JSTOR. https://doi.org/10.5951/jresematheduc.48.5.0488

Martin, D. B. (2009). Researching race in mathematics education. Teachers College Record, 111(2), 295–338.

Martin, D. B. (2012). Learning mathematics while Black. The Journal of Educational Foundations, 26(1&2), 47–66.

McWilliams, J., & Penuel, W. (2016). Queer theory in the learning sciences. In *Power and Privilege in the Learning Sciences: Critical and Sociocultural Theories of Learning* (pp. 93–114). https://doi.org/10.4324/9781315685762

Mendoza, E., Hand, V., van Es, E. A., Hoos, S., & Frierson, M. (2021). 'The ability to lay yourself bare': centering rupture, inherited conversations, and vulnerability in professional development. *Professional Development in Education*, *47*(2-3), 243-256.

Shah, N. (2017). Race, ideology, and academic ability: A relational analysis of racial narratives in mathematics. *Teachers College Record*, *119*(7), 1–42.

Sullivan, F., Hillaire, G., Larke, L. & Reich, J. (2020). Using Teacher Moments During the COVID-19 Pivot. *Journal of Technology and Teacher Education, 28*(2), 303-313. Waynesville, NC USA: Society for Information Technology & Teacher Education. Retrieved February 22, 2023 from https://www.learntechlib.org/primary/p/216171/

Philip, T. M., Souto-Manning, M., Anderson, L., Horn, I., J. Carter Andrews, D., Stillman, J., & Varghese, M. (2019). Making justice peripheral by constructing practice as "core": How the increasing prominence of core practices challenges teacher education. *Journal of Teacher Education*, 70(3), 251–264. https://doi.org/10.1177/0022487118798324
Rosebery, A., Ogonowski, M., DiSchino, M., & Warren, B. (2010). "The coat traps all your body heat": Heterogeneity as fundamental to learning. The Journal of the Learning Sciences,19, 322–357.
Warren,B.,Vossoughi,S.,Rosebery,A.S.,Bang,M.,&Taylor,E.V.(2020). Multiple ways of knowing: Re-imagining disciplinary learning. In *Handbook of the cultural foundations of learning* (pp. 277–294). Routledge
Yeh, C., Ellis, M., & Mahmood, D. (2020). From the margin to the center: A framework for rehumanizing mathematics education for students with dis/abilities. The Journal of Mathematical Behavior, 58, 100758. https://doi.org/10.1016/j.jmathb.2020.100758
Yeh, C., & Rubel, L. (2020). *Queering Mathematics: Disrupting Binary Oppositions in Mathematics Pre-service Teacher Education* (pp. 227–243). Cham: Springer International Publishing. https://doi.org/10.1007/978-3-030-44292-7_11

PROBLEM-SOLVING AT A DIVERSE SCHOOL: TEACHERS CHARACTERIZE THEIR EXPERIENCES AND THEIR MULTILINGUAL LEARNERS' EXPERIENCES

Richard Kitchen
University of Wyoming
rkitchen@uwyo.edu

"Martinez Elementary" teachers participated in professional development activities during the 2021-22 school year to learn about the Discursive Mathematics Protocol (DMP), a problem-solving based instructional protocol designed specifically for use with multilingual learners (MLs). These activities included training sessions designed to help the teachers learn about the DMP's dual focus on mathematical reasoning and the mathematics register. In this paper, a socio-political framework is used to examine interview data collected from Martinez Elementary teachers to learn how they characterized their experiences and their students' experiences when they implemented the DMP during problem-solving lessons. This study contributes to the research literature by providing insights about how teachers and students at a diverse school characterized their experiences involving problem-solving based instruction.

Keywords: Problem Solving, Equity, Inclusion, and Diversity.

In the study presented here, we use narratives collected from teachers at "Martinez Elementary" to examine their experiences and their students' experiences with problem-solving lessons that were implemented with the aid of an instructional protocol designed specifically for use with multilingual learners (MLs). The protocol, referred to as the "Discursive Mathematics Protocol" (DMP), builds on Pólya's (1945/1986) iconic problem solving heuristic by incorporating research-based "language practices" (LPs) grounded in theories of academic language development and acquisition (Moschkovich, 2015) and "essential teaching practices" (ETPs) (National Council of Teachers of Mathematics [NCTM], 2014). The DMP is intended to be used as a guide by teachers during problem-solving lessons to support students to simultaneously learn the language of mathematics and develop their mathematical reasoning through the implementation of the LPs and ETPs, respectively. The Martinez Elementary teachers' narratives are examined through a socio-political theoretical perspective.

Martinez Elementary is a highly diverse elementary school located in rural, northern New Mexico, USA that serves economically underserved communities populated almost entirely by Hispanic and Native American families (primarily Diné, also known as Navajo). The term "Hispanic" is commonly used in New Mexico to denote individuals and communities of Spanish descent. During the 2021-22 school year, Martinez Elementary teachers participated in professional development (PD) activities to learn about the DMP. These activities included attending PD sessions designed to help the teachers learn about the DMP and how to implement it, participating in both planning and debriefing sessions on six occasions (three times in the fall and three times in the spring) prior to problem-solving lessons during which they used the DMP, and observing instruction in which the use of the DMP was modeled for them during a problem-solving lesson with their students. The research questions addressed in this study are: (1) How did Martinez Elementary teachers characterize their experiences learning about and implementing the DMP during problem-solving lessons? (2) How did teachers characterize how their students experienced problem-solving lessons in which the DMP was used? Prior to presenting the research findings, instructional interventions for MLs and the DMP are introduced. Brief reviews of the research literature are then provided on problem-solving based instruction and on the historical lack of access that minoritized and economically underserved students like those who attend Martinez Elementary have had to problem-solving based mathematics instruction in the United States.

Mathematics Instructional Interventions for Multilingual Learners

There are a preponderance of interventions designed for use with MLs with learning disabilities (e.g., See Kong & Swanson, 2019). Interventions also exist for "mathematically promising" MLs (Cho et al., 2015). Only a few interventions such as the SIOP (Sheltered Instruction Observation Protocol) Model (Echevarría et al., 2011; Short, 2017) are designed specifically for use with MLs. SIOP can be used in any content area, not just mathematics. Through Dynamic Strategic Math (DSM), teachers modify vocabulary used in mathematical problems to support individual MLs' language proficiency and provide scaffolded instruction so that students can develop strategies to solve problems (Orosco et al., 2011). Culturally appropriate problem-solving instruction (CAPSI) includes explicit vocabulary instruction, schema-based instruction, and video modeling (Luevano & Collins, 2020). Baird et al. (2020) describe a mathematics intervention for MLs referred to as Project Mathematics and English Language Development (MELD). In MELD, significant scaffolds are provided to MLs such as student glossaries, illustrations, and content that is explicitly connected to students' home cultures. The DMP is unique among instructional protocols designed for MLs because of its dual focus on developing students' mathematical reasoning (not simply the learning of math skills) *and* supporting students' learning of the mathematical language.

Introducing the Discursive Mathematics Protocol

Early iterations of the Discursive Mathematics Protocol (DMP) were created more than a decade ago to address the lack of instructional protocols designed specifically to support mathematics instruction of MLs. The DMP takes advantage of Pólya's (1945/1986) heuristic for structuring problem-solving lessons by incorporating the four stages of the heuristic: (1) Understand the problem, (2) Devise a plan to solve the problem, (3) Carry out the plan to solve the problem, and (4) Reflect back on the plan and solutions derived. The language practices (LPs) and essential teaching practices (ETPs) are then integrated throughout the four stages of Pólya's (1945/1986) heuristic. The LPs are three-fold and are intended to provide teachers with insights about how to intentionally model language to help students communicate and learn (Lee et al., 2013) the language of mathematics. They include: Using instructional strategies designed to develop everyday English language skills (LP#1), engaging students in mathematical discourse (LP#2), and developing language specific to the discipline of mathematics (LP#3). It should be noted that the focus of LP#1 is the development of everyday English skills because in the United States, English is the language of instruction. In settings where English is not the language of instruction, LP#1 could focus on everyday language skills of the language of instruction in that setting.

The three ETPs included in the DMP are derived from the "Mathematics Teaching Practices" described in *Principles to actions: Ensuring mathematical success for all* (National Council of Teachers of Mathematics [NCTM], 2014), a policy document intended to provide guidance for U.S. teachers and policymakers in mathematics education. The ETPs are intended to support the development of students' mathematical reasoning. Mathematical reasoning, broadly defined as "the capacity to think logically and to justify one's thinking" (NCTM, 2014, p. 7), involves searching for similarities and differences, validating, and exemplifying (Jeannotte & Kieran, 2017). The three ETPs included are: Use and connect mathematical representations (ETP#1), pose purposeful questions (ETP#2), and elicit and use evidence of student thinking (ETP#3). There are many relationships among the LPs and ETPs. For example, posing purposeful questions (ETP#2) is a strategy that teachers can use to initiate discourse (LP#2) with and among students (Kooloos et al., 2020). For additional information about the DMP, see Kitchen et al. (in review).

Problem-Solving Based Instruction

In problem-solving based instruction, much emphasis is placed on students making sense of mathematics by making connections, communicating, and using representations to characterise their ideas (Gamoran et al., 2003). A prominent feature of problem-solving based instruction is the value placed upon mathematical conversations or discourse. During mathematical discourse, the teacher seeks to foster and engage in dialogue with her students (Herbel-Eisenmann & Cirillo, 2009). Through discourse, teachers support students to make sense of mathematical ideas, construct arguments, and justify mathematical claims (Moschkovich, 2015). Engaging students in mathematical problem-solving demands much of teachers. For instance, to help inform instruction during problem-solving lessons, teachers should look for opportunities to elicit and use evidence of students' thinking (Davidson et al., 2019).

Minoritized and Economically Underserved Students Access to Problem-Solving

In the United States, problem-solving based instruction in mathematics has generally not been a priority at diverse schools attended largely by minoritized and economically underserved students such as MLs (Davis & Martin, 2008; Kitchen, 2020). Low academic expectations have historically been the norm at schools attended by minoritized and economically underserved students (Kitchen, 2020; Kitchen et al., 2021). It is also the case that teachers at these schools often do not have the content and pedagogical expertise needed to support the development of their students' mathematical reasoning (Kitchen et al., 2007). Moreover, teachers who work at these schools may not believe that their students can learn via demanding instructional formats such as problem-solving based instruction (Boaler, 2002). There also tends to be an emphasis at these schools on the instruction of procedures over meaning making and understanding (Davis & Martin, 2008). Students who attend schools with such traditional instructional emphases are inclined to develop the belief that they cannot learn mathematics or that mathematics is just not for them (Kitchen, 2003). When professional development activities were initiated at Martinez Elementary to inform the school's teachers about the DMP, it was apparent that teachers at the school were generally not implementing problem-solving based instruction with their minoritized and economically underserved students. The theoretical perspective and research methods used in the study are introduced next.

Socio-political Theoretical Perspective

Gutiérrez' (2013) framed the "socio-political turn" that has taken place in the mathematics education research community as the need to "transform mathematics education in ways that privilege more socially just practices" (p. 40). From a socio-political theoretical perspective, educational policies and practices are studied from the viewpoint that differential access to educational opportunities is grounded in differences based on racialized and classed experiences (Battey, 2013). In the United States, White and Asian middle class and upper-middle class students have historically had more opportunities to learn challenging mathematical content than minoritized and economically underserved students (Schmidt & McKnight, 2012). A socio-political lens places the economic, social, cultural, and political contexts of teaching and learning in the foreground when considering phenomena such as whether minoritized students and economically underserved students such as MLs have access to rigorous mathematics instruction (Chirinda et al., 2022). A socio-political theoretical perspective was used in this study to examine the research participants' narratives vis-à-vis their experiences implementing the DMP with their minoritized and economically underserved students. It was also used to examine teachers' narratives regarding how their students responded to problem-solving based instruction in which the DMP was used.

Methods and Data Sources

Research strategies employed in narrative inquiry (Riessman, 1993) were used to solicit and examine Martinez Elementary teachers' narratives collected from focus group interviews conducted with them. The use of narrative inquiry is an intentional strategy used by social scientists to acknowledge the need to understand people's perspectives through their experiences (Pinnegar & Daynes, 2007). In this approach, I make use of stories told by the participants to make sense of underlying meanings that they communicated via their narratives (Riessman, 1993). Teachers' narratives were examined through a socio-political lens to understand teachers' perspectives. Using this lens, particular attention was paid to the teachers' narratives regarding educational opportunities afforded to them and to their students in mathematics.

Participants

All of the teachers at Martinez Elementary, not including instructional aides, were invited to participate in a focus group interview at the conclusion of the 2021-22 school year. Out of the 20 teachers at the school, 11 elected to participate in the focus group interview. Seven of the eight teachers whose narratives are featured here had taught for at least eight years. Five of these eight teachers had taught exclusively at Martinez Elementary or at a school in the town where Martinez Elementary is located. Thus, the narratives featured here are from teachers who had significant teaching experience at Martinez Elementary or the community it served. In addition, "Janine," "Angelo" and "Gabriel" (all teacher names used throughout are pseudonyms). had been recruited from their home country of the Philippines to teach at Martinez Elementary. At the time this study was undertaken, the school district in which Martinez Elementary is located recruited teachers from the Philippines to teach in the district because of teacher shortages they were experiencing. Similar to other teachers at Martinez Elementary, these three Filipino teachers had minimal experiences with problem-solving based instruction both as pre-tertiary students themselves and as teachers.

Data Collection

Semi-structured interviews were carried out via Zoom with participating teachers in April 2022. Four focus groups of teachers were interviewed, with each group consisting of 2-3 teachers from across the varying grade-levels found at the school (grades 4-6). Teachers were randomly placed in a group, though priority was given to having teachers from differing grade-levels in each group. Each interview was recorded and lasted between 60 and 75 minutes. Ten interview questions were asked, four with a pre-determined probe question (also known as a follow-up question). The following is an example of a question asked: "How helpful did you find following the four stages of Pólya's problem-solving heuristic during the problem-solving lessons?" A goal of the interviews was to engage participants in a conversation through the use of open-ended questions that prompt the teachers "to construct answers," (Riessman, 1993, p. 54) in collaboration with the interviewer, a goal of narrative inquiry research.

Data Analysis

Participating teachers' narratives were examined through a socio-political lens. I analyzed the teachers' narratives using a thematic analysis approach in which the focus is more on what is said rather than on how it is told (Riessman, 1993). I engaged in an iterative process of examining themes that emerged across the interviews, reflecting upon, and then clarifying participants' views. With a thematic analysis, the goal is to first collect the stories, then to inductively group similar stories, and finally to select key narratives for illustration purposes (Riessman, 1993).

Results

Two major themes were identified related to teachers' experiences implementing the DMP during problem-solving lessons and what they learned from using it: (1) Participants experienced a paradigmatic shift implementing the DMP, and (2) Participants' instruction changed through their experiences implementing the DMP. One major theme was identified about how teachers characterized their students' experiences vis-à-vis problem-solving lessons in which the DMP was used: Students were empowered. I now share narratives for the two themes that address the first research question, followed by the theme found to address the second research question.

Paradigmatic Shift

Teachers shared in their narratives that learning how to implement the DMP during problem-solving lessons was a paradigm shift for them; many had not been trained about how to implement problem-solving based instruction. Jim shared how teaching problem-solving using the DMP as a guide meant moving away from instruction in which students are first shown how to compute or solve a problem:

> I struggled at first to implement the DMP. I definitely like struggled the first couple times even wrapping my head around what we were doing. It was a break from the I do, we do, you do together, you do alone, which was kind of the bedrock since I started teaching. And so, that was a hard shift at first for me to even grasp. And then, once I kind of saw how it worked, I started implementing it in any subject that I could.

Both Suzie and Gloria shared what a big change it was for them to teach problem-solving. Suzie said, "I think it was kind of different because of the way we've taught for so long." Gloria shared that,

> … even though it was math, it was almost like going in somewhere and someone speaking a foreign language, that's how we felt at first. It was completely changing a lot of the ways we had been told we have to teach, which ended up being a good thing, but at the front, it was kind of, you know, different.

From a socio-political perspective, these narratives reflect how teachers at Martinez Elementary had limited opportunities themselves to experience and learn about problem-solving based instruction in a teacher education program or through PD. However, as they continued to learn about the DMP and implement more tasks, they became more comfortable using problem-solved based instruction as their students adapted to their instruction.

Changed Instruction

Gloria talked about how the DMP PD sessions and support provided had empowered her as an instructor: "It was really empowering for me to know that we could give them that power and let them [the students] work through a problem, and I think that's what I took out of it the most." Alana described how she learned about the importance of giving students the space to struggle to solve problems, rather than focusing simply on covering content:

> It helped me slow down too as a teacher and think that sometimes it's okay for them [her students] to struggle, it's okay if my kids don't have standard algorithms. Let them solve it [the problem], and then we can always come back and talk about it, instead of what's the more efficient way achieving our goal, or whatever.

Alana also provided some insights about how using the DMP impacted her instruction:

> [I started to] really explain and make sure that they understood the task [a feature of the DMP], because sometimes I feel like we just give the task and say, 'Okay, you should know how to do this.' Also, making sure that they all have the vocabulary, the math register they need, was super helpful. And giving them the opportunity to solve it in any way they choose, because when we teach in class, it seems like we're teaching a certain skill or objective, and so we want them to do it this way. But these tasks gave them the opportunity to just use what they had, use their knowledge.

Through the DMP trainings, Jim learned about the value of constructing and using what he termed "bite-sized" questions: "Yeah, I'm now using bite-sized questions, that's what I'm going to call it." The second ETP in the DMP is posing purposeful questions, something that Jim admitted he rarely did in his mathematics lessons prior to learning about the DMP. Posing purposeful questions and prompts involves more than simply eliciting information from students; such questions invite students to explain and potentially justify their thinking (Davidson et al., 2019). Analogous to the previous section, from a socio-political perspective the teachers at Martinez Elementary had had limited opportunities to learn about problem-solving based instruction (Kitchen, 2020). For them, learning about asking purposeful questions during mathematics lessons was a transformational experience that promoted their students regularly sharing their mathematical ideas with them and their students in class. Moreover, there was a sense of empowerment among the teachers to improve their mathematics instruction because their abilities, confidence, and motivation had increased (Kitchen, 2020) through the use of the DMP.

Students Were Empowered

When reflecting upon his experiences implementing the DMP, Angelo discussed how his students had learned strategies that they could use to solve problems:

> We can again say that our kids really improved their mathematical thinking in attacking a word problem. [When you] compare how students did in the past with task number one with how they did with the last tasks, they now have the skills to attack problems. For me, as their teacher, it's [engaging in problem-solving] been effective for our kids.

Similarly for Suzie, she was pleased that her students were learning how to think through engaging more in mathematical problem-solving:

> I saw kids do things that I don't think I would have seen, like if we just gave the regular problems or whatever. You start seeing kids really thinking differently, you're seeing the kids

> open up and show how they do it, and then I feel like the kids start feeling more confident and trying different things, too.

Sara shared how the COVID-19 pandemic had negatively affected students' learning and how engaging students in problem-solving via the DMP had made a difference in terms of helping students have more confidence in mathematics: "I think it's starting to build that confidence in these kids. That's a great thing to see [after the damage caused by the pandemic]. Gloria described how the changes in her instruction changed how her students responded to mathematics:

> Instead of me just getting up there and rushing through the lesson and this is how you do it, hopefully you'll remember it [based on what I show you to do]. I thought that [solving problems] was a little bit more helpful for my students and they seem to enjoy it a lot more. And then that way, as I was walking around the room during [problem-solving lessons], I could just prompt here and there, asking students to explain to me what you're doing just like we did with the big math tasks that you did, they liked it. By the end of the year, every single kid, even my low math kids, they are all asking can I do it on the board. Can I show them how to do this. They gained so much confidence this year, compared to any other year.

Rather than showing students how to solve a problem, Gloria believed she was now able to empower her own students to solve challenging problems without providing lots of instructional scaffolds. Alana agreed with Gloria, and shared the following:

> [I found] giving them the opportunity to solve it in any way they choose [was also empowering for her and her students], because when we teach in class, it seems like you know we're teaching a certain skill or objective, and so we want them to do it this way. But these tasks gave them the opportunity to just use their knowledge. I loved how kids went up [in front of the class to the board] and there would be four or five different ways [that her students solved the task]. And the kids, I think, were really empowered by that, especially the lower kids, I mean the kid that maybe sat there and did repeated addition 25 times. Like they got it correct, and so the fact they got the chance to go up there, I think, was big.

Alana's narrative also highlights how some students at Martinez Elementary were not be mathematically challenged, but were being relegated to continually practicing math skills. Gabriel expressed satisfaction with the PD he had experienced during the year to learn about the DMP and how it had affected his students:

> It's been a thrilling experience for me as a teacher doing the DMP. For some kids, this [type of instruction] is new to them, and they will do almost all the tasks, they will almost do all the work. [They're used to] teachers just presenting at the board, giving examples. But this time [with the DMP], they get to think, find answers by themselves, and explain to their classmates. So, just what Angelo said, at first they were somewhat confused about what to do [to solve a problem], but as the time goes by, every time that we implement those tasks they are getting better now and getting the procedure, the process of the DMP and it has helped them a lot. We saw the great progress of our students, not only in presenting their ideas, but with how they solved problems and how they do that on their own. So, the growth of the students is so amazing, the jump of their thinking skills, the progress. And it's also shown in our [test] scores that students can now construct their sentences really well, express their ideas to their classmates. Overall, it has been an amazing experience.

As reflected in Gabriel's response, he believed that students' learning had progressed through the course of the academic year and that this growth was also demonstrated through improved scores on a mathematics assessment that the school district administered. Janine shared how her students' experiences during problem-solving lessons had progressed:

> The tasks we solved early in the year, they were like hesitant to answer. As time has gone by, they're used to it. Now, they want to solve [the tasks] and they like to share with their group. They really love it!

Discussion

Though the DMP is designed for use specifically with MLs, the teachers at Martinez Elementary used it during problem-solving lessons that they implemented with all their students, including their MLs, in all their mathematics classes on six occasions during the 2021-22 school year. Through learning about the DMP and how to use it during problem-solving lessons, the teachers became empowered to teach using new instructional strategies. The teachers expressed how much they learned from incorporating the DMP as a guide during problem-solving lessons and even how, in Alana's case, it had influenced her mathematics instruction in general. To change their instruction, teachers experienced a paradigmatic philosophical shift to come to value problem-solving based instruction. This shift resulted in teachers generally focusing less on showing students how to do mathematics. Instead, teachers were learning how to engage students in mathematics inquiry, something that many of them had never experienced as students themselves. From a socio-political perspective, the teachers' narratives help to illuminate the limited opportunities that they had prior to participating in the DMP trainings to experience and learn about problem-solving based instructional formats. Given that all but one of the teachers whose narratives were included here had significant teaching experience, it was the case that teachers at Martinez Elementary generally lacked experience with problem-solving based instruction. Teachers at Martinez Elementary were simply not well-acquainted with this type of instruction. According to the teachers, it was also the case that students at Martinez Elementary were empowered through their experiences in the problem-solving based lessons. Teachers talked about how students learned mathematics and new problem solving strategies from these lessons, enjoyed these lessons, wanted to share their new mathematical insights with their peers, and were mathematically empowered through their experiencing with this novel instructional approach. From a socio-political perspective, the diverse student population at Martinez Elementary had also rarely experienced problem-solving based instruction. According to the teachers, these experiences were transformational for some students, for others it was a welcome relief from the sort of skills-based mathematics instruction that they had encountered for years.

Given that teachers at schools primarily attended by minoritized and economically underserved students such as MLs in the United States may not believe that their students can learn via problem-solving based instruction (Boaler, 2002), it is important to introduce instructional protocols such as the DMP to support teachers at these schools to improve their instruction. While examples exist in the research literature of teachers who induct their students into problem-solving based learning environments (See, for example, Lampert, 2001), research is needed that provides practical insights about how to implement and sustain problem-solving based instruction in schools such as Martinez Elementary attended largely by minoritized and economically underserved students (Kitchen, 2020; Kitchen et al., 2021). This study contributes to this research area (See Boaler, 2011; Kitchen et al., 2009) by providing insights about how teachers who had limited experiences with problem-based instruction who teach diverse students responded to using a problem-solving based instructional protocol. The teachers' narratives also demonstrate that teachers who have had little experience implementing problem-solving based instruction can, in fact, learn how to do so when consistent supports are provided to them to do so. Lastly, this study contributes to the research literature by providing evidence that minoritized and economically underserved students benefit from their teachers having significant supports to implement problem-solving based instruction in their mathematics classes.

References

Baird, A. S., Garrett, R., & August, D. (2020). Math and English language development: MELDing content and academic language for English learners. NABE Journal of Research and Practice, 10(1), 1-12. https://doi.org/10.1080/26390043.2019.1653051

Battey, D. (2013). Access to mathematics: A possessive investment in whiteness. Curriculum Inquiry, 43, 332–359. https://doi.org/10.1111/curi.12015

Boaler, J. (2002). Learning from teaching: Exploring the relationship between reform curriculum and equity. Journal for Research in Mathematics Education, 33(4), 239-58.

Boaler, J. (2011). Changing students' lives through the de-tracking of urban mathematics classrooms. Journal of Urban Mathematics Education, 4(1), 7–14.

Chirinda, B., Kitchen, R., Castellón, L. B., & Matute, K. (2022). Teaching mathematics in post-apartheid South Africa: The perspectives of teachers of Black students. Research in Mathematics Education. https://doi.org/10.1080/14794802.2021.2024086

Cho, S., Yang, J., & Mandracchia, M. (2015). Effects of the M3 curriculum on mathematics and English proficiency achievement of mathematically promising English language learners. Journal of Advanced Academics, 26(2), 112-142. https://doi.org/10.1177/1932202X15577205

Davidson, A., Herbert, S., & Bragg, L. A. (2019). Supporting elementary teachers' planning and assessing of mathematical reasoning. International Journal of Science and Mathematics Education, 17(6), 1151–1171. https://doi.org/10.1007/s10763-018-9904-0

Davis, J. & Martin, D. B., (2008). Racism, assessment, and instructional practices: Implications for mathematics teachers of African American students. Journal of Urban Mathematics Education, 1(1), 10-34.

Echevarría, J., Richard-Tutor, C., Canges, R., & Francis, D. (2011). Using the SIOP model to promote the acquisition of language and science concepts with English learners. Bilingual Research Journal, 34(3), 334–351.

Gamoran, A., Anderson, C. W., Quiroz, P. A., Secada, W. G., Williams, T., & Ashman, S. (2003). Transforming teaching in math and science. New York: Teachers College Press.

Gutiérrez, R. (2013). The sociopolitical turn in mathematics education. Journal for Research in Mathematics Education, 44(1), 37-68.

Herbel-Eisenmann, B. A., & Cirillo, M. (Eds.). (2009). Promoting purposeful discourse: Teacher research in mathematics classrooms. Reston, VA: National Council of Teachers of Mathematics.

Jeannotte, D., & Kieran, C. (2017). A conceptual model of mathematical reasoning for school mathematics. Educational Studies in Mathematics, 96(1), 1–16. https://doi.org/10.1007/s10649-017-9761-8

Kitchen, R. S. (2003). Getting real about mathematics education reform in high poverty communities. For the Learning of Mathematics, 23(3), 16-22.

Kitchen, R. (2020). The power of their ideas: Leveraging teachers' mathematical ideas in professional development. International Journal of Mathematical Education in Science and Technology. https://doi.org/10.1080/0020739X.2020.1847337

Kitchen, R., Castellón, L. B., Matute, K., & Mosqueda, E. (in review). Advancing an instructional framework for Emergent Bilinguals targeting mathematical reasoning and the language of mathematics. Mathematical Thinking and Learning.

Kitchen, R. S., DePree, J., Celedón-Pattichis, S. & Brinkerhoff, J. (2007). Mathematics education at highly effective schools that serve the poor: Strategies for change. Mahwah, NJ: Lawrence Erlbaum Associates.

Kitchen, R., Martinez-Archuleta, M., Gonzales, L., & Bicer, A. (2021). Actualizing change after experiencing significant mathematics PD: Hearing from teachers of colour about their practice and mathematical identities. Education Sciences, 11(11), 710. https://doi.org/10.3390/educsci11110710

Kitchen, R. S., Roy, F. C., Lee, O., & Secada, W. (2009). Comparing teachers' conceptions of mathematics education and student diversity at highly effective and typical elementary schools. Journal of Urban Mathematics Education, 2(1), 52-80.
Kong, J. E., & Swanson, H. L. (2019). The effects of a paraphrasing intervention on word problem-solving accuracy of English Learners at risk of mathematics disabilities. Learning Disability Quarterly, 42(2), 92-104. https://doi.org/10.1177/0731948718806659
Kooloos, C., Oolbekkink-Marchand, H., Kaenders, R., & Heckman, G. (2020). Orchestrating mathematical classroom discourse about various solution methods: Case study of a teacher's development. Journal für Mathematik-Didaktik, 41, 357-389. https://doi.org/10.1007/s13138-019-00150-2
Lampert, M. (2001). Teaching problems and the problems of teaching. Yale University Press.
Lee, O., Quinn, H., & Valdés, G. (2013). Science and language for English language learners in relation to next generation science standards and with implications for Common Core State Standards for English Language Arts and Mathematics. Educational Researcher, 42(4), 223-233. https://doi.org/10.3102/0013189X13480524
Luevano, C. & Collins, T. A. (2020) Culturally appropriate math problem-solving instruction with English language learners. School Psychology Review, 49(2), 144-160. https://doi.org/10.1080/2372966X.2020.1717243
Moschkovich, J. (2015). Academic literacy in mathematics for English learners. Journal of Mathematical Behavior, 40, 43-62. https://doi.org/10.1016/j.jmathb.2015.01.005
National Council of Teachers of Mathematics. (2014). Principles to actions: Ensuring mathematical success for all. Author.
Orosco, M. J., Swanson, H. L., O'Connor, R., & Lussier, C. (2011). The effects of dynamic strategic math on English language learners' word problem solving. The Journal of Special Education, 47(2), 96-107. https://doi.org/10.1177/0022466911416248
Pinnegar, S., & Daynes, J. (2007). Locating narrative inquiry historically. In D. J. Clandinin (Ed.), Handbook of narrative inquiry: Mapping a methodology (pp. 1–34). Thousand Oaks, CA: Sage.
Pólya, G. (1986). How to solve it: A new aspect of mathematical method. Princeton University Press. (Original work published in 1945)
Riessman, C. K. (1993). Narrative analysis. Newbury Park, CA: Sage Publications, Inc.
Schmidt, W. H., & McKnight, C. C. (2012). Inequality for all: The challenge of unequal opportunity in American schools. New York: Teachers College.
Short, D. J. (2017). How to integrate content and languge learning effectively for English language learners. Eurasia Journal of Mathematics Science and Technology Education, 13(7b), 4237-4260. https://doi.org/10.12973/eurasia.2017.00806a

SIMULATING A PEDAGOGY OF ENACTMENT MAY WORK FOR SOME EARLY NUMERACY TEACHERS BUT NOT OTHERS

Arielle Orsini
Concordia University
arielle.orsini@concordia.ca

Jenna Rice
Carleton University
jennarice@cmail.carleton.ca

Helena P. Osana
Concordia University
helena.osana@concordia.ca

Julie Houle
Lester B. Pearson School Board
jhoule02@lbpearson.ca

Anne Lafay
Concordia University
anne.lafay@univ-smb.fr

The Numeracy Kit for Kindergarten 5-Year-Olds (NyKK-5) is a tool to support kindergarten teachers' numeracy practices. Based on a pedagogy of enactment, the NyKK-5 was designed to simulate the types of in-class supports that have been identified as critical elements of effective mathematics professional development (PD). Two groups of teachers received PD on children's early numeracy: The NyKK-5 condition was provided the NyKK-5 and the theory condition was not provided any pedagogical tools. Students in the NyKK-5 and theory classrooms demonstrated greater numeracy growth relative to students in a comparison condition. Teacher logs and classroom observations suggested that the amount of time spent engaging students in mathematics was more predictive of student learning than the NyKK-5 by itself, but the amount of teaching experience may define one condition under which use of the tool is optimized.

Keywords: Professional Development; Early Childhood Education; Number Concepts and Operations

Young children's number sense is predictive of their future mathematical development, and kindergarteners who fail to acquire early numeracy skills tend to fall behind their peers as they advance through school (Aunola et al., 2004; Jordan et al., 2010). As such, teachers who provide a strong numeracy foundation in the early years play an important role in their students' future academic success (Presser et al., 2015; Starkey et al., 2004). Early childhood educators, however, need support to establish these numeracy foundations in their classrooms (Ginsburg et al., 2008). In the present study, we investigated the impact of a professional development (PD) initiative for kindergarten teachers centered on a "pedagogy of enactment" (Grossman & McDonald, 2008, p. 189). Our goal was to examine the effect of the professional development on children's numeracy over the course of the kindergarten year and to use descriptions of the teachers' classroom practice to explain the findings.

Situating Early Numeracy Professional Development in Practice

Early numeracy education centers on rich teacher-student conversations around numeracy concepts, where teachers capitalize on teachable moments during play and continually adjust instruction to respond to students' in-the-moment, educational needs (Ginsburg & Ertle, 2008; Jacobs & Empson, 2016). Creating this kind of learning environment requires a profound knowledge of students' numeracy development (Ball et al., 2008; Even & Tirosh, 1995), which enables teachers to notice and interpret students' understanding, appropriately respond in the moment, and make informed instructional decisions to advance their learning (Ghousseini & Sleep, 2011; Ginsburg & Ertle, 2008; Jacobs & Empson, 2016; Stahnke et al., 2016).

Although attending to children's mathematics thinking is a key characteristic of effective PD (Carpenter et al., 1989; Hawes et al., 2021), it is not sufficient: PD must be sustained and situated in the practice of teaching (Ball & Cohen, 1999; Borko et al., 2010). Such a pedagogy of enactment approach to PD entails *in situ* supports during the act of teaching, which are effective because they enhance teachers' ability to incorporate newly-acquired PD content into their daily instructional routines (Ball & Forzani, 2009; Hiebert & Morris, 2012; McDonald et al., 2013). Regrettably, such PD is often resource intensive and rarely sustainable in the long run. In the absence of prolonged in-class support from school leaders and mathematics coaches (see Gibbons et al., 2017, for an example), teachers need alternative resources to help them make instructional decisions in the moment, particularly within messy classroom environments that contain many demands on their attention. We posit that usable devices, such as a toolkit, may act as a suitable proxy for these in-class supports.

We developed the Numeracy Kit for Kindergarten 5-year-olds (NyKK-5) to act as a surrogate for the in-context supports that have been identified in previous research as an essential component of effective professional development (e.g., Ghousseini & Sleep, 2011). The NyKK-5 is a collection of activities and corresponding materials based on a set of core early numeracy skills that have been found to be predictive of children's achievement in the first grade and beyond (Jordan et al., 2007; Jordan et al., 2009; Navarro et al., 2012). Teachers can use the kit to enact numeracy activities in the classroom and formatively assess their students' mathematical thinking. It is portable and designed for teachers to use on the fly to plan their lessons, interpret their students' thinking, and make decisions about instructional moves that are responsive to their students' learning needs (Jacobs & Empson, 2016).

Present Study

Our primary research objective was to investigate whether the use of the NyKK-5 in kindergarten classrooms adds value to children's numeracy growth beyond PD content on children's numeracy development alone. We assigned teachers to one of three PD conditions: (a) NyKK-5: PD on children's thinking about core numeracy skills and on the contents and use of the NyKK-5; (b) theory: PD on the development of children's thinking about core numeracy skills, but without the NyKK-5; (c) comparison: a non-numeracy PD. We assessed children's numeracy growth over the school year to determine whether differences emerged between conditions. Because the NyKK-5 was designed to support teachers' in-the-moment decisions about their instructional practice, we predicted that the growth in students' learning would be greater in the NyKK-5 condition relative to the other two conditions. We also predicted that the students in the theory classrooms, because of the focus on children's thinking, would experience greater numeracy growth than the students in the comparison condition.

The second objective was to explain any condition effects using data on teachers' classroom practices. Teachers completed daily teacher logs and researchers conducted classroom observations on the teachers' enactment of PD-related material. These data were collected to lend insight into how teachers responded to the NyKK-5 and the conditions under which its effects could be maximized.

Method

Participants

Participating teachers and their students were recruited from six public schools in a large urban school board in Canada. The eight participating kindergarten teachers all identified as women and had between 2 to 29 years of teaching experience.

The student sample consisted of 104 kindergarteners. Because of absence (1 student), lack of assent (2 students), and missing data on the numeracy assessment (5 students), there were 96 students in the final sample. The participants had a mean age of 66.8 months (*SD* = 4.00) at pretest and 72.9 months (*SD* = 3.93) at posttest (41.1% girls; 58.9% boys; 0% other gender). Parents of participating students reported their household income in Canadian dollars on a demographic survey. We used these data as a proxy measure for family socioeconomic status. Almost a quarter (24%) of the participants did not respond or provided unusable responses to the question. Of those who did respond, 2.7% had an income less than $20,000, 9.6% had an income between $20,000 and $40,000, 11.0% had an income between $40,000 and $60,000, 5.5% had an income between $60,000 and $80,000, 6.8% had an income between $80,000 and $100,000, and 64.4% had an income over $100,000.

Design and Procedure

We used a quasi-experimental pretest-intervention-posttest research design where the teachers were assigned to one of three professional development conditions: (a) the NyKK-5 condition (4 teachers; *n* = 40 students) received professional development on children's numeracy and were given the NyKK-5 to use in their classrooms; (b) the theory condition (2 teachers; *n* = 27 students) received the same professional development on children's numeracy, but were not given the NyKK-5; and (c) the comparison condition (2 teachers; *n* = 29 students) received professional development on children's general cognitive development and accompanying activities, but with no focus on numeracy. Because of logistical constraints, we were unable to fully randomize the assignment of teachers to conditions.

We assessed the kindergarteners' numeracy skills at the beginning (October–November 2021) and end (April–May 2022) of the school year. The teachers began applying the resources they received in the PD in January 2022. Teachers completed daily teacher logs on their engagement in PD-related classroom activities for 17 weeks between January and April. Two research assistants also visited each teacher's classroom three times between January and April for classroom observations.

NyKK-5

The NyKK-5 is a collection of activities and manipulatives in a plastic case (dimensions: 15" x 12"x 5") that teachers can easily carry around in the classroom. In the case are colored plastic boxes, each corresponding to one of six specific numeracy skills: subitizing, counting, number knowledge, non-verbal calculation, story problems, and part-whole combinations (Jordan et al., 2007). The boxes in the NyKK-5 contain a variety of activity cards for each target skill, including sample questions to assess it and suggestions for follow-up prompts. Matching boxes contain manipulatives corresponding to several of the activity cards. The activities can be used in a variety of settings, including in one-on-one conversations with children, in the context of small group work (e.g., math centers), or with the whole class. Furthermore, the kit was designed to support responsive teaching (Jacobs & Empson, 2016) – that is, teachers can use it for formative assessment and to select follow-up instructional activities in response to students' mathematical thinking. Sample scenarios were provided during the professional development to illustrate the types of activities that could be selected as a function of student thinking.

Professional Development

The third and fourth authors facilitated all PD sessions in the study.

NyKK-5. Teachers in the NyKK-5 condition participated in three full-day PD sessions for a total of 18 hours. The content focused on the development of children's thinking in the six numeracy domains reflected in the NyKK-5. The teachers examined the contents of each box in the kit, explored how the contents addressed the six target numeracy skills, and reflected on how to use each component of the kit in their classrooms.

Theory. Teachers in the theory condition participated in two full-day PD sessions for a total of 12 hours. They received identical PD to the NyKK-5 condition but were not provided the NyKK-5 or any materials for classroom use.

Comparison. Comparison teachers participated in one 6 hr session on children's general cognitive development in the domains of creativity, collaboration, communication, and critical thinking. No information on children's numeracy was provided. Teachers were given a box of assorted LEGO® bricks and selected activities from the LEGO® Education program to support the cognitive development of young children in their classrooms (Hugo, 2016).

Measures

Student Numeracy Skills. We assessed children's early numeracy skills using a modified version of the Preschool Early Numeracy Screener-Brief Version (PENS-B; Purpura et al., 2015). The PENS-B is a measure of students' global numeracy that includes items related to the various skills shown in previous research to be critical for students' early mathematical development (e.g., Jordan et al., 2007). The skills assessed verbal counting, one-to-one counting, cardinality, counting a subset, subitizing, numeral comparison, set comparison, number order, numeral identification, set-to-numeral, story problems, number combinations, relative size, and ordinality. The PENS-B comprises 25 items, with a maximum possible score of 29. The verbal counting item is scored from 0 to 5 based on the highest number to which the participant can count, and the remaining items are scored as 0 or 1 (incorrect or correct, respectively).

Teacher Logs. On each day over 17 weeks, teachers reported whether they engaged in PD-related activities and if so, they selected from a list which specific PD skills they targeted. Teachers in the NyKK-5 and theory conditions chose from the following numeracy skills: subitizing, counting, number knowledge, non-verbal calculation, story problems, and part-whole combinations. Teachers in the comparison condition chose all that applied from creativity, communication, critical thinking, and collaboration.

Classroom Observations. Two trained research assistants observed each teacher in the classroom engage in PD-related activities three times between January and April. Each observation lasted 30 min comprising four blocks of 7 min each. Each block began with a 3-min time sampling observation of the teachers' pedagogical moves. The time sampling consisted of 5-s intervals (36 in total over the 3 min) with observations conducted in every odd-numbered interval. Using a prepared checklist, the observers selected from the following mathematics teaching moves: used declarative statements about mathematics, modeled mathematical activity to their students, asked their students questions about the topic, and listened and observed their students engage in a mathematical task. Non-mathematics behaviors were classified as "other."

The observers spent the next 3 minutes taking field notes to describe the activity observed during the preceding 3-min observation period. The observers then took a 1-min break before starting the next observation block. At the end of the four observation blocks, each observer took additional field notes to document any methodological challenges. The data used in the subsequent analyses comprise the observations of the teachers' interactions with their students during 216 coding intervals (3 observation sessions x 4 blocks x 18 time-sampling intervals). The observers had an interrater reliability of 89.5% across all 12 time-sampling blocks.

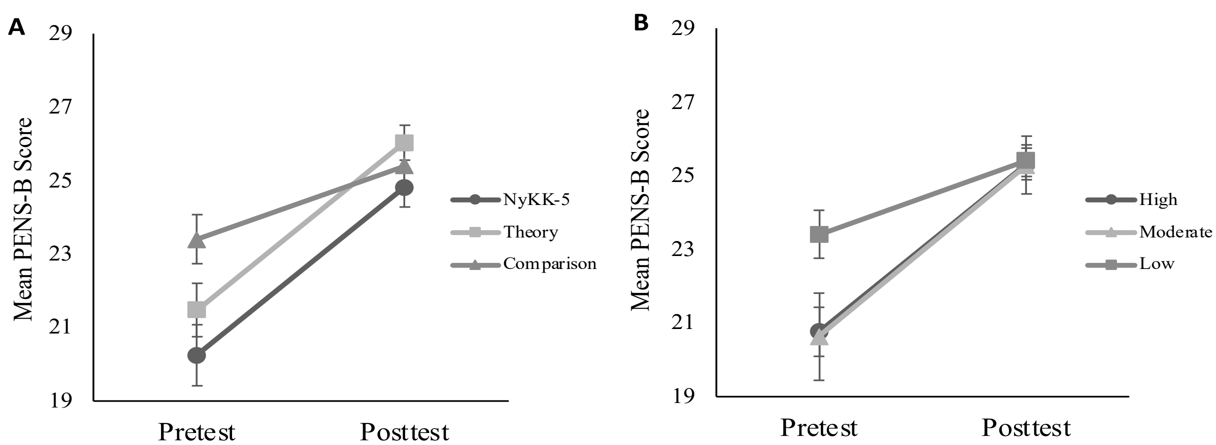
Results

In this section, we report on the effects of PD condition on students' numeracy growth over the course of the kindergarten year. This is followed by an examination of the teachers' classroom practices, as measured by the teacher logs and classroom observations, to augment our account of students' numeracy development in relation to the PD delivered and teacher individual differences.

We first conducted a preliminary analysis to determine whether children's household income should be included in subsequent models testing the factors related to students' numeracy growth. A chi-square test of homogeneity revealed that there were no differences in the proportion of students from higher-income households (> \$100,000) and lower-income households (< \$100,000) between conditions, $\chi^2(2, N = 73) = 2.13$, $p = 0.34$.

Students' Numeracy Growth

Our first objective was to investigate whether the NyKK-5 added value to children's numeracy gains beyond the theory on children's numeracy development alone. Means and standard deviations on the PENS-B by time and condition are presented in Panel A of Figure 1. A one-way analysis of variance (ANOVA) on the PENS-B gain scores revealed a statistically-significant effect of condition, $F(2,93) = 6.41$, $p = .002$, $\eta^2 = .24$. Follow-up Least Significant Difference (LSD; Levin et al., 1994) pairwise comparisons indicated that both the NyKK-5, $t(93) = 3.27$, $p = .002$, and the theory conditions, $t(93) = 2.96$, $p = .004$, improved more over time than the comparison condition, but no difference was found in students' numeracy gains between the NyKK-5 and theory conditions, $t(93) = 0.02$, $p = .98$. These results suggest that the NyKK-5 had no observable added value to children's numeracy gains beyond the PD that covered children's numeracy alone.



Note. Panel A: by condition; Panel B: by mathematics engagement.

Figure 1: Mean PENS-B Growth by Condition and Mathematics Engagement

Teachers' Classroom Practice

Teacher Logs. From the teacher log data, we computed the proportion of days each teacher reported engaging in PD-related activities (i.e., total number of days out of 78 days total). On average, teachers in the NyKK-5 condition engaged in 48.5 days of PD-related activity (62.2%), teachers in the theory condition engaged in 50.5 days (64.7%), and teachers in the comparison condition 46.5 days (59.6%). The teachers' self-reported engagement in PD-related activities appeared largely similar across all three conditions.

Next, we examined the specific numeracy skills teachers in the NyKK-5 and theory conditions targeted. For each teacher in both conditions, we computed the proportion of times each numeracy skill was targeted out of the total number of numeracy skills targeted and then computed average proportions by condition (see Figure 2). As can be observed, the proportion of times the teachers in the NyKK-5 and theory conditions engaged in each type of numeracy skill was comparable. Both groups spent most of their time targeting subitizing, counting, and number knowledge. Similarly, both conditions rarely targeted non-verbal calculation and focused on story problems and part-whole combinations even less so.

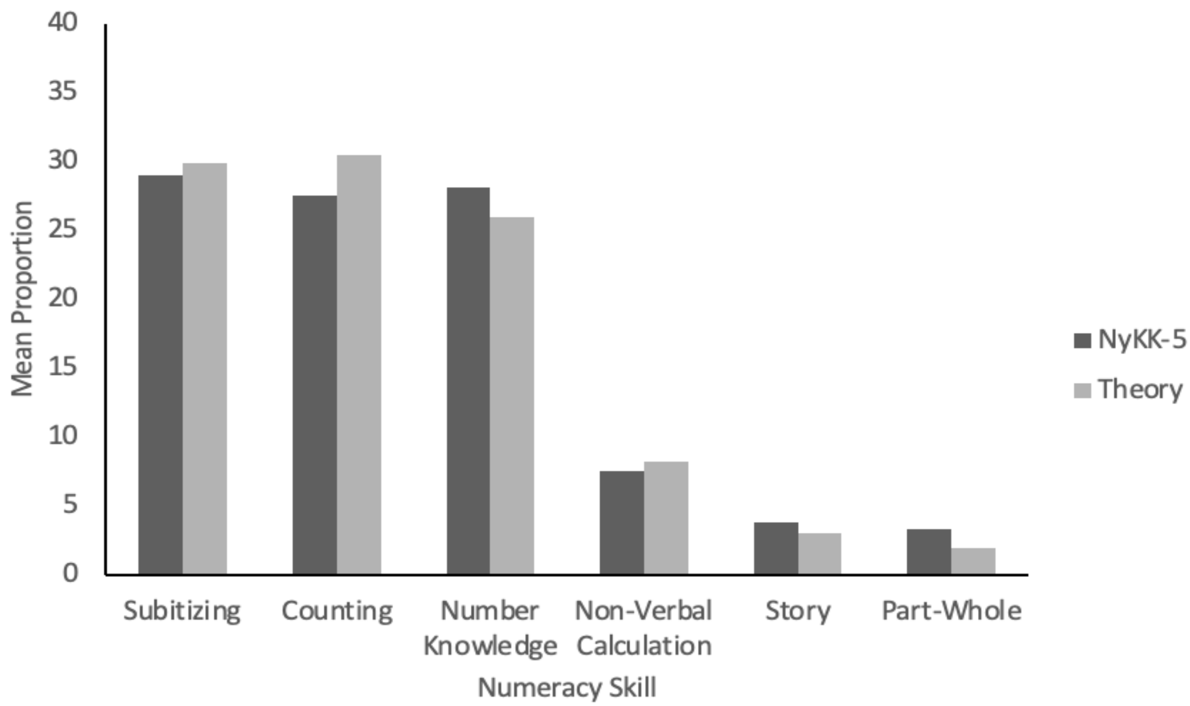
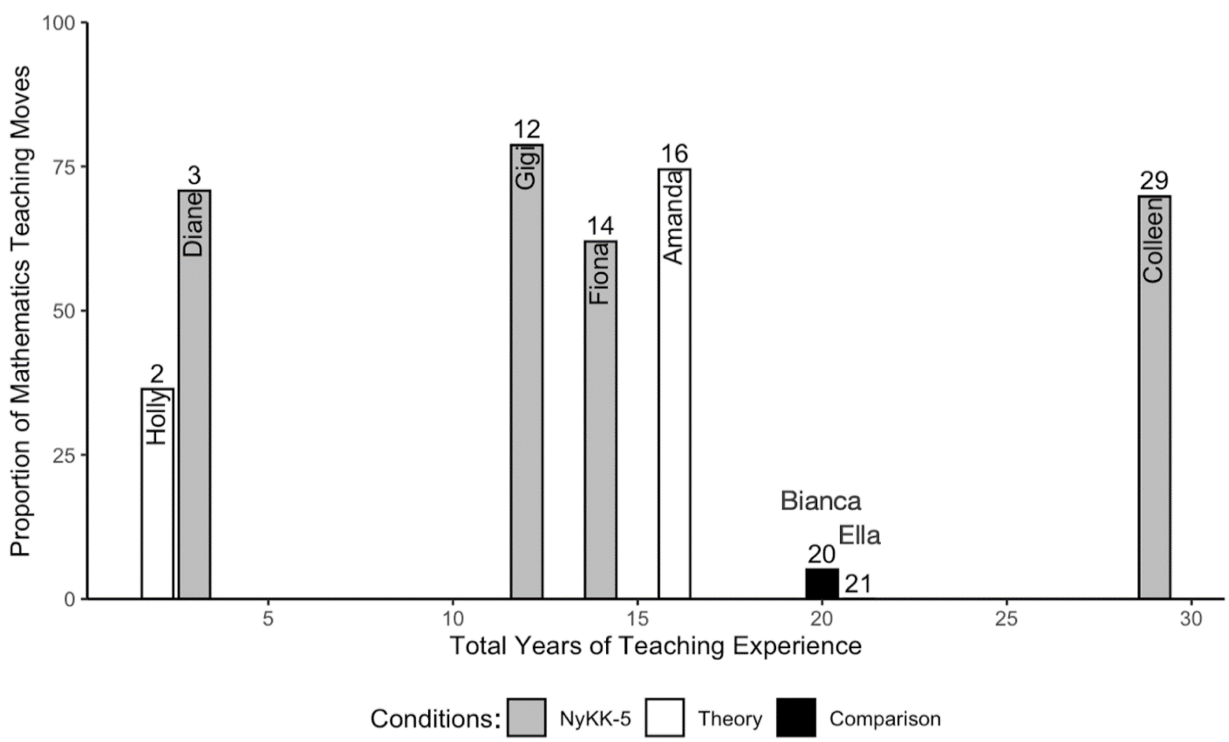


Figure 2: Mean Proportion Teacher Engagement in Each Numeracy Skill by Condition

Classroom Observations. For each teacher, we calculated the proportion of time that each teacher spent engaging in any type of mathematics teaching move out of the total number of behaviors observed across all 12 time-sampling blocks (i.e., 4 blocks x 3 observation sessions). We then regrouped the teachers into three mathematics engagement categories based on the ranges of proportions of mathematics teaching moves out of the total number of moves (math-related and non-math related): high (>50%), moderate (50–25%), and low (<25%) mathematics engagement. Both participants in the comparison condition fell in the low mathematics engagement group (0% and 5%), which was to be expected because the teachers were not introduced to any mathematics-related PD. All participants in the NyKK-5 condition demonstrated high mathematics engagement (62%, 70%, 71%, and 79%). Meanwhile, one participant in the theory condition displayed high mathematics engagement (75%) and the other displayed moderate mathematics engagement (36%).

A one-way ANOVA to test students' numeracy growth revealed a significant effect of mathematics engagement, $F(2,93) = 6.41$, $p = .002$, $\eta^2 = .24$ (see Figure 1, Panel B). Follow-up LSD pairwise comparisons showed that students of teachers with high mathematics engagement and the teacher with moderate mathematics engagement had greater numeracy gains than students of low mathematics engagement teachers, $t(93) = 3.42$, $p < .001$ and $t(93) = 2.52$, $p = .01$, respectively. No differences were observed between the high and moderate engagement groups, $t(93) = 0.10$, $p = .92$. These results indicate that regardless of whether they used the NyKK-5 or not, teachers' level of mathematics engagement in the classroom is related to students' numeracy gains.

Next, we attempted to explain the observed variability in teachers' level of mathematics engagement. Figure 3 presents the observed levels of mathematics engagement for each teacher by number of years of teaching experience and condition. Two important patterns emerge from examining all three variables together. First, across the sample, the level of mathematics engagement was related to the number of years of teaching experience, except in the comparison condition. Second, an examination of the data within condition revealed that all the teachers in the NyKK-5 condition were observed spending over 60% of their classroom interactions with their students on mathematics, despite varying years of teaching experience (i.e., ranging from 3 to 29 years). In contrast, however, a greater variability in mathematics engagement was observed in the theory condition. Specifically, Holly, with two years of experience, displayed a moderate level of engagement, whereas Amanda, with 16 years of experience, displayed a higher level of engagement. Although the sample is too small to draw any definitive conclusions, taken together, these findings may suggest that a tool such as the NyKK-5, designed to simulate in-class PD supports for teachers, could compensate for a lack of teaching experience.



Note. Teacher names are pseudonyms. Ella was in the comparison condition; the proportion of her teaching moves was 0%.

Figure 3: Proportion of Mathematics Engagement by Years of Total Teaching Experience

Discussion

The primary objective of the present study was to assess the impact of a pedagogical tool for teachers, the NyKK-5, on kindergarteners' numeracy development over the course of the school year. We provided professional development to kindergarten teachers on early numeracy with a focus on the development of children's thinking. One group of teachers was provided the NyKK-5 and another was not. The NyKK-5 was designed to simulate the types of in-class supports for teachers that have been identified as critical elements of effective mathematics PD. In particular, the design of the kit and its accompanying PD aimed to support teachers' reflections on their students' learning during the act of teaching (see Gibbons et al., 2017), thereby helping them to connect their newly-acquired knowledge of children's numeracy to their pedagogical actions in real time. In contrast, the teachers without the kit were left to their own devices to make such connections. The teacher logs and observational data generated a description of what was happening in the teachers' classrooms, which was then used to uncover potential explanations for any condition effects.

Contrary to our expectations, we found that the students in the classrooms of the teachers who used the NyKK-5 improved to the same extent as those whose teachers who were not provided the kit. This finding suggests that teachers' knowledge of children's numeracy, regardless of whether they were provided additional instructional tools for use in the classroom, is related to students' numeracy gains over the course of the kindergarten year. This finding is not new; teachers with knowledge about children's thinking are able to interpret what they know, prepare for common student misconceptions, and make instructional decisions to further support mathematical knowledge, reasoning, and problem solving (Ball et al., 2008; Even & Tirosh, 1995; Jacobs & Empson, 2016).

Regarding the secondary research objective, the teacher data added considerable nuance to this finding. First, despite the same degree of attention to all six numeracy skills in the NyKK-5 and theory classrooms, we found that the level of mathematics engagement varied and was more predictive of students' growth than use of the NyKK-5 by itself. Further, an analysis of the within-condition patterns of mathematics engagement and previous teaching experience suggested that the NyKK-5 may be particularly helpful for certain teachers, such as those with less teaching experience. This finding contributes to research in mathematics teacher education by raising the possibility that pedagogical tools over and above knowledge about how children's numeracy develops – and in particular, tools that are situated in practice and encourage teachers' reflection on their pedagogical actions (Grossman & McDonald, 2008) – may be useful under certain conditions but perhaps not all. More specifically, although the small sample precludes definitive conclusions, even teachers with relatively little teaching experience were able to leverage the NyKK-5 to maximize their mathematics engagement in the classroom. Without the tool, the teacher with only two years of experience had little to help her connect her newly-learned knowledge about children's thinking to her classroom practice, something for which the more-experienced teacher was able to compensate (Schoen et al., 2019).

Despite the correlational nature of the analyses and the small sample, the present study has implications for continued research in teacher education. An important implication is that a pedagogical tool for use in the classroom that serves as a proxy for *in situ* supports for teachers, such as the NyKK-5, may be an effective way to support teachers who experience barriers in engaging students in mathematics, such as a lack of experience. A replication with a larger sample of teachers and a randomized design is thus an important consideration for further investigation. Other teacher individual differences that may account for the variance in students' numeracy growth, such as their knowledge of mathematics itself (Oppermann et al., 2016) and level of confidence in their ability to teach numeracy in the early years (Hadley & Dorward, 2011; Ramirez et al., 2018), should also be examined in future studies.

References

Aunola, K., Leskinen, E., Lerkkanen, M.-K., & Nurmi, J.-E. (2004). Developmental dynamics of math performance from preschool to grade 2. *Journal of Educational Psychology*, *96*(4), 699–713. https://doi.org/10.1037/0022-0663.96.4.699

Ball, D. L., & Cohen, D. K. (1999). Developing practice, developing practitioners: Toward a practice-based theory of professional education. In L. Darling-Hammond & G. Sykes (Eds.), *Teaching as the learning profession: Handbook of policy and practice* (pp. 3–32). Jossey-Bass.

Ball, D. L., & Forzani, F. M. (2009). The work of teaching and the challenge for teacher education. *Journal of Teacher Education, 60*(5), 497–511. https://doi.org/10.1177/0022487109348479

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389–407. https://doi.org/10.1177/0022487108324554

Borko, H., Jacobs, J., & Koellner, K. (2010). Contemporary approaches to teacher professional development. In P. Peterson, E. Baker, & B. McGaw (Eds.), *International Encyclopedia of Education* (3rd ed., pp. 548–556). Elsevier. https://doi.org/10.1016/B978-0-08-044894-7.00654-0

Carpenter, T. P., Fennema, E., Peterson, P. L., Chiang, C. P., & Loef, M. (1989). Using knowledge of children's mathematics thinking in classroom teaching: An experimental study. *American Educational Research Journal*, *26*(4), 499–531. https://doi.org/10.2307/1162862

Even, R., & Tirosh, D. (1995). Subject-matter knowledge and knowledge about students as sources of teacher presentations of the subject-matter. *Educational Studies in Mathematics*, *29*(1), 1–20. https://doi.org/10.1007/BF01273897

Ghousseini, H., & Sleep, L. (2011). Making practice studyable. *ZDM: The International Journal on Mathematics Education*, *43*(1), 147–160. https://doi.org/10.1007/s11858-010-0280-7

Gibbons, L. K., Kazemi, E., Hintz, A., & Hartmann, E. (2017). Teacher time out: Educators learning together in and through practice. *NCSM Journal of Mathematics Education Leadership*, *18*(2), 28-46. Retrieved from https://tedd.org/wp-content/uploads/2017/05/JMEL-2017-Teacher-Time-Out.pdf

Ginsburg, H. P., & Ertle, B. (2008). Knowing the mathematics in early childhood mathematics. In O. N. Saracho & B. Spodek (Eds.), *Contemporary perspectives on mathematics in early childhood education* (pp. 45–66). Information Age.

Ginsburg, H. P., Lee, J. S., & Boyd, J. S. (2008). Mathematics education for young children: What it is and how to promote it. *Social Policy Report*, *22*(1), 1–24. https://doi.org/10.1002/j.2379-3988.2008.tb00054.x

Grossman, P., & McDonald, M. (2008). Back to the future: Directions for research in teaching and teacher education. *American Educational Research Journal, 45*(1), 184–205. https://doi.org/10.3102/0002831207312906

Hadley, K. M., & Dorward, J. (2011). The relationship among elementary teachers' mathematics anxiety, mathematics instructional practices, and student mathematics achievement. *Journal of Curriculum & Instruction*, *5*(2), 27–44. https://doi.org/10.3776/joci.2011.v5n2p27-44

Hawes, Z., Merkley, R., Stager, C. L., & Ansari, D. (2021). Integrating numerical cognition research and mathematics education to strengthen the teaching and learning of early number. *British Journal of Educational Psychology*, *91*(4), 1073–1109. https://doi.org/10.1111/bjep.12421

Hiebert, J., & Morris, A. K. (2012). Teaching, rather than teachers, as a path toward improving classroom instruction. *Journal of teacher Education*, *63*(2), 92–102. https://doi.org/10.1177/0022487111428328

Hugo, S. (2016). *365 things to do with LEGO bricks: Lego fun every day of the year*. DK Children.

Jacobs, V. R., & Empson, S. B. (2016). Responding to children's mathematical thinking in the moment: An emerging framework of teaching moves. *ZDM Mathematics Education*, *48*(1–2), 185–197. https://doi.org/10.1007/s11858-015-0717-0

Jordan, N. C., Glutting, J., & Ramineni, C. (2010). The importance of number sense to mathematics achievement in first and third grades. *Learning and Individual Differences*, *20*(2), 82–88. https://doi.org/10.1016/j.lindif.2009.07.004

Jordan, N. C., Kaplan, D., Locuniak, M. N., & Ramineni, C. (2007). Predicting first-grade math achievement from developmental number sense trajectories. *Learning Disabilities Research & Practice*, *22*(1), 36–46. https://doi.org/10.1111/j.1540-5826.2007.00229.x

Jordan, N. C., Kaplan, D., Ramineni, C., & Locuniak, M. N. (2009). Early math matters: Kindergarten number competence and later mathematics outcomes. *Developmental Psychology*, *45*(3), 850–867. https://doi.org/10.1037/a0014939

Levin, J. R., Serlin, R. C., & Seaman, M. A. (1994). A controlled, powerful multiple-comparison strategy for several situations. *Psychological Bulletin*, *115*(1), 153–159. https://doi.org/10.1037/0033-2909.115.1.153

McDonald, M., Kazemi, E., & Kavanagh, S. S. (2013). Core practices and pedagogies of teacher education: A call for a common language and collective activity. *Journal of Teacher Education*, *64*(5), 378–386. https://doi.org/10.1177/0022487113493807

Navarro, J. I., Aguilar, M., Marchena, E., Ruiz, G., Menacho, I., & Van Luit, J. E. (2012). Longitudinal study of low and high achievers in early mathematics. *British Journal of Educational Psychology*, *82*(1), 28–41. https://doi.org/10.1111/j.2044-8279.2011.02043.x

Oppermann, E., Anders, Y., & Hachfeld, A. (2016). The influence of preschool teachers' content knowledge and mathematical ability beliefs on their sensitivity to mathematics in children's play. *Teaching and Teacher Education*, *58*, 174-184. https://doi.org/10.1016/j.tate.2016.05.004

Presser, A. L., Clements, M., Ginsburg, H., & Ertle, B. (2015). Big math for little kids: The effectiveness of a preschool and kindergarten mathematics curriculum. *Early Education and Development*, *26*(3), 399–426. https://doi.org/10.1080/10409289.2015.994451

Purpura, D. J., Reid, E. E., Eiland, M. D., & Baroody, A. J. (2015). Using a brief preschool early numeracy skills screener to identify young children with mathematics difficulties. *School Psychology Review*, *44*(1), 41–59. https://doi.org/10.17105/SPR44-1.41-59

Ramirez, G., Shaw, S. T., & Maloney, E. A. (2018). Math anxiety: Past research, promising interventions, and a new interpretation framework. *Educational Psychologist, 53*(3), 145–164. https://doi.org/10.1080/00461520.2018.1447384

Schoen, R. C., LaVenia, M., Chicken, E., Razzouk, R., & Kisa, Z. (2019). Increasing secondary-level teachers' knowledge in statistics and probability: Results from a randomized controlled trial of a professional development program. *Cogent Education*, *6*(1), 1613799. https://doi.org/10.1080/2331186X.2019.1613799

Stahnke, R., Schueler, S., & Roesken-Winter, B. (2016). Teachers' perception, interpretation, and decision-making: A systematic review of empirical mathematics education research. *ZDM Mathematics Education*, *48*(1–2), 1–27. https://doi.org/10.1007/s11858-016-0775-y

Starkey, P., Klein, A., & Wakeley, A. (2004). Enhancing young children's mathematical knowledge through a pre-kindergarten mathematics intervention. *Early Childhood Research Quarterly*, *19*(1), 99–120. https://doi.org/10.1016/j.ecresq.2004.01.002

SUCCESSES AND CHALLENGES FROM EARLY CAREER MATHEMATICS TEACHERS

J. Vince Kirwan
Kennesaw State University
jkirwan@kennesaw.edu

Belinda P. Edwards
Kennesaw State University
bedwards@kennesaw.edu

Despite working towards a common vision for teacher preparation, newer teachers experience successes and challenges in their teaching. This study investigated the experiences of three third year teachers to identify what they found successful or challenging in their teaching. Findings indicated that successes or challenges were contextually dependent upon the individual, though common themes across experiences existed.

Keywords: Professional Development, Teacher Knowledge, Preservice Teacher Education, Informal Education

National standards documents identify a vision of what teacher preparation should be, indicating that preservice teachers need preparation centered on knowledge of mathematics, how students learn mathematics, and mathematics pedagogy aligned with effective and equitable teaching practices (AMTE, 2019; CBMS, 2012). However, teaching mathematics requires integrating and using this knowledge (Barker et al., 2019), which can be challenging and mostly learned once a newer teacher is positioned in their own classroom (Feiman-Nemser, 2001). Scholars have described teachers with less than three years of experience as novices (Arbaugh et al., 2015; Lampert, 2010) or advanced beginners (Berliner, 2004). Research in mathematics education has investigated the classroom practices of veteran teachers (Ball & Forzani, 2011; Borko, 2004), providing limited insight into experiences of secondary mathematics teachers with less than three years of experience.

Literature Review

Berliner (2004) developed a theory describing a progression that teachers move through as they build their expertise. We describe the first two stages here. The first stage, the novice stage, is experienced by first year teachers. Teachers in the novice stage are inflexible, desire conformity and compliance, and use rules absolutely. Novices are gaining experience and learning common structures and expectations of the school they work in. The second stage, the advanced beginner stage, is often experienced by second- and third-year teachers. Teachers in this stage begin to leverage their experiences to inform their classroom sense-making and decisions. Advanced beginners recognize contextual similarities across their experience and begin to develop a more flexible, nuanced collection of practical knowledge for teaching.

Classroom management is a common challenge newer teachers encounter (e.g., Evertstein & Weinstein, 2006). McCormack, Gore, and Thomas (2006) described classroom management as actions teachers take to establish and maintain an orderly classroom learning environment that supports meaningful academic learning. They suggested that newer teachers abandoned research-based instructional practices in favor of textbook-driven instruction because they had not yet built the more flexible and practical knowledge needed to deal with extreme student behavior. Similarly, Hover and Yeager (2004) found that newer teachers' instructional approaches frequently did not facilitate group work or student-engaged lessons because they did not want to risk losing control during the lesson. Notice these perspectives of classroom management focused on the teacher exerting inflexible control and compliance (Berliner, 2004).

Research has also indicated that teachers' instructional practices evolve during the first few years of their teaching careers (Berliner, 2004). When following ten mathematics teachers through their first two years of teaching, Grossman et al. (2000) found that teaching practices learned in methods coursework were not evident until the teacher's second and third years of teaching. Similarly, Ensor (2001) found that despite first year mathematics teachers having knowledge of and being able to talk about research-based practices, they did not facilitate such practices in their own classrooms. Decisions were often justified in terms of student behavior issues and their beliefs about students' learning abilities. Taken together, these findings suggest that much of the learning from teacher preparation might develop through experience (Ward et al., 2011) and the activity of teaching over time (Berliner, 2004; Rhoads & Weber, 2016).

In this study, we explored experiences of third year teachers working in secondary school mathematics classrooms. Specifically, we address the following questions.

4. What aspects of teaching do third year teachers talk about when prompted to share their successes and challenges?
5. Were these aspects of teaching described as successful or challenging by third year teachers?

Methods

The three participants in this study graduated in the same cohort from a one-year Master of Arts in Teaching (MAT) program from a state university in the southeastern region of the United States. Participants had already earned an undergraduate degree in an area related to mathematics and were seeking state certification and a master's degree. The MAT focused primarily on general and mathematics pedagogy. After graduating, each participant requested one of the authors serve as their university mentor as required by their new teacher induction program. The first author served as a mentor for one participant, and the second author for two participants. All participants were female and in their third year of teaching when data was collected. One participant, Rachel, taught 8th grade in a small rural school serving a nearly all White student population. The other two participants, Kelly and Lola, taught Algebra and Geometry high school courses in large suburban schools. The courses taught were composed of predominately Black and Latinx student populations.

Data was collected during the entire 2019 – 2020 school year, including midway through the spring semester in which the COVID-19 pandemic began. Data came from regularly scheduled conversations, typically once every two weeks. Conversations focused on successes and challenges participants had encountered during the past couple of weeks, what made these events successful or challenging, and how they saw their preparation helping or hindering in addressing them. Kelly engaged in these conversations solely through email, Lola with a combination of email and audio recorded conversations in-person, and Rachel through a combination of text and video messaging though the phone application, WhatsApp.

Analysis

Once data collection was complete, all data sources were transcribed into a single chronological transcript for each participant. Each transcript had the Data Reduction Method (Miles, Huberman, & Saldana, 2014) applied to it to identify sections of it where the participant was responding to a prompt or question regarding successes or challenges in their classroom practice. If the participant shared multiple topics in responding to this question, each topic was treated as its own section. This resulted in a total of 66 sections across all three transcripts. Next, each section was read to identify episodes—a portion of the section that was describing a single idea, issue, or thought. Some sections in the transcript contained a single episode, whereas other sections contained multiple episodes. The total number of episodes across all three transcripts was 123. These episodes served as the unit of analysis for this study and were coded using an inductive and deductive coding process.

Topics newer teachers talk about. The deductive coding process began with a list of topics that were used in the mathematics methods for teaching courses all participants had taken during their university preparation program. All participants had taken a sequence of two mathematics methods courses taught by the first author during their preparation program. Each episode was read and if a topic from the methods topic list appeared to match, it was tagged with this topic. If an episode did not seem to align with a topic on this list, it was left untagged. All episodes that were tagged with the same topic were then read to develop a description that could be used to identify them. The collection of these descriptions was then refined and applied back onto the collection of all episodes to identify those they matched with. This cycle of applying these codes and refining their descriptions was iterated until the descriptions solidified and both authors agreed with all coding. This process was quite similar to the Constant Comparative Method (Glasser & Strauss, 1967), but not exactly the same as we began by deductively applying a list of codes (i.e., methods topics) onto the episodes rather than having them emerged from the episodes (i.e., inductive development of codes).

For all episodes that were not associated with one of the tags from the methods topic list, an inductive coding process was used—the Constant Comparative Method (Glasser & Strauss, 1967). Episodes were read and sorted into categories containing a single theme. A description was developed for each category and these descriptions were then applied back onto the episodes to ensure the description captured only the desired category's episodes. The cycle of developing a description and applying it back onto the collection of all episodes was iterated until the descriptions stabilized and both authors were in agreement with all coding.

Classifying topics talked about as successful or challenging. Each episode was also classified as being perceived as successful or challenging using the following process. First, if a participant named the experience as successful or challenging, it was classified that way. Next, if a participant did not name the experience directly, the following descriptions were used.

- Successful – describing what a newer teacher a) can do or accomplish, b) feels confident about, or c) expresses a desirable behavior or result
- Challenge – describing what a newer teacher a) cannot yet do or accomplish, b) does not feel confident about, or c) expresses an undesirable behavior or result

Note that these descriptions sought to consider the episode from the perspective of the teacher as whether they observed the event as being successful or challenging. Viewing these experiences from other's perspectives (e.g., administrator) may have resulted in a different classification. Third, if the episode did not fit within these descriptions, the episode was examined to see if it contained an actionable plan. If so, the episode was classified as a success. If an actionable plan was not described, then the episode was classified as a challenge. It is important to note that a small number (5) of episodes contained evidence of both success and challenge descriptions. For these episodes, they were classified as both a success and a challenge.

Findings

In this section we report our key finding from the analysis conducted. These findings are separated into two sections, each corresponding to one of our two research questions.

Research Question One

Recall that our first research question investigated what aspects of teaching newer teachers talked about when asked to share their successes and challenges. The way participants spoke about their successes and challenges rarely referenced aspects of student mathematical thinking and learning. Rather, mathematics and the mathematics classroom appeared to serve as context to the space where participants operated within.

Our analysis indicated that newer teachers talked about a variety of topics. Although there were 123 episodes coded in total, the table below includes topics with counts of five or more. Including only these topics was done to a) focus on the most common topics talked about and b) work within space limitations. These 77 episodes accounted for 63% of all of the episodes coded.

Table 1: Descriptions and Frequencies of Common Topics Teachers Talked about for All Participants

Topic	Description	Frequency
Classroom Management*	The teacher describes a situation focusing on non-mathematical undesirable individual student behaviors, sometimes including a management decision.	22
Engagement*	The teacher identifies or describes an external student behavior regarding participating in an aspect of mathematics lesson or learning mathematics, often as an aspect in need of cultivation.	19
Interpreting and Making Sense of Student Thinking*	The teacher hypothesizes about an underlying reasoning, rationale, or sensibleness for student behaviors, statements, or actions (mathematically related or not) that initially seemed nonsensical.	8
Pragmatic Teaching Strategies*	The teacher describes a specific pedagogical course of action for supporting student's learning that occurred outside of class time.	7
"Unit"/Long-Term Planning	The teacher describes an aspect of preparing for teaching that is not focused on an individual lesson, but planning that focuses on sequences of lessons.	6
Teacher Student Rapport	The teacher describes an experience focused on developing the relationship between the teacher and student(s), or a situation that highlighted the utility of an existing teacher-student relationship.	5
Motivation*	The teacher identifies or describes an internal student disposition towards school or learning mathematics, often as an aspect in need of cultivation.	5
Differentiation*	The teacher describes a situation where they are responsible for teaching at topic and making it accessible for varying ability levels.	5

Note. An asterisk (*) indicates a topic that was examined in the mathematics pedagogy courses completed by participants during their university MAT program.

The most common topic newer teachers talked about was Classroom Management, which was consistent with other research findings (e.g., McCormack et al., 2006). Episodes classified this way contained statements focusing on undesirable student behavior that was non-mathematical in nature. For example, consider the excerpt below from Kelly.

> I have one student that sleeps every day. I attempt to wake him up at least five times a class. I have emailed and called the parent but have never gotten a response. He has met with the counselor and still no luck.

Notice the focus is on responding to this student's behavior to change it. Kelly stated specific actions she had taken, though it has not resulted in the behavior she desired, which epitomizes the description of Classroom Management. Although Classroom Management focused on student behavior, it was not the only topic that did so.

The second most common topic newer teachers talked about was Engagement. Notice both Classroom Management and Engagement focus on an external characteristic—student behavior. What distinguishes Engagement from Motivation is that Motivation is an internal disposition (i.e., student controlled) whereas Engagement is an external action (i.e., teacher influenced). For example, consider the Engagement excerpt below from Rachel.

> I had half of my Algebra class not do their homework one day this week. I was prepared and had the homework assignment on paper. I had them do it at lunch, and I had about four of them refuse to do any of it. I had several in another class tell me that they just guessed on a test question instead of actually trying. We were doing test corrections, trying to learn from our mistakes. And multiple admitted they didn't try.

Notice Rachel's decision to be prepared and have the homework assignment on paper so students can make test corrections during lunch, continuing to press students to engage mathematically. She described this in conjunction with student behaviors regarding choosing (not) to work with mathematics, illustrating the description of Engagement.

In addition to their aggregates, we also separated these counts by participant (see Table 2).

Table 2: (Relative) Frequencies of Common Topics Teachers Talked about Separated by Participant

	Kelly	Lola	Rachel
Classroom Management*	3 (23%)	12 (24%)	7 (12%)
Engagement*	2 (15%)	9 (18%)	8 (13%)
Interpreting and Making Sense of Student Thinking*	---	3 (6%)	5 (8%)
Pragmatic Teaching Strategies*	---	7 (14%)	---
"Unit"/Long-Term Planning	---	1 (2%)	5 (8%)
Teacher Student Rapport	---	3 (6%)	2 (3%)
Motivation*	2 (15%)	3 (6%)	---
Differentiation*	2 (15%)	2 (4%)	1 (2%)
Totals	9 (68%)	40 (80%)	28 (46%)

Note. Totals are less than 100% because participants spoke about topics that had counts of less than five, which were not included in this manuscript as described above Table 1.

The topics that every participant talked about were Classroom Management, Engagement, and Motivation. Although these were talked about by each participant, notice how often they talked about each was not the same. For example, Rachel talked about Classroom Management (12%) about half as often as either Lola (24%) or Kelly (23%). Additionally, notice how much more common it was for Kelly to talk about Differentiation (15%) in contrast to Lola (4%) or

Rachel (2%). Although participants were more consistent in how often they talked about Engagement, it still varied from 13 – 18%.

This separation by participant also identifies variation in how commonly each participant talked about these most prevalent topics. For example, notice how these topics constituted 80% of what Lola talked about, whereas they were 46% of what Rachel spoke on. Moreover, we can see that some topics were unique to a single participant (i.e., Lola's Pragmatic Teaching Strategies) or a pair (e.g., Rachel and Lola both speaking about Interpreting and Making Sense of Student Thinking). Overall, data suggest the relevance of each topic to the individual varied.

Research Question Two

Recall that our second research question investigated if the topics newer teachers talked about were described as challenging or successful to them. Our analysis indicated that newer teachers varied in what they talked about as challenging or successful. Although there were 128 codes of challenges or successes (recall that five episodes contained evidence of both success and challenge descriptions), the table below includes topics with counts of five or more. Including only these topics was done to a) focus on the most common topics talked about as successful, challenging, or both, and b) work within space limitations.

Table 3: Frequencies of Challenges and Successes for Common Topics Teachers Talked about for All Participants

Topic	Challenge	Success
Classroom Management*	12	12
Engagement*	14	7
Interpreting and Making Sense of Student Thinking*	5	3
Pragmatic Teaching Strategies*	2	5
"Unit"/Long-Term Planning	2	4
Teacher Student Rapport	---	5
Motivation*	4	1
Differentiation*	3	2
Totals	42	39

The total ratios of challenges to success for the most common topics was balanced. Although ratios varied depending upon the topic, some topics were more often talked about as successful. For example, consider the excerpt below from Rachel on the topic of Teacher Student Rapport, which was only spoken about in terms of successes.

> I've started writing thank you notes every Friday. I've done it three straight weeks now. I write one to a student in every class. I'm trying to find value in every student, so they know I care. I've gotten a couple hugs and several 'thank yous' so I think it's going well so far.

Notice Rachel's focus on building connections with students, exemplifying the description of Teacher Student Rapport. She identified the desired result (i.e., recognition from students, states "I think it's going well"), aligning with the description of a success.

Although multiple topic's ratios were skewed towards successful, not every topic talked about was referred to this way. For example, consider the excerpt below from Lola on the topic of Interpreting and Making Sense of Student Thinking.

> I want my students to benefit from the projects and reviews that I am giving to improve their grades, but the group that I want to take these opportunities is not. I know that students have

many things going on besides academics though, and by this time of the year they are exhausted.

In this excerpt, Lola stated a possible reason for why the group of students did not take advantage of the opportunity to improve their grade, typifying the description of Interpreting and Making Sense of Student Thinking. This also aligns with the description for a challenge via the undesirable result (i.e., the group of students not capitalizing on this provided opportunity).

In addition to their aggregates, we also separated these counts by participant (see Table 4).

Table 4: Ratios of Challenges to Successes for Common Topics Teachers Talked about Separated by Participant

	Kelly	Lola	Rachel
Classroom Management*	3:1	6:6	3:5
Engagement*	2:0	5:4	7:3
Interpreting and Making Sense of Student Thinking*	---	3:0	2:3
Pragmatic Teaching Strategies*	---	2:5	---
"Unit"/Long-Term Planning	---	0:1	2:3
Teacher Student Rapport	---	0:3	0:2
Motivation*	2:0	2:1	---
Differentiation*	1:1	1:1	1:0
Totals	8:2	19:21	15:16

Separating these counts by participant highlighted some differences. Although the overall challenge to success ratio was balanced (42:39) and that Lola and Rachel's ratios were nearly balanced, Kelly's ratio was heavily skewed towards challenges. Of Kelly's successes, one episode was Classroom Management classified as both a challenge and a success, and the other was an episode about Differentiation. Although investigating Kelly's episodes did not provide insight as to why this skew towards challenges occurred, we wondered if Kelly was still in Berliner's (2004) "novice" stage, or perhaps limited resources (Ward et al., 2011) influenced this skew.

We also noticed some topics had variation in the ratio between challenges and successes between participants, whereas others were consistent. For example, Classroom Management was nearly always challenging for Kelly (3:1), balanced for Lola (6:6), and more frequently successful for Rachel (3:5). However, this variation was not always the case as Teacher Student Rapport was only talked about successfully by Lola and Rachel, and Motivation was nearly always challenging for Kelly and Lola. Overall, the most common topics participants talked about varied as to if they were successful or challenging. The only exceptions to this were Teacher Student Rapport and Motivation as noted.

Discussion

Newer teachers encountered both challenges and successes in their experiences and topics they spoke about. A common theme with the two most frequently talked about topics (i.e., Classroom Management, Engagement) was the focus of exerting control and compliance on student behavior, which agrees with previous research findings (e.g., Hover & Yeager, 2004). Although teachers can influence someone's behavior, they cannot directly control it. One question we wondered about was how newer teachers view the distinction between the authority they hold as a teacher versus their locus of control as a human. To what extent do they recognize the boundary between direct control versus influence when working with students? Also, how does a newer teacher learn to flexibly contextualize their classroom management decisions from a novice to an advanced beginner (Berliner, 2004)?

Some of the topics the year three teachers spoke about were explicitly incorporated into their preparation program (e.g., interpreting and making sense of student thinking, differentiation) and identified in a common vision for teacher preparation (e.g., AMTE, 2019).The way and extent to which teacher preparation influences newer teacher's practice and decision making is based on the classroom settings, school structures related to mentoring, and their goals and visions for teaching (Hammerness, 2003; Jansen et al., 2018) as well as their development as a newer teacher (Berliner, 2004). Similar to contextualized teacher knowledge identified by Feldman and Herman (2015), this would help to explain why participants did not always talk about the same topics, nor why they experienced the same kinds of successes or challenges with them. The relevancy of a topic to each newer teacher depended upon their situational context, which also agrees with Leatham (2006) who concluded that practice is shaped by context.

Given the influence of varying school contexts beginning teachers will work within, we wondered how and to what extent this might be accounted for in teacher preparation. Moreover, given the evolution of newer teachers during the first few years of their careers, are teacher preparation programs designed to primarily serve teachers through their induction phase, or the phase after they become established classroom teachers? Although the vision provided in standards documents (e.g., AMTE, 2019) identifies what a well-prepared newer teacher should know, we wondered to what extent these components are pragmatic given the variation in teacher preparation programs and development that newer teachers experience.

Acknowledgements

This work was supported by a grant from the Education Economics Center at Kennesaw State University.

References

Arbaugh, F., Ball, D. L., Grossman, P., Heller, D. E., & Monk, D. (2015). Deans' corner views on the state of teacher education in 2015. Journal of Teacher Education, 66, 435–445.

Association of Mathematics Teacher Educators [AMTE] (2019). Standards for Preparing Teachers of Mathematics. Available online at amte.net/standards

Ball, D. L., & Forzani, F. M. (2011). Building a common core for learning to teach and connecting professional knowledge to practice. American Educator, 35(2), 17–21, 38–39.

Barker, D. D., Lannin, J. K., Winsor, M. S., & Kirwan, J. V. (2019). Integrating knowledge for instruction: A tale of two teachers. The Mathematics Enthusiast, 16(1), 331–358.

Berliner, D. C. (2004). Describing the behavior and documenting the accomplishments of expert teachers. Bulletin of Science, Technology & Society, 24(3), 200–212.

Borko, H. (2004). Professional Development and Teacher Learning: Mapping the Terrain, Educational Researcher, 33(8), 3–15.

Conference Board of the Mathematical Sciences [CBMS]. (2012). The mathematical education of teachers II. Providence, RI and Washington, DC: American Mathematical Society and Mathematical Association of America.

Ensor, P. (2001). From preservice mathematics teacher education to beginning teaching: A study in recontextualizing, Journal for Research in Mathematics Education, 32(3), 296-320.

Feldman, A., & Herman, B. C. (2015). Teacher contextual knowledge. In R. Gunstone (Ed.), Encyclopedia of science education (pp. 1020–1021). Springer.

Evertstein, C. M., & Weinstein, C. S. (2006). Classroom management as a field of inquiry. In C. M. Evertstein & C. S. Weinstein (Eds.), *Handbook of classroom management: Research, practice, and contemporary issues* (pp. 3–15). Lawrence Erlbaum Associates.

Feiman-Nemser, S. (2001). Helping novices learn to teach. Journal of Teacher Education, 52(1), 17–30.

Glaser, B. G., & Strauss, A. L. (1967). The discovery of grounded theory: Strategies for qualitative research. New Brunswick, NJ: Aldine Transaction.

Grossman, P.L., Valencia, S., Evans, K., Thompson, C., Martin, S., & Place, N. (2000). Transitions into teaching: Learning to teach writing in teacher education and beyond. Journal of Literary Research, 32, 631–662.

Hammerness, K. (2003). Learning to hope, or hoping to learn? The role of vision in the early professional life of teachers. Journal of Teacher Education, 54(1), 43–56.

Hover, S. D., & Yeager, E. A. (2004). Challenges facing beginning history teachers: An exploratory study. International Journal of Social Education, 19(1), 8–26.

Jansen, M., Gallivan, H., & Miller, E. (2018). Early-career teachers' instructional visions for mathematics teaching: impact of elementary teacher education. Journal of Mathematics Teacher Education, 23, 183–207.

Lampert, M. (2010). Learning teaching in, from, and for practice: What do we mean? Journal of Teacher Education, 61(1-2), 21–34.

Leatham, K. (2006). Viewing mathematics teachers' beliefs as sensible systems. Journal of Mathematics Teacher Education, 9(1), 91-102.

McCormack, A., Gore, J., & Thomas, K. (2006). Early career teacher professional learning. Journal of Teacher Education, 34(1), 95–113.

Miles, M. B., Huberman, A. M., & Saldana, J. (2014). Qualitative data analysis: A methods sourcebook. Thousand Oaks, CA: Sage Publications.

Ward, C. J., Nolen, S. B., & Horn, I. S. (2011). Productive friction: How conflict in student teaching creates opportunities for learning at the boundary. International Journal of Educational Research, 50, 14–20.

TEACHERS' ATTITUDES OF INSTRUCTIONAL GESTURES IN MATH AND SCIENCE LEARNING

Rebecca Boncoddo
Central Connecticut University
Boncoddo@ccsu.edu

Mitchell J. Nathan
University of Wisconsin-Madison
mnathan@wisc.edu

Teruni Lamberg
University of Nevada, Reno
terunil@unr.edu

Travis Olsen
University of Nevada, Las Vegas
travis.olson@unlv.edu

A teacher's role in helping students understand mathematics and engage in sense-making is crucial. Therefore, we investigated how teachers viewed instructional gestures after they participated in a content and pedagogy based professional development project. The findings revealed that teachers thought about gestures as a means of communication and cognition. Implications for supporting teachers' use of instructional gestures to help students to make mathematical connections through integrated content-based professional development are discussed.

Keywords: Teacher Noticing, Teacher Beliefs, Mathematical Representations, Instructional Activities and Practices

Teachers instinctively use gestures to communicate information, along with speech, during instruction (Alibali & Nathan, 2012). Similarly, students also utilize gestures as they engage in math lessons (Arzarello et al., 2009). Many teachers do not explicitly think about how gestures support communication, thinking, and learning as they emerge naturally during instruction (Schenck et al., 2022). However, making teachers aware of their own gestures may allow them to think about how to strategically leverage gestures in their instruction (Sung et al., 2022). Little research exists on how to support teachers to effectively use gestures in their instruction (Georgiou & Ioannou, 2019; Roth, 2001). However, we believe that research on gestures cannot stand alone, and needs to be connected to content based professional development for teachers to really leverage their own gestures to support student learning.

Sherin, Jacobs and Phillips (2010) pointed out that there is a need for future research to understand how to support teachers to notice, understand, and endorse students' gestural and verbal participation. Sherin recommends that future research should build on the current work on teachers' noticing of children's mathematical thinking (Jacobs, Lamb, & Philipp, 2010; Mason, 2002; Sherin & Han, 2004; Sherin & van Es, 2009). This process involves understanding what teachers attend to and how teachers interpret gestures to support learning. Therefore, we designed a content based professional development in mathematics and science for K-8 teachers and explicitly embedded research on how gestures support cognition and student learning. We investigated teachers' attitudes about the utility of gestures for both instruction and student learning as they engaged in the professional development.

Method

The data sources include teacher journal responses at the end of a week-long professional development, video recordings of the professional-development sessions, and field notes.

Twenty-six elementary and middle-school educators from a Western State participated in the study as part of a larger effort to improve primary and middle grade integrated STEM (science, technology, engineering, and mathematics) instruction. Twenty-one teachers completed the journal responses during their participation in a week-long professional development session focused on supporting teachers to learn the Common Core standards along with Next Generation science standards. Specifically, the math content focused on measurement and data analysis and was embedded in the Next Generation Science Standards: Science & Technology, Physical Science, and Personal and Social Perspectives (NGSS) with a particular focus on nanotechnology.

The goal of the professional development was to increase teacher STEM content knowledge and pedagogical content knowledge using a design research approach (Cobb, Confrey, DiSessa & Lehrer, 2003). The design of this PD included an interdisciplinary team made up of mathematics educators, mathematicians, cognitive scientists, nanotechnology scientists, science educators, district leaders, and regional professional development teams that were part of the larger initiative. Encouraging teachers to become aware and use instructional gestures to support student learning was an explicit goal of the professional development. Therefore, researchers from mathematics education and the Learning Sciences developed video-based modules that provided examples and outlined the research principles embedded in the professional development design. PD activities explicitly discussed the role of instructional gestures in teaching after teachers viewed the videos. These activities were designed to meaningfully integrate research-based principles into teachers' classroom practices and to raise teachers' awareness and understanding of the utility of instructional gestures for students' STEM learning. The specific design used for professional development is outlined in Figure 1.

The PD design incorporated four phases: Teachers as Learners, Pedagogy, Examining Student Thinking, and Connect to Curriculum and Planning (Author, date). The research on gestures was specifically embedded in the context of Teachers as Learners and Examining Student Thinking. Teachers viewed 4 video recordings on the research on gestures, and the facilitator-researchers engaged the teachers in guided discussions. Teachers had the opportunity to explicitly discuss what they saw in the videos in small and whole-group sessions. Teachers also had chances throughout the week-long PD to think about what it meant to use gestures in their teaching practices and to attend to the gestures of others. At the end of the week, as part of the PD, teachers were prompted to journal about their views of instructional gestures.

> *Reflect on how you use gestures and when you communicated your ideas or listened to someone's explanation. How did this influence your ability to make sense of what others were saying or thinking? What are your thoughts about how you might use gestures in the classroom?*

To analyze and interpret these data, we used a case study approach (Yin, 2014). The data were analyzed using the constant comparative method (Strauss & Corbin, 2018). Video data from these PD sessions was transcribed and examined for emerging themes. Constant comparative methodology incorporates four stages: "(1) comparing incidents applicable to each category, (2) integrating categories and their properties, (3) delimiting the theory, and (4) writing the theory" (Glaser & Strauss, 1967, p. 105). The data from the video analysis data was triangulated with the data from the journal entries and from observers' daily field notes.

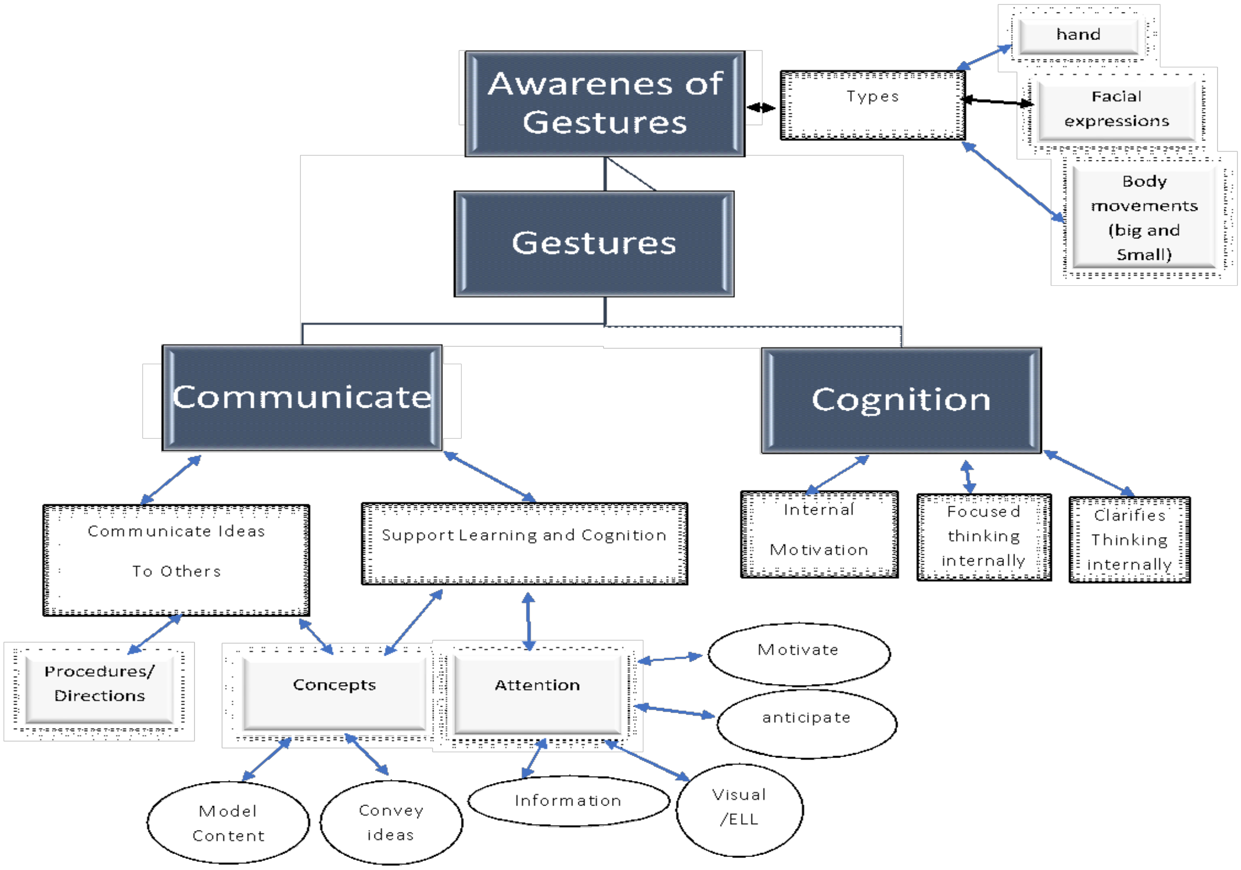


Figure 1. How teachers thought about gestures in their journal responses (Author, date).

Results

Issues Related to Gestures that Emerged within the PD Session

The videotape data of teachers discussing the gesture videos and their involvement in the participatory activities around their STEM teaching practices revealed issues that teachers thought about concerning instructional gestures. For example, in one instance, teachers engaged with a module on how instructional gestures can help students make mathematical connections across different visual representations. It explained that instructional gestures are small, situated actions intended to help comprehension and learning by complementing and reinforcing the information that is presented in the speech and images was explained. A classroom video clip example of a teacher using gestures to explain how to solve an algebraic equation was embedded in this module. This video showed how the properties of an equation relate to the properties of a balance scale. It also depicted how one can maintain balance (equivalence) when one "removes" identical terms from both sides of an equation much like one can remove identical objects from two sides of a balance scale. The teachers were asked to discuss what they observed in small groups and share their thoughts. Many issues emerged in the discussion as teachers explored how to incorporate instructional gestures into their own teaching practices. A few select issues that illustrate the deep pedagogical engagement that emerged are these:

- What role did actions with physical models play in helping students think about actions with formal representations?
- Should a teacher teach students how to gesture an idea or concept, or should gestures naturally emerge to represent thinking?
- Gestures can be used when a speaker's thinking and speech is still catching up, since we
- can often gesture about an idea before we can verbalize it clearly to others.

Discussion

Throughout the professional development sessions teachers became more aware of the use of instructional gestures even though nearly all of them shared that they naturally gestured during instruction. They grew in their awareness of the value of gestures to convey students' thinking, even before students have the language to clearly express themselves verbally. They considered how gestures help make operations on formal representations more meaningful (such as symbol manipulation), and how they help to relate one visual representation to another. They explored distinctions between the importance of small gestures along with grand gestures that included facial expressions and nodding. They also thought about the role of technology and the context in which gestures were used to build common ground with their students (Nathan & Alibali, 2011)

More broadly, teachers thought about gestures as an additional means of communication that can support student cognition (Alibali & Nathan, 2007) and as practical, accessible, yet powerful ways to focus students' attention during instruction to relevant locations in the learning environment. Teachers reported that instructional gestures motivated students to stay focused and engaged. Additionally, instructional gestures were useful to clarify their own thinking.

Gestures were viewed as helpful to learners by clarifying misconceptions (e.g., what "both sides of the equation" actually refers to), pointing to help visual learning, and focusing attention to find information. It establishes common ground by linking unfamiliar things (e.g., algebra equations) to familiar things (balance scales) and helps to externalize someone's thinking. The listener can more easily follow and anticipate one's reasoning, which can help to understand and diagnose one's thinking.

It should be noted that the PD introduced the research on gestures within the context of teacher learning of math and science content. Teachers made connections to their own learning as well as reflected on how this might impact their teaching and student learning. The journal entries revealed that the teachers became aware of the role of gestures in their learning and thought about how they would use gesturing in the classroom to support student learning.

The findings suggest that teachers should be introduced to research on gestures in the context of their own teaching practices and learning experiences to support teacher noticing. This will aid teacher to make sense of the research, be able to apply it in a personal way.

References

Alibali, M. W. & Nathan, M. J. (2007). Teachers' gestures as a means of scaffolding students' understanding: Evidence from an early algebra lesson. In Goldman, R., Pea, R., Barron, B. J., and Derry, S. (Eds.) Video Research in the Learning Sciences (pp. 349-365). Mahwah, NJ: Erlbaum.

Alibali, M. W. & Nathan, M. J. (2012). Embodiment in mathematics teaching and learning: Evidence from learners' and teachers' gestures. Journal of the Learning Sciences, 21(2), 247-286. DOI: 10.1080/10508406.2011.611446.

Alibali, M. W., Nathan, M. J., Wolfgram, M. S., Church, R. B., Jacobs, S. A., Johnson Martinez, C., & Knuth, E. J. (2014). How teachers link ideas in mathematics instruction using speech and gesture: A corpus analysis. Cognition and instruction, 32(1), 65-100.

Arzarello, F., Paola, D., Robutti, O., & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. Educational Studies in Mathematics, 70, 97-109.

Cobb, P., Confrey, J., DiSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. Educational Researcher, 32(1), 9-13.

Church RB, Goldin-Meadow S. The mismatch between gesture and speech as an index of transitional knowledge. Cognition. 1986;23:43–71. Doi: 10.1016/0010-0277(86)90053-3.

Georgiou, Y., & Ioannou, A. (2019). Teachers' Concerns about Adopting Technology- Enhanced Embodied Learning and Their Mitigation through Professional Development. Journal of Technology and Teacher Education, 27(3), 335–370.

Goldin-Meadow S. Hearing gesture: How our hands help us think. Cambridge, MA: Harvard University Press; 2003.

Jacobs, V. R., Lamb, L. L., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. Journal for Research in Mathematics Education, 41(2), 169-202.

Lamberg, T., Gillette-Koyen, L., & Moss, D. (2020). Supporting teachers to use formative assessment for adaptive decision making. Mathematics Teacher Educator, 8(2), 37-58.

Mason, J. (2002). Researching your own practice: The discipline of noticing. Routledge.

Nathan, M. J., & Alibali, M. W. (2011). How gesture use enables intersubjectivity in the classroom. Integrating gestures.

Novack M, Goldin-Meadow S. Learning from gesture: How our hands change our minds. Educ Psychol Rev. 2015 Sep;27(3):405-412. doi: 10.1007/s10648-015-9325-3. PMID: 26366048; PMCID: PMC4562024.

Roller, M. R., & Lavrakas, P. J. (2015). Applied qualitative research design: a total quality framework approach. New York: The Guilford Press.

Roth, W. M. (2001). Gestures: Their role in teaching and learning. Review of Educational Research, 71(3), 365–392. https://doi.org/10.3102/00346543071003365

Schenck, K., Walkington, C., & Nathan, M. J. (2022). Groups that move together, prove together: Collaborative gestures and gesture attitudes among teachers performing embodied geometry. In S. Macrine and J. Fugate (Eds.) Movement matters: How embodied cognition informs teaching and learning (pp. 131-145). MIT Press.

Shein, P. P. (2012). Seeing with two eyes: A teacher's use of gestures in questioning and revoicing to engage English language learners in the repair of mathematical errors. Journal for Research in Mathematics Education, 43(2), 182-222.

Sherin, M. G., & Han, S. Y. (2004). Teacher learning in the context of a video club. Teaching and Teacher Education, 20(2), 163-183.

Sherin, M., Jacobs, V., & Philipp, R. (Eds.). (2010). Mathematics teacher noticing: seeing through teachers' eyes (1st ed.). Routledge. https://doi.org/10.4324/9780203832714

Sherin, M., & Van Es, E. A. (2009). Effects of video club participation on teachers' professional vision. Journal of Teacher Education, 60(1), 20-37.

Strauss, A. & Corbin, J. (1998). Basics of qualitative research: Grounded theory procedures and techniques.
Newbury Park, CA: Sage.

Sung, H., Swart, M., & Nathan, M. J. (2022, April). Teaching teachers teaching students: How embodied cognition can help pre-service teachers assess student's mathematical thinking. Paper presented at the 2022 Annual Meeting of the American Educational Research Association, San Diego, CA. AERA.

Yin, R. (2014). Case study research design and methods (5th Ed.).

TRACING THE CO-EVOLUTION OF TEACHER LEARNING BETWEEN PROFESSIONAL DEVELOPMENT AND CLASSROOM PRACTICE

Hala Ghousseini
University of Wisconsin
Ghousseini@wisc.edu

Burcu Alapala
University of Wisconsin
alapala@wisc.edu

Rahul Panda
University of Wisconsin
rpanda3@wisc.edu

How teachers can experience adaptive professional development (PD) experiences is still understudied in the literature on teacher learning, which for the most part reflects an emphasis on learning outcomes rather than the process of learning. In this study, we use a situated perspective on teacher learning to investigate the coevolution of a teacher's sensemaking about facilitating classroom discussions between her classroom practice and the school-based PD experiences. The study contributes to an understanding of how the process of teacher learning can be supported through continuous, adaptive professional learning experiences and of the co-evolution of teacher learning between settings of practice.

Keywords: Adaptive Professional Development, Teacher Learning, Classroom Discourse

Mathematics classroom discussions are characterized by social and intellectual demands, where teachers not only support students' knowledge and skills but also their identities as problem solvers. During these discussions students engage collaboratively in a process where they make claims and justify them using reasoning that is based in disciplinary practices and in their existing knowledge and cultural and linguistic resources. Facilitating this work is complex and often non-routine: it requires from teachers a capacity to improvise in the midst of contingent interactions, relying on professional judgment to steer instruction productively to support mathematical goals and an inclusive mathematics learning community. Mathematics teachers, for example, must make judgments about how to respond to students individually and in groups, drawing on specialized knowledge of both mathematics and student thinking to further instructional objectives. All the while, they must attune their practice to students' needs, treating students as sensemakers and providing all students access to cognitively demanding tasks. Learning to do this complex and responsive work of teaching requires models of professional learning that nurture teachers' adaptive expertise and pedagogical reasoning in relation to teachers' own contexts of practice. How teachers experience such adaptive professional learning experiences is still understudied in the literature on teacher learning, which for the most part reflects an emphasis on learning outcomes rather than the process of learning (Walkoe & Luna, 2019). In this study, we aim to investigate how a teacher's sensemaking about facilitating classroom discussions between her classroom practice and the school-based professional development (PD) co-evolve.

The extant literature on teacher learning emphasizes the importance of designing professional learning opportunities that are adaptive and responsive to teachers' local contexts and their sensemaking (Ghousseini & Kazemi, in press). Transformative teacher learning in mathematics can be supported through sustained, connected professional development experiences that are close to teachers' own practice where they have opportunities to make sense of new knowledge and instructional practice through ongoing collaboration and inquiry around purposive activities (Silver et al., 2009; Lefstein et al., 2020). Among the PD models aligned with this vision of teacher learning is job-embedded professional development where teachers are supported to make sense of the complexities of teaching during their workday or through experiences that are integrated in the contexts of their own practice (Kazemi et al., 2021; Zepeda, 2014). Typical approaches to job-embedded professional development include coaching and modeling of research-based instructional strategies, and co-teaching practices (Semon et al., 2020). Althauser (2015), for instance, describes a job-embedded PD that engaged teachers in aligning their curriculum with state math standards and in peer teaching where they implemented teaching strategies and assessments that they had developed with the support of their district's curriculum specialist. Most studies of job-embedded PD, however, focus on outcomes such as student learning or teacher practices and their views of such experiences (Althauser, 2015; Dennis & Hemmings, 2018). Missing from the literature are examinations of how teachers learn through job-embedded PD. Walkoe and Luna (2019) affirm the absence of such studies from the literature and argue, "Questions that address the process of teacher learning are largely absent in studies of teacher learning. These include important questions such as: What mechanisms contributed to observed outcomes? What resources did teachers bring to bear and how were they utilized in PD activities? What goals emerged for teachers as they engaged in PD?" (p. 285).

Our investigation of teacher learning in this paper is motivated by these questions. Using the case study of a teacher who participated in an adaptive, job-embedded PD focused on learning to facilitate classroom math discussions, we address this question: How did the teacher's sensemaking co-evolve between her classroom practice and the school-based PD experiences? By examining her process of recontextualization (van Oers, 1998) of goals and resources negotiated with other participants from the PD through her own practice, we aim to contribute to an understanding of teacher learning from PD opportunities closely tied to their practice. The study also contributes to the theme of this conference in the way our results bring insights related to the problems of practice teachers face in facilitating student-centered classroom mathematics discussions.

Teacher Sensemaking About Argumentation-Based Discussions as a Process of Recontextualization

The literature on argumentation repeatedly asserts that argumentation is more than an activity—it rests on establishing a classroom culture for argumentation over time through intentional norm setting, activity selection, and guidance (Knudsen et al. 2018). Like all ambitious teaching, developing a culture for mathematical argumentation is complex and creates many intellectual and pedagogical demands for teachers in the way they have to intentionally (1) choose and structure mathematical problems or routines; (2) develop norms for what counts as acceptable arguments; and (3) provide language supports and use discourse structures to engage students in the practices of argumentation (Makar et al., 2015).

We use a situative perspective on teacher learning (Greeno, 2005) to frame sensemaking about facilitating argumentation as a teacher's opportunities to wrestle with new ideas and figure out how and why they work within their own teaching and learning spaces. In the context of adaptive professional development that is embedded in teachers' practice, teacher sensemaking involves opportunities to attend to problems of practice through experimenting with and revisiting ideas, drawing in the process on a wide array of resources and tools to support emergent ideas or goals (Kazemi et al., 2021). Researchers have called for attending to the dynamic boundaries and relationships between various settings that influence and explain teacher learning, looking at the coevolution of participation between classroom practice and PD (Kazemi & Hubbard, 2008). In this study, we focus on this form of coevolution of participation through the lens of continuous sensemaking and recontextualizing of learning (van Oers, 1998) across settings of teacher learning through adaptive PD.

The literature on teacher learning typically treats the idea of recontextualizing as teachers taking constructed discursive resources from one context and reconstructing them to enact in a secondary context that has unique conditions and rules (Ensor, 2001; Marchant et al., 2021). This view of context explains teachers' actions in terms of using knowledge, tools, and experience gained in one context and applying them to a new scenario. In contrast, Van Oers (1998) characterizes recontextualization as a process of continuous sensemaking in the way a context is treated as not a static setting that teachers are placed into, but rather dynamically constructed by them in situ as a particular activity setting. This is accomplished by determining one's goal, examining prior experiences, and "finding out which means are available, investigating which actions make sense to perform in order to achieve the goal chosen, and by relating motive, goal, object, means etc…" (van Oers, 1998, pp. 481-482). We posit that this continuous process of recontextualization provides teachers with opportunities to learn. Viewed from a social practice perspective, adaptive PD can enable a process of continual contextualizing when participants engage in mutual activities where they experiment with new possibilities for action and adjust their contributions to shared activities as they move between settings of learning over time.

Methods

Participants and Context

The professional development model, we refer to as Learning Labs (LLs), was embedded in an elementary school located in the United States' Midwest. The school is diverse with 40% Latinx students. The LLs were co-designed and co-facilitated by three instructional coaches from the school and three teacher educators from our research team. Our co-design work engaged and involved eight teacher participants in learning to facilitate argumentation-based mathematics discussions. Each LL consisted of four phases: new learning, planning, enactment, and debrief (Kazemi et al., 2021). In the new learning phase participants engaged in a reading, analysis of artifacts, or discussion of problems of practice related to the practices of giving explanations or justifications and supporting broad student participation. Building on their new learning, then the participants planned a lesson together to be enacted in one of the participant teachers' classes. During this planning, all participants shared ownership of designing the math task, crafting tasks and working on questions that elicited and pressed for students' thinking. The enactment phase involved inquiry into practice where teachers both did the work of teaching in response to students' performance and had opportunities to pause the lesson to consider problems of practice. In the debrief phase, the participants reflected on the lesson and areas of improvement and identified future collective goals for the labs and for their own practice. Between consecutive LLs, the teachers and coaches met weekly to co-design, implement, and reflect on lessons deliberately enacted by the teachers to contextualize some goals related to the PD in their own practice. They reflected on these lessons using video-stimulated recall interviews (VSRIs).

In this paper, we use an instrumental case study (Stake, 2005) of one teacher participant, Karla (pseudonym), to inform our understanding of a particular phenomenon, namely, how teachers' sensemaking about ambitious teaching can co-evolve between participation in PD experiences and their own classroom practice. Karla is a white female 5th grade teacher who was an active participant in the LLs, often highlighting her own experiences and vocalizing her sensemaking. Her propensity to reflect publicly on the way her sensemaking was coevolving between LLs and her own practice influenced our choice of her for this case study.

Data Collection and Analysis

For this paper, our data consisted of (1) transcripts of three interviews conducted with Karla (one prior to the start of the LLs and two at the end of each year); (2) Eight transcripts of LLs debriefing phases spanning over years 1 & 2; (3) Four audio records of video stimulated recall interviews (VSRIs) that Karla completed with her school-based coach. The VSRIs focused on classroom discussions that Karla recorded between LLs. Our data analysis at one level focused on identifying the problems of practice that Karla grappled with across contexts and the array of tools and resources she used over time to address them. We tracked these problems and resources longitudinally as suggested in Figure 1, across both her participation in LLs as well as in her interviews and VSRIs with the instructional coach.

Interview 1	LL1	VSRI (1 & 2)	LL (2-4)	Interview 2	LL 5	VSRI 3	LL (6-8)	VSRI 4	Interview 3
Year 1					Year 2				

Figure 1: Karla's data sources over two years

We started with the first interview to determine Karla's initial practice with respect to leading mathematics discussions, and to identify the nature of questions and challenges she faced while promoting them in her classroom. We recognized a problem of practice when Karla reported on classroom interactions that she experienced as troublesome, challenging, confusing, recurrent, unexpectedly interesting, or otherwise worthy of comment" (Horn & Little, 2010, p. 189). This first analysis surfaced a central problem of practice for Karla related to *bringing more student participation and voices during whole class discussions*. A prior analysis of the problems of practice that participants jointly identified over the course of the LLs (Cordero-Siy et al., 2021) allowed us to trace the continuity of Karla's sensemaking about this problem of practice through her participation in the LLs. Starting with LL1, we identified instances where she contributed to the participants' joint work on framing challenges related to participation. We focused on moments where she elaborated the nature of a challenge and provided insights from her experience about how to address the challenge and the reasoning behind it. Focusing on her participation in LLs' debriefs was constructive to our understanding of her sensemaking as we saw alignment between Karla's challenge of broadening participation in her classroom discussions and the Lab participants' joint sensemaking. We also identified the materials and resources (such as practices or classroom tools) that participants proposed to address the problem of practice during their joint sensemaking. We continued using these analytic lenses across consecutive data sources spanning both LLs and Karla's classroom practice: we tracked the evolution over time of the way Karla reframed various challenges pertaining to the problem of practice she is addressing, and revisited her understanding or use of a resource or a tool to address the problem of practice she is focusing on. Figure 2 represents this co-evolution between LLs and her practice.

Results

Karla's initial practice with discussions. In her initial interview (Int.1), Karla described her classroom mathematics discussions as typically happening in small groups where students share different strategies and explain their thinking to each other. She reported avoiding whole class discussions because she would "hear from the same five voices, so it's better when we're in small groups." This concern for "varied student voices" (Int. 1) was a problem of practice that persisted in Karla's thinking and sensemaking throughout her participation in the PD in Years 1-2. It was motivated by Karla's overarching commitment to creating a learning environment in which "all students identify as someone who can learn math" (Int. 1). Leveraging more student voices, however, was complicated by what Karla referred to as *student vulnerability in the face of power relationships in the classroom*. She stated, "some students see themselves as leaders and others as followers." Leaders, in her view, are ones perceived to give better explanations or to be more fluent in English. This situation influenced students' hesitation to participate in classroom discussions, especially in whole group.

Karla's continual commitment to attending to this problem of practice led her to recontextualize ideas, tools, and resources across these contexts. Reflecting on the consequences of these experiences with her coach and with other participants led her to a more nuanced understanding of the challenge of student participation that went beyond the binary of small group vs. whole group. For space limitations, we present this evolution in her thinking based on her participation in the first year of PD.

The co-evolution of Karla's sensemaking between LLs and her practice. A recurring theme in the LLs was creating classroom communities for all students to engage with mathematical ideas (Cordero-Siy et al., 2021). This theme aligned with Karla's challenge of leveraging more student voices, where we found evidence in Karla's VSRIs of her continual pursuit of creating spaces for students to share their own ideas and engage with each other's mathematical thinking. Accordingly, she drew from various pedagogical tools leveraged during LLs to support the pursuit of this goal. As represented in Figure 2 below, over the course of the LLs participants leveraged various pedagogical resources and tools to facilitate equitable participation. For example, in LLs 1 and 2 participants reflected on factors that may be shaping student willingness to talk—the physical setting of the classroom, the nature of the task, and students' familiarity with routines that support their sharing of ideas. Accordingly, they reasoned about possible resources and tools that could build on these factors constructively. Among the resources they considered were instructional tasks that focus on number relations rather than the right answer, using more turn-and-talk participation structure, and supporting student communication with visuals and tools like small whiteboards.

In our analysis of Karla's VSRIs with her coach, we noted the way she was drawing on these resources and problematizing their use at the same time. She revealed in her first VSRI the way the use of white boards and turn-and-talk to facilitate student sharing did not fully resolve the issue of inequitable participation as only a few students were willing to share their thinking. Karla pondered whether calling on students to paraphrase their partner's ideas could enhance the efficacy of these classroom routines and tools. Student sharing with the help of these resources, she opined, still faced the tension of putting students in a vulnerable position when 'cold-calling' students—in case they could not articulate or did not understand their partner's ideas. Moreover, while partner talk and work on a whiteboard may have supported student talk in small groups, Karla explained that inequitable participation persisted in the whole group setting. Students did not engage with others' solution strategies when shared with the whole class. This concern found resonance in the subsequent LL2 when participants noticed similar patterns of participation. During the debrief, Karla noted then "But that's something I struggle with in my class, and I wonder if it's pacing, the activity went too long. That [students] all had such rich partner to partner conversations then when I ask someone to get up and explain it, it's like I already know this because I just talked about it with my partner." (LL2)

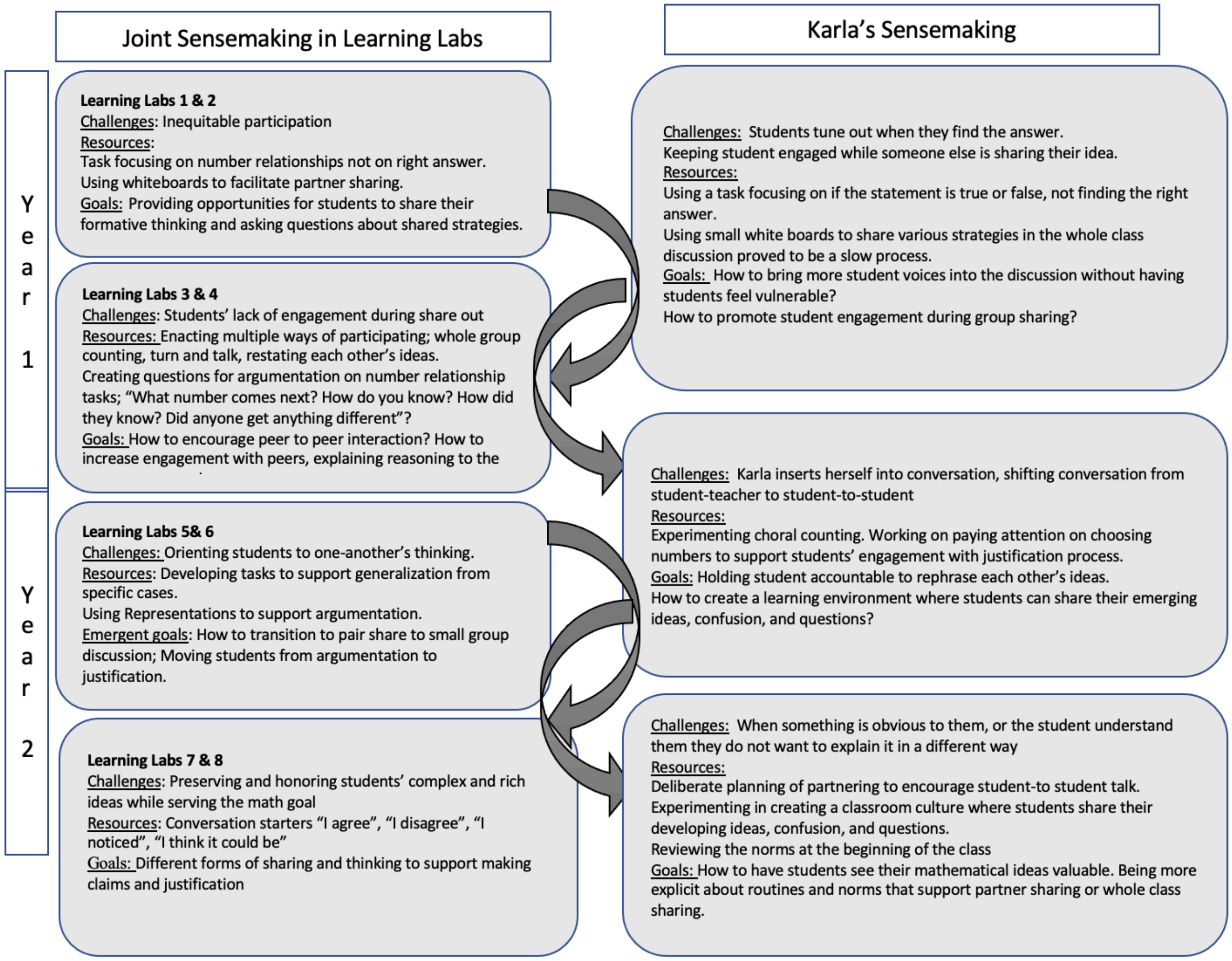


Figure 2: The Co-evolution of Challenges, Resources and Goals between Learning Labs and Karla's Practice

The participants' joint sensemaking in LL2 led them to identify this resource to address this problem: crafting tasks that focus on reasoning rather than a perceived *right answer*. Taking a cue from the group's suggestion, Karla experimented with the use of math tasks in her class where students needed to prove why a math statement was True/False. Reflecting on her work with her coach in the subsequent VSRI, she noted that unlike previous occasions when students tuned out after they had found an answer to a problem, students showed greater engagement, likely precipitated by the nature of the mathematical task whose central objective was not finding a final result, but the *process*. Yet, despite the changes she observed in overall engagement, the challenge of participation in her view continued to persist: some students still felt uncomfortable sharing in the whole-class setting, and Karla debated holding students accountable to sharing their partner's thinking as it may ruin the 'feeling of ... like, having a safe space to talk about math' (VSRI-2).

The subsequent two LLs (3 & 4) offered Karla opportunities to further think about student participation in whole class discussions. As participants grappled with students' hesitation to share during whole class discussions, they considered how *orienting students to each other's* thinking can support them in understanding each other's ideas and in representing those ideas to the whole class (LL3). They agreed on the importance of *monitoring students' positioning as sensemakers and creating connections between students' ideas* while orienting them to each other's work. In LL4, the participants further considered how being explicit about *the norms and practices of participation* in small and whole group discussions can bring in more students' voices. They also agreed on the importance of modeling and highlighting forms of participation that can be productive for everyone's learning. In her own teaching, Karla adapted these strategies to address her dilemma of surfacing mathematical ideas from students who are generally reluctant while ensuring these students don't feel vulnerable: She created a participation structure protocol to support students' sharing of each other's ideas, which she described in Interview 2 as:

> I still struggle with the small group, but the large group, we have a protocol we follow. So, you all get think time and then you talk to a neighbor and then someone shares out and then we think about what they said. We've done those steps several times. We've done them with the lab and then I've repeated them, so we have set steps that we do in large group. So that feels really good. It sets it up for some good discussions to happen because there's a lot of voices being heard. You're talking in a small group. You feel safe when you're talking to a partner. When you're sharing out, you feel safe because you can share your partner's idea. don't know, It's like when you're one voice of many in many it's not as ... You would think it would be scarier to share with the large group, but the way we've set up all the protocols is, "Just share an idea that you heard at your table." So, it's not your idea. It doesn't have to be your idea. (Interview 2, lines 94-100)

Through her lesson debriefs, Karla reported that the use of this protocol in her practice was productive for creating spaces where all students could engage in mathematics. She explained that such supports could disrupt students' negative mathematical identities, where students' ideas are positioned as valuable. Karla shared her classroom experiences with the participants in the subsequent LL4, elaborating her view of the consequences of students' acknowledging each other's contributions, "With Brian, who has this image of himself that he's not good at math, and he was the first one in the group to say four is half of eight. And like in a small group, I can hear that and point it out to him. And I often do it. You know you were the first one to say it, right? And you helped the whole group understand that" (LL4).

By the end of the first year of PD, Karla's reflections on the challenge of bringing more voices into the discussion moved away from binaries of whole group vs. small group. Her reflections on this challenge allowed her to consider an array of ways students can be supported to work with each other in both group structures. In the second year of the study, we continued seeing this shift in Karla's perspective due her continual sensemaking about strategies and pedagogical tools to create a safe environment for students—like tasks that invite multiple ways of participation and focus on mathematical thinking processes and creating norms for partner and whole-class sharing. Having noticed increased partner sharing and variety of student voices in the whole class discussions with the help of her pedagogical toolbox, Karla's problem of practice started shifting from student participation goals to supporting discussions in various group structures where she had minimal teacher intervention.

Discussion

In this study of the co-evolution of a teacher's participation between PD and her own practice, we aimed to attend to teacher learning as a process of continuous sensemaking rather than an outcome. We portrayed Karla's learning beyond simply taking up tools from PD to her practice where we used evidence from her participation in both contexts to show how the recontextualization of ideas and tools can lead to richer deliberations about problems of practice and ambitious teaching. Karla's iterative recontextualization of strategies from LLs to her practice and the additional questions she considered about practice suggest the way tools can mediate activity not only by translating abstract conceptual problems for beginners into a series of concrete steps, but also through the affordances (and sometimes constraints) it contributes to the development of their sensemaking and learning.

References

Althauser, K. (2015). Job-embedded professional development: Its impact on teacher self-efficacy and student performance. *Teacher Development*, *19*(2), 210-225.

Cordero-Siy, E., Prough, S., Alapala, B., & Ghousseini, H. (2021). Identifying and recontextualizing problems of practice in learning to facilitate discussions with argumentation. In D. Olanoff, K. Johnson, & S. Spitzer, (Eds.), *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Philadelphia, PA.

Dennis, D.; & Hemmings, C. (2018). *Making the simple more complex: the influence of job-embedded professional development in supporting teacher expertise in reading*. https://doi.org/10.1111/lit.12172

Ensor, P. (2001). From preservice mathematics teacher education to beginning teaching: A study in recontextualizing. *Journal for Research in Mathematics Education*, *32*(3), 296-320.

Ghousseini, H., & Kazemi, E. (In Press). Professional learning tasks through job-embedded teacher professional development. In G. Stylianedes, J. Cai, & P. Kenney (eds.), *On the learning and teaching of mathematics: Research studies in honor of Edward A. Silver.*

Gomez Marchant, C. N., Park, H., Zhuang, Y., Foster, J. K., & Conner, A. (2021). Theory to practice: Prospective mathematics teachers' recontextualizing discourses surrounding collective argumentation. *Journal of Mathematics Teacher Education*, *24*(6), 671-699.

Greeno, G. (2005). Learning in Activity. In R. Sawyer (Ed.), *The Cambridge Handbook of the Learning Sciences* (Cambridge Handbooks in Psychology, pp. 79-96). Cambridge: Cambridge University Press. doi:10.1017/CBO9780511816833.007

Horn, I. S., & Little, J. W. (2010). Attending to problems of practice: Routines and resources for professional learning in teachers' workplace interactions. *American Educational Research Journal, 47*(1), 181-217.https://doi.org/10.3102/0002831209345158

Kazemi, E., & Hubbard, A. (2008). New directions for the design and study of professional development: Attending to the co-evolution of teachers' participation across contexts. *Journal of Teacher Education*, *59*(5), 428-441.

Kazemi, E., Ghousseini, H., Cordero-Siy, E., Prough, S., McVicar, E., & Fox, A. (2021). Supporting teacher learning about argumentation through adaptive, school-based professional development. *ZDM Mathematics Education*, *53*, 435-448. https://doi.org/10.1007/s11858-021-01242-5

Knudsen, J., Stevens, H., Lara-Meloy, T., Kim, H. J., & Shechtman, N. (2018). *Mathematical argumentation in middle school: The what, why, and how*. Corwin Press.

Lefstein, A., Vedder-Weiss, D., & Segal, A. (2020). Relocating research on teacher learning: Toward pedagogically productive talk. *Educational Researcher*, *49*(5), 360-368.

Makar, K., Bakker, A., & Ben-Zvi, D. (2015). Scaffolding norms of argumentation-based inquiry in a primary mathematics classroom. *ZDM*, *47*, 1107-1120.

Semon, S.; Lane, D.; Jones, P., & Smith, S.M. (2020). Job-embedded professional development: implementing co-teaching practices in general education classrooms. *International Journal of Inclusive Education.* https://doi.org/10.4324/9781315719693

Silver, E. A., Clark, L. M., Ghousseini, H. N., Charalambous, C. Y., & Sealy, J. T. (2007). Where is the mathematics? Examining teachers' mathematical learning opportunities in practice-based professional learning tasks. *Journal of Mathematics Teacher Education*, *10*(4), 261-277.

Stake, R. (2005). *Multiple case study analysis*. New York: The *Guilford Press*.
Walkoe, J. D., & Luna, M. J. (2020). What we are missing in studies of teacher learning: A call for microgenetic, interactional analyses to examine teacher learning processes. *Journal of the Learning Sciences*, *29*(2), 285-307.
Van Oers, B. (1998). The fallacy of decontextualization. *Mind, culture, and activity*, *5*(2), 135-142.
Zepeda, S. J. (2014). *Job-embedded professional development: Support, collaboration, and learning in schools*. Routledge. https://doi.org/10.4324/9781315719693

UNDERSTANDING MATHEMATICS COACHES' OPPORTUNITIES TO ENGAGE WITH EQUITY DURING THEIR OWN PROFESSIONAL DEVELOPMENT

Evthokia Stephanie Saclarides
University of Cincinnati
saclares@ucmail.uc.edu

Juan Manuel Gerardo
University of Cincinnati
gerardjm@ucmail.uc.edu

Keywords: instructional leadership; professional development; equity, inclusion, and diversity

In response to calls to promote more equitable mathematics teaching practices (Gutiérrez, 2009), mathematics coaching has been named as a promising professional development (PD) structure to support teachers to develop equity-centered pedagogies (Marshall & Buenrostro, 2021). Yet, "coaching may fall short if coaches are insufficiently prepared for such complex, sensitive work" (Marshall & Buenrostro, 2021, p. 602) and few studies have explored how coaches' own professional development might prepare them to support equitable teaching and learning practices at their school sites. Thus, the overarching purpose of this investigation is to better understand how mathematics coaches engage with equity while doing the math during their own district-sponsored professional development.

Our study took place in a relatively well-resourced public school district, pseudonymously referred to as Hamilton, which is located in a southeastern, metropolitan area of the United States. We partnered with 12 elementary mathematics coaches and one district administrator. Data sources included video data, accompanying transcripts, and field notes for six doing the math sessions, as well as 15 transcribed interviews completed with a subset of participants.

To better understand our participants' opportunities to engage with equity while doing the math we used Gutiérrez's (2009) equity framework as our analytic lens. Gutiérrez (2009) conceptualizes equity along four dimensions: (1) access; (2) achievement; (3) identity; and (4) power. Access and achievement comprise the dominant axis while identity and power comprise the critical axis. Both authors engaged in multiple rounds of qualitative coding (Creswell, 2013) to identify instances in which the coaches discussed each dimension of equity during their own professional development as it might apply to supporting teaching and learning at their respective schools.

Across the doing the math sessions, our analysis showed that the mathematics coaches primarily had opportunities to engage with the dominant equity axis, while they less frequently engaged with the critical equity axis. Furthermore, discussions centering on the access dimension seemed to dominate the coaches' equity-centered conversations. When engaged in conversations about access, the mathematics coaches primarily discussed potential modifications that could be made to mathematics tasks to: (1) provide an entry point for students, or (2) extend the task for students.

Our study adds much needed research describing mathematics coaches' own professional learning opportunities to engage with equity. Future research might consider exploring how coaches can be explicitly supported to discuss equity issues related to power and identity during their own professional development.

References

Creswell, J. W. (2013). *Research design: Qualitative, quantitative, and mixed methods approaches* (4th ed.). Thousand Oaks, CA: Sage.

Gutiérrez, R. (2009). Framing equity: Helping students 'play the game' and 'change the game'. *Teaching for Excellence and Equity in Mathematics, 1*(1), 2009, 5-7.

Marshall, S. A., & Buenrostro, P. M. (2021). What makes mathematics teacher coaching effective? A call for a justice-oriented perspective. *Journal of Teacher Education, 72*(5), 594-606. https://doi.org/10.1177/00224871211019024

USING A FORMATIVE EVALUATION FRAMEWORK TO VALIDATE A TEACHING OBSERVATION TOOL

EL USO DE UN MARCO DE EVALUACIÓN FORMATIVA PARA LA VALIDACIÓN DE UNA HERRAMIENTA DE OBSERVACIÓN DE LA ENSEÑANZA

Kathleen Melhuish
Texas State University
melhuish@txstate.edu

Brittney Ellis
Texas State University
b_e107@txstate.edu

Alejandra Sorto
Texas State University
sorto@txstate.edu

Teaching observation protocols serve purposes beyond research, such as providing formative feedback for teachers' growth in their practice. Such observation tool usage requires different approaches for validation. For this reason, we developed the formative teaching evaluation framework adapted from the student-assessment literature. We used this framework as a guide toward collecting and organizing evidence of practitioners using the Math Habits Tool as a means for formative assessment of teaching. This report focuses on how we designed surveys and interviews to collect validity evidence related to the formative teaching evaluation intentions of the MHT. Overall, we established that the MHT was being used as intended by teachers and school leaders, and we provide details about the development and analysis procedures we took toward validating this observation tool for practitioner use.

Keywords: Measurement; Professional Development

Teaching observation tools are often positioned as both research tools and tools to provide formative feedback to support teacher reflection and growth (Boston et al., 2015). Yet, as noted by Yee et al. (2022), these protocols tend to focus on how "they are intended to be used to measure teaching behaviors" (p. 219) without attention to how they can be used for professional growth. Common observation protocols have focused validation efforts on measurement elements such as achieving inter rater reliability, establishing criterion-related validity, or validation through panels or experts (e.g., Gleason, et al., 2017; Hill et al., 2012; Schoenfeld, et al., 2018). While these traditional validation approaches are critical, we suggest there is also a need to explicitly collect validity evidence related to intended use beyond research contexts. That is, if these tools are developed to also provide formative feedback for teachers, then it is crucial that we attend to the viability of this usage type.

In this report, we present the *formative teaching evaluation* framework adapted from Nicol and Macfarlane-Dick's (2006) literature-based categories of formative assessment for students. We used this framework to provide a means to collect and organize evidence of an observation tool's formative evaluation usage. We share data from the Math Habits Tool observation protocol project, focusing on how surveys and interviews were used to collect validity evidence from users related to the formative teaching evaluation intentions of the instrument.

Formative Evaluation and Teacher Feedback

Schools have become increasingly data-driven (Marsh & Farrell, 2015) with teacher observation by school leaders or peers serving as the most common data source for evaluating teaching (Firestone & Donaldson, 2019; Steinberg & Donaldson, 2016). Such evaluation provides summative information as well as a mechanism to encourage instructional growth; that is, "while observations were initially conceived as tools for evaluation, such protocols are now seen as key levers for the improvement of teaching" (Hill & Grossman, 2013, p. 372). Hill and Grossman have suggested that observation needs certain qualities to productively support instructional growth, such as content, observers, and time.

Instructional growth necessitates a reflection process that cannot be achieved through experience and training alone (Loughran, 2002). Productive reflection goes beyond summative lenses and requires nuanced, formative lenses for consideration of how instructional practice is working and for whom (Zeichener & Liston, 1996). Instructional practice needs to be decomposed from broad practice to specific practices (e.g., Grossman et al., 2009) and align with subject-specific goals. Evaluation focused on broad practice without specific ties to content has been found to have insufficient details needed to reflect on and change practice (e.g., Rigby et al., 2017). Hunter and Springer (2022) identified four critical observation feedback characteristics from the literature: "a) aligns with an improvement area, (b) discusses the feedback's evidential basis, (c) sets specific improvement goals, and (d) includes actionable next steps" (p. 380). That is, the quality of observation is about the content and whether the feedback provides a roadmap to be actionable toward change in practice. This aligns with Hill and Grossman's argument that a small grain-size is needed to support such actions and planning, a theme consistent with Firestone and Donaldson's (2019) larger literature synthesis.

One of the cross-cutting benefits of observation is creating a shared language for discussing ideas around instruction (Firestone & Donaldson, 2019); however, the relationship between the observer and teachers can have a strong impact on whether a particular observation is viewed as supportive for professional growth. In Paufler et al.'s (2020) study, they found that teachers diverged on whether they thought an evaluation system was primarily serving the purpose of judgment or formative feedback with attention to the way the system was communicated and how meetings with the observer played out. While the literature suggests that observation and feedback can provide teachers with support for instructional growth, the potential for formative evaluation is not always met by the incorporation of observation and feedback alone.

The Formative Teaching Evaluation Framework

Literature on formative assessment centers on assessing students (e.g., Black & Wiliam, 2010). To assess formative teaching evaluation goals, we developed a framework composed of formative assessment attributes. In prior work, we started creating this framework by adapting Nicol and Macfarlane-Dick's (2006) synthesis on formative assessment for students (Melhuish & Thanheiser, 2017). We suggest that formative assessment for teaching meets the following criteria:

1. helps clarify what quality instruction is (goals, criteria, expected standards).
2. facilitates the development of reflection on teaching.
3. delivers high-quality information to teachers about their teaching and student learning.
4. encourages dialogue around teaching and learning.
5. encourages positive motivational beliefs and self-esteem.

6. provides opportunities to close the gap between current and desired instructional practice.
7. provides information to observers (including school leaders) that can be used to help shape their support for teachers.

To meet these goals, formative assessment tools must reflect teachers' efforts at enacting critical components of teaching, provide details capturing the nature of these efforts, and offer direction for change and growth. That is, observation needs to be specific, actionable, and provide a language that can support robust conversation and reflection.

Study Background and Methods

The Math Habits Tool

This study is situated in a larger project aimed at designing and validating the Math Habits Tool (MHT). The tool components overlay with the instructional triangle (Figure 1; Cohen et al., 2003; Hawkins, 2002). The components were developed iteratively by appealing to literature on student-centered classrooms focused on justifying and generalizing and professional development with in-service teachers and school leaders. It meets the recommendations of explicitly situating observed components in mathematical activity and using a small grain-size.

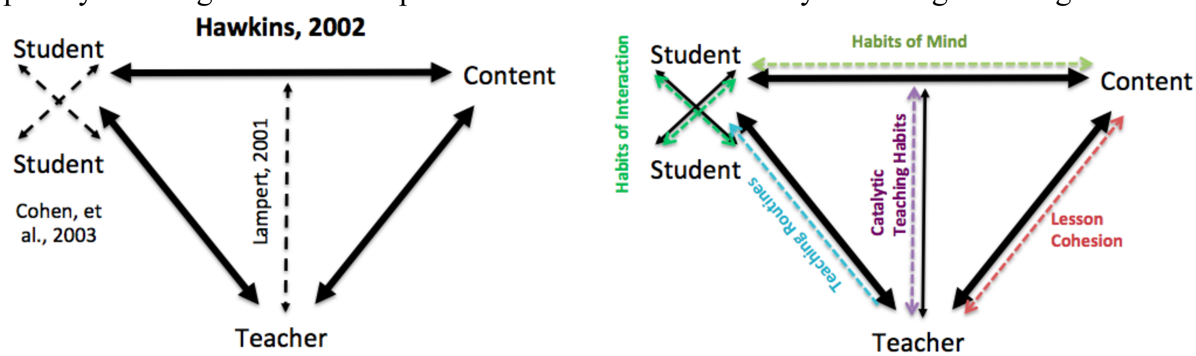


Figure 1. Instructional Triangle and Components of the MHT

Interpretation and Use Statement. Aligned with Carney et al. (2022), we provide the following interpretation and use statement. The Math Habits Tool measures standards-based instructional elements including teacher moves and routines and student math habits of mind and interaction that have been identified as important in the literature. Each component is measured via observation with a timestamp. The tool was designed for the full K-12 content spectrum for planning, observation, and reflection with teachers, principals, coaches, and within professional learning communities. It can be used to document an entire lesson or a brief drop-in segment. Rather than scores, the instrument provides timelines and frequencies for different activities. Analyzing patterns between students and teachers found in the timelines can be used to plan for teacher moves that may support increased student engagement in justification and generalization. Intended use is formative for planning, observing, and reflecting, not summative; that is, the certain occurrences or frequencies of activities should not be used to evaluate teachers.

Surveys, Interviews, Participants

The survey was designed to better understand how K-12 stakeholders (e.g., teachers, coaches, administrators/principals) the MHT, and verify whether the MHT is being used as a formative assessment tool (rather than for teacher evaluation). To address the second goal, we created multiple-choice survey items using the criteria for formative assessment for teaching framework as a guide. Two related versions were created and administered for teachers and school leaders, respectively. The survey included one open-response question intended to collect qualitative data about how practitioners perceived the purpose of the MHT. The survey was completed by 243 K-12 stakeholders (213 teachers, 30 school leaders) from a total of 53 different schools ranging from elementary to high school across the United States. All stakeholders had used the MHT and their experiences levels using the tool varied.

Survey Analysis. A principal component analysis was used to determine the underlying survey structure which led to the identification of three factors (elaborated in the results). We also considered correlation of all items to further bolster validity of the framework's construct coherence. The open-ended item responses regarding users' perceived purpose of the tool were qualitatively coded by research team members using an iterative process of open-coding, meeting to discuss codes, arriving at condensed themes, and applying these themes to the data.

Follow-Up Interviews. After survey analysis was completed, we interviewed 16 practitioners. A cluster analysis was conducted on the Likert scale responses to identify teacher profiles using a k-means clustering algorithm with a Euclidean distance metric. We ran the algorithm with two through eight clusters and used a D-B index to determine the best number of clusters. This analysis revealed five clusters (see Table 1). Participants who indicated agreement to be contacted for a follow-up interview were selected from each cluster, totaling 11 teachers. Five school leaders (principals/coaches) who volunteered were also interviewed. In the follow-up interviews, we asked practitioners to elaborate on their survey responses, share further information about their perceived purpose of the tool, and explain in detail how they used the tool with reference to particular parts (e.g., which parts were most used, which were least used) and contexts (e.g., who they used the tool with). A set of analytic notes were created after each interview to identify whether responses seemed consistent with surveys and how participants explained their use of the tool in terms of planning, teaching, and reflection.

Table 1: Description of the five clusters of survey respondents and dimensions of the survey

	Affective Items	Observation and Collaboration Items	Planning and Reflection Items
Cluster 1	Low	Low	Low
Cluster 2	Mid	High	Mid
Cluster 3	High	High	High
Cluster 4	Low	Low	Mid
Cluster 5	High	Low	Mid

Results

The survey results provided validity evidence that the MHT was meeting formative assessment goals for the majority of the respondents.

Surveys: Closed-Form Items

A scree plot indicated three distinct dimensions underlying the formative assessment survey (Figure 2). A set of sample items related to each dimension can be found in Table 2. Notice that individual activities were most prevalent with all teachers stating that they used the tool to plan for how to engage students in particular habits of mind and interactions. The observation and collaborative function occurred less frequently, yet indicated prevalent usage. This is not surprising or misaligned with intended usage because the time allotted for observation would be smaller than the potential tool usage as an individual reflection tool. Finally, we note that the MHT also seems to be supporting (at least to some degree) affective and understanding goals with majority of teachers agreeing that the tool supported instruction for justifying/generalizing and promoting confidence to refine instruction. We suggest that the closed-form results helped to bolster our validity of usage claims and provides a blueprint for other instrument developers.

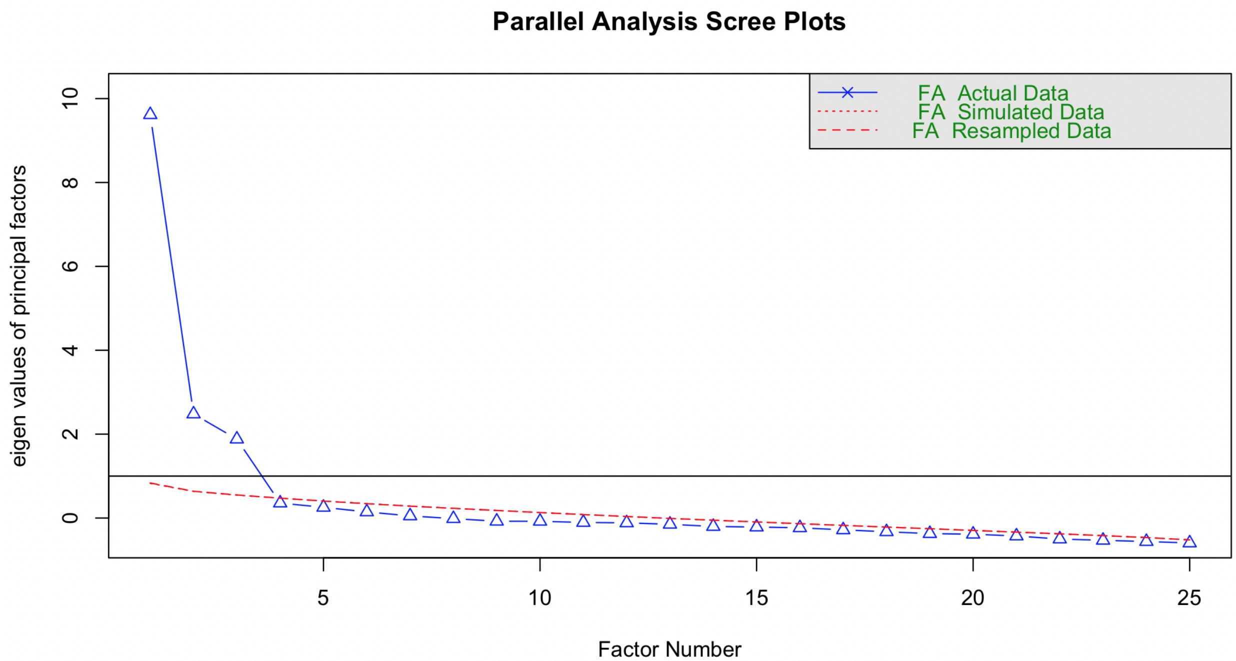


Figure 2: Scree plot. Note the three dimensions above the simulated data.

Table 2: Survey Items and Dimension

	Dimension	Percentages
I use the Math Habits Tool to plan how to engage students in habits of mind/interaction.	Individual	Never (0%) Only with PD (13%) Rarely/Occasionally (32%) Weekly/Daily (55%)
I use the Math Habits Tool to reflect on whether enacting teaching routines I planned to implement went as expected.	Individual	Never (4%) Only with PD (18%) Rarely/Occasionally (48%) Weekly/Daily (30%)
I use the Math Habits Tool to talk with school leaders (e.g., coaches, principals) about classroom observations.	Observation and Collaboration	Never (14%) Only with PD (25%) Rarely/Occasionally (59%) Weekly/Daily (2%)
I use the Math Habits Tool to observe teachers' classrooms with other teachers	Observation and Collaboration	Never (34%) Only with PD (34%) Rarely/Occasionally (31%) Weekly/Daily (1%)

By using the Math Habits Tool, I feel confident in refining my instruction.	Affect and Understanding	Strongly Disagree (3%) Disagree (9%) Agree (57%) Strongly Agree (31%)
Using the Math Habits Tool has helped me better understand how instruction can support students in justifying/generalizing.	Affect and Understanding	Strongly Disagree (3%) Disagree (3%) Agree (35%) Strongly Agree (59%)

Surveys: Open-Ended Item

The survey also asked teachers and school leaders to respond to the question: *In your own understanding, what is the purpose of the Math Habits Tool?* This question was used to better situate the users' responses on the closed-form items and gather additional evidence of how they conceptualized the tool's purpose. From the coding process, five main themes were identified: *common language and framework*, *planning and reflection on practice, observation and feedback, mechanism to promote students' mathematical reasoning,* and *a mechanism to support students' mathematical communication with each other.* Regarding common language and framework, stakeholders articulated that the MHT provided a specific language for both students and teachers to use together as well as a structure that reflects important elements of instruction. One teacher noted that the tool is a "quick reference for best learning/teaching practices in math," which can support student learning. A school leader further explained that the tool provides a "common language and conceptual grasp for a department of math teachers." This came up in 60% of the school leader responses (18 total) and 36% of the teacher responses (76 total).

The tool as a mechanism to promote students' mathematical reasoning was a salient theme for teachers, with 140 teachers (66%) noting this purpose in their responses. For example, a teacher wrote, "the purpose of the [tool] is to engage all students in mathematical ideas and understanding by using representations and making connections to build justifications and generalizations." While less of a focus for school leaders (10 total, 35%), many mentioned this alongside other purposes. For instance, one school leader stated the purpose of the tool was to "ground the work you do planning for lessons, discussions, tasks [and] connect things in a way that leads students to math conjectures and generalizations," which speaks to the tool as a framework, planning tool, and a tool to support discussions. A less salient theme for teachers was observation and feedback use, with 12 teachers (5.63%) and 13 school leaders (43.33%) mentioning this purpose, which aligns with our findings in the prior section.

Teachers disclosed either planning or reflecting on practice as the purpose of the tool; ideally, the goal would be for planning and reflection together to be viewed as a purpose. For instance, a teacher mentioned, "I use this to reflect on what was in my lesson, and I use this also to think about planning my upcoming lessons and activities." Overall, about 20% of teachers (41 total) and 27% of school leaders (8 total) mentioned planning and/or reflection on teaching. Finally, the tool as a mechanism to support students' mathematical communication with each other was brought up in 68 teachers' (32%) and 12 school leaders' (40%) responses. This was often talked about as facilitating discourse about mathematics in the classroom, to "engage students in purposeful conversations about math." Due to the nature of the question, practitioners may have thought to record only one purpose of the tool. For this reason, having practitioners expand on their responses or add other purposes in the interviews was critical for validation.

Interviews

Semi-structured interviews provided a space for open conversations with practitioners to collect additional evidence of how the tool was being used. It was a crucial part of our process to determine whether the tool was being used for formative assessment, and to confirm whether it was *not* being used for other purposes, such as summative evaluation. All interviewees expressed formative assessment uses with school leaders, as exemplified in the following quote from a K-5 mathematics coach:

> …sometimes the tool is more for me and then I just filter it [into] just general teacher language when I'm talking to the teacher, or sometimes I will say 'Okay let's look at the tool together,' since this is a teacher who is very familiar with it, and we can use that to do some of the reflection, but it's all really…how can I help this teacher to identify a goal for themselves to increase their practice.

This coach explained multiple purposes including common language, collaborating, reflecting, and planning. They went on to explain, "I think from the teacher's perspective… just the amount of formative assessment data you get from utilizing the tool with your students and just understanding how they're thinking, how they're making connections…." They voiced the collection and use of formative assessment that focused teachers' attention to what students are thinking. Another coach directly explained that they are "careful that… when we go in with admin it doesn't feel like another evaluation," recognizing that the tool should support learning goals but not summative evaluation.

In interviews, teachers also reflected that the MHT was not being used for summative evaluation. One 6th grade teacher elaborated that it is, "less of like a 'are you good, are you not?' Why is it good and just very vague. It makes it very concrete, easily talked about versus evaluative." Overall, teachers elaborated on the role the tool played in their teaching, their planning and reflection, and their collaboration in the school, and emphasized the communication role of the tool. For example, a high school teacher stated, "I also think it's a great onboarding tool for administrators who don't have experience in math classrooms specifically or even if they do, it's a great tool that launches conversation between teachers and administrators." They continued to discuss planning, saying, "But, like everything else about the planning with… those things in the tool in mind gave me life and joy and gave my students success and confidence and so that's why it's ingrained in me."

We purposefully selected teachers from clusters that had survey responses reflecting less achievement of formative assessment goals. From these interviews, we identified several ways teachers voiced that the MHT was not meeting formative assessment needs. One concern from a middle school teacher was a misalignment between the mathematics emphasized in the observation tool (such as justifying and generalizing) and the mathematics emphasized in the school's curriculum. She noted that the "philosophy of the curriculum is very rote math" and the tool was hard to use in the later parts of the year that were more focused on procedures. Another issue that arose was burn out, especially since the pandemic. This same teacher noted, "it was just like one more thing to make me feel like I was not a good teacher, and so I think that can be very, very discouraging." This comment provides negative evidence in relation to the affect and understanding dimensions (*encourages positive motivational beliefs and self-esteem* from the framework). In this case, she viewed the MHT as just one more thing to balance and imposing an ambitious image of instruction that was hard to enact.

Finally, some teachers noted that while they used the content of the tool in their thinking when planning, they no longer relied on the physical tool. A teacher in a 6th grade math and science Spanish immersion classroom stated that, "In my mind, it's a bridge to take us from traditional math classrooms where the teacher is delivering information, students are note taking, students regurgitate said information and we all move on and then students learn how to hate math." They continued explaining that "when I'm planning, I'm thinking about these things, but not through the [physical] tool." One explanation for this was that the school provided "lots of initiatives" and so they had other tools (besides the MHT) to provide the support they needed.

Discussion

As researcher tools become adapted to new contexts, it is important to validate for purposes beyond research. Our goal in this report was to share some of our efforts to validate an observation tool for a practitioner setting. To meet this aim, we developed the formative evaluation of teaching framework and a corresponding survey to administer to teachers and school leaders. Throughout this work, we take the general stance that tool validation should be done through the accumulation of different types of evidence to build a robust validity argument (Kane, 1992). While we have done substantial validation from a research perspective, in alignment with best practices suggested by other researchers, we found a need to consider the validity of the instrument for use by teachers and school leaders. That is, we wanted to lay out a systematic way to collect evidence of whether the MHT was successful in its formative evaluation intentions for practitioners. To do this, we collected evidence from many end users (teachers/school leaders) via a survey, and then additional evidence from detailed interviews with a targeted subset of practitioners who completed the surveys.

By using the formative evaluation of teaching framework as a guide while collecting evidence of the MHT's formative evaluation usage, we found that the MHT was being used as intended overall. The open-ended questions and interviews provided additional details and reflected some of the literature-based arguments for quality observation tools including providing a shared language (e.g., Firestone & Donaldson, 2019), tying into specific content (e.g., Rigby, et al., 2017), and providing actionable means to plan for and reflect on instruction (e.g., Hunter & Springer, 2022). We also explored how the same observation tool may not meet its formative assessment goals for all instructors. Our results indicated that there may be misalignment between curriculum and observation emphasis, competing school initiatives, and inferred quality evaluations that could interfere with the MHT's formative evaluation potential.

The ways in which school leaders use an observation tool as formative feedback may impact how teachers take up using the tool on their own and with other teachers. In the surveys and interviews, we noticed that teachers mentioned higher use of the MHT with professional development support. All the teachers and school leaders in our study had some professional development support around the use of the tool. Additional research is needed to see how the MHT formative evaluation use can be met with different degrees of support. Overall, we see this study as contributing an image of a positive step for validating tools, but additional work is needed for expansion to other contexts.

Acknowledgments

This material is based upon work supported by the National Science Foundation under Grant No. 1814114. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

References

Black, P., & Wiliam, D. (2010). Inside the black box: Raising standards through classroom assessment. Phi delta kappan, 92(1), 81-90.

Boston, M., Bostic, J., Lesseig, K., & Sherman, M. (2015). A comparison of mathematics classroom observation protocols. Mathematics Teacher Educator, 3(2), 154-175.

Cohen, D. Raudenbush, S. Ball, D. (2003). Resources, instruction, and research. Educational Evaluation and Policy Analysis, 25, 119-142.

Firestone, W. A., & Donaldson, M. L. (2019). Teacher evaluation as data use: what recent research suggests. Educational Assessment, Evaluation and Accountability, 31, 289-314.

Gleason, J., Livers, S., & Zelkowski, J. (2017). Mathematics classroom observation protocol for practices (MCOP2): A validation study. Investigations in Mathematics Learning, 9(3), 111-129.

Hawkins, D. (2002). The informed vision. New York, NY: Algora Publishing

Hill, H. C., Charalambous, C. Y., Blazar, D., McGinn, D., Kraft, M. A., Beisiegel, M., ... & Lynch, K. (2012). Validating arguments for observational instruments: Attending to multiple sources of variation. Educational Assessment, 17(2-3), 88-106.

Hill, H., & Grossman, P. (2013). Learning from teacher observations: Challenges and opportunities posed by new teacher evaluation systems. Harvard educational review, 83(2), 371-384.

Hunter, S. B., & Springer, M. G. (2022). Critical feedback characteristics, teacher human capital, and early-career teacher performance: A mixed-methods analysis. Educational Evaluation and Policy Analysis, 44(3), 380-403.

Loughran, J. J. (2002). Effective reflective practice: In search of meaning in learning about teaching. Journal of teacher education, 53(1), 33-43.

Marsh, J. A., & Farrell, C. C. (2015). How leaders can support teachers with data-driven decision making: A framework for understanding capacity building. Educational Management Administration & Leadership, 43(2), 269-289.

Melhuish, K. M., & Thanhesier, E. (2017). Using formative evaluation to support teachers in increasing student reasoning. In L. West & M. Boston (Eds.), Annual perspectives in mathematics education 2017: Reflective and collaborative processes to improve mathematics teaching (pp. 183–199). National Council of Teachers of Mathematics.

Nicol, D. J., & Macfarlane‐Dick, D. (2006). Formative assessment and self‐regulated learning: A model and seven principles of good feedback practice. Studies in higher education, 31(2), 199-218.

Paufler, N. A., King, K. M., & Zhu, P. (2020). Promoting professional growth in new teacher evaluation systems: Practitioners' lived experiences in changing policy contexts. Studies in Educational Evaluation, 65, 100873.

Schoenfeld, A. H., Floden, R., El Chidiac, F., Gillingham, D., Fink, H., Hu, S., ... & Zarkh, A. (2018). On classroom observations. Journal for STEM Education Research, 1, 34-59.

Steinberg, M. P., & Donaldson, M. L. (2016). The new educational accountability: Understanding the landscape of teacher evaluation in the post-NCLB era. Education Finance and Policy, 11(3), 340-359.

Rigby, J. G., Larbi-Cherif, A., Rosenquist, B. A., Sharpe, C. J., Cobb, P., & Smith, T. (2017). Administrator observation and feedback: Does it lead toward improvement in inquiry-oriented math instruction?. Educational Administration Quarterly, 53(3), 475-516.

Yee, S., Deshler, J., Rogers, K. C., Petrulis, R., Potvin, C. D., & Sweeney, J. (2022). Bridging the gap between observation protocols and formative feedback. Journal of mathematics teacher education, 25(2), 217-245.

USING DESIGN-BASED RESEARCH TO EXPLORE A MODEL OF PROFESSIONAL DEVELOPMENT

Rachael Eriksen Brown
Penn State Abington
reb37@psu.edu

Chandra Hawley Orrill
UMass Dartmouth
corrill@umassd.edu

Kun Wang
UMass Dartmouth
kwang1@umassd.edu

Keywords: Professional Development, Research Methods

Professional development (PD) aims to support teachers in improving their teaching practices as well as strengthening their content knowledge. This poster shares how we are using design-based research (DBR; Cobb et al., 2003; Collins et al., 2004; DBRC, 2003) to create a model of PD that intends to engage teachers playfully in content and practice.

DBR is iterative research about an innovation as well as theory using input from stakeholders as well as systematic research questioning the theory and success of the innovation. In this case, we generated a working theory about teacher knowledge (Weiland et al., 2021) and through this we conjectured how to best engage teachers in learning. We first consulted with two advisory groups about what we call our Model of Coherent Learning for Teaching. One group was teachers and the other researchers. This stakeholder feedback led to a modification of the model to focus on content instruction (two-thirds) and applying learning to instruction (one-third). Next, we created a pilot PD experience to understand the innovation through capturing small group discussions during the PD. This led to our current understanding of our model.

The Model of Coherent Learning for Teaching includes three stages: Exploration, Connection, and Application. Exploration playfully engages teachers with focused mathematical ideas including conjecture testing and mathematical argument. This stage has been consistent in our PD work over the past decade and past research suggests this type of engagement is successful for teacher learning (Brown, 2009; Burke, 2017; Orrill & Brown, 2023). Connection is focused on mathematical structure and representation. This stage surfaced as necessary through analysis of video of the pilot PD. Finally, the Application stage is where teachers directly connect with practice by considering how we can support students to have similar experiences playing with content and representations. The final stage was suggested through our stakeholder feedback as well as through previous work where there was evidence that teachers struggled to do this with PD ideas (e.g., Orrill & Kittleson, 2015). Throughout, teachers are working in small groups and the facilitator leads full group discussions using questioning and highlighting key mathematical ideas.

This poster will highlight the evolution of the PD model as it is tied to iterations of DBR. This will include the current model, where it came from, and how we are using DBR to further develop it.

Acknowledgments

This material is based upon work supported by the National Science Foundation under Grants No. DRL-1125621, DRL-1621290, and 2201125. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

References

Brown, R. E. (2009). Community building in mathematics professional development. (Unpublished doctoral dissertation). University of Georgia, Athens, GA.

Burke, J. P. (2017). Case study of mathematical playfulness in an adult mathematics classroom setting. (Unpublished doctoral dissertation). University of Massachusetts Dartmouth, Dartmouth, MA

Cobb, P., Confrey, J., DiSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. Educational Researcher, 32(1), 9–13.

Collins, A., Joseph, D., & Bielaczyc, K. (2004). Design research: Theoretical and methodological issues. Journal of the Learning Sciences, 13(1), 15–42.

Design-Based Research Collective (DBRC). (2003). Design-based research: An emerging paradigm for education inquiry. Educational Researcher, 32(1) 5–8. http://www.jstor.org/stable/3699927

Orrill, C. H., & Brown, R. E. (2023). Using design-based research to develop a professional development model. In M. J. Spector, B. B. Lockee, & M. D. Childress (Eds.), Learning, design, and technology. Springer. https://doi.org/10.1007/978-3-319-17727-4_177-1

Orrill, C. H., & Kittleson, J. (2015). Translating learning into practice: Considering the relationship between teachers' professional development and teaching. Journal of Mathematics Teacher Education, 18(3), 273-297. doi: 10.1007/s10857-014-9284-5

Weiland, T., Orrill, C. H., Nagar, G. G., Brown, R. E., & Burke, J. (2021). Framing a robust understanding of proportional reasoning for teachers. Journal of Mathematics Teacher Education, 24, 179-202. https://doi.org/10.1007/s10857-019-09453-0

WHAT AND HOW EXPERIENCED AND NOVICE COACHES NOTICE: A FRAMEWORK TO ANALYZE COACH NOTICING

Julie M. Amador
University of Idaho
jamador@uidaho.edu

Ryan Gillespie
University of Idaho
rgillespie@uidaho.edu

Jennifer Kruger
University of Rochester
jkruger@warner.rochester.edu

Adam Hanan
University of Idaho
hana3274@vandals.uidaho.edu

We examined what and how experienced (mentor coaches) and novice coaches (coach participants) noticed as they analyzed a vignette of a coaching interaction between a coach and a teacher. We modified the van Es (2011) Learning to Notice Framework for a coaching context to analyze What and How coaches notice. We collected data from ten mentor coaches, who were experienced coaches and ten coach participants who were more novice coaches. We compared pre and post noticing for coach participants based on a two-year professional development model and compared the noticing to the mentor coaches. Findings indicate coach participant noticing for What and How coaches notice collectively shifted toward a greater focus on the teacher, included more interpretation, and was more specific from the beginning of the professional development to the end. The mentor coach noticing, on average, was more teacher-focused, interpretative, and specific than those of the coach participants.

Keywords: Noticing, Professional Development, Coaching, Online Professional Development, Vignette

Coaching is a professional development process used to support teachers to improve their instruction (West & Staub, 2003). Within mathematics education, content-focused coaching (e.g., West & Cameron, 2013) is one common coaching model that involves iterative cycles in which a coach works one-on-one with a teacher with a focus on mathematical learning goals for students. Coaching cycles are typically comprised of three sequential components: a pre-conference discussion to plan a lesson; a collaboratively taught lesson; and a post-conference discussion to debrief the lesson (Bengo, 2016; West & Staub, 2003). When coaching, the coach is charged with responding to multiple simultaneous obligations, such as supporting the teacher to design a high-quality lesson, providing the teacher with assistance to learn mathematics content and pedagogy, and establishing a trusting and productive relationship with the teacher. Given the complexity of coaching, it is important to focus on how coaches learn to coach (Stein et al., 2022) and what they notice in the process of coaching. Research on noticing (e.g., Mason, 2002) has shown that attending to stimuli and interpreting what is attended to is important when making productive instructional decisions.

The purpose of this paper is to provoke consideration about what is important to notice in a mathematics coaching context and illuminate critical events coaches and mentor coaches (those with more experience) notice. Additionally, we were interested in analyzing how coaches notice aspects of coaching conversations. Researchers of teacher noticing have traditionally classified more advanced forms of noticing as those focused on students' thinking, as compared to teacher pedagogy (e.g., Jacobs et al., 2010; van Es, 2011). Building on teacher noticing, the intent is to better understand the critical events that coaches notice and consider coach noticing in coordination with existing teacher noticing frameworks. The following question is answered: When analyzing a coaching conversation between a coach and a teacher (a) *What and How* do coaches (coach participants and mentor coaches) notice? We situate the study within an innovative online mathematics video coaching experience, adapted from an in-person modality to an online modality to support coaches (Choppin et al., 2020).

Theoretical Framing and Related Literature

We theoretically frame this paper with noticing, adhering broadly to noticing as the concept of attending to and interpreting that which is important in each context. The work of Jacobs et al. (2010) foregrounds students' mathematical thinking as the focal aspect, signifying the importance for teachers to attend to and interpret students' thinking and then make responsive decisions based on the interpretation. Cognizant of the work of Jacobs and colleagues, we draw heavily on the concepts of noticing and learning to notice of Sherin and van Es (2009). We consider noticing to be a learned skill (i.e., van Es, 2011) with levels of traceable progression to denote more advanced forms of noticing from more rudimentary forms of noticing. We also recognize various avenues through which one could notice, such as variation in noticing the actor, variety in topic, and differing approaches in the stance of noticing (e.g., Sherin and van Es, 2009). Considerate of the purpose of our project—to support and train mathematics coaches—we consider the applicability of the Learning to Notice Student Mathematical Thinking Framework (van Es, 2011) in a context beyond that of a student and teacher in a classroom: that of coaching in contrast with that of teaching. We argue that the concept of learning to notice is as applicable in a coaching conversation as it is in a teaching conversation and emphasize transferability of noticing beyond the classroom context.

The actions of coaching and what coaching entails varies based on context, coaching model, and the coaches and teachers taking part. In a study on conversational behaviors of early childhood coaches, Jayaraman et al. (2015) highlight the work of Trivetter (2009), noting that adults learn best when they are active participants, can apply their learning in an immediate context, and have multiple opportunities to practice their learning and reflect. However, the core of coaching and what the role encompasses varies depending on context and intended function. Russell et al. (2020) recognized that roles for coaches are not all the same, but the process typically includes creating intensive teacher learning opportunities that are job embedded, which occur through workshops, one-on-one work, and professional learning communities. Gibbons and Cobb (2017) described specific activities in which coaches take part. They noted that coaching can include analyzing classroom videos, facilitating book studies, visiting classrooms, co-designing instruction, conducting action research, examining student work, and more. These activities can vary in duration and intensity, with some lasting an entire school year and others being single opportunities. In many cases, the divergence in coaching activities (e.g., Johnson et al., 2018; Russell et al., 2020) can be attributed to different models of coaching. In the mathematics education context, common models include instructional coaching (Knight, 2007), cognitive coaching (Garmston, Linder, & Whitaker, 1993; Denton & Hasbrouck, 2009), and content-focused coaching (West & Staub, 2003).

Content-focused coaches strive to deepen a teacher's understanding of the content they teach and content-specific pedagogy needed to foster student learning (West & Staub, 2003). According to West and Staub (2003), the art of content-focused coaching lies in balancing "direct assistance" coaching moves that provide teachers with specific instruction or guidance and coaching moves "that invite teacher contributions" such as self-reflection throughout these three phases (p. 15). Learning to coach in this way, and to coach in ways that ultimately support teacher learning is challenging. Baker and Knapp (2019) recognized the importance of supporting coaches and developed a tool, the Decision-Making Protocol for Mathematics Coaches, to help coaches be purposeful in their work. They argued that mathematics cannot be separated from the work of coaching. Results from their work indicated that support may be necessary to help coaches with their interactions with teachers. Analysis of coaching interactions has shown that coaching actions can support teacher learning (Gibbons & Cobb, 2016). Although Gibbons and Cobb (2016) focused on coaches' actions and not how coaches talk with teachers, they identified particular aspects of coaches' planning practices that supported teaching, such as identifying long-term goals, assessing instructional practice, and identifying paths for teacher development, among others. Their work and others (e.g., Sailors & Shanklin, 2010) have shown that coaching can have positive outcomes; however, more research is needed on how to support coaches and on how coaches learn and perceive coaching interactions. To know how to support coaches, data on coach thinking and how coaches approach their work is necessary. Parallel to teacher noticing, we consider coach noticing to be an essential aspect of a coach's practice that may relate to a coach's ability to effectively support their teachers. Therefore, knowing what and how coaches notice as they analyze coaching conversations can illuminate how coaches approach and make sense of coaching situations, a foundational step needed to better support coach development. Just as classroom teachers analyze students' thinking and work, often based on video of teaching episodes, we argue a similar importance for coaches analyzing teachers' thinking based on coaching interactions.

Methods

In our content-focused coaching model (Choppin et al., 2021), coaches engaged teachers in online coaching cycles that included a planning meeting, lesson implementation, and a debriefing meeting. We then used video from the planning discussions of the coaching cycles to provide professional development to those learning to coach. For the purposes of this study, we engaged coaches (termed *coach participants*) and those teaching coaches (termed *mentor coaches*) in analyzing segments from transcriptions of three different planning discussions—we refer to both groups collectively as *participants*. We refer to the transcribed segments from the planning discussions as *vignettes* to determine what and how coaches notice.

Participants

Participants included 10 mentor coaches and 10 coach participants. Mentor coaches were part of the project team and had experience as mathematics coaches. Their role was to provide professional learning opportunities to those learning to coach. Coach participants were those enrolled in our professional development experiences as participants (see Amador et al., 2021) to learn how to support teachers through content-focused coaching cycles.

Data Collected

Data came from interviews with each of the 20 participants. Participants were interviewed about their coaching background and practices. The interviews also included the focal section for this analysis, which we term *vignette analysis*. Prior to the interview, each participant was given three different vignettes, which were actual transcripts from coaching conversations we recorded as part of a prior professional development project to support teachers. We used transcripts from our prior project intentionally, as they reflected authentic coaching conversations. Each of the three vignettes was from a different coach; we purposefully selected coaches for the vignettes who had varying discourse patterns (see Gillespie et al., 2019). The vignettes represented approximately 3-5 minutes of coaching conversation. For the purpose of this paper, we focus on only one of the three vignettes. During the interview, participants—both mentor coaches and coach participants—being interviewed were given time to review the vignettes. In addition to the transcript of the coaching conversation that included 14 talk turns back and forth between the coach and teacher, participants were given the following overview:

In this vignette, the coach and teacher are planning a lesson about fractions in a 4th grade classroom. The lesson involves a task in which students are asked to use visual strategies to determine the amount of brownies a person would receive if seven brownies were shared between four friends.

For the interview process, participants had copies of the entire transcript of the vignette and were given as much time as they needed to review the transcript and respond to prompts. Figure 1 shows an excerpt from the transcript participants were provided.

1 Coach 1:	Let's start with what's our goal. What are we trying to achieve by doing this task?
2 Teacher 1:	I think the goal is to see how they can divide eight brownies, or a group of brownies into seven parts, no, four parts equally. Taking what they have, seven, and how can these four people all get the same amount?
3 Coach 1:	That's zooming in on this problem itself. If we try to step out, one step, mathematically speaking, what are we trying, what's the big idea that we're trying to get at with the kids?
4 Teacher 1:	The big idea for them is to make sure that they can visually represent the division in the fractions and support their answers that way.

Figure 1. Vignette excerpt

Participants were asked a series of questions about the vignette. For the purposes of this study, we focus on one question they were asked because it was specific to noticing and designed to broadly elicit what they notice: (a) What is your overall reaction and what did you notice as you read the vignette? The prompt was intentionally open-ended to gather participants' initial noticing about the coach and teacher interactions in the vignette.

In this paper, we focus on coach participant responses at the point they entered the two-year professional development project (pre data, n=10) and responses to the same vignettes at the point they exited the two-year professional development project (post data, n=10). We also analyzed responses from the mentor coaches at one point in the project (n=10), intentionally not interviewing them pre/post as they were leading the professional development. The purpose of including the mentor coaches in the analysis was to provide a baseline of the type of noticing that occurred for these vignettes by those with considerable experience teaching and coaching mathematics. There were 10 coach participants, resulting in 20 analyzed responses (including pre and post) and 10 mentor coaches, resulting in 30 total coded responses.

Data Analysis

To answer the research question, about what and how do coaches (coach participants and mentor coaches) notice, we worked from the Learning to Notice Student Mathematical Thinking Framework (van Es, 2011) and modified it for the purposes of analyzing a coaching conversation, as compared to the implementation of a lesson. Figure 2 shows the modified framework, termed the Coach Noticing Framework.

	Level 1 Baseline	*Level 2 Mixed*	*Level 3 Focused*	*Level 4 Extended*
What Coaches Notice	Attend to whole conversation, participant engagement or behavior, the math task or related lesson plan broadly	Primarily attend to coach's action Begin to attend to teacher's mathematical thinking or pedagogical reasoning	Attend to teacher's mathematical thinking or pedagogical reasoning	Attend to the relationship between teacher's mathematical thinking or pedagogical reasoning and coach's action
	Level 1 Baseline	*Level 2 Mixed*	*Level 3 Focused*	*Level 4 Extended*
How Coaches Notice	Form general impressions of what occurred Provide descriptive and evaluative comments Provide little or no evidence to support analysis	Form general impressions and highlight noteworthy events. Provide primarily evaluative with some interpretive comments Begin to refer to specific events and interactions as evidence	Highlight noteworthy events Provide interpretive comments Refer to specific events and interactions as evidence Elaborate on events and interactions	Highlight noteworthy events Provide interpretive comments Refer to specific events and interactions as evidence Elaborate on events and interactions Make connections between events and principles of teaching and learning On the basis of interpretations, propose alternative coaching solutions.

Figure 2. Coach Noticing Framework

For clarification to Figure 2, to be coded as a Level 3 for *What*, the response had to focus on specific aspects of the teacher's mathematical thinking (meaning the teacher in the vignette). To be coded as a Level 4 for *What*, we considered comments about the teacher's content knowledge to be encompassed in the meaning of "teacher's mathematical thinking.

Data from the coach participants and mentor coaches were entered into a spreadsheet and blinded for participant type and for whether or not the response was pre or post for the coach participants. Four researchers each independently coded each of the vignette responses for the 30 total participants, assigning only one level for *What Coaches Notice* and one level for *How Coaches Notice*, based on the Coach Noticing Framework. The entire utterance of the coach in response to prompts about what was noticed was coded. The highest noticing level obtained at any point in the utterance was the code assigned, similar to the coding process of other researchers analyzing noticing (i.e., Jacobs et al., 2010). The four researchers then met and reconciled each of their codes until they agreed on all 30 responses. Data were then unblinded by participant type and whether the response was at the beginning of the project (pre) or end (post) for coach participants. Descriptive statistics were then calculated, with attention on the 10 mentor coaches and 10 coach participants. Themes in the levels of responses and types of responses were then identified. For example, all responses receiving a Level 1 code for *What* were reviewed and memos were written about similarities. This was done for each of the eight options (i.e., *How Coaches Notice* Level 1-4; *What Coaches Notice*, Level 1-4).

Findings

From analyzing 20 pre and post participation responses from coach participants and 10 responses from mentor coaches, we identified three noteworthy trends.

Coach Participant and Mentor Coach Noticing

First, based on averages, the mentor coaches noticed at higher levels than the coach participants for *What Coaches Notice* and *How Coaches Notice*. The baseline noticing for coach participants (pre) for *What Coaches Notice* was 1.6/4.0, compared to a baseline noticing for mentor coaches at 2.2/4.0. The baseline noticing for coach participants for *How Coaches Notice* was 1.9 for coach participants (pre) compared to 2.4 for the mentor coaches. For both *What Coaches Notice* and *How Coaches Notice*, the mentor coaches as a group demonstrated higher levels of noticing. Despite the difference in overall average, we found variability within each group. Four of the ten coach participants and three of the ten mentor coaches noticed at a Level 1 for both *How Coaches Notice* and *What Coaches Notice*, indicating that despite the increased coaching experience, the mentor coaches were not all focused on teachers' reasoning (*What*) or focused on noticing in a way that emphasized elaborated interpretation and connections.

To illustrate the Level 1 findings there were evident in both participant groups, we provide an example. Coach Arnold, a coach participant, shared the following during the pre-interview in response to the question, "What is your overall reaction and what did you notice as you read the vignette?":

> One thing that really stuck out to me about that one was that they were spending a lot of time kind of anticipating what kids would do and what felt hard and how they would tackle that, which I thought was important. It was really focused on the goal.

We coded this statement as "Level 1 - Baseline" for *What Coaches Notice* because the statement from the participant coach broadly described the focus of the vignette conversation without any attention to the coach actions or teacher thinking. Similarly, we coded this statement as "Level 1 - Baseline" for *How Coaches Notice*. This score was based on the participant coach providing only descriptive and evaluative comments, without any attempt at interpreting events in the vignettes using evidence.

Of notable difference between the coach participant group and mentor coach group was the demonstration of higher levels of noticing. For the baseline data, all coach participants noticed at a Level 1 or Level 2 for both *What Coaches Notice* and *How Coaches Notice*. No coach participant reached a Level 3 or Level 4. In contrast, five of the mentor coaches noticed at a Level 3 or Level 4 for either *What Coaches Notice* or *How Coaches Notice*, indicating half of the mentor coaches demonstrated more advanced noticing. In fact, two of the mentor coaches noticed at a Level 4 for both *What Coaches Notice* and *How Coaches Notice*. When comparing the two different participant groups, the range in noticing from the mentor coach group compared to similarities in How coach participants noticed, within the group, is a notable point. And although mentor coaches on average noticed at higher levels, higher levels of noticing were not evident for all mentor coaches, which speaks to the challenge of learning of notice effectively, even for those with considerable experience.

Changes in Participant Coach Noticing

Second, when comparing the pre and post noticing levels for coach participants, increases were seen in the averages for *What Coaches Notice* and *How Coaches Notice*. For the coach participants, the average level of noticing for *What Coaches Notice* increased from 1.6 to 1.9 from pre to post and the average level of noticing for *How Coaches Notice* increased from 1.9 to 2.0. Most coach participants noticed at a Level 2 or higher for the post response. Table 1 shows the breakdown for pre and post for coach participants for both *What Coaches Notice* and *How Coaches Notice*.

Table 1: Comparison of Coach Participant Noticing

	Pre-participation Responses		Post-participation Responses	
Participant Coach	What	How	What	How
Dyson	1	1	2	1
Stevens	2	2	2	4
Bell	1	1	1	2
Butler	1	2	2	1
Clark	2	2	2	2
Glover	1	1	1	1
Logan	2	3	3	3
Rice	1	2	3	1
Howard	2	2	1	2
Snyder	2	2	2	3
Participant Coach Average	1.6	1.9	1.9	2

The following is an example of a Level 2 response for *What* and *How* coaches notice:

> I really thought it went well. You know? I highlighted how the coach starts right away with, "Let's start with, what's our goal?" It's very goal-focused. I like that the coach used the word "our" instead of "your". It kind of takes ownership. You know? They're working together as a team. The coach also did a great job letting the teacher talk and having the wait time to really let the teacher get their thoughts out. The coach wasn't jumping in and trying to give the answers. The one question the coach asked was like, "What's the big idea?" Which I thought was great. "Let's talk about what we think the kids are going to do with this problem. What misconceptions might pop up." I just thought that the coach did a really good job in drawing stuff out of the teacher that the teacher might not have necessarily thought of.

In this example, for *What Coaches Notice*, the coach participant primarily focuses on the coach and the actions of coach, with some indication about the teacher. The response is primarily evaluative, with comments such as, "the coach did a really good job", with some initial interpretation such as indicating that that particular pronoun used demonstrated shared "ownership." Responses that this level were common in the post data, with 7 of 10 coach participants noticing at a Level 2 or higher for *What Coaches Notice* and 6 of 10 noticing at a Level 2 or higher for *How Coaches Notice*.

Third, we found inconsistent changes from pre to post when analyzing coach participant data with respect to whether or not noticing improved from the beginning of the two-year professional development to the end. Results at the individual coach participant level, comparing changes from pre to post in noticing indicated that for *What Coaches Notice*, 4 of 10 coach participants increased their noticing level, 5 remained the same, and 1 decreased. For *How Coaches Notice*, 3 of 10 coach participants increased their noticing level, 5 had the same level pre and post, and 2 decreased. Only one coach participant increased their noticing level for both *What Coaches Notice* and *How Coaches Notice* from the pre to the post data collection. These findings indicate that even though the average level of noticing for *What Coaches Notice* and *How Coaches Notice* increased from the pre to the post interviews, the data were inconsistent at the participant level.

Despite the variability, there were coach participants noticing at higher levels at the end of the project. The following is an example from a coach participant coded as a Level 3 for *What Coaches Notice* and Level 3 for *How Coaches Notice*, from the post data set:

> I noticed that, at the beginning, the coach asked, "What's the goal," and I think the teacher—I don't remember exactly—but sets a mathematical goal. The coach pushed her more to think—like a mathematical thinking goal or a bigger, overarching idea goal? I liked that there was that push to really think about a deeper goal for her instruction. I noticed, at some point, they were talking about misconceptions, and it seemed like—I think the coach—the teacher said, "A misconception?" and then the coach validated, like, "Right." Then the teacher kept talking. I thought that was valuable, and I think that the coach's input wasn't too, it wasn't trying to steer the teacher into a different direction. It was more carrying the conversation.

In this example, the teacher's thinking is mentioned, even quoted at one point, and the emphasis is on the interaction between the coach and teacher. The coach participant makes interpretations about the coach's intent with the conversation, "to steer the teacher into a different direction" and discusses how the coach pushes the teacher toward increased mathematical thinning and emphasis on the big mathematical idea.

Discussion and Conclusion

We consider the shift of coach participant noticing toward higher levels as a group to be noteworthy; however, we also highlight the lack of change for some participants and the shift to lowers levels for others to be worth consideration. Of note, we recognize that no aspect of the three-part professional development process was focused specifically on noticing, although the emphasis was on supporting coach participants to learn how to support teachers, thus an emphasis on considering teachers and making responsive decisions—aspects akin to noticing (e.g., Jacobs et al., 2010; van Es & Sherin, 2002). Likewise, neither the coach participants nor the mentor coaches had the Coach Noticing Framework (Figure 2) during the professional learning activities. We conjecture that if we had specifically focused the professional development on the framework, then we would have seen more dramatic shifts in noticing. Given that the framework was not available for participants, we consider the shifts that occurred noteworthy and consider the changes favorable for many of the participants. Additionally, we conder the modification of the van Es (2011) Learning to Notice Student Mathematical Thinking Framework to be a contribution to the field of coaching and noticing, as the framework provide ways for the field to analyze and consider noticing in a coaching context, specifically, content-focused coaching (e.g., West & Staub, 2003).

Acknowledgments

This material is based upon work supported by the National Science Foundation under Grant #2006263. We acknowledge the contributions of Jeffrey Choppin and Cynthia Callard.

References

Amador, J. M., Bragelman, J., & Superfine, A. C. (2021). Prospective teachers' noticing: A literature review of methodological approaches to support and analyze noticing. Teaching and Teacher Education, 99. https://doi.org/10.1016/j.tate.2020.103256

Baker, C., & Knapp, M. (2019). The decision-making protocol for mathematics coaching: Addressing the complexity of coaching with intentionality and reflection. Mathematics Teacher Educator, 7(2), 27-43. https://doi.org/10.5951/mathteaceduc.7.2.0027

Bengo, P. (2016). Secondary mathematics coaching: The components of effective mathematics coaching and implications. Teaching and Teacher Education, 60, 88-96.

Cherasaro, T.L., Broderson, R.M., Reale, M.L., & Yanoski, D.C. (2016). Teachers' response to feedback from evaluators: What feedback characteristics matter? U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance, Regional Educational Laboratory Center, Washington, DC. https://files.eric.ed.gov/fulltext/ED570162.pdf

Choppin, J., Amador, J., Callard, C., Carson, C., Gillespie, R., Kruger, J., Martin, S., & Foster, G. (2021). A three-part synchronous online model for middle grades mathematics teachers' professional development. In K. Hollebrands, R. Anderson, & K. Oliver (Eds.), Online Learning in Mathematics Education.(pp. 167- 186). Springer.

Corbin, J., & Strauss, A. (2008). Basics of qualitative research: Techniques and procedures for developing grounded theory (3rd ed.), SAGE Publications. https://doi.org/10.4135/9781452230153

Costa, A. L., & Garmston, R. J. (2016). Cognitive coaching: Developing self-directed leaders and learners. Rowan & Littlefield.

Denton, C. A., & Hasbrouck, J. (2009). A description of instructional coaching and its relationship to consultation. Journal of Educational and Psychological Consultation, 19(2), 150-175. https://doi.org/10.1080/10474410802463296

Garmston, R., Linder, C., & Whitaker, J. (1993). Reflections on cognitive coaching. Educational Leadership, 51(2), 57-61.

Gibbons, L. K., & Cobb, P. (2016). Content-focused coaching: Five key practices. The Elementary School Journal, 117(2), 237-260. https://doi.org/10.1086/688906

Gibbons, L. K., & Cobb, P. (2017). Focusing on teacher learning opportunities to identify potentially productive coaching activities. Journal of Teacher Education, 68(4), 411-425. https://doi.org/10.1177/0022487117702579

Ippolito, J. (2010). Three ways that literacy coaches balance responsive and directive relationships with teachers. The Elementary School Journal, 111(1), 164-190. https://doi.org/10.1086/653474

Jacobs, V., Lamb, L., & Philipp, R. (2010). Professional noticing of children's mathematical thinking. Journal for Research in Mathematics Education, 41(2), 169–202. https://doi.org/10.5951/JRESEMATHEDUC.41.2.0169

Jayaraman, G., Marvin, C., Knoche, L., & Bainter, S. (2015). Coaching conversations in early childhood programs. Infants & Young Children, 28(4), 323-336. https://doi.org/10.1097/IYC.0000000000000048

Johnson, S. R., Pas, E. T., Bradshaw, C. P., & Ialongo, N. S. (2018). Promoting teachers' implementation of classroom-based prevention programming through coaching: The mediating role of the coach–teacher relationship. Administration and Policy in Mental Health and Mental Health Services Research, 45(3), 404-416.

Knight, J. (2007). Instructional coaching: A partnership approach to improving instruction. Corwin.

Landis, J. R., & Koch, G. G. (1977). The measurement of observer agreement for categorical data. Biometrics, 33, 159–174.

Russell, J. L., Correnti, R., Stein, M.K., Thomas, A., Bill, V., & Speranzo, L. (2020). Mathematics coaching for conceptual understanding: Promising evidence regarding the Tennessee math coaching model. Educational Evaluation and Policy Analysis, 42(3), 439-466. https://doi.org/10.3102/0162373720940699

Sailors, M., & Shanklin, N. L. (2010). Introduction: Growing evidence to support coaching in literacy and mathematics. The Elementary School Journal, 111(1), 1-6. https://doi.org/10.1086/653467

Sherin, M. G. & van Es, E. A. (2009). Mathematics teachers' "learning to notice" in the context of a video club. School of Education and Social Policy, 24(2), 244-276. https://doi.org/10.1016/J.TATE.2006.11.005
Stein, M. K., Russell, J. L., Bill, V., Correnti, R., & Speranzo, L. (2022). Coach learning to help teachers learn to enact conceptually rich, student-focused mathematics lessons. Journal of Mathematics Teacher Education, 25(3), 321–346. https://doi.org/10.1007/s10857-021-09492-6
van Es, E. A. (2011). A framework for learning to notice student thinking. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), Mathematics teacher noticing: Seeing through teachers' eyes (pp. 134-151). Routledge.
West, L., & Cameron, A. (2013). Agents of Change. Heinemann.
West, L., & Staub, F. C. (2003). Content-focused coaching: Transforming mathematics lessons. Heinneman.

WHAT DOES IT MEAN FOR GEOMETRY TEACHERS TO IMPROVE A LESSON? A MULTIMODAL ANALYSIS

Gil Schwarts
University of Michigan
gils@umich.edu

Patricio Herbst
University of Michigan
pgherbst@umich.edu

Soobin Jeon
University of Michigan
sbjeon@umich.edu

Amanda Brown
University of Michigan
amilewsk@umich.edu

A central goal of lesson-centered professional development programs (PD) for mathematics teachers is to learn by constructing an artifact, for example, by designing and improving a lesson plan together. That leads to the questions, what does it mean, for mathematics teachers, to improve a lesson? And how can improvements be accounted for in the analysis of the resulting artifacts, especially when these are multimodal? This paper lays the groundwork for answering such questions by drawing on empirical data from a lesson-centered PD program for secondary geometry teachers. We show how semiotic choices were made to convey that the teacher would need to support students when geometry instruction moves from a construction task to a proof, by (1) addressing students' confusion; and (2) creating a shared language to discuss diagrams. We relate these findings to teachers' professional growth and the conference theme.

Keywords: Geometry and Spatial Reasoning; Problem-Based Learning; Professional Development; Research Methods.

Objectives of the Study

As part of the emphasis on practice in professional development programs (PD) (Cohen & Ball, 1999) for mathematics teachers (MTs), lesson-centered PD – that focus on collaborative work around one lesson (e.g., Morris & Hiebert, 2011) – have risen in popularity. A key activity in these programs is the iterative process of revising artifacts such as mathematical tasks, storyboarded lessons, vignettes, or lesson plans (e.g., in Lesson Study programs; Lewis et al., 2009). Such activities are aimed at the professional growth of the MTs through their involvement in the construction and revision of artifacts (building on the *constructionism* approach proposed by Papert & Harel, 1991). While numerous studies have sought to account for MTs' professional growth when they design lessons together (see Huang & Shimizu, 2016), it is less common to find studies that analyze the revised artifacts themselves. Thus, some questions remain unexplored, in the context of a lesson-centered PD: What does it mean, for MTs, to improve a lesson? How can improvements be accounted for in the analysis of the resulting artifacts, especially when these are multimodal? And how can changes in artifacts be linked with MTs' professional growth, if at all? These lines of inquiry align with recent calls for theorizing teachers' multimodal professional learning (Lefstein et al., 2020). By multimodal we allude to communication that relies on two or more of the communication modalities including written, oral, visual, kinesthetic, and gestural. With the proliferation of lesson-centered PDs, and particularly such that include storyboarding or vignette-making, there is a need for a framework that: 1) captures and compares multimodal semiotic choices that MTs make while creating and revising instructional artifacts (analyzing written and visual modes), and 2) relates these choices to MTs' oral arguments to provide a multimodal characterization of teachers' professional learning (analyzing, in addition, oral and perhaps gestural expressions). To lay the ground for developing such a framework, we focus on the first component by comparing two artifacts from a lesson-centered PD in which secondary MTs created versions of a geometry lesson, represented through storyboards of cartoon characters. We ask:

How do changes made over time to the storyboard of a geometry lesson evince the kinds of improvements that MTs were attending to in the context of a lesson-centered PD?

In this study, storyboarded lessons serve as visual lesson plans that MTs created and constantly revised over two years. In this sense, our use of the word *improvement* is based on the assumption that all revisions made by MTs were intentional. We do not view these revisions as objective improvements, but rather as expressions of what the MTs considered improvement.

Theoretical Framing

This section describes how we theorize revisions in storyboards, and how we relate such revisions to MTs' opportunities for professional growth. Our focus on improvements in storyboarded lessons leads us to take a closer look at visual as well as written semiotic choices. By semiotic we mean the "intricate web of signs signifying signifiers" (Radford et al., 2008, p. vii). That is, we are looking for the meanings that are made through expressive choices in the two modalities (visual and written). To decode visual and linguistic meanings, we draw on aspects of systemic functional linguistics (SFL) (Halliday & Matthiessen, 2004; Matthiessen et al., 2010). While SFL originally offered a lens to explore meaning in language, various adaptations to other semiotic media (e.g., in children's picture books by Painter et al., 2013; or in film by O'Halloran, 2004) have allowed Herbst et al. (in press) to extend it to the interpretation of storyboarded lessons.

In SFL, a system can be thought of as a set of options that are available to construe particular meanings. Herbst et al. (in press) discuss several systems that use multimodal resources to construe meanings in the design of storyboarded lessons: Orientation, Visibility, Temporality, and Individuality. The system of *orientation* relates to the choices designers of storyboards make to tell viewers where to focus their attention. Similar to how cameras are used in filmmaking, this system includes the choices of which points of view will be used, and when the storyboard will zoom-in and out (see Figures 2b and 2c for different point of views). Another resource to orient viewers is the use of captions (see Figure 1a) that provide textual information which is not represented in speech bubbles, such as "five minutes later". A second system is *visibility*, which supports the construal of meanings by making classroom experience more or less visible to the viewer, to convey "what is important for the viewer to see" (ibid, p. 11). For example, the choice to inscribe a diagram on the board versus invoking it as present without actually providing it (e.g., by using the legend "diagram here"), or to transcribe a question that was posed by the teacher but only invoke how students responded (e.g., by having student captions "blah blah"). *Temporality* supports the development of a sense of timeliness and duration in the representation of a lesson (e.g., by ordering frames in a certain way, ordering speech bubbles within a frame, or using the gutter between frames to move time forward). A fourth system is *individuality*, which alludes to choosing how characters or artifacts will be made distinct from each other (e.g., by choosing how characters look like). To further interpret some of the linguistic choices we draw on Christie's differentiation (2002) between *instructional* and *regulative* registers in classroom discourse. The instructional register refers to the language, symbols, and visual representations involved in representing the mathematical content of the lesson (e.g., "Reflect point A over line 1", Figure 1b), while the regulative register refers to the language, symbols, and visuals the teacher use to tell students what they are requested to do (e.g., “Read the instructions”, Figure 1a). The systems noted above (orientation, visibility, temporality, individuality) characterize yet a third register, which projects the regulative and instructional registers when a lesson is represented for an audience (as opposed to enacted by its participants for instruction’s sake), and that Herbst et al. (in press) name the *transactive* register.

As noted above, our second goal is to relate improvements in artifacts to MTs’ professional growth. To do that, we draw on Clarke and Hollingsworth’s (2002) interconnected model for professional growth, which consists of four domains of the teacher’s work that interact with each other: the external domain (e.g., adoption of a new textbook), the personal domain (e.g., knowledge, beliefs, dispositions), the domain of practice (e.g., the actual teaching of a lesson), and the domain of consequences (e.g., students’ interest in the subject). Our prior research showed how the work MTs do when improving storyboards is a form of “*virtual professional experimentation*” (Milewski et al., 2018, p. 111) that takes place in the domain of practice, but has consequences in other domains, especially in the internal domain (see Herbst et al., 2020). In other words, we showed how differences that can be traced during virtual professional experimentation are related to processes of professional growth. In this paper, we draw on the assumption that it is worthwhile to better understand the differences that can be accounted for in the domain of practice (e.g., improving the storyboarded lessons here), hypothesizing that observable changes in the domain of practice suggest that professional learning occurs.

Methods

Context and Data Collection

Between 2015 and 2017, a two-year practice-based PD was implemented to engage secondary MTs in designing and improving storyboarded representations of lessons that use reform-oriented instructional tasks. The goal they pursued was to demonstrate for other teachers how these tasks could be used in classrooms and the finished storyboards were meant to become part of a library of practice. Sixteen MTs from six schools in a Midwestern state were divided into four groups (two for geometry and two for algebra). In Year 1, each group was assigned four novel tasks and developed four storyboarded lessons (one for each task). In Year 2, representatives from the groups taught the lessons in their classrooms and later shared their experiences in the PD. These experiences were used to inform further revisions and improvements of the storyboarded lessons. The blended PD included yearly face-to-face meetings, monthly video-conference meetings, and weekly online forums. For this study, we focus on one geometry lesson, developed by six geometry teachers who had between 12 and 26 years of teaching experience. The main task of the lesson aims at students’ understanding of composite reflections over two intersecting lines. Assuming that students had previously engaged with construction tasks that required transformations of objects in the coordinate plane, and in addition practiced proving, the lesson goal was to connect plane transformations and geometric proofs. The first part of the task (hereafter, construction task) gives a line and a point outside the line. Students are asked to draw a point on the line, connect the two points, and reflect the resulting segment across the line. This construction leads them to find out that the angle measures (between the segment and the line) are preserved and congruent (see the diagram on the board in Figure 1a). The following task (hereafter, proof task) reads: “The composite of two reflections over two intersecting lines produces a rotation centered at the intersection of the lines that is double the rotation of the first line to the second line.” The attached diagram shows two intersecting lines and a point next to one of the lines (see the diagram on the board in Figure

1b). In a previous analysis of the interactions that occurred in this PD (Milewski et al., 2019) we found that MTs had trouble framing this task (because it is neither a proof nor a construction) when implementing the lesson in their classrooms (in the beginning of Year 2). Our goal for this analysis was to explore how coping with this challenge was apparent, if at all, in the improvement of the lesson over time. The data collected includes video records of group meetings and classroom implementations, forum logs and all artifacts, including versions of the lessons storyboard. The sources used for the current data analysis are two versions of the lesson storyboard from the end of Year 1 and Year 2 (comprising 12 and 20 frames, respectively).

Data Analysis

Our multimodal analysis looks at the use of written and visual modalities (Christie, 2002; Herbst et al., in press) drawing on SFL (Halliday & Matthiessen, 2004), and also informed by principles of visual data analysis (Kress & Van Leeuwen, 2020). Similar to the steps in the content analysis of written texts, visual data analysis comprises image description, segmentation, coding, memo writing, and iterative interpretation of these. In more detail, we have conducted the following steps: (1) storyboards are made of a sequence of frames. To find improvements, we compared the frames and sought for places where MTs made changes to frames or added frames. At this stage we found that no changes were made in the first nine frames of the storyboard, and so we focused on the 10th frame onward. We described what is happening in each frame. (2) Based on the descriptions we wrote; we identified which frames in one storyboard were comparable to the frames in the other. We deemed the frames comparable when the action taking place in each frame was similar (e.g., the teacher launches the task), and defined the pairs of comparable frames as our unit of analysis – how "same" periods in the lesson were represented in the different storyboards. For example, we found that a period that was represented with one frame in Year 1 was represented with three frames in Year 2. We also identified new periods in Year 2 that had no equivalences. This process of segmentation included creating tables that displayed the frames side by side and detailed the differences found between them. (3) We identified the semiotic changes in the comparable sets of frames, in relation to the semiotic registers. By analyzing the semiotic choices using the theoretical framing presented above, we found themes of improvements in the lesson (similar to the identification of themes in textual content analysis), on which we elaborate below.

Findings

The participants' semiotic choices elaborate the transition between a task in which students were asked to make a *construction* and a subsequent task in which they were asked to do a *proof*. As mentioned above, the lesson started with nine frames that narrate how the teacher launches the construction task and how the class makes sense of it. These frames were identical in Year 1 and Year 2 (Y1 and Y2, hereafter). The lesson refinement starts when the teacher launches the proof task, and is illustrated by the following improvements: (1) representing the teacher's address of students' confusion, and (2) representing the teacher's displaying of a student-generated diagram

Improvement #1: Representing the Teacher's Address of Students' Confusion

The first improvement is illustrated by three periods in the lessons that were revised between Y1 and Y2 (Figures 1, 2 and 3). In the following, we analyze each period by comparing and contrasting frames. The first change is the launch of the proof task (Figure 1): in Y1 the teacher asks the students to read the task instructions (Figure 1a), while in Y2 the teacher reads them (Figure 1b). This change is apparent in several semiotic choices: In terms of orientation, the additions to the caption in Y2 orient the viewers to notice that the teacher is the one who reads the instructions. Another difference is in the use of registers in the teacher's speech bubble. In Y1 (Figure 1a) the teacher uses only the regulative register (asks students to to "read", "tell", and "show"), whereas in Y2 (Figure 1b) the teacher uses also the instructional register when reading the instructions ("Reflect point A over line 1 to A'"), that is, the viewers can see what mathematical actions the students are asked to do. While the storyboarded lesson in Y1 glosses over the details of the task, the Y2 lesson draws the viewer's attention to the details of the task, to make sense of the students' confusion that follows. Furthermore, there is a change in the system of visibility with the different display of a diagram in each frame. In Y1 (Figure 1a) the diagram on the board is a remnant of the previous phase of the lesson that appeared on the previous frames as well; in Y2, the teacher projects a new diagram which coheres with the proof task described in the teacher's speech bubble, illustrating a multimodal representation of the teacher's use of the instructional register. This diagram was handed to students as part of the task sheet.

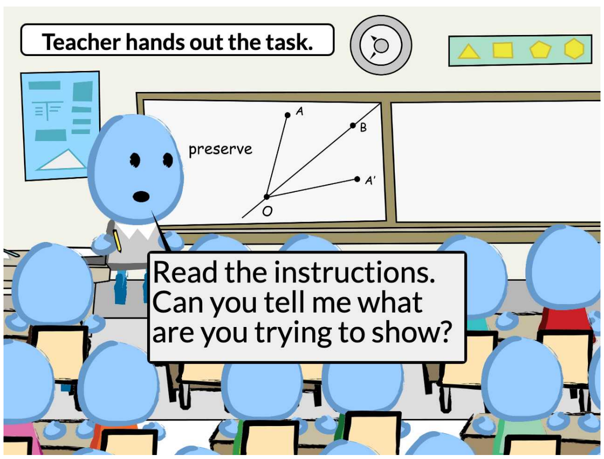


Figure 1a. Y1: Frame 10

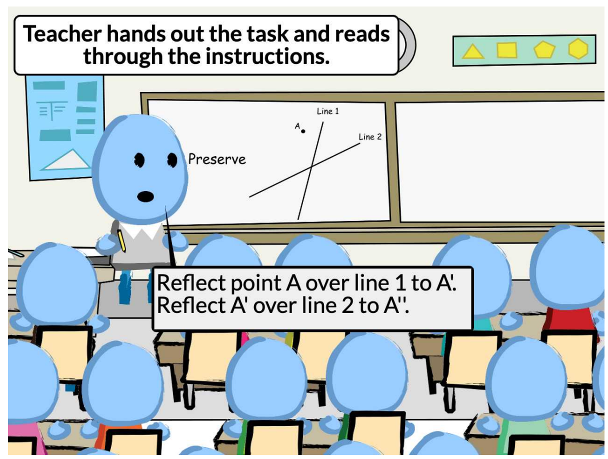


Figure 1b. Y2: Frame 10

Figure 1. Launch of the task.

The second comparison (Figure 2) elaborates what happens in the lesson once the task is read (either by the teacher or the students). In Year 2, the teacher asks the students to solve, and the viewers are told that the students are confused. MTs used several semiotic resources to draw attention to this confusion, changing or adding frames (including a different point of view), facial expressions, caption, and text. While in both Figures 2a and 2b the orientation is the student's point of view, focusing on the teacher – frame 11 in Y2 (Figure 2b) is zoomed in, the teacher is looking at the students in silence, which creates a dramatic pause in the lesson. This is followed by a frame (Figure 2c) where the orientation shifts to one similar to the teacher's point of view facing the students, but that includes the teacher as one being observed. For the first time in this storyboard, several students have different facial expressions (a choice to represent students' individuality). These facial expressions are explained by the caption, "some groups are confused." The students' confusion is presumably induced by the shifting from construction to proof without explicitly discussing it. The changed representation of time (temporality) – the elaboration of the period after the task was issued accomplished by adding an additional frame – lends even more emphasis to this challenging shift. In terms of visibility and temporality, some actions that could possibly happen in the lesson are not narrated: In Figure 2a, viewers do not see if the students reacted to the teacher's question in the previous frame (Figure 1a). In Figure 2b, viewers do not see who added points A' and A'' to the diagram, yet they may guess that it was the teacher as they are in control of the board. The lack of dialogue adds more focus to other components of the storyboard (e.g., facial expressions, caption).

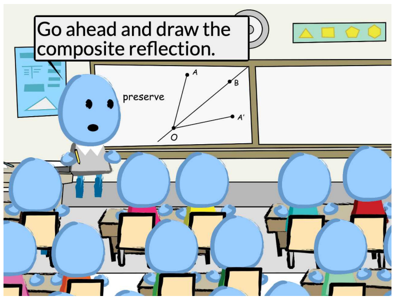


Figure 2a. Y1: Frame 11

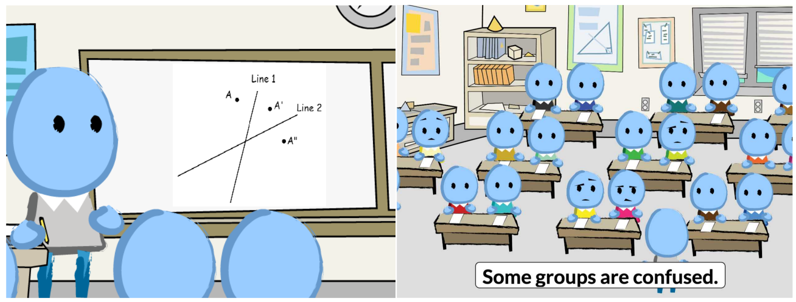


Figure 2b and 2c. Y2: Frames 11 and 12

Figure 2. The period in the lesson after the task was read.

The third comparison (Figure 3) shows the teacher walking around the room as the students engage in the proof task. A salient improvement made in this moment is the addition of dialogue and the camera's focus and zoom-in on the students and the teacher's faces (Figures 3b, 3c), as opposed to focusing only on the teacher in Figure 3a. Individuality is construed not only by using different facial expressions and color of the shirts, but also by distinguishing the students' interactions with the teacher: Yellow answers the teacher's question hesitantly, alluding to the angle as "something," while Pink seems more convinced and talks about angles. These individual characteristics display the heterogeneity of the class, since the students are not represented as a monolithic whole. In addition, some linguistic choices were made to convey the interpersonal teacher-students relationships: In Figure 3b, the teacher asks "what are *we* trying to show?", suggesting that making sense of the task is a joint effort. In Figure 3c, the caption says that the teacher "addresses confusion", while the teacher is asking a question – which means that asking questions is the teacher's method of addressing confusion (instead of, for example, answering).

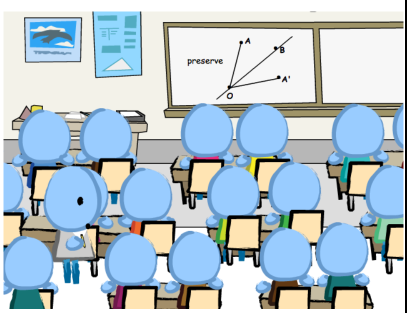


Figure 3a. Y1: Frame 12

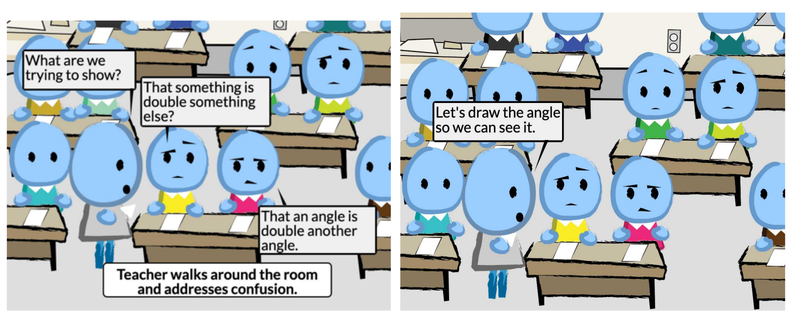


Figure 3b and 3c. Y2: Frames 13 and 14

Figure 3. Teacher walks around the classroom.

Improvement #2: Representing the Teacher's Displaying of a Student-Generated Diagram
The second improvement is the added representation of a discussion around student-generated diagrams in the storyboarded lesson in Y2 (see Figure 4). While this part of the lesson was not represented in Y1, it was represented using six frames in Y2. In the following, we analyze the first three frames of this period. The teacher is inviting a group to share their work, which the teacher deliberately selected when walking around the room (Figure 4a). Then, the teacher labels the diagram, aiming at creating a shared language (Figures 4b, 4c). Comparing these frames to the Y1 storyboard, which lacked representation of how the task is handled, suggests that over Y2 the MTs decided that the intricacies of the discussion around student work should be spelled out through storyboarding. Since each student drew their own diagram, the lesson improvement is manifested in detailing the processes in which the teacher turns this diversity of expressions into a coherent discussion. In what follows, we detail how semiotic choices led us to conclude this.

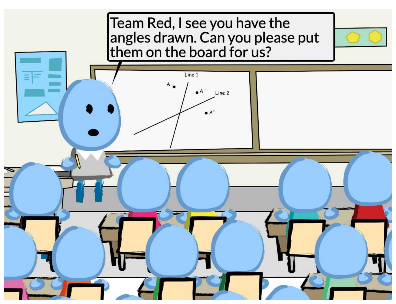


Figure 4a. Y2: Frame 15

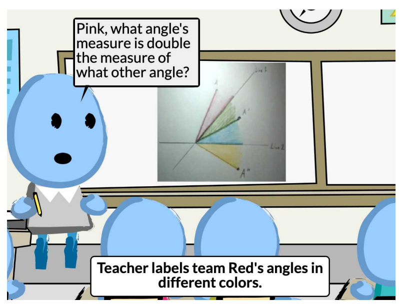


Figure 4b. Y2: Frame 16

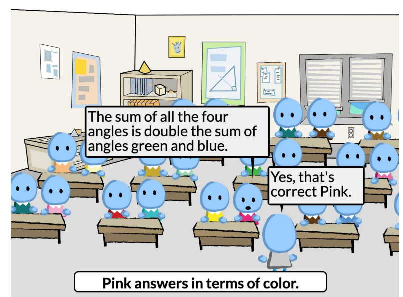


Figure 4c. Y2: Frame 17

Figure 4. Teacher labels and presents a student-generated diagram.

The change in orientation between Frame 14 (Figure 3c) and Frame 15 (Figure 4a) indicates the transition from work time to classroom discussion. In Frame 15, The speech bubble indicates the teacher's specific request for Team Red to share their work, rather than inviting any student to do so. The teacher words ("for us") aims at building interpersonal connections. The teacher is also attending to the specific mathematics that the class should focus upon: "the angles drawn". which is another evidence for the teacher's language becoming more subject-specific. In Frame 16 (Figure 4b) we see changes in orientation by zooming in on the teacher and the diagram and adding a caption. The caption provides a sense of timeliness, explaining that the diagram was already put on the board and that the teacher was the one who colored it. This draws viewers' attention to the actions that the teacher is making to support the discussion. In terms of visibility, the choice to include the diagram on the board means that it is important for the viewers' understanding of the course of the lesson. The teacher's speech bubble shows that they ask Pink directly to use the colors in the diagram to answer a question they were previously asked, in Frame 13 (Figure 3b). The repetition of the same question, yet with more precise fashion, draws viewers' attention to the importance of precision in the process of proof. In the next frame, the orientation changes, which lets viewers see that the students' facial expressions are all the same, possibly suggesting they are no longer confused. The caption "Pink answers in terms of color," together with Pink's speech bubble, illustrate how coloring the diagram was a key decision for overcoming the confusion: By using the colored diagram, Pink is able to say what is to be proven. The teacher reassures this by saying "Yes, that's correct, Pink." Pink's emerging ability to use precise language suggests that the teacher's moves in Frames 10-17 were productive.

Discussion

This study set out to examine what MTs perceive as improvements in a representation of a geometry lesson. By comparing two versions of a storyboarded lesson, we identified some of the semiotic changes that MTs made over time which allow us to articulate what was improved in the lesson. We found that the improved representation of this lesson, as was reflected in the changes in the artifact, included not only a detailed depiction of how the teacher launches a task, but also of the students' reaction to the task and how the teacher addresses their confusion or other complexities that the task brings up. We identified two themes which overall suggest that the MTs were interested in representing the shift from a construction situation to a proof in a greater depth and resolution. In particular, the MTs improved the lesson in ways that drew viewers' attention to the students' confusion and the teacher's moves in response to the confusion (first improvement), and to the teacher's way of creating a shared language about the diagram during the whole-class discussion (second improvement). These foci were reflected in the following semiotic choices: addition of frames, inclusion of different points of view (camera positions), zooming in on the teacher or students, addition of captions that oriented viewers to notice particular details, addition of students' facial expressions and speech bubbles, and more.

Furthermore, a noticeable change is that the students' perspective on the task is represented in the Y2 version. The lesson revision through storyboarding revealed an aspect of improvement that is often subtle in spoken/written discourse: The added frames showed that an enhanced noticing of students' perspectives (including their puzzlement and quandaries) is an essential component of the teacher's expertise. This finding, related to the conference theme regarding MTs' attention to student engagement, reminds us that a novel task coupled with the teacher's good will are not sufficient to ensure productive student engagement in a problem-based lesson. The Y2 lesson suggests that the anticipation of student confusion is crucial for deciding on the moves and scaffolds that the teacher uses. Another feature of the suggested improvements is the increased specification of the mathematical content in the teacher's talk, illustrated by the increased use of the instructional register. We argue that using a more subject-specific language when storyboarding suggests an increased engagement with the mathematical details of the lesson. Thus, it points to how, during the PD, the domain of practice shapes and is shaped by the participants' internal domain, since their tacit knowledge about launching tasks and address students' confusion was shared with others and represented in the artifact. We argue that this kind of work with MTs enables them to attend more to "the detailed and intricate nature of practice" (Ball & Forzani, 2009, p. 507), thus, it contributes to their professional growth. This illustrates how professional learning environments can help MTs address student engagement.

Theoretically, this paper contributes an analytic description of semiotic choices, drawing on methods from SFL and its extensions. The question we raised – how to account for changes in artifacts and relate them to professional growth? – is yet to be fully answered. We aim for this work to serve as a foundation for future analyses of storyboarded lessons and other artifacts like lesson plans. This would be a fruitful area for further work that uses multimodal data and multiple teacher groups and lessons to conceptualize MTs' professional growth. The main limitation of this study is the focus on one lesson. Although the corpus includes additional data that could be analyzed similarly, we chose to focus on one lesson to ensure that we provide enough contextual details and are able to present the analyzed frames. Despite this limitation, this study offers insights into how MTs operationalize the goal set to them of improving a lesson.

Acknowledgments

The writing of this paper has been supported by James S. McDonnell Foundation Grant 220020524 and NSF grant DRL 2201087. The data analyzed was collected with support of a Mathematics and Science Partnership grant from the State of Michigan Department of Education to the Macomb Intermediate School District (APR # MI50804, Deborah Ferry, P.I.). All opinions are those of the authors and do not necessarily represent the views of the sponsors.

References

Ball, D., & Forzani, F. M. (2009). The work of teaching and the challenge for teacher education. *Journal of Teacher Education*, *60*(5), 497-511. https://doi.org/10.1177/0022487109348479

Christie, F. (2002). *Classroom discourse analysis: A functional perspective*. Continuum.

Clarke, D., & Hollingsworth, H. (2002). Elaborating a model of teacher professional growth. *Teaching and Teacher Education*, *18*(8), 947–967. https://doi.org/10.1016/S0742-051X(02)00053-7

Cohen, D. K., & Ball, D. L. (1999). *Instruction, capacity, and improvement.* Consortium for Policy Research in Education, University of Pennsylvania, Graduate School of Education. https://repository.upenn.edu/cpre_researchreports/8

Halliday, M. A. K., & Matthiessen, C. M. I .M. (2004). *An introduction to functional grammar* (3rd ed.). Hodder Arnold.

Herbst, P., Chazan, D., & Schleppegrell, M. (in press). How digital storyboards support the transaction of practice: The semiotic infrastructure of representations of practice. In B. Pepin et al. (Eds.), *Handbook of digital resources in mathematics education*. Springer.

Herbst, P., Ko, I., & Milewski, A. (2020). A heuristic approach to assess change in mathematical knowledge for teaching geometry after a practice-based professional learning intervention. *Research in Mathematics Education, 22*(2), 188-208 https://doi.org/10.1080/14794802.2019.1704851

Huang, R., & Shimizu, Y. (2016). Improving teaching, developing teachers and teacher educators, and linking theory and practice through lesson study in mathematics: an international perspective. *ZDM – Mathematics Education*, *48*, 393-409. https://doi.org/10.1007/s11858-016-0795-7

Kress, G., & Van Leeuwen, T. (2020). *Reading images: The grammar of visual design*. Routledge.

Lefstein, A., Louie, N., Segal, A., & Becher, A. (2020). Taking stock of research on teacher collaborative discourse: Theory and method in a nascent field. *Teaching and Teacher Education*, *88*, 102954. https://doi.org/10.1016/j.tate.2019.102954

Lewis, C. C., Perry, R. R., & Hurd, J. (2009). Improving mathematics instruction through lesson study: A theoretical model and North American case. *Journal of Mathematics Teacher Education*, *12*, 285-304.

Matthiessen, C., Teruya, K., & Lam, M. (2010). *Key terms in systemic functional linguistics*. A&C Black.

Milewski, A., Bardelli, E., & Herbst, P. (2019, April). Framing a task as both construction and proof: How do teachers manage? [Paper presentation]. *National Council of Teachers of Mathematics (NCTM) Conference 2019*, San Diego, California.

Milewski, A., Herbst, P., Bardelli, E., & Hetrick, C. (2018). The role of simulations for supporting professional growth: Teachers' engagement in virtual professional experimentation. *Journal of Technology and Teacher Education*, *26*(1), 103-126.

Morris, A. K., & Hiebert, J. (2011). Creating shared instructional products: An alternative approach to improving teaching. *Educational Researcher*, *40*(1), 5-14. https://doi.org/10.3102/0013189X10393501

O'Halloran, K. L. (2004). Visual semiosis in film. In K. O'Halloran (Ed.), *Multimodal discourse analysis: Systemic functional perspective* (pp. 109-130). Continuum.

Painter, C., Martin, J. R., & Unsworth, L. (2013). *Reading visual narratives: Image analysis of children's picture books*. Equinox Publishing Ltd.

Papert, S., & Harel, I. (1991). Situating constructionism. In S. Papert, & I. Harel (Eds.), *Constructionism* (pp. 1-11). Ablex Publishing Corporation

Radford, L. (2008). The ethics of being and knowing: Towards a cultural theory of learning. In L. Radford, G. Schubring, & F. Seeger (Eds.), *Semiotics in mathematics education* (pp. 215–234). Brill Sense.

WORTHWHILE PROBLEMS: HOW TEACHERS EVALUATE THE INSTRUCTIONAL SUITABILITY OF CONTEXTUAL ALGEBRA TASKS

Cody L. Patterson
Texas State University
codypatterson@txstate.edu

Mai Bui
Texas State University
mtb104@txstate.edu

Lino Guajardo
Texas State University
lrg74@txstate.edu

Carlos Acevedo
Texas State University
cia24@txstate.edu

Brandi Rygaard Gaspard
Texas State University
brr102@txstate.edu

Rebecca McGraw
University of Arizona
rmcgraw@arizona.edu

We investigate the beliefs that influence middle and high school algebra teachers' appraisals of contextual problems having diverse mathematical and pedagogical features. We asked six teachers to analyze six contextual algebra tasks and indicate how they would apportion instructional time among the six tasks based on their structure, pedagogical features, and connections to the real world. We recorded small-group discussions in which teachers shared their responses to this activity, and qualitatively analyzed their discussions for evidence of beliefs that influenced their appraisals of the tasks. The teachers' beliefs about contextual problems attended to task authenticity, opportunities for mathematical activity, obligations of tasks, and pedagogy and access. Our preliminary findings can inform future efforts to equip teachers with contextual tasks that develop students' algebraic reasoning and problem solving.

Keywords: Teacher Beliefs, Algebra and Algebraic Thinking, Professional Development

Multiple K–12 mathematics curriculum recommendations and standards documents call for students to learn to apply mathematics in settings beyond school. The National Council of Teachers of Mathematics' *Principles and Standards for School Mathematics* indicate that instructional programs should equip students to "Recognize and apply mathematics in contexts outside of mathematics" (NCTM, 2000). The Common Core State Standards for Mathematical Practice state that "proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace" (NGA & CCSSO, 2010, MP.4); the Texas Essential Knowledge and Skills include a process standard with similar verbiage about applying mathematics to "problems arising in everyday life, society, and the workplace" (TEA, 2012).

While policy documents have underscored the importance of students learning to apply mathematical ideas beyond school, research has also highlighted the potential of contextual problems to help students develop and understand new mathematical ideas. The instructional theory of Realistic Mathematics Education suggests the use of "realistic" situations (which may or may not arise from "real-world" contexts) to stimulate the development of mathematical tools and concepts (Gravemeijer, 2005; Van den Heuvel-Panhuizen & Drijvers, 2020). We view both perspectives on contextual tasks – that mathematics can provide windows into the larger world, and that the world can inspire mathematical thinking and growth – as essential for developing mathematics courses that equip students for success in college, career, and civic life.

Prior research has investigated preservice and inservice teachers' dispositions regarding "real-world" connections in mathematics classes, including connections to issues of social injustice (*e.g.*, Gainsburg, 2008; Girnat & Eichler, 2011; Simic-Muller et al., 2015). Additionally, Sevinc and Lesh (2018) have investigated preservice teachers' views about realistic mathematics problems and suggested interventions that can help teachers think critically about contextual problems in textbooks from a modeling perspective. However, research on how teachers select and allocate time to contextual problems more broadly – including problems with a tighter focus on curricular content – is relatively scarce. We hope to contribute to understanding of teachers' dispositions toward the use of contextual problems by examining the beliefs about mathematics learning, instructional practice, and obligations of mathematics teaching that might influence teachers' decisions to select a contextual task for classroom use.

Theoretical Framework

A fundamental assumption of our collaboration with secondary algebra teachers is that students benefit when they have opportunities to learn mathematics through problems that authentically reflect the relevance of algebra content beyond the classroom. We are guided by the practicality ethic, which indicates that teachers will only adopt a change proposal if they view it as practical: that is, if the proposal has instrumental content, is congruent with existing practice, and offers benefits commensurate with costs of implementation (Doyle & Ponder, 1977). We therefore seek to understand the beliefs and practical constraints that guide teachers' selection of classroom tasks, including beliefs about the benefits and costs associated with implementation of authentic contextual tasks.

Palm (2006, 2008) describes several dimensions of authenticity for contextual mathematics tasks: whether a task describes an event that is reasonably likely to occur; whether the question posed is one that would likely occur naturally; whether the methods for solving the problem are congruent with a realistic purpose for finding the answer; and whether representations of information in the problem are realistic. For simplicity, in our study we condense these dimensions down to two: authenticity of the event presented, and information provided, and authenticity of the (mathematical) processes involved in solving the problem. In keeping with Vos (2018), we view authenticity as a social construct; rather than imposing our own meaning on task authenticity as researchers, we leave it to our teacher-participants to define what it means for a task to be authentic to real-world considerations. Brantlinger (2022) acknowledged that contextualized mathematics must be in tune with students' experiences, interests, and potential futures in order to be received as authentic. McGraw and Patterson (2019) investigated how authenticity features of contextual tasks influenced secondary teachers' negotiation of problem spaces as they worked on these tasks. A key finding of this work is that features of the task context can widen or constrain opportunities for mathematical thinking; therefore, considerations about task authenticity and about opportunities for mathematical thinking are intertwined.

In a study of elementary teachers creating modeling tasks, Turner et al. (2022) found that teachers tended to start with a context personal to students and build the problems from the context, rather than thinking of a hypothetical situation to fit the mathematics intended (p. 17). This ensured the context was realistic to real-world experiences, and the anticipated mathematics of the problem was evoked in an authentic way. Teachers highlighted the obligation of a modeling task to "deepen their [students'] critical awareness", as their tasks included creating models to highlight environmental and community issues (p. 11). Along with obligation to the students' wider awareness of community issues in creating their modeling problems, teachers attended to grade-level content and standardized test demands, which sometimes generated conflict in how they framed the mathematics within the modeling task. Based on the literature and our own prior experiences in mathematics teacher education, we anticipate that secondary teachers experience similar tensions between relevance to students' lives and adherence to curricular content when selecting contextual problems for classroom use.

Guided by this framework, we aim to address the following research questions:

1. In what ways does task authenticity influence teachers' evaluations of the instructional suitability of contextual tasks in algebra?
2. What other considerations influence teachers' judgments about instructional suitability and the amount of time that should be allocated to contextual tasks in algebra?

Method of Study

The data for this study were drawn from a 2022 workshop conducted as part of the [blinded] project, which studies the teaching of algebra in grades 7–9 and provides professional development (PD) for secondary teachers. Participants in this study included six mathematics teachers from different urban K–8 academies and high schools in the southern United States.

Data Collection

To investigate participating teachers' beliefs about contextual problems, we designed a set of six *Worthwhile Problems* (Table 1) with different task authenticity features and varying alignments with algebra curricular content.

Table 1: The Six Worthwhile Problems

Muffins The Edison High School Marching Band is selling muffins to raise money for an upcoming trip. The cost in dollars of producing and selling x muffins is given by the function $C(x) = 50 + 0.25x$, and the revenue in dollars earned from selling x muffins is given by the function $R(x) = 2x$. How many muffins must the band sell in order to make a profit of 1400 dollars?	*Braking* A comprehensive review of transportation policy by Jack D. Jernigan and Meltem F. Kodaman, written for the Virginia Transportation Research Council, indicates that for an average car traveling s miles per hour, the formula $d = s^2/20$ gives the braking distance d (in feet) needed to come to a stop. What would be the braking distance needed for a car traveling 80 miles per hour to come to a stop? If we saw a set of skid marks 125 feet long made by a mid-size sedan, and knew that no collision occurred at the scene, what could we infer about the car?	*Population* The World Population Clock announced that the world human population reached 8 billion on November 15, 2022. Based on the following data from the World Bank, when should we expect the world population to reach 9 billion? [Table shows world population each decade from 1970 to 2020]
Unemployment The following graph, shown on Fox News in 2010, shows the number of jobs that were lost before, during, and after the Great Recession of 2007 to 2009. What does the graph show clearly? What does it not show so clearly? In what way(s) is the graph misleading?  Image source: https://www.businessinsider.com/fox-news-charts-tricks-data-2012-11	*Ramp* Use the information below to create a wheelchair ramp design for a door at your school or another building that could use one. Laws for new construction and modifications to existing structures are created so that public facilities are as barrier- free as possible. For example, portions of the American Disability Association (ADA) guidelines provide directions for building ramps or using existing space as ramps. Some of these guidelines are: • Slope and Rise: The least possible slope shall be used for every ramp. The maximum slope of a ramp in new construction shall be 1:12. The maximum rise for any run shall be 30 in (760 mm) [several other technical specifications are given]	*Wrapping* Taylor is wrapping holiday gifts for their family. Each gift comes in a rectangular box of height h inches, width w inches, and length L inches. Write a formula in terms of h, w, and L for the total area of wrapping paper needed to cover each box on all sides.

In a PD session led by the first author, six participants first worked on the six problems individually and thought about the mathematical work involved in each problem. Each teacher then used an online tool to create a pie chart showing how they would divide up the total amount of time (100%) reserved for contextual problems in their class among the six "types" of contextual problems exemplified by the tasks. Participants were encouraged to apportion time among "types" of contextual problems – focusing on features such as the relationship between the task and the real world, and the mathematical thinking required, rather than specific mathematics content objectives – and not to worry about standards or testing. They then met in groups of three for 45 minutes to discuss their time allocation to each problem before returning to a whole-group discussion. These conversations were recorded and transcribed for analysis.

Data Analysis

Thematic analysis (Braun & Clarke, 2006) was used to analyze the data. All six authors served as coders. In the first phase, each coder watched the two small-group videos to gain familiarity with the discussions. Each coder took notes on beliefs that appeared to influence teachers' evaluation of contextual tasks to generate an initial set of codes. All coders then met and discussed these initial codes, including examples of each from the transcripts. After reviewing the codes to eliminate redundancy and overlap, we arrived at a set of 16 final codes. We looked for common themes and organized codes into four domains: Task Authenticity (A), Mathematical Activity (M), Obligations of Tasks (O), and Pedagogy and Access (P) (Table 2).

Table 2: Coding Scheme for Teachers' Beliefs About Contextual Problems

Theme 1: Task Authenticity
A1. A task should authentically reflect the information (and representation of information) a person would have when working in the context described.
A2. A task should authentically reflect the (mathematical) processes a person might use in the context described to solve a real problem.
Theme 2: Mathematical Activity
M1. A task should address mathematics content.
M2. A task should encourage students to engage in mathematical practices and processes.
M3. Students should learn to build, critique, and analyze mathematical models, and analyze ways in which mathematical models approximate (or fail to approximate) reality.
M4. Tasks that allow students to engage in creative thinking are beneficial.
Theme 3: Obligations of Tasks
O1. Students should be exposed to tasks that foreshadow how mathematics appears or is used in the world outside of school.
O2. Mathematics class should help students to be prepared for classes in other academic disciplines.
O3. Mathematics class should prepare students for future mathematics courses.
O4. Tasks that reflect the expectations and format of standardized achievement tests might support student success on these tests.
O5. Students should be exposed to contexts that intersect with social justice concerns.
Theme 4: Pedagogy and Access
P1. Tasks that relate to students' lived experiences are beneficial for students.
P2. Tasks that offer "hands-on" experience are beneficial for students.
P3. Generally, less time should be allocated to tasks that have low cognitive demand.
P4. Teachers should make an effort to manage the language load of tasks.
P5. A task that allows multiple approaches or solutions is beneficial because it allows access to the task for more students.

In the third phase, individual coder re-read the two small-group transcripts and assigned code(s) to each talk turn in a separate spreadsheet. A talk turn may receive one code, more than one code, or no code depending on its content. For example, consider a part of a group conversation about the *Braking* problem:

Denise: It's like, okay, it says 80, put the 80 in there and that was it. Yeah. And not that I didn't think it was relevant. Why may I spend time on that was my thing on that.

Frances: Well, you know, they deal with that in physics.

Denise stated that she would not spend too much instructional time on this problem because it is too simple, while Frances argued that it was relevant to physics, a course many students take in high school. Their turns of talk were coded P3 and O2, respectively. Then each coder collapsed all the belief codes that they assigned each participant for the discussion of each task. For example, Denise's beliefs on *Braking* included P3 and other codes for other talk turns.

In the fourth phase, six coders met to compare the codes for each participant at the task level. If a code appeared in at least five coders' results, we kept it as a consensus code for subsequent analysis. We discarded any codes that showed up in fewer than three coders' results for the same task. For each code that had agreement from at least three coders, we returned to the relevant discussion excerpts and discussed our reasons for assigning (or not assigning) the code until we reached consensus of at least five researchers (or failed to do so and discarded the code). At the end of this process, each teacher had a set of belief codes associated with each task. We report on these consensus codes in the following section.

Results and Analysis

Table 3 shows the consensus codes associated with each teacher and each task, along with the percentage of time for contextual tasks each teacher allocated to each task. In this section we share some key insights gained from our analysis, along with some illustrative examples from the small-group discussion transcripts.

Table 3: Consensus Belief Codes from Small-Group Discussions

TASK	YELLOW GROUP			PINK GROUP		
(% of time allocated)*	Benjamin	Danielle	Viola	Denise	Felipe	Frances
Muffins	M2 (11.9)	O4, P3 (14.3)	A1, A2, O1, P1 (10)	P5 (10)	O1 (25)	M1, O1 (11.9)
Braking	M1, P3 (23.8)	A1, A2, M1, M2, O1, O4, P4, P5 (19)	O1, P1 (15)	M1, P3 (5)	(15)	O1, O2, P3 (23.8)
Population	P5 (23.8)	M3 (28.6)	O1, O5 (20)	M1, O4, O5 (25)	M3 (25)	M1 (23.8)
Unemp.	M2, P3 (8.3)	M1 (4.8)	M2, O4 (25)	M1, O1, P1 (20)	M2 (15)	(8.3)
Ramp	(23.8)	M1, O1, O5, P1, P2 (23.8)	O1, P1 (10)	M1, O1, P2 (35)	O1 (20)	O1 (23.8)
Wrapping	M2, P3 (8.3)	A2, M2, M3, P2 (9.5)	(15)	A2, M2, O4, P2 (5)	A2 (0)	A2, M1, M2, P2 (8.3)

* Benjamin and Frances worked together and completed a single pie chart, so their time allocations are identical.

Authenticity

Considerations of task authenticity influenced teachers' evaluations of some of the six tasks, most notably *Wrapping*. Teachers in both groups (Danielle and Denise) pointed out that *Wrapping* appeared to be a standard surface area calculation task, but that in a real-world scenario, a person wrapping gifts would need to account for overlap and waste (A2). Felipe suggested that the authenticity of the task could be salvaged by changing the context:

> A much better surface area question, theoretically, well, because that'd be lateral surface area, would be painting a room. … Not even paint a room, you have a box. How much paint are you going to need to paint the whole box? … Yeah, that would be more realistic because it's not like, oh, you're going to need a little bit more. It's just cover it. It's just exact.

In some cases, considerations of authenticity seemed to influence teachers to defend choices made in the tasks provided. Benjamin suggested that *Braking* could be improved by giving students a graph of braking distance as a function of speed and asking them to write a formula for the quadratic function in vertex and standard form; further questioning by the first author confirmed that Benjamin was evaluating the task primarily based on its affordances for helping students learn algebra content (M1). However, Danielle rebutted,

> You don't always want to be given a graph because as Viola said, in all the careers that you could use this in, just like this problem where it's giving you research back to, "This is where this equation came from." Everybody in career paths are not generating these equations and these functions, they're using them to determine other things.

This suggests that Danielle considered authenticity of the information in the task (A1) and of the mathematical processes one would use (A2), as compared to what one might encounter in an analogous professional context, as a key factor in determining the ideal task structure.

Opportunities for Matheatical Activity

Participants endorsed some tasks, and rejected others, based on opportunities they appeared to provide for students to learn Algebra I content and engage in mathematical practices.

In some cases, judgments based on the opportunity to engage with grade-level mathematics content (M1) stood in tension with judgments based on opportunities to engage in mathematical processes and practices (M2). Viola viewed *Unemployment* as an opportunity for students to critically analyze a graph; however, Danielle rejected the task for Algebra I because she did not perceive a clear alignment to algebra standards and content (though she did suggest that she would likely allocate more time to the task in a statistics course). While Benjamin recognized the value of the critical thinking in the task (M2), he suggested that the type of function in the graph would not lend itself to sophisticated mathematical work: "… it's critical thinking, this would be good as a warmup right there … But then I look at the graph because I know it better. It's linear. Come on." As a general principle, Benjamin seemed to evaluate tasks primarily based on the opportunities they created for students' mathematical thinking, and assigned lower priority to tasks whose cognitive demand he considered inappropriately low for algebra students (P3).

One remarkable pattern was that despite concerns about authenticity, teachers in both groups envisioned opportunities for *Wrapping* to engage students in mathematical practices (M2). Danielle and Benjamin pointed out that the task could engage students in constructing and measuring a net for a solid to determine the surface area; we consider this an instance of using representations strategically to solve problems. Denise pointed out that one virtue of the task is that students can approach it without knowing the formula for surface area, and indicated that she valued students' gaining confidence in their ability to think about novel problems when they do not have relevant procedures readily available more than their fluency in formulas.

Obligations of Mathematics Tasks

At various times, participants appeared to attach positive judgments to tasks that contained important mathematics and also fulfilled other civic or professional obligations that they associated with mathematics teaching. For example, when discussing *Ramp*, Denise and Felipe pointed to the applicability of the task to other professional domains (O1):

Denise:	And so when they're asking, when are we ever going to use this, that's exactly a time where, which makes it totally authentic.
Felipe:	And it's also like because math leads into construction and engineering, so it's a real world [application].

Their comments suggest that they valued the task for its potential to inform students about where they might use algebra in the future (O1). This was a theme in discussion of multiple tasks.

Additionally, teachers occasionally recognized and praised connections of the tasks to matters of social justice (O5). Denise conspicuously connected *Population* to the politics of replacement theory, which had been in the news around the time of this activity:

Denise:	I would take that to social justice too.
Felipe:	Yeah. So it's like what could we eventually start decreasing in population?
Denise:	Oh no, I don't mean that kind of social justice. I mean that certain populations are afraid that they're being outnumbered now, so they're trying to... strengthen their power. We definitely go there.
Frances:	You can also compare our population to other countries.
Denise:	Oh, we can do that too. Yeah. Well, and then of the different groups.

We found it notable that despite Felipe's and Frances' bids to connect the problem to less politically heated issues such as global population decrease and different countries' population growth, Denise found the opportunity to connect the task explicitly to an issue related to the politics of racism and xenophobia compelling. This was a remarkable example of a more general belief on Denise's part that tasks should broaden students' awareness of the world around them and illustrate the role of mathematics in the world beyond school.

Pedagogy and Access

Some teachers identified features of tasks that they considered pedagogically advantageous or likely to broaden students' access to mathematical thinking. We took particular note of instances in which teachers spontaneously suggested alternatives to tasks that might accentuate these pedagogical advantages or heighten accessibility and relevance for students. Frances imagined giving her students the materials from the *Wrapping* problem to work on hands on (P2) and reflected back on a past activity where her students grappled with related ideas, "so they had to come up with their measurements to make a little... and they made little boxes, big boxes. So it was kind of interesting." Danielle suggested a similar modification, "You know what I think would be fun with this? Bring in a bunch of different boxes and ask them, 'Which box do you think is going to take the most wrapping paper?' And they all vote and then each group has a box and they have to figure out how much each would give".

Several teachers valued tasks for their potential to be related to students' lived experiences, making the mathematics of the tasks more motivating and accessible for students (P1). Viola pointed out that her own students were conducting fundraisers for an organization they had just started and therefore could relate to the content of *Muffins.* She also suggested that while students could picture an automobile accident and imagine the skid marks made by a braking vehicle in *Braking*. Of *Unemployment*, Denise suggested that some of her junior students might be trying to find jobs and might find the topic of job availability relatable (P1); she then revised her thinking and surmised that students might not think about their own situation in such a broad social context but could be guided to do so (O1).

Discussion

The goal of our study was to gain insight into how mathematics teacher educators might help teachers select and design contextual tasks that make meaningful connections between algebra and students' lives while creating opportunities for rich mathematical thinking and problem solving. Using the *Worthwhile Problems* activity as a springboard for discussion, we were able to identify four distinct domains of belief that influence teachers' judgments of which types of contextual problems are worthy of instructional time and students' attention. Furthermore, we saw that teachers often take on an "authorship" role when faced with a task that has perceived deficiencies relative to their priorities and professional commitments, envisioning alternative versions of the task that better meet students' needs and curricular goals. We saw this in Felipe's and Danielle's reimagined versions of *Wrapping*, Benjamin's suggested modifications to *Braking*, and Denise's refocusing of *Population* on current social justice issues.

We recognize some limitations of our study associated with our selection of "Worthwhile Problems" for the activity. For example, teachers may have specific beliefs and dispositions associated with mathematical modeling (Sevinc & Lesh, 2018) that inform the complexity and ambiguity that they are willing to tolerate in classroom tasks; we did not have any tasks open-ended enough to help us identify these boundaries on teacher practice. Additionally, while the *Worthwhile Problems* activity helped us map teacher beliefs that influence approval or rejection of contextual tasks, it provided limited insight into how teachers prioritize different beliefs that may be in conflict. Because a sound understanding of teacher beliefs and their implications for classroom practice requires attention to the network within which these beliefs operate (Leatham, 2006), we plan to revise the activity to include more focused discussion prompts that ask teachers to prioritize among different values and beliefs and make design choices.

Overall, our study results suggest that teachers respond positively to tasks that make connections to the world beyond the mathematics classroom, especially when these tasks authentically reflect the mathematical questions and processes that people would pursue in other coursework, careers, and civic life. Teachers sometimes moderate these positive judgments when they perceive that tasks do not contain sufficient opportunities to reinforce grade-level content or develop students' mathematical practice, as seen with *Unemployment*. In future work, we hope to continue refining our map of teacher priorities for contextual tasks and better understand how teachers manage tensions between authenticity concerns about tasks and the need to connect tasks to curricular content, which is often abstract. With additional insight, we expect to be able to help researchers and educators design mathematical problems that are culturally relevant and address contemporary issues of social injustice while providing opportunities for mathematical thinking that teachers expect to see in classroom tasks.

Acknowledgment

This work is supported by the National Science Foundation through the Discovery Research PreK-12 (DRK-12) program (Award #1908825). Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

References

Brantlinger, A. (2022). Critical and vocational mathematics: Authentic problems for students from historically marginalized groups. Journal for Research in Mathematics Education, 53(2), 154–172. https://doi.org/10.5951/jresematheduc-2019-0025

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. Qualitative research in psychology, 3(2), 77–101.

Doyle, W., & Ponder, G. A. (1977). The practicality ethic in teacher decision-making. Interchange, 8(3), 1–12.

Gainsburg, J. (2008). Real-world connections in secondary mathematics teaching. Journal of Mathematics Teacher Education, 11, 199–219.

Girnat, B., & Eichler, A. (2011). Secondary teachers' beliefs on modelling in geometry and stochastics. Trends in Teaching and Learning of Mathematical Modelling: ICTMA14, 75–84.

Gravemeijer, K. P. E. (1994). Developing realistic mathematics education. CD-b Press/Freudenthal Institute, Utrecht.

Leatham, K. R. (2006). Viewing mathematics teachers' beliefs as sensible systems. Journal of Mathematics Teacher Education, 9, 91–102. https://doi.org/10.1007/s10857-006-9006-8

McGraw, R., & Patterson, C. L. (2019). Establishing Problem Spaces and Boundaries during Small Group Work on a Contextualized Problem. Journal of Mathematics Education, 12(2), 1–17.

National Council of Teachers of Mathematics. (2000). Principles and Standards for School Mathematics.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). Common Core State Standards.

Palm, T. (2006). Word problems as simulations of real-world situations: A proposed framework. For the learning of mathematics, 26(1), 42–47.

Palm, T. (2008). Impact of authenticity on sense making in word problem solving. Educational Studies in Mathematics, 67, 37–58. https://doi.org/10.1007/s10649-007-9083-3

Sevinc, S., & Lesh, R. (2018). Training mathematics teachers for realistic math problems: a case of modeling-based teacher education courses. ZDM, 50, 301–314. https://doi.org/10.1007/s11858-017-0898-9

Simic-Muller, K., Fernandes, A., & Felton-Koestler, M. D. (2015). " I just wouldn't want to get as deep into it": Preservice teachers' beliefs about the role of controversial topics in mathematics education. Journal of Urban Mathematics Education, 8(2). https://doi.org/10.21423/jume-v8i2a259

Texas Education Agency. (2012). Texas Essential Knowledge and Skills for Mathematics.

Turner, E. E., Bennett, A. B., Granillo, M., Ponnuru, N., Roth Mcduffie, A., Foote, M. Q., ... & McVicar, E. (2022). Authenticity of elementary teacher designed and implemented mathematical modeling tasks. Mathematical Thinking and Learning, 1 – 24. https://doi.org/10.1080/10986065.2022.2028225

Van den Heuvel-Panhuizen, M., & Drijvers, P. (2020). Realistic mathematics education. Encyclopedia of mathematics education, 713–717.

Vos, P. (2018). "How real people really need mathematics in the real world"—Authenticity in mathematics education. Education Sciences, 8(4), 195. https://doi.org/10.3390/educsci8040195

TEACHING PRACTICE AND CLASSROOM ACTIVITY

ALIGNING TEACHERS' MOVEMENTS AND IDENTITIES

Maria Eloisa Nuguid
Purdue University
mnuguid@purdue.edu

Keywords: Teacher Educators, Informal Education, Cognition.

Objectives

To address the need of making mathematics meaningful to all learners, I explore the research question: How do teachers' identities align with their movements in a small group mathematics station? I argue that teachers communicate parts of their identities through movement. Directly aligning with the theme, body-based movement such as gesturing facilitates learning, meaning-making, rapport-building, and enhances students' engagement.

Theoretical Framework

I use frameworks of embodied cognition and identity as performance. Embodied cognition is a process through which an organism navigates its surroundings using its body and connects new ideas and representations to prior experiences, facilitating meaning-making (Nathan, 2021; Núñez et al., 1999). Examples include gesturing such as pointing or tracing a triangle in the air (Alibali & Nathan, 2012). I focus on a sociocultural lens of embodied cognition to account for both individual and collaborative interactions (Danish et al., 2020). Next, identity as performance is "the performance and the recognition of the self…by telling stories, joining groups, [and] acting in a particular way at a particular time" (Butler, 1988; Darragh, 2016, p. 29). Both the teacher's performance and input from students inform her identity.

Methods

I apply a qualitative case study methodology (Merriam & Tisdell, 2016) to provide intensive and holistic description and characterize movement and identity. After listening and participating in a read-aloud in a kindergarten classroom, kindergarteners separated into integrated stations around the classroom including a mathematics station. Using qualitative methods, I found through transcription, coding, and thematic analysis of video and interview data that I collected that participants made both conscious and subconscious choices with actions. Through an interpretivist perspective, I aligned parts of my participants' semi-structured interviews about their identities with the movements that they performed during facilitation to extract meaning.

Results and Implications

My first participant identified as a teacher and mother. Through past experiences of teaching elementary students, some of her conscious actions aligned with her identity as a teacher like lightly touching a students' forearm to redirect attention, forming shapes with her hands, and folding her hands in her lap to model respect. My second participant identified as both a teacher and teacher educator, and similarly she used conscious movements to connect the activity to real life contexts while also noting in an interview that some movements were subconscious. For future directions, I plan to study facilitation in informal science, technology, engineering, and mathematics (STEM) learning spaces.

References

Alibali, M. W., & Nathan, M. J. (2012). Embodiment in Mathematics Teaching and Learning: Evidence From Learners' and Teachers' Gestures. *The Journal of the Learning Sciences*, *21*(2), 247–286. https://www.jstor.org/stable/23266406

Butler, J. (1988). Performative Acts and Gender Constitution: An Essay in Phenomenology and Feminist Theory. *Theatre Journal*, *40*(4), 519. https://doi.org/10.2307/3207893

Danish, J. A., Enyedy, N., Saleh, A., & Humburg, M. (2020). Learning in embodied activity framework: a sociocultural framework for embodied cognition. *International Journal of Computer-Supported Collaborative Learning*, *15*(1), 49–87. https://doi.org/10.1007/s11412-020-09317-3

Darragh, L. (2016). Identity research in mathematics education. *Educational Studies in Mathematics*, *93*(1), 19–33. https://doi.org/10.1007/s10649-016-9696-5

Merriam, S. B., & Tisdell, E. J. (2015). *Qualitative research: a guide to design and implementation* (Fourth edition). John Wiley & Sons.

Nathan, M. J. (2021). *Foundations of embodied learning: a paradigm for education*. Routledge.

Núñez, R. E., Edwards, L. D., & Matos, J. F. (1999). Embodied Cognition as Grounding for Situatedness and Context in Mathematics Education. *Educational Studies in Mathematics*, *39*(1/3), 45–65. https://www.jstor.org/stable/3483160

AN EXPLORATION OF INTEGRATING MATHEMATICAL MODELING IDEAS IN TEACHING APPLICATION OF FUNCTIONS

Li Siqing
St. Paul's College
Siqing.li@hotmail.com

Lili Zhou
California State University, Los Angeles
Lzhou18@calstatela.edu

This paper presents an instructional design that integrates mathematical modeling concepts to teach the application of functions. The design emphasizes exploratory teaching materials, real-world scenarios, student-driven tasks, and technologies, which have shown potential in aiding students to attain a better understanding of function-related concepts, fostering their engagement in class, and enhancing their problem-solving and self-exploration skills. To validate the efficacy of the new design, we implemented a curriculum and interviewed students who experienced and did not experience the design. Using thematic analysis, we analyzed the transcripts and found that the design could improve students' performance across several measures, although some challenges were also identified.

Keywords: Application of Functions; Mathematical Modeling.

Introduction

Functions are a foundational mathematical concept woven throughout secondary curricula. High school students are expected to apply functions through modeling to solve problems. However, traditional instruction in the Application of Functions chapter in the Mathematics textbook of the People's Education Press Version (PEP, 2017) has some drawbacks that may hinder student development. Firstly, the textbook heavily focuses on teacher-centered instruction (Fu, 2020), with questions primarily centered around simple calculations and basic conceptual understanding. For example, studies found students memorize formulas without fully understanding equations (Zhao, 2022) or have difficulty visualizing function graphs (Chen, 2022). To better prepare students for future careers, a shift towards a more student-centered education system is necessary (Roytblat, 2022). Secondly, the textbook lacks real-world scenarios that are relevant to functional models (Wuolle, 2016). Thirdly, the textbook demonstrates a lack of utilization of technology (Weng & Li, 2020). Research supports the notion that teachers need to acquire proficiency in leveraging technology to effectively meet the evolving demands of teaching (Powers & Blubaugh, 2005).

Mathematical modeling serves as a bridge between mathematics and the real world (Meerschaert, 2013). It involves refining real-world problems, expressing them mathematically, solving the models, and translating the solutions back to their original contexts (Williams, 2013). Mathematical modeling is well-suited to address the aforementioned issues, as it uses mathematics to represent, analyze, predict, and provide insights into real-world phenomena (Garfunkel et al., 2016). Moreover, mathematical modeling takes a student-centered approach where students choose and investigate their own problems (Borba & Villarreal, 2005). Technology also plays a key role in mathematical modeling by providing essential support to make mathematical modeling a more accessible mathematical activity for students (Ang, 2010).

We have developed an instructional design that incorporates modeling ideas and utilizes a variety of exploratory teaching materials. This design addresses three key areas of instruction: student-centered learning, the integration of real-world scenarios for building models, and the utilization of a computer-aided learning environment. In this study, we aim to investigate how this modeling-based design and inquiry-based approach can influence student performance and perception on function teacher practices through an inquiry-based approach.

Theoretical Background

Mathematical modeling has long been integrated into mathematics teaching (Frejd, 2012; Frejd & Bergsten, 2018; Hernandez-Martinez, 2018; Stillman, 2011; Zubi et.al., 2019). It introduces new practices (Cai et al., 2014), helps apply math to understand science (Wake, 2014), develops skills for higher education and careers (Fletcher et al., 2018), and improves thinking and problem-solving (MOE, 2017). Modeling can be used in any course (Hernández, 2016).

Using real-world contexts in teaching can increase student engagement (Gainsburg, 2008), but focus is needed on developing critical thinking, not just motivation and concept mastery. Resources, ideas and training help incorporate real-world connections, but critical thinking and literacy development are lacking.

Student-centered approaches are increasingly popular, shifting from teacher-led instruction to student ownership of learning (Din & Wheatley, 2007). Student-centered activities improve non-academic areas (Shah & Kumar, 2020); less student-centered approaches show academic improvements (Shah & Kumar, 2020).

Technology serves multiple functions, including as a tool for doing mathematics, providing an environment for practicing skills, and developing understanding (Drijvers et al., 2010). Teachers use technologies like graphing calculators, interactive math environments, geometry applications, spreadsheets and the internet to develop lessons (Powers & Blubaugh, 2005). Teacher candidates use additional technologies like statistics software and CAS to enhance teaching (Powers & Blubaugh, 2005).

Design

For analyzing the effectiveness of the design, we selected 100 high school freshmen who had already learned the definitions, analytical expressions, and graphs of functions such as linear, quadratic, exponential and logarithmic functions from a local high school in South China, using convenience sampling. These students had no prior experience with curricular integrating mathematical modeling. To teach the application of functions, we split the students into two groups: an experimental group and a control group. The settings of the groups were almost identical, including having a same teacher. The two groups had essentially the same level of understanding of functions (as measured by a quiz). The only difference was that the experimental group used our design while the control group did not.

After two classes teaching/learning the application of functions, we collected data through 30-minute semi-structured interviews with students. Using stratified sampling, we selected 25 students from both groups to invite to the interviews. In the study, we focused on the six tasks used in the interview: 1. Describing their feelings in the class; 2. Describing what functions are; 3. Interpreting the meaning of some functions; 4. Plotting the graphs of some functions; 5. Computing the solutions of word problems about functions; 6. Building mathematical models in real-world scenarios and computing solutions. During the interviews, following up questions were used to elicit details about how they comprehend, apply, and transfer knowledge of functions.

We transcribed the interviews verbatim and then analyzed the transcriptions using thematic analysis to answer the research question. We coded segments of the interviews by running a keyword analysis of students' responses to each task. The most common terms were used as codes. We then identified three overarching categories from the codes for all open-ended tasks: (a) positive; (b) neutral; (c) negative. Responses that did not contain those keywords were individually coded to the categories. The teachers' responses were analyzed independently.

Findings

Due to the word limitation, in this proposal, we only report key findings, which include: a) higher levels of engagement and positive attitudes towards the class among students in the experimental group; b) better performance in graphing functions and a greater inclination to explore different mathematical models among students in the experimental group; and c) higher scores in solving textbook function problems among the control group.

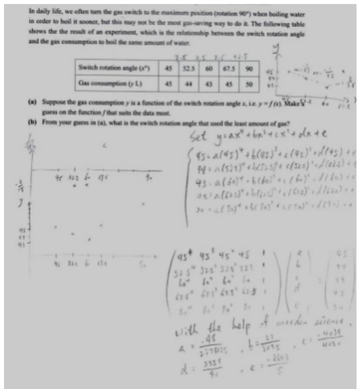
In task 1, students in the experimental group tended to express a positive attitude towards the class. The most frequently used keywords extracted from their interviews were 'interesting' and 'high technologies'. Students in the control group, however, preferred using relatively negative words like 'abstract' and 'boring'. The keyword that commonly appeared in both groups was 'difficult'.

In task 2, students were asked to define function. We found that 17 out of 25 students in the experimental group could realize that a function is a relation between two sets of elements, while only 11 students in the control group could.

In tasks 3 and 4, we arbitrarily selected functions, such as linear, quadratic, logarithmic, and exponential functions, and asked students to describe the properties of these functions and plot their graphs. Both groups performed similarly in task 3, with little difference (16 students from the control group and 15 from the experimental group). But students in the experimental group did better in task 4. The numbers of students who could correctly plot graphs in the experimental and control groups were 17 and 12, respectively. Remarkably, 4 students in the experimental group said they could use Python to plot graphs.

In task 5, students were asked to solve function problems from textbooks. Students in the experimental group could obtain an average score of 6.1 out of 10, while the control group obtained a higher average score of 6.8.

Further, in task 6 students were provided with some real-world scenarios. Although both groups thought open-ended problems were difficult to start, students in the experimental group were more exploratory. They tried different mathematical models for their given scenarios and could come up with potential solutions when not knowing how to solve the problems immediately. For example, a student in the experimental group tried to build a quartic function for a problem, as shown in Fig. 1, and students in experimental group claimed they could write a Python program to fit the function. In contrast, students in the control group were capable of plotting a graph by connecting all data points directly, imitating the example question in the textbook. They were less likely to explore new models. Fig. 2 below showed a classical model built by students in the control group for the same problem.



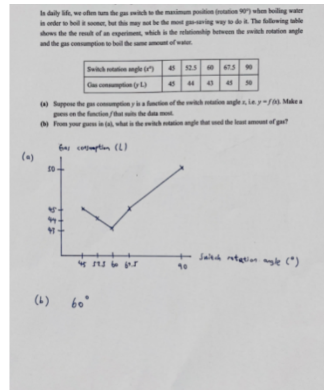


Fig. 1: An Experimental Group Sample   Fig. 2: A Control Group Sample

Discussion and Conclusion

In the study, we made an instructional design illuminated by the notion that integrating mathematical modeling into mathematics education can improve students' performance. By analyzing the interviews of students and teachers who had applied the design, we found that the design has both strengths and weaknesses.

On the positive side, with the new design, students showed a greater ability to learn the application of functions. Students in the experimental group could not only clearly describe their understanding of what functions are and plot function graphs but also apply functions to solve problems in their daily lives (Williams, 2013). The results showed that the design can increase students' in-depth learning of functions and greatly improve their problem-solving and self-exploratory skills. Moreover, students in the experimental group were more engaged in the class. They showed significant interest and were very active, which may have benefited from the use of new technologies, including hardware like screen sharing and software like GeoGebra and Python (Powers & Blubaugh, 2005).

However, students in the traditional group performed better at solving word problems, which is crucial in the current exam-oriented education system. Students may not desire an in-depth understanding of functions; they may only need an efficient way to obtain high scores on exams.

In summary, with the development of the times and the deepening of educational reform, students will not just be trained as "exam machines." Our new design would benefit students in their long-term career development, although we still need to consider the trade-off between in-depth understanding and efficiency.

References

Ang, K. C. (2010). Teaching and learning mathematical modelling with technology. *In Proceedings of the 15th Asian Technology Conference in Mathematics*, Beijing, China, pp.19-29.

Borba, M. C., & Villarreal, M. E. (2005). *Humans-with-media and the reorganization of mathematical thinking: Information and communication technologies, modeling, visualization and experimentation* (Vol. 39). Springer Science & Business Media.

Bruner, J. S. (1974). From communication to language—A psychological perspective. *Cognition*, *3*(3), 255-287.

Chen H. (2022). Gao zhong shu xue han shu jiao xue zhong shu xue si xiang fang fa de ying yong [Application of Mathematical Thinking Method in Mathematical Function Teaching in Senior High School]. *New Curriculum* (10),102-103.

Din, F. S., & Wheatley, F. (2007). A literature review of the student-centered teaching approach: National implications. In *National Forum of teacher education Journal* (Vol. 17, No. 3, pp. 1-17).

Drijvers, P., Boon, P., & Van Reeuwijk, M. (2010). Algebra and technology secondary school algebra: Revising topics and themes and exploring the unknown.
Frejd, P. (2012). Teachers' conceptions of mathematical modelling at Swedish Upper Secondary school. *Journal of Mathematical Modelling and Application*, *1*(5), 17-40.
Frejd, P., & Bergsten, C. (2018). Professional modellers' conceptions of the notion of mathematical modelling: ideas for education. *ZDM*, *50*, 117-127.
Fu, G. (2020). The knowledge-based versus student-centred debate on quality education: Controversy in China's curriculum reform. *Compare: A Journal of Comparative and International Education*, *50*(3), 410-427.
Gainsburg, J. (2008). Real-world connections in secondary mathematics teaching. *Journal of Mathematics Teacher Education*, *11*, 199-219.
Garfunkel, S. A., Montgomery, M., Bliss, K., Fowler, K., Galluzzo, B., Giordano, F., ... & Zbiek, R. (2016). *GAIMME: Guidelines for assessment & instruction in mathematical modeling education*. Consortium for Mathematics and Its Applications.
Gilbert, J. K. (2006). On the nature of "context" in chemical education. *International journal of science education*, *28*(9), 957-976.
Hernández, M. L., Levy, R., Felton-Koestler, M. D., & Zbiek, R. M. (2016). Mathematical modeling in the high school curriculum. *The mathematics teacher*, *110*(5), 336-342.
Hernandez-Martinez, P., & Vos, P. (2018). "Why do I have to learn this?" A case study on students' experiences of the relevance of mathematical modelling activities. *ZDM*, *50*, 245-257.
Kaiser, G. (2020). Mathematical modelling and applications in education. *Encyclopedia of mathematics education*, 553-561.
Meerschaert, M. M. (2013). Chapter 3-Computational Methods for Optimization. *Mathematical Modeling*, 57-112.
PEP (People's Education Press). (2017). *Pu tong gao zhong ke cheng biao zhun shi yan jiao ke shu shu xue bi xiu yi* [Ordinary high school curriculum standard experiment textbook Mathematics Compulsory I]. Beijing: People's Education Press. (In Chinese)
MOE (Ministry of Education of China). (2017). *High school mathematics curriculum standards (Experimental version)*. Beijing: People's Education Press. (in Chinese).
Powers, R., & Blubaugh, W. (2005). Technology in mathematics education: Preparing teachers for the future. *Contemporary Issues in Technology and Teacher Education*, *5*(3), 254-270.
Roytblat, R. (2022). *Experiences of Teachers Using Student-Centered Approaches in China* (Doctoral dissertation, Walden University).
Shah Ph, D., & Kumar, R. (2020). Literature Review of The Learner Centered Teaching. *Shah, RK (2020). Literature Review of Learner Centered Teaching. International Journal of Research in Education and Psychology (IJREP)*, *6*(4), 22-45.
Stillman, G. (2011). Applying metacognitive knowledge and strategies in applications and modelling tasks at secondary school. *Trends in teaching and learning of mathematical modelling: ICTMA14*, 165-180.
Tawfik, A. A., Law, V., Ge, X., Xing, W., & Kim, K. (2018). The effect of sustained vs. faded scaffolding on students' argumentation in ill-structured problem solving. *Computers in Human Behavior*, *87*, 436-449.
Wake, G. (2014). Making sense of and with mathematics: The interface between academic mathematics and mathematics in practice. *Educational Studies in Mathematics*, *86*, 271-290.
Weng, J., & Li, H. (2020). Early technology education in China: A case study of Shanghai. *Early Child Development and Care*, *190*(10), 1574-1585.
Williams, H. P. (2013). *Model building in mathematical programming*. John Wiley & Sons.
Wuolle, S. R. (2016). *How and why teachers use real world connections in the secondary mathematics classroom*. Simon Fraser University, Education: Faculty of Education.
Zhao C. (2022). *Teaching Design and Practice of "Application of Function" Unit in High School Mathematics Under Core Literacy.* http://doi.org/10.27332/d.cnki.gshzu.2022.001002
Zubi, I. a. M. I., Peled, I., & Yarden, M. (2019). Modelling Tasks and Students with Mathematical Difculties. In *ICME-13 monographs. Springer International Publishing*. http://doi.org/10.1007/978-3-030-14931-4_12

APPROACHING SIMILARITY BY USING GEOMETRIC REPRESENTATIONS: ENHANCING HIGH SCHOOL STUDENTS' LEARNING EXPERIENCES

Mayra L. Ortiz Galarza, Ph.D.
University of Texas Rio Grande Valley
mayra.ortizgalarza@utrgv.edu

Luis M. Fernandez, Ph.D.
University of Texas Rio Grande Valley
luis.fernandez01@utrgv.edu

Objectives and Purpose of the Study

This study aims to contribute to the efforts that enhance the teaching and learning of trigonometric functions by proposing a novel approach to teach critical concepts in trigonometry that are non-intuitive (i.e. similarity, ratio, and proportion) (Gur, 2009).

Theoretical Framework

A conceptual corridor (Confrey, 2006) was established to explicate students' learning trajectories while exploring similarity. A grounded theory framework (Charmaz, 2010; Cobb, et al., 2001) was used to analyze the conceptual corridor followed by students.

Design

A design study was engineered to create opportunities for students to develop mathematical understanding and competence. The design study focused on investigating students' development of meta-representational competence (Boester & Lehrer, 2008). The study took place in a public high school located in a US-Mexico border city; involved twenty-three secondary students enrolled in a Geometry class. The "Colored Triangles" activity (Figure 1) was designed to investigate students' understandings of mathematical concepts such as patterns and the geometric definition of similarity, co-variation, and ratio and proportion.

Summary of Results and Implications

Approaching trigonometric concepts such as similarity and ratios through multiple representations resulted on making the learning process more interesting and motivated students to sustain their interest in mathematics. The learning trajectory (shown by students when immersed in an inquiry-based activity to explore the concept of similarity) made evident the importance of visual representations of mathematical objects. The "Colored Triangles" activity provided students with opportunities to learn about ratios, proportions, and reinforce their knowledge about similar triangles. Implications of the study highlight the importance of using geometric representations of mathematical objects when teaching trigonometric concepts.

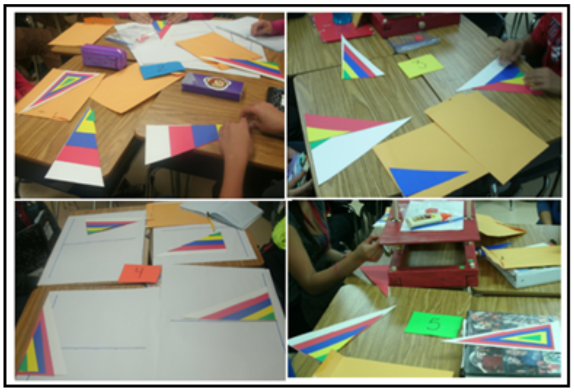


Figure 1: Colored Triangles Modeling Activity.

References

Boester, T., & Lehrer, R. (2008). Visualizing algebraic reasoning. Algebra in the early grades, 211-234.

Charmaz, K. (2010). Constructing grounded theory: A practical guide through qualitative research. SAGE Publications Ltd, London.

Cobb, P., Stephan, M., McClain, K., & Gravemeijer, K. (2001). Participating in classroom mathematical practices. The Journal of the Learning Sciences, 10(1&2), 113-163.

Confrey, J. (2006). The evolution of design studies as methodology. In *The Cambridge handbook of the learning sciences* (pp.135-152). New York, NY: Cambridge University Press.

DIFFERENT PROCEDURES TO FIND NUMBERS IN AN INTERVAL WITH HIGH SCHOOL STUDENTS

DIFERENTES PROCEDIMIENTOS PARA HALLAR NÚMEROS EN UN INTERVALO CON ESTUDIANTES DE BACHILLERATO

Mayra Zulay Suárez Rodríguez
Cinvestav
mayra.suaez@cinvestav.mx

Ana Isabel Sacristán
Cinvestav
asacrist@cinvestav.mx

In this report we show an investigation of how high-school students learn strategies to find intermediate numbers in an interval to understand the property of numerical density. Research has shown that some high-school students have difficulty in understanding this property. To mitigate this difficulty, we proposed a Hypothetical Learning Trajectory, using school mathematics topics, for high-school students to learn about numerical density. Our study showed that some students recognized that there is an infinite quantity of intermediate numbers in an interval; however, all the participants had difficulty understanding why there is no successor in a set other than the natural or integer numbers.

Keywords: Learning Trajectories and Progressions, Mathematical Representations, Number Concepts and Operations, Design Experiments

Introduction and Research Questions

The number density property has been poorly understood by students of all school grades (Vamvakoussi & Vosniadou, 2010): some believe that there is no number in an interval and others believe that there is a finite quantity of numbers in an interval, in sets other than those of the natural numbers or integers. The density property means that a number can be found in an interval; this implies that: a) there is no number that has a successor, and b) there is an infinite quantity of numbers in an interval. For example, in the research carried out by González-Forte et al. (2021), students of 5th and 6th grade of elementary school (10-12 years old) and from 1st to 4th grade of high school (12-16 years old) answered that there was a finite amount of numbers in the interval between 3.49 and 3.50; some students even mentioned that there was only one number in that interval. There are also situations where students believe that the extremes of an interval (not natural, not integer) are consecutive; therefore they believe that there is not any other number in that interval (Vamvakoussi & Vosniadou, 2010). For example, in the study carried out by Cabarcas and Soler (2017), 9th grade students (around 15 years of age) believed that a rational number had an immediate predecessor and successor. They believed that the successor of 14/4 was 15/4 because their numerators are consecutive with the same numerator; and that there is a finite quantity of numbers in an interval of the rational numbers.

On the other hand, in the studies carried out by Neumann (1998) and Vamvakoussi and Vosniadou (2010), some students said that there are only decimal numbers between decimals, but not fractions; in the same way, they believed that there are fractions between fractions, but not decimal numbers. In Neumann's (1998) study, students had difficulty accepting that there are fractions between 0.3 and 0.6. Vamvakoussi and Vosniadou (2010) concluded, then, that students not only decided what type of symbolic representation the numbers belonging to an interval should have, but also how many there should be. For example, in their research, a student stated that there could be more numbers between 0.001 and 0.01 if the decimal numbers (the intermediate ones) were expressed in fractions and that thus there could be an infinity. For this reason, Duval (1995/2004) emphasizes for students to engage with different representations of a mathematical object from an early age, since learning mathematics implies the use of different semiotic registers (arithmetic, algebraic, geometric, etc.) of representation (fractional, decimal, etc.).

In order for a student to understand about number density –which implies understanding that there is no successor for a decimal, rational or real number–, we designed and put into practice a Hypothetical Learning Trajectory (HLT) (Simon, 1995), following the principles of Design-Based Research (DBR) (Cobb and Gravemeijer, 2008). A HLT is a sequence of activities that hypothesizes on how to support a student in their learning process of a mathematical concept (Simon, 1995). For our HLT, we used different topics from school mathematics that would allow a student to understand number density, as suggested by McMullen and Van Hoof (2020).

We present below the aims of a study that began in 2020 and of which we have already given some results in Suárez-Rodríguez and Sacristán (2022):

a) Study the previous conceptions of the students participating in the study regarding the property of numerical density. For this, we used a diagnostic questionnaire (as a pre- and post-test).

b) Analyze the actions of the students during the development of the implementation of the activities proposed in the HLT, in relation to the property of numerical density and the property of the discrete of natural numbers.

Theoretical framework

Thinking about the discrete and the dense

Discrete mathematics is the basis of everything that corresponds to natural numbers or countable sets, while continuous mathematics is the basis of everything related to continuity (Levin, 2021; "Discrete Mathematics", 2021, pars. 2 -3). Hence, as Vamvakoussi and Vosniadou (2010) explain, only the natural numbers have the property of the discrete, because every natural number has a successor in terms of order relationships. That is, these authors emphasize that the terms "discrete" and "dense" are used with respect to the usual order relationship, thus the natural numbers are discrete, the rational numbers are dense, and the real numbers are dense and continuous. The property of the discrete is related to the natural numbers in the sense that any natural number has a successor (Vosniadou & Vamvakoussi, 2010); as indicated by Peano (1889/1979) in his axioms for the natural numbers, these numbers have a successor. On the other hand, Vamvakoussi and Vosniadou (2004) developed some categories based on the results of their research to find out how much students know about numerical density and the discrete property (Table 1).

Table 1: Categories of thinking about the discrete and the dense

Naive discreteness [Naive thinking about the discrete]	The thinking is that there is no number between two *consecutive false* rational numbers. Vamvakoussi and Vosniadou (2004) coined this expression to refer to the idea that a successor of a rational number exists.
Advanced discreteness [Advanced thinking about the discrete]	The thinking is that there is a finite quantity of numbers between two consecutive false rational numbers.
Discreteness–density [Mixed thinking between discrete and dense]	In some cases, the thinking is that between two rational numbers there is an infinity of numbers; and in other cases, that there is a finite number.

Naive density [Naive thinking about density]	There is an understanding that there is an infinity of numbers in an interval, but this situation is not justified by using the density property. The symbolic representation of the extremes of an interval influences this way of thinking; it is believed there can only be an infinite number of decimal numbers between decimals and an infinity of fractions between fractions, but not an infinity of fractions between decimals or otherwise.
Sophisticated density [Advanced thinking about density]	There is a sophisticated understanding of the density property; that is, it is understood that there is an infinite number of numbers between two rational numbers, regardless of their symbolic representation and this is justified through the use of the density property.

Semiotic registers of representation

Learning mathematics constitutes a space for the analysis of cognitive activities (conceptualization, reasoning, problem solving and text comprehension) that require the use of expressive and representational systems that are different from those of natural language or images (Duval, 1995/2004). As Duval states, representational systems constitute semiotic registers or registers of semiotic representation. These registers allow an individual to communicate ideas that are transformed into others without changing their meaning (Moreno & Sacristán, 1996). However, Duval (2006) recommends also using non-semiotic registers as contexts in which one can work with materials such as matches, sticks, basins, etc. We consider, then, the importance of using various representational registers to support the learning and understanding of the property of number density.

Methodological Framework and Methodology

Hypothetical Learning Trajectory (HTL)

The expression "hypothetical trajectory of learning" refers to a prediction that a researcher makes as to the path where learning might proceed (Simon, 1995). A trajectory is hypothetical because the "real" learning trajectory is not yet known in advance, and it becomes real when the students go through the proposed activities. Simon (1995) describes three components that a HLT must include: a) learning objectives, understood as a set of aims that are expected to be achieved for the student to build new knowledge, b) learning or instructional activities that follow a sequence, and c) hypothetical learning process, in which conjectures are proposed to support the student's learning process. The researcher can make modifications to any or all of the three components of a trajectory in order for them and the students to build and improve their knowledge.

Design and development of HLT activities

As we mentioned, our research methodology follows the guidelines of Design-Based Research (DBR). The DBR is a methodological approach in which the researcher carries out a systematic study of the design, development and assessment of educational interventions in different cycles, and for this, hypotheses are suggested through a HLT (Cobb & Gravemeijer, 2008). In this report we show some results of one of two cycles of research activities (the other is being carried out during the year 2023). In this first cycle, four students between the ages of 15 and 17 from Colombia participated: Angie, Paola and Néstor who were in high-school and Violeta, who was in the last year of middle-school. Some activities were carried out in person and others virtually. The design corresponding to the first cycle of activities involved three phases: The students answered the same questionnaire in both the first and third phases, and in the second, they solved HLT activities.

First and third phase. *(Pretest and post-test).* The students answered a questionnaire of four questions related to the number of numbers that an interval can have. These questions were based on those proposed by Suárez-Rodríguez and Figueras (2020) and Vamvakoussi and Vosniadou (2004). We wanted to know how much students know about numerical density –before and after the implementation of the HLT– according to the categories elaborated by Vamvakousi and Vosniadou (2004) (Table 1). The pretest lasted approximately 30 minutes, while the post-test, 10 minutes. In this report we present the answers to the question: How many numbers are there between 0 and 1?

Second phase. *(HLT activities).* The HLT includes four activity sessions in which a hypothesis was proposed for each one (Table 2). We wanted to propose hypotheses related to various topics of school mathematics which we considered could support students to learn and understand about numerical density. Each session lasted about 1h30'.

Table 2: HLT sessions with their corresponding hypotheses

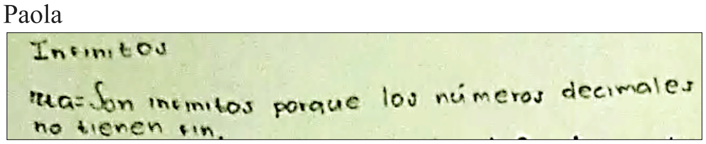
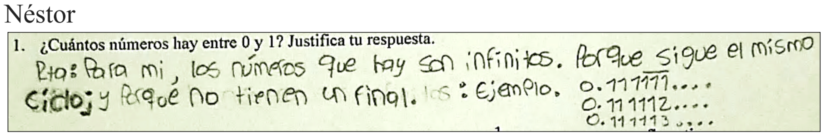
Sessions	*Hypotheses*
Session 1. Initial approaches to number density.	From two situations: one related to everyday life and the other related to a hypothetical scenario, it is thought that the student can have initial approaches to the property of density.
Session 2. Approach to number density through the similarity of triangles.	It is considered that by using triangle similarity. students can learn about the density property of rational numbers.
Session 3. Approach to number density through arithmetic progressions and geometric progressions.	It is possible that by finding arithmetic and geometric halves in an interval, students can understand the density property of rational numbers in the set of real numbers.
Session 4. Approach to numerical density through the property of continuity.	Using the continuity property, students may understand the density property of irrational numbers in the set of real numbers.

Results

First phase: Pre-test results

Table 3 shows the answers of the students to the question: "How many numbers are there between 0 and 1?" We classified those answers according to the categories proposed by Vamvakoussi and Vosniadou (2004) (Table 1).

Table 3: Students responses to a pretest question

Responses	*Types of thinking*
Paola  Infinitos Rta= Son infinitos porque los números decimales no tienen fin. *Infinities* *They are infinite because decimal numbers have no end.*	*Naive density* *[Naive thinking about density]*
Néstor  1. ¿Cuántos números hay entre 0 y 1? Justifica tu respuesta. Rta: Para mi, los números que hay son infinitos. Porque sigue el mismo ciclo; y porque no tienen un final. Ejemplo. 0.111111.... 0.111112.... 0.111113.... *For me, the numbers that there are are infinite. Because it follows the same cycle; and because they don't have an end. Example:* $0.111\overline{111}$ ... 0.111112 ... 0.111113 ...	*Sophisticated density* *[Advanced thinking about density]*

Violeta 1. ¿Cuántos números hay entre 0 y 1? Justifica tu respuesta. No hay Ningun numero Porque el 0 esta antes del 1 *There is no number because 0 is before 1.*	*Naive discreteness* *[Naive thinking about the discrete]*
Angie hay infinitos, debido que existe el 0.1, 0.2, 0.3(...). *There are infinities, because there are 0.1, 0.2, 0.3 (...).*	*Naive density* *[Naive thinking about density]*

We observe in Table 3 that Paola and Angie show a tendency to have a *Naive thinking about density*, since both stated that there are indeed an infinite quantity of numbers between 0 and 1. However, they did not give a process to find these intermediate numbers in the interval. Néstor also indicated that there is an infinite quantity of numbers between 0 and 1. He used decimal writing as a semiotic representation in an arithmetic register and wrote examples of numbers in periodic decimal writing. That is, Néstor justified his answer by means of a process of potential infinity of adding decimals to the decimal part of a periodic number as a way of finding intermediate numbers between 0 and 1. Violeta manifested a *Naive thinking about the discrete*: For her, the numbers 0 and 1 are consecutive. It is possible that she is only considering 0 and 1 as natural numbers, so she would be in a natural number domain knowledge system.

Second phase: Results of HLT activities

Next, we describe some student activities during sessions 1, 3 and 4 (Table 2) of the HLT.

Session 1. In this session, students solved two activities. In one of them, the students did a brief reading about the beginnings of the long jump as a sport and answered five questions. In Figure 1 we show Paola's response to the question: "According to the reading, which athlete records are between 5.40 and 7.70?" She responded appropriately, pointing out all the numerical records of the athletes, between 5.40 and 7.70 (5.41, 6.20, 6.375, 7.05, 7.40, 7.51, 7.54, 7.60 and 7.605). Subsequently, the students answered the question: "Do you think that 7.40 has a successor? Explain your answer." Figure 2 shows their responses.

According to the reading, which records, of the athletes, are between 5.40 and 7.70?

5,41 - 6,20 - 6.375 - 7,05 - 7,40 - 7,51 - 7.54 - 7,60 - 7,605

Rta: Todos los registros se encuentran entre 5.40 y 7.70.

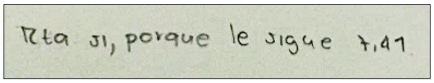
All records are between 5.40 and 7.70

Figure 1: Paola's response (Session 1)

We observe in Figure 2 that Paola, Violeta and Angie pointed to 7.41 as the "successor" of 7.40. Néstor not only writes 7.41, but also writes other numbers like 7.42 and 7.43 (numbers with two decimal places) and numbers like 7.401, 7.402, and 7.403 (numbers with three decimal places) as "successors" to 7.40. In general, we consider that students believe that there is a successor to a decimal number that depends on the number of decimal places (i.e. "the successor" of a number with one decimal place is another that must have a decimal place. However, we do not we know what could happen with a number like 7.99 (number with two decimal places); possibly students would write 7.100 (a number with three decimal places).

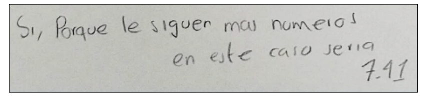
Do you think 7.40 has a successor? Explain your answer.

Paola



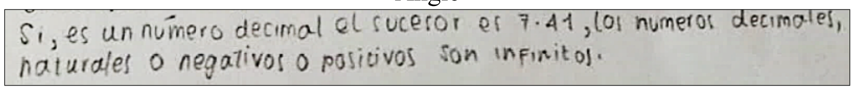
Yes, because it is followed by 7.41

Violeta



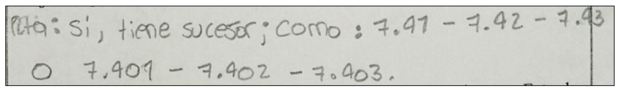
Yes, because it is followed by more numbers [,] in this case it would be 7.41

Angie



Yes, it is a decimal number, the successor is 7.41, the decimal, natural or negative or positive numbers are infinite.

Néstor



Yes, it has a successor; as 7.41-7.42-7.43 or 7.401-7.402-7.403

Figure 2: Responses of the four students (Session 1)

Session 3. In this session, activities related to arithmetic progressions and to geometric progressions were carried out. The hypothesis is for students to encounter other ways to find numbers in an interval, in order to understand the density property of rational numbers. Students were asked to find five arithmetic means between 4 and 22 from the general expression of the nth term of the sequence: $u = a + (n - 1)d$, where a is the first term, n the number of terms and d the difference between one term and the next. Figure 3 shows the task carried out by Néstor where he found the five arithmetic means between 4 and 22 (7, 10, 13, 16 and 19) with $d = 3$. Later, the students found other arithmetic means when they were asked to reduce the difference in half (i.e. $d = 1.5 = 3/2$). Figure 4 shows the work done by Angie, where she found the arithmetic means in decimal writing: 5.5, 7.0, 8.5, 10, 11.5, 13, 14.5, 16, 17.5, 19 and 20.5.

With the previous information [reading about the long jump], find five arithmetic means between 4 and 22.

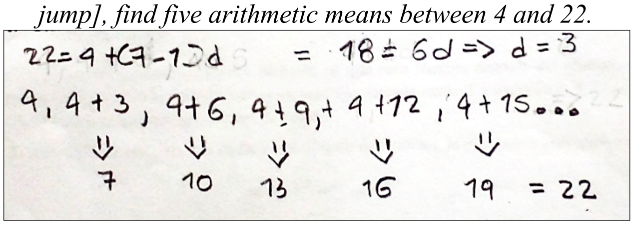


Figure 3: Nestor's response (Session 3)

If d were halved, that is, d=1.5, or d=3/2, what would be the new arithmetic means between 4 and 22?

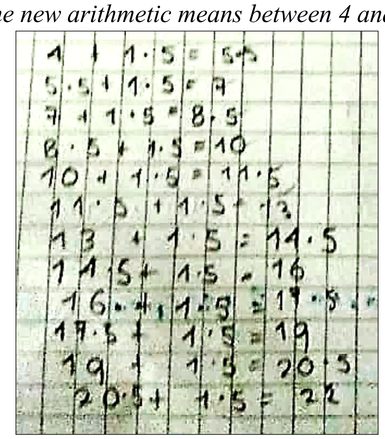
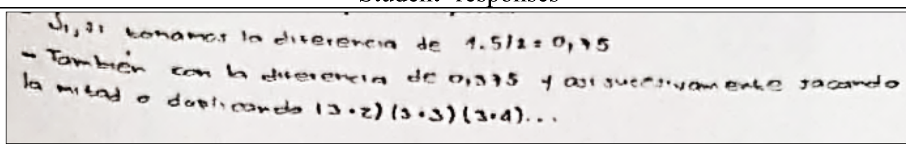
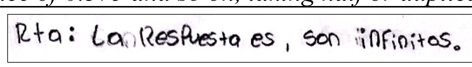
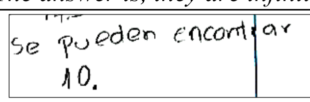
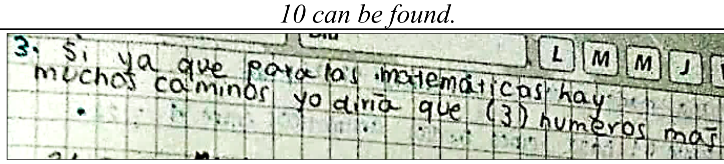


Figure 4: Angie's response (Session 3)

Subsequently, the students answered the question: “Can you find more numbers between 4 and 22 (different from the numbers found in the previous questions)? How many? Explain your answer”. In Table 4 the answers of the students are shown; we tried to classify them according to the categories by Vamvakoussi and Vosniadou (2004) (Table 1).

Paola mentions that with half the difference (*1.5*), more terms can be found (*1.5/2 = 0.75*); likewise, with half of 0.75 which is 0.375, and so on. So her answer is in the category *Advanced thinking about density*, since she justifies a process to find more numbers. It is worth mentioning that Paola not only indicates that with the process of finding half the difference, more numbers can be found, but also through “doubling”. However, it seems that she has an inappropriate concept of the term doubling, since she writes *3.2, 3.3, 3.4*,..., that is, she refers to adding two tenths, three tenths, four tenths, etc. to the number *3* (the difference of the arithmetic progression). Néstor exhibits a *Naive thinking about density* when he affirms that an infinity of numbers can be found. Violeta answers that “10 can be found”, that is, a finite quantity of numbers between 4 and 22, which we consider is *Advanced thinking about the discrete*. Also Angie shows *Advanced thinking about the discrete* in this question, since she indicates a finite quantity: 3. It is not clear what Angie means by “for mathematics there are many paths”. It is possible that she refers to the fact that there are several solutions to solve a problem and that the method she chose (which we do not know) leads her to say 3.

Table 4: Responses of the four students (Session 3)

Can you find more numbers between 4 and 22 (different from the numbers found in the previous questions)? How many? Explain your answer	
Student’ responses	Types of thinking
 Si, si tomamos la diferencia de 1.5/2 = 0,75 - También con la diferencia de 0,375 y así sucesivamente sacando la mitad o duplicando (3.2) (3.3) (3.4)... *-Yes, if we take the difference of 1.5/2 = 0.75* *-Also with the difference of 0.375 and so on, taking half or duplicating (3.2) (3.3) (3.4)...*	*Paola* *Advanced thinking about density*
 Rta: La Respuesta es, son infinitos. *The answer is, they are infinite.*	*Néstor* *Naive thinking about density*
 se pueden encontrar 10. *10 can be found.*	*Violeta* *Naive thinking about the discrete*
 3. Si ya que para las matemáticas hay muchos caminos yo diría que (3) numeros mas *Yes, since for mathematics there are many paths I would say (3) more numbers*	*Angie* *Naive thinking about the discrete*

Session 4. In this session, an activity was designed based on one proposed by Tovar (2011). The hypothesis is that a student can understand the density of irrationals in the real numbers, from the continuity property of the number line. Students constructed a square of side length 1 on the number line and rotated the diagonal of the square on the line clockwise (see the segment $\overline{AX}$ in Figure 5; Violet’s example).

The students zoomed in several times with GeoGebra and noticed that the “right end” of the diagonal segment did not coincide with any rational number on the line, thus showing that an irrational number (in this case $\sqrt{2}$) can be found in a given interval. Subsequently, they wrote four intervals that enclosed the point that did not coincide with any rational. Figure 6 shows the task performed by Violeta after completing the construction of the square with side 1 (Figure 5). Violeta writes four intervals that contain point X. It is possible that she has forgotten to write the decimal point (used in Colombia). The intervals are: (1.414, 1.4144), (1.4136, 1.4146), (1.4132, 1.4148), and (1.413, 1.415).

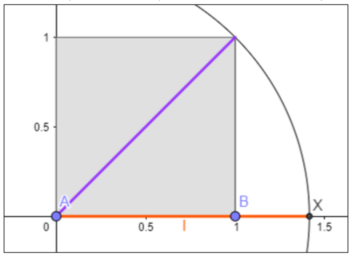


Figure 5: Violet's graph in GeoGebra (S4)

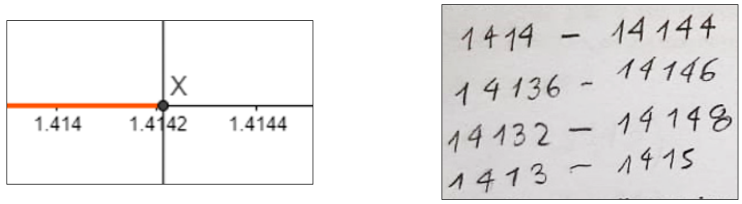
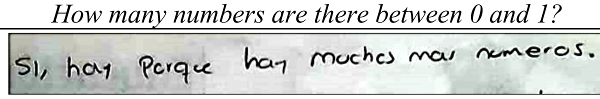


Figure 6: Graph in GeoGebra and intervals written by Violeta that enclose point X (Session 4)

Third phase: Post-test results

The students showed an evolution in their responses in the post-test, as compared to the answers in the pre-test. For example, Violeta wrote: "Yes there are because there are many more numbers" in her answer to the question: "How many numbers are there between 0 and 1?" (Figure 7). Contrary to what Violeta wrote in her pre-test to this same question: "There is no number because 0 is before 1" (Table 3). We consider that Violeta has been able to understand that there are numbers between 0 and 1 Although she indicates that "there are many more numbers" we do not know if she refers to a finite or infinite quantity of numbers. In the cases of Paola and Néstor, they preserved the same idea that there are infinite numbers in a given interval, both in the pre-test and in the post-test. However, during the pre-test, Paola did not justify how to find intermediate numbers in an interval, while in the post-test she was already able to justify a method to find these numbers. Angie also mentioned the existence of an infinite number of intermediate numbers in an interval during the development of the pre-test; however, during the post-test she talked about an infinite number of "intervals" and did not justify the reason for her answer.

How many numbers are there between 0 and 1?



Yes, there is because there are many more numbers.

Figure 7: Violeta's response to the post-test

Conclusions

The study sought ways to promote in high-school students procedures to find intermediate numbers in an interval that implied learning and understanding of numerical density in various activities. These HLT activities included school mathematics topics such as arithmetic and geometric progressions, the construction of a square with side length 1, similarity of triangles, as first approaches to the density property of decimal numbers.

We observed that the students did not seem to interpret "successor" correctly. Apparently, the idea of a successor for the students was a number greater than the one given, or the one that follows it in a subset of numbers with the same characteristics (same number of decimal places). For this reason, it is necessary to reconsider how the questions were posed, related to the existence of a successor in a set other than those of the natural numbers or of the integers. That is, we must (re)design the activities so that students may realize the non-existence of a successor in the sets of rational numbers (expressed as fractions, and/or as decimal numbers), and of the real numbers.

Acknowledgments

To Conacyt and Cinvestav for financing this research study.

References

Cabarcas, B., y Soler, C. (2017). Aproximación a la propiedad de densidad del conjunto de los números racionales desde las representaciones en el registro como fracción y el registro decimal [Tesis de Maestría, Pontificia Universidad Javeriana]. https://repository.javeriana.edu.co/handle/10554/37869

Cobb, P., & Gravemeijer, K. (2008). Experimenting to support and understand learning processes. In A.E. Kelly, R.A. Lesh y J.Y. Baek (Eds.). *Handbook of design research methods in education. Innovations in Science, Technology, Engineering and Mathematics Learning and Teaching*, (pp. 68-95). Lawrence Erlbaum Associates.

Duval, R. (2004). *Semiosis y Pensamiento Humano. Registros Semióticos y Aprendizajes Intelectuales*. Universidad del Valle.

Duval, R. (2006). Un tema crucial en la educación matemática: La habilidad para cambiar el registro de representación. *La gaceta de la RSME, 9.1*, 143-168.

González-Forte, J. M., Fernández C., Van Hoof, J. y Van Dooren, W. (2021). Formas de responder sobre la densidad de los números racionales de estudiantes de primaria y secundaria. En Diago, P. D., Yáñez D. F., González-Astudillo, M. T. y Carrillo, D. (Eds.), *Investigación en Educación Matemática XXIV* (pp. 319 – 326). Valencia: SEIEM.

Levin, O. (2021). *Discrete Mathematics: An Open Introduction.* (3ª ed.). Independently Published.

Matemática discreta. (2021, 30 de octubre). *Wikipedia, La enciclopedia libre*. https://es.wikipedia.org/w/index.php?title=Matem%C3%A1tica_discreta&oldid=139389762.

McMullen, J., & Van Hoof, J. (2020) The role of rational number density knowledge in mathematical development. *Learning and Instruction*, 65, 101228. https://doi.org/10.1016/j.learninstruc.2019.101228

Moreno, L., & Sacristán, A. (1996). Representaciones y Aprendizaje. *Investigaciones en Matemática Educativa*, 277-289. Grupo Editorial Iberoamérica.

Neumann, R. (1998). Students' ideas on the density of fractions. In H. G. Weigand, A. Peter-Koop, N. Neil, K. Reiss, G. Törner, & B. Wollring (Eds), *Proceedings of the annual meeting of the Gesellschaft fur Didaktik der Mathematik* (pp. 97-104). http://webdoc.sub.gwdg.de/ebook/e/gdm/1998/

Peano, J. (1979). *Los principios de la aritmética. Expuestos según un nuevo método* (Trad. J. Velarde). Pentalfa Ediciones. (Trabajo original publicado en 1889).

Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, 26, 114–145. https://doi.org/10.2307/749205

Suárez-Rodríguez, M., & Figueras, O. (2020). An approach to density in decimal numbers: a study with pre-service teachers. In A.I. Sacristán, J.C. Cortés-Zavala & P.M. Ruiz-Arias, (Eds.). *Mathematics Education Across Cultures: Proc. 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Mexico* (pp. 803-819). Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020-122

Suárez-Rodríguez, M., & Sacristán, A. I. (2022). Different ways of learning number density: a hypothetical trajectory with high school students. In A. E. Lischka, E. B. Dyer, R. S. Jones, J. Lovett, J. Strayer & S. Drown, (Eds). *Proceedings of the 44th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 911-928). Middle Tennessee State University.

Tovar, J. (2011). *Un acercamiento al concepto y completitud de los números reales* [Tesis de maestría, Universidad Nacional de Colombia]. https://repositorio.unal.edu.co/bitstream/handle/unal/10182/johnedinsontovargil.2011.pdf?sequence=1&isAllowed=y

Vamvakoussi, X., & Vosniadou, S. (2004). Understanding the structure of the set of rational numbers: A conceptual change approach. *Learning and Instruction*, *14*, 453-467. doi:10.1016/j.learninstruc.2004.06.013

Vamvakoussi, X. y Vosniadou, S. (2010). How many decimals are there between two fractions? Aspects of secondary school students' understanding of rational numbers and their notation. *Cognition and Instruction*, *28*(2), 181-209. https://doi.org/10.1080/07370001003676603

DIFERENTES PROCEDIMIENTOS PARA HALLAR NÚMEROS EN UN INTERVALO CON ESTUDIANTES DE BACHILLERATO

DIFFERENT PROCEDURES TO FIND NUMBERS IN AN INTERVAL WITH HIGH SCHOOL STUDENTS

Mayra Suárez-Rodríguez
Cinvestav
mayra.suarez@cinvestav.mx

Ana Isabel Sacristán
Cinvestav
asacrist@cinvestav.mx

En este informe mostraremos una investigación de cómo estudiantes de bachillerato aprenden estrategias para hallar números intermedios en un intervalo con el fin de comprender sobre la propiedad de densidad numérica. Investigaciones han mostrado cómo algunos estudiantes que terminan su bachillerato tienen dificultad para comprender sobre esta propiedad. Para mitigar esta dificultad propusimos una Trayectoria Hipotética de Aprendizaje en la que incluyera temas de la matemática escolar que nos permitiera plantear hipótesis de cómo un estudiante puede aprender sobre densidad numérica. La investigación reflejó que algunos estudiantes de bachillerato reconocen que existe una infinidad de números en un intervalo, sin embargo, todos los participantes tuvieron dificultad para comprender por qué no existe un sucesor en un conjunto que no sea el de los números naturales o enteros.

Palabras clave: Trayectorias de aprendizaje y progresiones, representaciones matemáticas, conceptos y operaciones numéricas, experimentos de diseño

Introducción y Preguntas de Investigación

La propiedad de densidad numérica ha sido poco comprendida por estudiantes de todas las etapas escolares (Vamvakoussi y Vosniadou, 2010): algunos creen que no existe un número en un intervalo y otros creen que hay una cantidad finita de números en un intervalo, en conjuntos diferentes a los de los números naturales o enteros. Dado un intervalo, la propiedad de densidad numérica indica que se puede encontrar un número en él, lo que implica que: a) no existe un número que tenga sucesor (inmediato), y b) existe una infinidad de números en un intervalo. Por ejemplo, en la investigación realizada por González-Forte et al. (2021), estudiantes de 5to y 6to de primaria (10-12 años) y de 1° a 4° de secundaria (12-16 años de edad) respondieron que había una cantidad finita de números en el intervalo entre 3.49 y 3.50; incluso, algunos estudiantes mencionaron que solo existía un número en dicho intervalo. También se presentan situaciones donde estudiantes creen que los extremos de un intervalo (no natural, no entero) son consecutivos, por ello creen que no hay otro número en él (Vamvakoussi y Vosniadou, 2010). Por ejemplo, en el estudio realizado por Cabarcas y Soler (2017), estudiantes de noveno grado (alrededor de 15 años de edad) creían que un número racional tenía un antecesor y un sucesor, inmediatos. Ellos creían que el sucesor de 14/4 era 15/4 porque sus numeradores son consecutivos con un mismo numerador.

Por otro lado, en los trabajos de Neumann (1998) y de Vamvakoussi y Vosniadou (2010) se ha encontrado que algunos estudiantes tienden a decir que solo hay números decimales entre decimales, pero no fracciones; de igual manera, que hay fracciones entre fracciones, pero no números decimales. En el estudio de Neumann (1998), los estudiantes tuvieron dificultad para aceptar que entre 0.3 y 0.6 hay fracciones. Vamvakoussi y Vosniadou (2010) concluyeron, entonces, que los estudiantes no solo decidían qué tipo de representación simbólica debían tener los números que pertenecen a un intervalo sino cuántos debía haber. Por ejemplo, en el trabajo de estas autoras, un estudiante manifestaba que podía haber más números entre 0.001 y 0.01 si los números decimales (los intermedios) se expresaban en fracciones y que así podía haber una infinidad. Por ello, Duval (1995/2004) recalca que un estudiante pueda manejar diferentes representaciones de un objeto matemático desde edades tempranas, pues aprender matemáticas implica que haya un uso de diferentes registros semióticos (aritmético, algebraico, geométrico, etc.) de representación (fraccionaria, decimal, etc.).

Con la finalidad de que un estudiante pueda comprender sobre densidad numérica –que implica comprender que no existe un sucesor para un número decimal, racional o real–, diseñamos y pusimos en práctica una Trayectoria Hipotética de Aprendizaje (THA) (Simon, 1995); que sigue los lineamientos de una Investigación Basada en Diseño (IBD) (Cobb y Gravemeijer, 2008). Una THA es una secuencia de actividades que siguen hipótesis de cómo apoyar a un estudiante en su proceso de aprendizaje sobre un concepto matemático (Simon, 1995). Para esta THA tomamos diferentes temas de la matemática escolar que permitieran comprender sobre densidad numérica, tal como lo sugieren McMullen y Van Hoof (2020). Presentamos a continuación objetivos que se propusieron para una investigación que se inició en 2020 y que en Suárez-Rodríguez y Sacristán (2022) ya habíamos mostrado algunos resultados:

a) Indagar sobre las concepciones previas de los estudiantes participantes en el estudio respecto a la propiedad de densidad numérica. Para ello se utiliza un cuestionario diagnóstico (pretest).
b) Indagar y evidenciar las actuaciones de los estudiantes durante el desarrollo de la puesta en marcha de las actividades propuestas en la THA, con relación a la propiedad de densidad numérica y a la propiedad de lo discreto de los números naturales.

Marco Teórico

Pensamiento sobre lo discreto y lo denso

La matemática discreta es la base de todo lo que corresponde con los números naturales o los conjuntos numerables, mientras que la matemática continua es la base de todo lo relacionado con la continuidad (Levin, 2021; “Matemática discreta”, 2021, parfs. 2-3). Como explican Vamvakoussi y Vosniadou (2010), la propiedad de lo discreto solo la cumplen los números naturales, ya que cada natural tiene un sucesor, en términos de relaciones de orden. Es decir, estas autoras hacen énfasis en que los términos “discreto” y “denso” se usan con respeto a la relación de orden habitual, así los números naturales son discretos, los números racionales son densos y los números reales son densos y continuos. La propiedad de lo discreto encierra todo lo vinculado con los números naturales en el sentido de que cualquier número natural tiene un sucesor (Vosniadou y Vamvakoussi, 2010); como lo indicó Peano (1889/1979) en sus axiomas para los números naturales, que estos números tienen un sucesor. Por otro lado, Vamvakoussi y Vosniadou (2004) elaboraron unas categorías con base en los resultados de su investigación para conocer qué tanto saben los estudiantes sobre densidad numérica y la propiedad de lo discreto (Tabla 1).

Tabla 1: Categorías de pensamiento sobre lo discreto y lo denso

Pensamiento ingenuo sobre lo discreto	Se considera que no hay otro número entre dos números racionales *consecutivos falsos* (expresión acuñada por Vamvakoussi y Vosniadou, 2004, para decir que existe un sucesor en los números racionales.)

Pensamiento avanzado sobre lo discreto	Se cree que hay un número finito de números intermedios entre dos números racionales consecutivos falsos.
Pensamiento entre lo discreto y lo denso	En algunas situaciones se piensa que entre dos números racionales hay una cantidad infinita de números, y en otros, que hay una cantidad finita.
Pensamiento ingenuo sobre lo denso	Se contempla que hay una infinidad de números en un intervalo, pero no se justifica la situación usando la propiedad de densidad. La representación simbólica de los extremos de un intervalo influye en la forma de pensar; se cree que solo puede haber una infinidad de decimales entre decimales, pero no fracciones; de igual modo sucede con fracciones entre fracciones.
Pensamiento sofisticado sobre lo denso	Hay una comprensión bastante sofisticada de la propiedad de densidad, es decir, se pone de manifiesto que se entiende que entre dos números racionales hay una infinidad de números, independientemente de su representación simbólica, y se justifica con la propiedad de la densidad.

Registros semióticos de representación

Aprender matemáticas constituye un espacio para el análisis de las actividades cognitivas (conceptualización, razonamiento, resolución de problemas y comprensión de textos) que requieren del uso de sistemas de expresión y de representación distintos a los del lenguaje natural o de las imágenes (Duval, 1995/2004). Estos sistemas de representación conforman *registros semióticos* o *registros de representaciones semióticas* (op. cit.). Estos registros permiten que un individuo pueda tener una comunicación de ideas que se transforman en otras sin cambiar su significado (Moreno y Sacristán, 1996). No obstante, Duval (2006) recomienda no solo limitarse a registros semióticos, sino también utilizar registros no semióticos como contextos en los que se pueda trabajar con materiales como cerrillas, palitos, cuencas, etc. Consideramos, entonces, la importancia de usar varios registros de representación para apoyar el aprendizaje y comprensión de la propiedad de densidad numérica.

Marco Metodológico y Metodología

Trayectoria Hipotética de Aprendizaje (THA)

La expresión "trayectoria hipotética de aprendizaje" se refiere a una predicción que hace un investigador en cuanto a la ruta por la cual el aprendizaje podría proceder (Simon, 1995). Una trayectoria es hipotética porque la trayectoria de aprendizaje "real" aún no se conoce de antemano, y se vuelve real cuando los estudiantes ponen en marcha las actividades propuestas (op. cit.). Simon (1995) elabora tres componentes que debe conformar una THA: a) *objetivos de aprendizaje*, entendidos como un conjunto de propósitos que se espera lograr para que el estudiante construya nuevos conocimientos, b) *actividades de aprendizaje*, que se conforma de actividades instruccionales siguiendo una secuencia, y c) *proceso hipotético de aprendizaje*, en el que se elaboran conjeturas para apoyar el proceso de aprendizaje del estudiante. El investigador puede realizar modificaciones en cualquiera o en todos los tres componentes de una trayectoria con el fin de que él como los estudiantes construyan y mejoren sus conocimientos (op. cit.).

Diseño y elaboración de las actividades de la THA

Como lo mencionamos, nuestra metodología de investigación sigue las pautas de una Investigación Basada en Diseño (IBD). La IBD es un enfoque metodológico en el que el investigador lleva a cabo un estudio sistemático de diseño, desarrollo y evaluación de intervenciones educativas en diferentes ciclos, y para ello se sugiere planteamientos de hipótesis a través de una THA (Cobb y Gravemeijer, 2008). En este informe mostraremos algunos resultados de uno de dos ciclos de actividades de la investigación (el otro se está llevando a cabo durante el año 2023). Para este primer ciclo participaron cuatro estudiantes con edades de entre

15 y 17 años, de Colombia: Angie, Paola y Néstor que cursaban el bachillerato y Violeta, el último año de la secundaria. Algunas actividades se llevaron a cabo de manera presencial y otras virtual. El diseño correspondiente al primer ciclo de actividades implicó tres fases: Los estudiantes respondieron un mismo cuestionario tanto en la primera como en la tercera fase, y en la segunda, resolvieron actividades de la THA.

Primera y tercera fase (pretest y postest). Los estudiantes respondieron a un cuestionario de cuatro preguntas relacionadas con la cantidad de números que puede tener un intervalo. Estas preguntas surgieron de las elaboradas por Suárez-Rodríguez y Figueras (2020) y Vamvakoussi y Vosniadou (2004). Se quiso conocer qué tanto saben los estudiantes sobre densidad numérica – antes y después de la implementación de la THA– de acuerdo con las categorías elaboradas por Vamvakousi y Vosniadou (2004) (Tabla 1). La duración del desarrollo del pretest fue de 30 minutos aproximadamente, mientras que la del postest fue de 10 minutos. En este informe mostraremos las respuestas de la pregunta: *¿Cuántos números hay entre 0 y 1?*

Segunda fase (actividades de la THA). La THA está compuesta por cuatro sesiones de actividades en las que se planteó una hipótesis para cada una (Tabla 2). Se quiso plantear hipótesis vinculadas con varios temas de la matemática escolar en las que consideramos que pueden apoyar al estudiante para aprender y comprender sobre densidad numérica. Cada sesión tuvo una duración de alrededor de 1h30'.

Tabla 2: Sesiones de la THA con sus hipótesis correspondientes

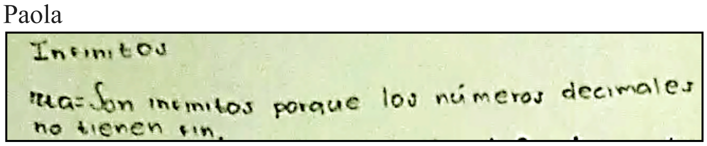
Sesión	*Hipótesis*
Sesión 1. Primeros acercamientos a la propiedad de densidad.	A partir de dos situaciones: una relacionada con la cotidianidad y otra vinculada con un escenario hipotético, se piensa que el estudiante puede tener sus primeros acercamientos a la propiedad de densidad.
Sesión 2. Acercamiento a la propiedad de densidad a través de la semejanza de triángulos.	Se contempla que usando semejanza de triángulos los estudiantes puedan aprender sobre la propiedad de densidad de los números racionales.
Sesión 3. Aproximación a la propiedad de densidad a partir de progresiones aritméticas y progresiones geométricas.	Es posible que hallando medios aritméticos y geométricos en un intervalo el estudiante comprenda la propiedad de densidad de los números racionales en el conjunto de los reales.
Sesión 4. Aproximación a la propiedad de densidad por medio de la propiedad de continuidad.	Se cree que usando la propiedad de continuidad los estudiantes comprenden la propiedad de densidad de los números irracionales en el conjunto de los reales.

Resultados

Primera fase: Resultados del pretest

En la Tabla 3 se muestran las respuestas de los estudiantes a la pregunta: *¿Cuántos números hay entre 0 y 1?* e intentamos clasificarlas según las categorías propuestas por Vamvakoussi y Vosniadou (2004) (Tabla 1).

Tabla 3: Respuestas de los estudiantes a una pregunta del pretest

Respuestas	*Tipos de pensamiento*
Paola 	*Pensamiento ingenuo sobre lo denso*

Néstor 1. ¿Cuántos números hay entre 0 y 1? Justifica tu respuesta. Rta: Para mi, los números que hay son infinitos. Porque sigue el mismo ciclo; y Porqué no tienen un final. los : Ejemplo. 0.111111.... 0.111112.... 0.111113....	*Pensamiento sofisticado sobre lo denso*
Violeta 1. ¿Cuántos números hay entre 0 y 1? Justifica tu respuesta. No hay Ningun numero Porque el 0 esta antes del 1	*Pensamiento ingenuo sobre lo discreto*
Angie hay infinitos, debido que existe el 0.1, 0.2, 03(...).	*Pensamiento ingenuo sobre lo denso*

Se observa en la Tabla 3 que Paola y Angie muestran una tendencia a tener un pensamiento ingenuo sobre lo denso, ya que ambas manifestaron que sí hay una infinidad de números entre 0 y 1. Sin embargo, ellas no explicaron un proceso para hallar estos números intermedios en el intervalo. Néstor también indicó que hay infinidad de números entre 0 y 1. Él usó la escritura decimal como representación semiótica en un registro aritmético y anotó ejemplos de números en escritura decimal periódica. Es decir, Néstor justificó su respuesta por medio de un proceso de infinito potencial de agregar cifras decimales en la parte decimal de un número de forma periódica como una manera de hallar números intermedios entre 0 y 1. Violeta manifestó un *pensamiento ingenuo sobre lo discreto*: al parecer, para ella, los números 0 y 1 son consecutivos. Es posible, que ella solo esté considerando 0 y 1 como números naturales, por lo que ella se encontraría en un sistema de conocimientos de dominio número natural.

Segunda fase: Resultados de las actividades de la THA

Se describen algunas actuaciones de los estudiantes de las sesiones 1, 3 y 4 (Tabla 2) durante el desarrollo de la THA.

Sesión 1. En esta sesión los estudiantes resolvieron dos actividades. En una de ellas los estudiantes realizaron una breve lectura sobre los primeros inicios del salto largo como deporte y respondieron cinco preguntas. En la Figura 1 mostramos la respuesta de Paola a la pregunta: *Según la lectura, ¿cuáles registros, de los atletas, se encuentran entre 5.40 y 7.70?* Ella responde de manera adecuada, señaló todos los registros numéricos, de los atletas, que se encuentran entre 5.40 y 7.70 (5.41, 6.20, 6.375, 7.05, 7.40, 7.51, 7.54, 7,60 y 7.605). Posteriormente, los estudiantes contestaron la pregunta: *¿Crees que 7.40 tenga sucesor? Explica tu respuesta.* En la Figura 2 se muestran las respuestas de ellos.

Según la lectura, ¿cuáles registros, de los atletas, se encuentran entre 5.40 y 7.70

5,41 - 6,20 - 6.375 - 7,05 - 7,40 - 7,51 - 7.54 - 7,60 - 7,605

Rta: Todos los registros se encuentran entre 5.40 y 7.70.

Figura 1: Respuesta de Paola (Sesión 1)

¿Crees que 7.40 tenga sucesor? Explica tu respuesta.

Paola

Rta si, porque le sigue 7,41

Violeta

Si, Porque le siguen mas numeros en este caso seria 7.41

Angie

Si, es un número decimal el sucesor es 7.41, los numeros decimales, naturales o negativos o positivos son infinitos.

Néstor

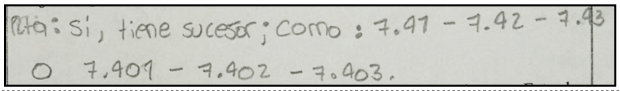


Figura 2: Respuestas de los cuatro estudiantes (Sesión 1)

Se observa en la Figura 2 que Paola, Violeta y Angie señalaron a 7.41 como el “sucesor” de 7.40. A diferencia de Néstor, que no solo escribe 7.41, sino también otros números como 7.42 y 7.43 (números con dos cifras decimales) y números como 7.401, 7.402 y 7.403 (números con tres cifras decimales) como “sucesores” de 7.40. En general, consideramos que los estudiantes creen que existe un sucesor para un número decimal que depende de la cantidad de cifras decimales (i.e. “el sucesor” de un número con una cifra decimal es otro que debe tener una cifra decimal. Sin embargo, no sabemos que podría ocurrir con un número como 7.99 (número con dos cifras decimales), creemos que los estudiantes podrían escribir 7.100 (número con tres cifras decimales).

Sesión 3. En esta sesión se llevó a cabo una actividad relacionada con progresiones aritméticas y otra con progresiones geométricas. Se planteó la hipótesis de que el estudiante pueda ver otra forma de hallar números en un intervalo, con el fin de que comprendiera la propiedad de densidad de los números racionales. Se les pidió a los estudiantes hallar cinco medios aritméticos entre 4 y 22 a partir de la expresión general del término n-ésimo de la sucesión: $u = a + (n - 1)d$, donde a es el primer término, n el número de términos y d la diferencia entre un término y el siguiente. En la Figura 3 se observa la tarea realizada por Néstor donde halló los cinco medios aritméticos entre 4 y 22 (7, 10, 13, 16 y 19) con $d = 3$. Después los estudiantes hallaron otros medios aritméticos cuando se les solicitó reducir la diferencia a la mitad (i.e. $d = 1.5 = 3/2$). En la Figura 4 se observa el trabajo realizado por Angie, donde halló los medios aritméticos en escritura decimal: 5.5, 7.0, 8.5, 10, 11.5, 13, 14.5, 16, 17.5, 19 y 20.5.

Con la información anterior [lectura sobre el salto largo] halla cinco medios aritméticos entre 4 y 22.

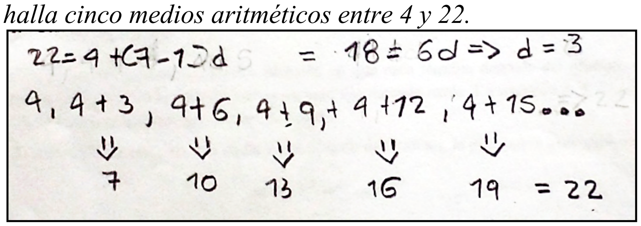


Figura 3: Respuesta de Néstor (Sesión 3)

Si d se redujera a la mitad, es decir, d=1.5, o d=3/2, ¿cuáles serían los nuevos medios aritméticos entre 4 y 22?

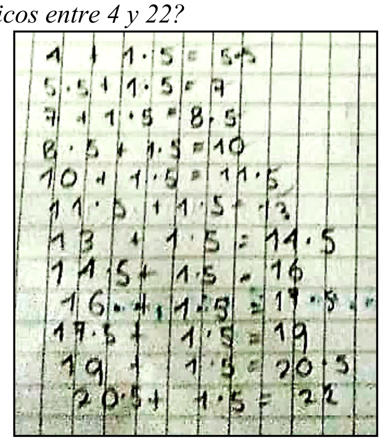


Figura 4: Respuesta de Angie (Sesión 3)

Posteriormente, los estudiantes respondieron a la pregunta: *¿Puedes hallar más números entre 4 y 22 (diferentes a los números hallados en las anteriores preguntas)? ¿Cuántos? Explica tu respuesta*. En la Tabla 4 se muestran las respuestas de los estudiantes y también intentamos clasificarlas según las categorías por Vamvakoussi y Vosniadou (2004) (Tabla 1).

Paola menciona que con la mitad de la diferencia (1.5) se pueden hallar más términos (1.5/2 = 0.75); de igual manera, con la mitad de 0.75 que es 0.375, y así de manera sucesiva. Por lo que su respuesta se encuentra en la categoría *pensamiento sofisticado sobre lo denso*, ya que justifica un proceso para hallar más números. Cabe mencionar que Paola no solo indica que con el proceso de encontrar la mitad de la diferencia se puede hallar más números sino también "duplicando". Sin embargo, al parecer, ella tiene un concepto no adecuado del término duplicación, ya que escribe 3.2, 3.3, 3.4, …, es decir, ella se refiere a sumar dos décimas, tres décimas, cuatro décimas, etc. al número 3 (que es la diferencia de la progresión aritmética). Néstor exhibe un *pensamiento ingenuo sobre lo denso* cuando afirma que se puede hallar una infinidad de números. Violeta responde que *"se pueden encontrar 10"*, es decir, una cantidad finita de números entre 4 y 22, por lo que exhibe un *pensamiento avanzado sobre lo discreto*. En esta pregunta, creemos que Angie muestra un *pensamiento avanzado sobre lo discreto*, ya que indica una cantidad finita: 3. No está claro lo que Angie quiere decir con "para las matemáticas hay muchos caminos". Es posible que ella se refiera a que hay varias soluciones de resolver un problema y que el método que ella escogió (que desconocemos) le lleva a decir 3.

Tabla 4: Respuestas de los cuatro estudiantes (Sesión 3)

¿Puedes hallar más números entre 4 y 22 (diferentes a los números hallados en las anteriores preguntas)? ¿Cuántos? Explica tu respuesta	
Respuestas de los estudiantes	Tipos de pensamiento
Si, si tomamos la diferencia de 1.5/2 = 0,75 - También con la diferencia de 0,375 y así sucesivamente sacando la mitad o duplicando (3.2) (3.3) (3.4)...	*Paola* Pensamiento sofisticado sobre lo denso.
Rta: La Respuesta es, son infinitos.	*Néstor* Pensamiento ingenuo sobre lo denso.
se pueden encontrar 10.	*Violeta* Pensamiento avanzado sobre lo discreto.
3. Si ya que para las matemáticas hay muchos caminos yo diría que (3) numeros más	*Angie* Pensamiento avanzado sobre lo discreto

Sesión 4. Para esta sesión se diseñó una actividad con base en una planteada por Tovar (2011). Se plantea la hipótesis de que el estudiante pueda comprender sobre la densidad de los irracionales en los reales a partir de la propiedad de continuidad de la recta numérica. Los estudiantes construyeron un cuadrado de lado de longitud uno sobre la recta numérica desde el origen, y rotaron la diagonal del cuadrado sobre la recta en el sentido de las manecillas del reloj (ver el segmento $\overline{AX}$ en la Figura 5; ejemplo de Violeta).

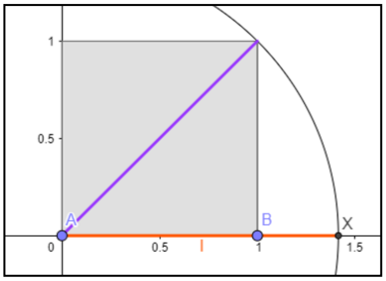


Figura 5: Gráfica de Violeta en GeoGebra (S4)

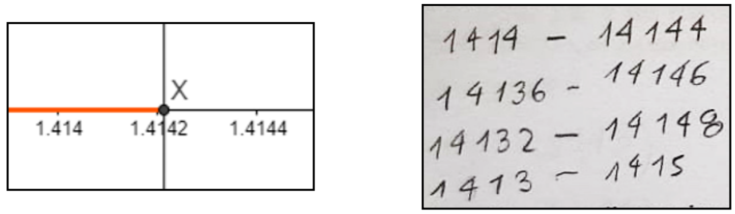


Figura 6: Gráfica en GeoGebra e intervalos escritos por Violeta que encierran al punto X (Sesión 4)

Al realizar varios zooms en GeoGebra, los estudiantes notaron que el “extremo derecho” del segmento de la diagonal no coincidía con algún número racional de la recta, mostrando así, que se puede encontrar un número irracional (en este caso $\sqrt{2}$) en un intervalo dado. Posteriormente, ellos escribieron cuatro intervalos que encerraran al punto que no coincidía con algún racional. En la Figura 6 se observa la tarea realizada por Violeta después de realizar la construcción del cuadrado de lado uno (Figura 5). Violeta escribe cuatro intervalos que contienen al punto X. Es posible que a ella se le haya olvidado escribir la coma decimal (que se usa en Colombia). Los intervalos son: (1.414, 1.4144), (1.4136, 1.4146), (1.4132, 1.4148) y (1.413, 1.415).

Tercera fase: Resultados del postest

Los estudiantes evidenciaron una evolución en sus respuestas al postest en comparación con las del pretest. Por ejemplo, Violeta escribió: “Sí hay porque hay muchos más números” en su respuesta a la pregunta: *¿Cuántos números hay entre 0 y 1?* (Figura 7). Contrario a lo que Violeta escribió en su pretest en esta misma pregunta, que indicó: “No hay ningún número porque el 0 está antes del 1” (Tabla 3), consideramos que Violeta ha podido comprender que hay números entre 0 y 1. Aunque ella indica que “hay muchos más números” no sabemos si se refiere a una cantidad finita o infinita de números. En los casos de Paola y Néstor, ellos preservaron la misma idea de que hay infinidad de números en un intervalo dado, tanto en el desarrollo del pretest como del postest. Sin embargo, durante el pretest Paola no justificó como encontrar números intermedios en un intervalo, mientras que en el postest ya pudo justificar un método para encontrar estos números. Angie también mencionó la existencia de una infinidad de números intermedios en un intervalo durante el desarrollo del pretest; sin embargo, durante el postest habló sobre infinidad de “intervalos” y no justificó el por qué de su respuesta.

¿Cuántos números hay entre 0 y 1?

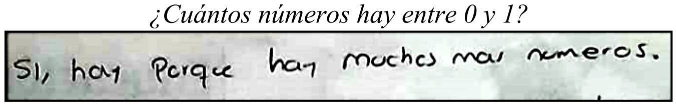


Figura 7: Respuesta de Violeta al postest

Conclusiones

Consideramos que el trabajo de investigación buscó maneras de promover, en estudiantes de bachillerato, procedimientos para hallar números intermedios en un intervalo que implicó en varias actividades un aprendizaje y una comprensión hacia la densidad numérica. Estas actividades de la THA incluyeron temas de la matemática escolar como progresiones aritméticas y geométricas, la construcción de un cuadrado de lado de longitud uno, semejanza de triángulos y primeros acercamientos a la propiedad de densidad de los números decimales.

Observamos que los estudiantes no parecían interpretar “sucesor” correctamente. Al parecer, la idea de un sucesor para los estudiantes era un número mayor al dado, o aquel que le sigue en un subconjunto de números con mismas características (misma cantidad de cifras decimales). Por ello, es necesario reconsiderar cómo plantear las preguntas relacionadas con la existencia de un sucesor en un conjunto que no sea el de los números naturales o el de los enteros. Es decir, debemos (re)diseñar las actividades que permita que los estudiantes se percaten de la inexistencia de un sucesor en los conjuntos de los números racionales (expresados como fracciones, y/o como números decimales), y reales.

Agradecimientos

A Conacyt y a Cinvestav por el financiamiento de este trabajo de investigación.

Referencias

Cabarcas, B., y Soler, C. (2017). Aproximación a la propiedad de densidad del conjunto de los números racionales desde las representaciones en el registro como fracción y el registro decimal [Tesis de Maestría, Pontificia Universidad Javeriana]. https://repository.javeriana.edu.co/handle/10554/37869

Cobb, P., & Gravemeijer, K. (2008). Experimenting to support and understand learning processes. In A.E. Kelly, R.A. Lesh y J.Y. Baek (Eds.). *Handbook of design research methods in education. Innovations in Science, Technology, Engineering and Mathematics Learning and Teaching*, (pp. 68-95). Lawrence Erlbaum Associates.

Duval, R. (2004). *Semiosis y Pensamiento Humano. Registros Semióticos y Aprendizajes Intelectuales*. Universidad del Valle.

Duval, R. (2006). Un tema crucial en la educación matemática: La habilidad para cambiar el registro de representación. *La gaceta de la RSME, 9.1*, 143-168.

González-Forte, J. M., Fernández C., Van Hoof, J. y Van Dooren, W. (2021). Formas de responder sobre la densidad de los números racionales de estudiantes de primaria y secundaria. En Diago, P. D., Yáñez D. F., González-Astudillo, M. T. y Carrillo, D. (Eds.), *Investigación en Educación Matemática XXIV* (pp. 319 – 326). Valencia: SEIEM.

Levin, O. (2021). *Discrete Mathematics: An Open Introduction.* (3ª ed.). Independently Published.

Matemática discreta. (2021, 30 de octubre). *Wikipedia, La enciclopedia libre*. https://es.wikipedia.org/w/index.php?title=Matem%C3%A1tica_discreta&oldid=139389762.

McMullen, J., & Van Hoof, J. (2020) The role of rational number density knowledge in mathematical development. *Learning and Instruction*, 65, 101228. https://doi.org/10.1016/j.learninstruc.2019.101228

Moreno, L., & Sacristán, A. (1996). Representaciones y Aprendizaje. *Investigaciones en Matemática Educativa*, 277-289. Grupo Editorial Iberoamérica.

Neumann, R. (1998). Students’ ideas on the density of fractions. In H. G. Weigand, A. Peter-Koop, N. Neil, K. Reiss, G. Törner, & B. Wollring (Eds), *Proceedings of the annual meeting of the Gesellschaft fur Didaktik der Mathematik* (pp. 97-104). http://webdoc.sub.gwdg.de/ebook/e/gdm/1998/

Peano, J. (1979). *Los principios de la aritmética. Expuestos según un nuevo método* (Trad. J. Velarde). Pentalfa Ediciones. (Trabajo original publicado en 1889).

Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, 26, 114–145. https://doi.org/10.2307/749205

Suárez-Rodríguez, M., & Figueras, O. (2020). An approach to density in decimal numbers: a study with pre-service teachers. In A.I. Sacristán, J.C. Cortés-Zavala & P.M. Ruiz-Arias, (Eds.). *Mathematics Education Across Cultures: Proc. 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Mexico* (pp. 803-819). Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020-122

Suárez-Rodríguez, M., & Sacristán, A. I. (2022). Different ways of learning number density: a hypothetical trajectory with high school students. In A. E. Lischka, E. B. Dyer, R. S. Jones, J. Lovett, J. Strayer & S. Drown, (Eds). *Proceedings of the 44th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 911-928). Middle Tennessee State University.

Tovar, J. (2011). *Un acercamiento al concepto y completitud de los números reales* [Tesis de maestría, Universidad Nacional de Colombia].

https://repositorio.unal.edu.co/bitstream/handle/unal/10182/johnedinsontovargil.2011.pdf?sequence=1&isAllowed=y
Vamvakoussi, X., & Vosniadou, S. (2004). Understanding the structure of the set of rational numbers: A conceptual change approach. *Learning and Instruction*, *14*, 453-467. doi:10.1016/j.learninstruc.2004.06.013
Vamvakoussi, X. y Vosniadou, S. (2010). How many decimals are there between two fractions? Aspects of secondary school students' understanding of rational numbers and their notation. *Cognition and Instruction*, *28*(2), 181-209. https://doi.org/10.1080/07370001003676603

CENTERING STUDENTS' ASSETS IN EARLY ELEMENTARY MATHEMATICS: TEACHERS' BELIEFS ABOUT MATHEMATICS, LANGUAGE, AND EMERGENT BILINGUALS

Brittany Caldwell
Vanderbilt University
Brittany.Caldwell@Vanderbilt.Edu

This study explored early grades teachers' professed beliefs about mathematics, language, and Emergent Bilinguals (EBs). The research question was: what are early grades teachers' professed beliefs about mathematics, language, student thinking, students' out-of-school experiences, and students' home and everyday language practices, in particular for EBs? The teachers in this study displayed varying degrees of asset-based responses (74%-100%) to the survey and discussed beliefs related to 1) students' backgrounds and experiences, 2) students' everyday and home languages, 3) mathematics vocabulary, and 4) supporting EBs. During interviews, teachers described their beliefs about students' assets (experiences and home/everyday language) in ways that aligned with 1) allowing students' assets in the classroom or 2) drawing on students' assets to support mathematics learning.

Keywords: Teacher Beliefs, Equity, Inclusion and Diversity, Elementary School Education

Many factors impact how teachers orchestrate their mathematics instruction. One factor that has been the focus of many studies in mathematics education is teachers' beliefs (e.g., Lee & Ginsburg, 2007; Raymond, 1997; Roesken-Winter, 2013; Staub & Stern, 2002; Stipek et al., 2001). Beliefs can impact teaching practice (Schoenfeld, 2002; Raymond, 1997), shaping the ways teachers facilitate mathematics instruction in early grades.

Teachers' beliefs about language and language learning can impact EBs' opportunities to participate in mathematics instruction. There are beliefs that may be unproductive for supporting the needs of EBs in the mathematics classroom. For example, a belief that mathematics should be taught by an English as a Second Language/ English Language Development (ESL/ELD) teacher may reflect a view that children need to learn English before they can participate in mathematics. This could also lead to placing a student in an ESL/ELD class with a teacher unprepared to teach mathematics. There is specific content and pedagogical content knowledge related to teaching mathematics. These may not be present from teachers with little training in mathematics and in a classroom where the main and often the only focus is teaching English.

Fernandes (2020) highlights a need for more research to look at teachers' beliefs about language and mathematics instruction with EBs. Given the need, I analyzed early grades teachers' beliefs about mathematics, language, and students' assets. I explored the research question: what are early grades teachers' professed beliefs about mathematics, language, student thinking, students' out-of-school experiences, and students' home and everyday language practices, in particular for EBs?

Literature Review

Teachers' beliefs are shaped by the setting and context where they teach. Lee and Ginsburg (2007) found that teachers in middle-socioeconomic status (SES) preschools were more likely to support activities relevant to the students' interests and were more focused on the social aspect rather than the academic aspect, of preschool. In these schools, teachers viewed their students as coming from homes with educational resources that prepared them for school. In comparison, teachers serving low-SES preschools were more likely to highlight the importance of academics and direct instruction in preschool. Aligned with deficit views, the teachers from low-SES preschools positioned their students as coming from disadvantaged homes and needing to catch up. This distinction is crucial for EBs in poor schools as they may not have opportunities to draw on their full set of resources to learn mathematics if their teachers hold deficit views of them. Teachers' deficit views of students can limit students' access to quality mathematics instruction and opportunities to learn mathematics with understanding (Lee & Ginsburg, 2007; Turner et al., 2012). Instead, teachers need to hold asset-based views of their students, including EBs, to fully support their mathematics learning. Research has highlighted the importance of 1) drawing on EBs' experiences and backgrounds to provide access to content (e.g. Aguirre et al., 2012; Turner et al., 2012), 2) leveraging students' home and everyday language to support student learning (e.g., Brenner, 1998; de Araju et al., 2018; Turner & Celedón-Pattichis, 2011; Moschkovich, 2013), and 3) teaching vocabulary as connected to concepts and not isolated (Moschkovich, 2013, Moschkovich, 2015a). When teachers hold beliefs that align with these recommendations, they have asset-based views of their students and may become more likely to provide opportunities for mathematics and language learning in the classroom.

Fernandes (2020) found that pre-service teachers' beliefs about the use of home languages in the classroom fell into one of four categories along a continuum that reflect beliefs about native language use in mathematics classrooms: no native language, limited use of native language, extensive use of native language, and bilingualism. In his study, two out of 31 participants were characterized as having beliefs related to no native language use in the classroom, whereas nearly half (n=15) were characterized as limited native language use (Fernandes, 2020). These two categories were highlighted to show an expansion of previous work (Ruiz, 1984) highlighting teachers' beliefs that focus on a deficit view (i.e., language as a problem) of students' native language. In contrast, participants who identified language as a resource shared beliefs related to extensive use of native language in the classroom (n=10) and bilingual use in the classroom (n=3) (Fernandes, 2020). The teachers' beliefs that were documented in this study were complex and included mixed views of students' native language use in the mathematics classroom.

Framework

In this study I looked at beliefs through the lens of Multiple Mathematical Knowledge Bases (MMKB) (Aguirre et al., 2012; Turner et al., 2012), and the language orientations construct (Fernandes, 2020). I drew on Philipp (2007) to define beliefs as "psychologically held understandings, premises, or propositions about the world that are thought to be true [...] as dispositions toward action" (p. 259).

The primary purpose of this study was to look at teachers' beliefs about language and mathematics for EBs. Therefore, I used the language orientations construct (Fernandes, 2020) with the associated MEELS instrument (Fernandes & McLeman, 2012) to design the data collection protocols and to look at the data. For the study's design, I adapted MEELS to develop a survey and inform the interview questions that I used for data collection. I used MEELS to include items related to native language use in the mathematics classroom, fairness of supporting EBS, and teaching strategies for EBs. The language orientations framework (Fernandes, 2020) was the primary lens I used to look at teachers' responses. This framework includes four orientations that range from viewing language as a problem to viewing language as a resource.

On the farthest side of the continuum where language is viewed as a problem, Fernandes (2020) highlighted pre-service teachers' orientations that reflected "no native language." In this group, the teachers felt there was no room for native language in a mathematics classroom. Next, pre-service teachers with a "limited use of native language" believed that it was acceptable for students to use their native language in some situations, but the goal was to replace their language with English in the mathematics classroom. Moving towards views of language as a resource, Fernandes (2020) found that some pre-service teachers held beliefs that reflected "extensive use of native language" where teachers supported any language use in the classroom and the goal was to learn mathematics regardless of language. The final group had beliefs that reflected and promoted "bilingualism," where the pre-service teacher promoted native language use because they believed that native languages support mathematics learning. While Fernandes (2020) only included native language in these constructs, I expanded this work to include everyday communication. This is because many researchers (e.g., Au & Kawakami, 1985; Brenner, 1998; de Araujo et al., 2018; Gutiérrez, Baquedano-López & Tejeda, 1999; Turner & Celedón-Pattichis, 2011) have found that inviting all of students' linguistic resources into the classroom, including native language and language practices (e.g., talk story, joking, storytelling, dialects), supports engagement and learning.

In addition to teachers' beliefs about language and mathematics, I also included opportunities for teachers to share their thoughts about students' experiences and interests related to mathematics outside of the classroom. MMKB includes students' "multiple understandings and experiences that have the potential to shape and support students' mathematics learning" (Turner et al., 2016, p. 49). One key feature of MMKB is students' interests and experiences outside of the classrooms, so I intentionally asked questions and included statements about this in my data collection protocols.

Methodology

This is a qualitative study where I used the frameworks of MMKB (Aguirre et al., 2012; Turner, et al., 2012) and language orientations (Fernandes, 2020) to design the study and analyze the data. Data for this study came from survey and interview responses from a group of experienced early grades teachers. My analysis of these responses drew on descriptive statistics and interpretive analysis. I focused on the teachers' words and their responses to statements and questions. One guiding assumption I have about the teachers in this study is that these teachers are professionals who navigate many things that can impact their instruction that may or may not align with their beliefs about mathematics instruction with EBs. I intentionally did not count or run statistical analyses on the data because of the small sample size (n=20) and because I purposefully selected these teachers in a fashion that made them not representative of the larger teaching force (e.g., in terms of years of teaching, experiences with professional development).

Participants

I recruited participants using convenience sampling from sources including mutual colleagues, individuals I knew, and a well-established social media platform. During recruitment, I identified specific criteria for eligibility. This included teaching for at least 5 years in an early grades classroom (Pre-K through 3rd grade) and current placement in an early grades classroom. I also told the teachers that the study was about mathematics and language. In mathematics education research, there are a plethora of studies done with pre-service teachers (e.g., Ambrose, 2004; Turner et al., 2016). Therefore, I intentionally selected experienced teachers (five or more years of teaching) to explore their beliefs and practices. Recruitment occurred during the 2020-2021 school year.

All twenty teachers responded to a survey online through google forms. Of these 20 participants, four participants were preschool teachers, three were kindergarten teachers, five were first grade teachers, one was the distance/online teacher for third grade, one teacher had a kindergarten-first grade combination class, and the remaining six did not identify their current grade as this was a question I added after some of the teachers had already filled out the survey. Given the eligibility criteria, all teachers had been teaching for over five years, ranging between 5 and 36 years. While most teachers reported being monolingual, six reported speaking and understanding Spanish and English, three reported speaking and understanding French and English, one reported familiarity with Mandarin, and one reported "other". Half of the participants reported being monolingual and the other half reported being multilingual.

Related to the students' demographics, the teachers reported on their students' EL[5] designation and family background to the best of their knowledge. While four teachers reported having no designated ELs in their class, others taught classes of only ELs. Of the 20 teachers, 16 of them had at least one EL in their class during the 2020-2021 school year.

Data Collection

There were two sources of data for this study that came from survey and interview responses. For the survey, I recruited 20 teachers as participants. The items on the survey included demographic information (e.g., years of teaching experience, professional development, student information and statements about mathematics, student learning, teaching, the role of language in learning math, and emergent bilinguals. Teachers responded using a 5-point Likert scale ranging from strongly agree to strongly disagree. Some survey items were adapted from MEELS (Fernandes & McLeman, 2012). MEELS was used by Fernandes (2020) as a data collection tool as he developed the language orientations framework. Fernandes (2020) used MEELS to identify pre-service teachers' beliefs about mathematics and language instruction with EBs (Fernandes, 2020).

From the participant group of 20, five agreed to be interviewed after taking the survey. Therefore, I interviewed these five teachers using a semi-structured interview. The five participants self-selected to participate in the interview and were not selected to be representative of the larger sample. Questions on the interview asked teachers to reflect on their mathematics teaching, students learning, views of the role of language in learning mathematics, and supporting emergent bilinguals.

Data Analysis

I first coded the responses to analyze the survey as aligning with an asset-based view. I used previous research to identify if agreement or disagreement with each statement reflected an asset-based view. In particular, I drew on the recommendations from research related to 1) students' experiences and backgrounds (e.g., Aguirre et al., 2012; Turner et al., 2012), 2) students' home and everyday language (e.g., Brenner, 1998; de Araujo et al., 2018; Turner & Celedón-Pattichis, 2011; Moschkovich, 2013; Moschkovich, 2015a), and 3) teaching

[5] In the survey Emergent Bilinguals (EBs) were referred to as "ELs" to remain consistent with terms that the teachers were familiar with. I mirror this language when I share the responses to align more closely with the teachers' responses. However, I use the term Emergent Bilinguals (EBs) throughout the rest of the paper to highlight an alternative term that frames young multilingual learners using an asset-based view and shifts away from the English-dominant way of labeling.

mathematics vocabulary (Moschkovich, 2013, Moschkovich, 2015a). After I determined if agreement or disagreement revealed an asset-based view for each statement, I coded each participants' response as aligning or not aligning to an asset-based view. I then calculated the percentage of asset-based responses for each participant. For this calculation, I only looked at non-neutral responses (responses that fell on either side of neutral) and divided the number of asset-aligning responses by the total number of non-neutral.

To further examine the results of the survey, I analyzed participants' answers to the interview questions. I went through each interview from beginning to end and transcribed each section related to beliefs about students' experiences, language, supporting EBs, and mathematics vocabulary. I then summarized teachers' responses and quotes using descriptive codes. I looked across the participants to identify themes and the details of how these five teachers talked about their beliefs related to mathematics, students' experiences, language, and supporting EBs. Since I only interviewed five of the 20 teachers, these descriptions are not reflective of my entire participant group and cannot be generalized to other groups. However, these descriptions offer detailed ways in which these five teachers described students' assets related to mathematics and can provide insights into the beliefs that impact instruction for EBs

Teachers' Beliefs

The teachers responded with at least 74% of their non-neutral responses in ways that reflect an asset-based view. Beyond what I found in the survey responses, the interviews with five of the 20 teachers clarified and provided more detailed descriptions of their beliefs particularly related to supporting EBs' mathematics learning. From the interviews, I found that teachers held beliefs about students' assets and teaching mathematics with EBs related to students' everyday and home language, students' backgrounds and experiences, mathematics vocabulary, and supporting EBs.

In this study of teachers' beliefs, I found that teachers held varying degrees of asset-based views of their EBs and described beliefs specific to students' backgrounds and experiences, students' everyday and home languages, mathematics vocabulary, and supporting EBs. In this section, I synthesize these findings and highlight how these teachers described their beliefs related to using students' assets in mathematics instruction in two ways: 1) allowing students' assets in the classroom and 2) drawing on students' assets for mathematics learning.

I developed these two categories in part from the MMKB framework. The MMKB framework (Aguiree et al., 2012; Turner et al., 2012; Turner et al., 2016) identified how pre-service teachers talked about using students' thinking and experiences in their lesson plans. My first category, allowing students assets in the classroom, is in part related to the "initial practices" category of the MMKB framework. The teachers in my study talked about using students' experiences as context for mathematics problems and that students could use home and everyday language to demonstrate that they understood mathematics. These beliefs acknowledge that students' experiences and language are valuable and important but not necessarily a central part of learning mathematics. My second category, drawing on students' assets for mathematics learning, in many ways reflects the "meaningful connections" and "incorporating" categories from the MMKB framework. This category more closely aligns with beliefs that math and language practices should be used to support mathematics learning that reflects what students do in their homes and communities.

Allowing Students' Assets in the Classroom

Consistently across the five teachers who were interviewed, descriptions about students' assets revealed that they all allowed students' assets in the classroom. For example, they all discussed ways that they would allow students to demonstrate their understanding or share an answer using any way they could. All the teachers also talked about using students' backgrounds and experiences as context for mathematics problems. Two of the five teachers did not provide explicit details of using the mathematical practices from the students' lives or drawing on students' linguistic resources to support learning. One teacher talked about using students' names, and another said that their students probably used technology in their homes. Previous work has characterized ways various types of everyday mathematical practices can be used in mathematics instruction including using context that is 1) based on assumptions, 2) reflects layering or mathematizing, or 3) uncovering mathematical activities (Aguirre et al., 2012; Turner et al., 2012). Some of the ways these teachers talked about contextualizing problems reflected surface-level uses of information, some of which were likely based on assumptions about students rather than informed by information that was gathered from them. While these teachers talked about students using their assets in the classroom, three of the five teachers had no clear connections to content or learning. These responses in some ways reflect asset-based views of their students and can potentially act as an entry point for teachers to incorporate students' assets into mathematics instruction more fully. For example, acknowledging that eliciting and attending to students' experiences outside of the classroom is a part of leveraging students' experiences in meaningful ways. Eliciting and attending to may act as an entry point for teachers to start to make these meaningful connections (Turner et al., 2012; Turner et al., 2016).

Drawing on Students' Assets for Mathematics Learning

Three of the five teachers (Ms. G, Ms. L, and Ms. C) discussed drawing on students' assets for learning mathematics and mathematics vocabulary beyond just allowing students' assets in the classroom. Related to students' backgrounds and experiences, Ms. G and Ms. L made clear connections to the student's home life as they talked specifically about the mathematical practices from students' homes (measurement in building and keeping track of score in bowling) and highlighted ways they used these practices in instruction (a unit on measurement and during discussions). When teachers identify the mathematics that students use at home, such as keeping track of allowance or keeping score in bowling, and use this information during instruction, they connect to ways students use mathematics outside of school (Turner et al., 2016). Ms. G and Ms. L made connections to mathematical practices that their students saw or used at home and talked about how they brought these practices into their instruction. This aligns with the third category of using MMKB, uncovering mathematical activities. For EBs, this approach to teaching mathematics can broaden access to mathematical content by making it familiar and positioning student' activities from their homes as a valuable resource for learning.

Regarding students' home and everyday language, Ms. G and Ms. C talked about how they encouraged students to learn mathematics content and vocabulary by drawing on students' linguistic resources. Both teachers talked about the practice of revoicing to support students as they learned "math words" (Ms. C) and "academic language" (Ms. G). Revoicing is one effective approach to supporting students' mathematics learning (de Araujo et al., 2018; Moschkovich, 2015b) as it provides opportunities for students to hear and use more precise mathematical language and can support participation in mathematical discussions (Moschkovich, 2015b) . When EBs have access to all their linguistic resources in the classroom, they have more opportunities to make meaning for mathematics content and language. In one study comparing two groups, the students from classrooms where home languages were accepted and encouraged outperformed students in classrooms where home languages were not accessible for classroom learning (Turner & Celedón-Pattichis, 2011). The importance of EBs having access to all their linguistic resources is frequently documented in the literature for student success (e.g., Brenner, 1998; de Araujo et al., 2018; Turner & Celedón-Pattichis, 2011; Moschkovich, 2013; Moschkovich, 2015a).

Conclusion

This study explored early grades teachers' professed beliefs about mathematics, student thinking, and students' early, out of school experiences with mathematics, particularly for EBs. The teachers in this study displayed varying degrees of asset-based responses (74%-100%) to the survey and discussed beliefs related to 1) students' backgrounds and experiences, 2) students everyday and home languages, 3) mathematics vocabulary, and 4) supporting EBs. During the interviews, teachers described their beliefs about students' assets (experiences and home/everyday language) in ways that aligned with either allowing students' assets in the classroom or drawing on students' assets to support mathematics learning.

This study corroborates and extends previous research (e.g., Fernandes, 2020; Turner et al., 2016). The research on teachers' use of students' MMKB (Aguirre et al., 2012; Turner et al., 2012; Turner et al., 2016) has highlighted characteristics of drawing on MMKB (students' thinking and mathematical experiences outside of school). The existing research on MMKB primarily focuses on how MMKB is integrated into instruction (Aguirre et al., 2012; Turner et al., 2012; Turner et al., 2016) and ways that curriculum creates "spaces" to draw on students MMKB (Land et al., 2018). This study adds to that research by providing examples of how teachers talk about using students' linguistic resources in addition to their experiences outside of school.

The findings specific to teachers' descriptions of allowing or drawing on students' home and everyday languages in the classroom (to demonstrate understanding or as a resource to make meaning) corroborates other findings related to teachers' beliefs. Like Fernandes' (2020) findings, most of his participants fit within the two middle categories, which reflected a combination of seeing students' language as a problem and seeing it as a resource. In my sample, 85% of the participants fit within these "blended" categories which is similar to his finding of 80% of his participants holding views with blended beliefs. In my sample and in the sample from Fernandes (2020) 15% of the teachers demonstrated views that align with a bilingual orientation. As discussed in the "drawing on students' assets for mathematics learning" category in this analysis, the teachers in this group discussed using the experiences and language of students to support mathematics content and language. The teachers who talked about students using their home language to make meaning for mathematics and for "math talk" or "academic language" also expressed beliefs that all linguistic resources should be used to learn. My findings corroborate the work by Fernandes (2020) in that the teachers' beliefs discussed in this chapter also fall within a continuum in the ways they view students' language in a mathematics classroom.

Differing from his study, I did not have any teachers that responded in ways that reflected a "no native language" orientation. Fernandes (2020) conducted his study with pre-service teachers. My study extends this work as it was done with veteran teachers. This analysis also extends the previous work as it looks at teachers' beliefs about assets more generally.

References

Aguirre, J. M., Turner, E. E., Bartell, T. G., Kalinec-craig, C., Foote, M. Q., McDuffie, A. R., Drake, C. (2012) Making connections in practice: How prospective elementary teachers connect to children's mathematical thinking and community Funds of Knowledge in mathematics instruction. *Journal of Teacher Education, 64(2),* 178-192.

Ambrose, R. (2004). Initiating change in prospective elementary school teachers' orientations to mathematics teaching by building on beliefs. *Journal of Mathematics Teacher Education, 7*, 91–119

Au, K. H. & Kawakami, A. J. (1985) Research Currents: Talk story and learning to read. *Language Arts, 62*(4). 406-411.

Brenner, M. E. (1998) Adding cognition to the formula for culturally relevant instruction in mathematics. *Anthropology and Education Quarterly, 29(2)* 214-244.

de Araujo, Z., Roberts, S. A., Willey, C., & Zahner, W. (2018) English learners in K-12 mathematics education: a review of the literature. *Review of Educational Research. 88(6)*. 879-919.

Fernandes, A. (2020) The language orientations of future mathematics teachers in the United States. *Journal of Research in Mathematics Education. 9*(1), 7-29. doi: 10.17583/redimat.2020.4297

Fernandes, A., & McLeman, L., (2012). Developing the mathematics education of English learners scale (MEELS). In L. R. Van Zoest, J- J. Lo, & J. L. Kratky (Eds.), Proceedings of the Thirty Fourth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (p. 591-597). Kalamazoo, MI: Western Michigan University.

Gutiérrez, K. D., Baquedano-López, P., & Tejeda,C., (1999) Rethinking diversity: Hybridity and hybrid language practices in the third space. *Mind, Culture, and Activity. 6*(4), 286-303. doi: https://doi.org/10.1080/10749039909524733

Lee, J. S. & Ginsburg, H. P. (2007). Preschool teachers' beliefs about appropriate early literacy and mathematics education for low- and middle-socioeconomic status children. *Early Education and Development, 18*(1). 111- 143, doi: 10.1080/10409280701274758

Moschkovich, J. (2013) Principles and guidelines for equitable mathematics teaching practices and materials for English language learners. *Journal of Urban Mathematics Education. 6(1).* 45-57.

Moschkovich, J.N. (2015a) Academic literacy in mathematics for English Learners. *The Journal of Mathematical Behavior*, http://dx.doi.org/10.1016/j.jmathb.2015.01.005

Moschkovich, J. (2015b) Scaffolding student participation in mathematical practices. *ZDM Mathematics Education. 47(7)*, 1067-1078. doi: 10.1007/s11858-015-0730-3

Philipp. R. A. (2007). Mathematics teachers' beliefs and affect. In F. K. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning: A project of the national council of teachers of mathematics.* (pp. 257-318). Charlotte, NC: Information Age Publishing.

Raymond, A. M. (1997). Inconsistency between a beginning elementary school teacher's mathematics beliefs and teaching practice. *The Journal of Research in Mathematics Education, 28*(5), 500-576. doi: https://doi.org/10.5951/jresematheduc.28.5.0550

Roesken-Winter, B. (2013) 11. Capturing mathematics teachers' professional development in terms of beliefs. In Li, Y. & Moschkovich, J. N. (Eds), *Proficiency and Beliefs in Learning and Teaching Mathematics: Learning from Alan Schoenfeld and Günter Törner.* (157-177). Rotterdam, The Netherlands: Sense Publishing.

Ruiz, R. (1984). Orientations in Language Planning. *NAEB Journal. 8*(2), 15-34.

Staub, F. C. & Stern, E. (2002) The Nature of Teachers' Pedagogical Content Beliefs Matters for Students' Achievement gains: Quasi-Experimental Evidence from elementary mathematics. *Journal of Educational Psychology*. 94(2), 344-355. doi: 10.1037//0022- 0663.94.2.344.

Stipek, D. J., Givvin, K. B., Salmon, J. M., & MacGyvers, V. L. (2001). Teachers' beliefs and practices related to mathematics instruction. *Teaching and teacher education*, *17*(2), 213- 226.

Turner, E., & Celedón-Pattichis, S. (2011). Mathematical problem solving among Latina/o kindergartners: An analysis of opportunities to learn. *Journal of Latinos and Education, 10*(2), 146-169.

Turner, E. E., Drake, C., Roth McDuffie, A., Aguirre, J. M., Bartell, T. G., & Foote, M. Q. (2012). Promoting equity in mathematics teacher preparation: A framework for advancing teacher learning of children's multiple mathematics knowledge bases. *Journal of Mathematics Teacher Education, 15*(1), 67–82.

Turner, E. E., Foote, M.Q, Stoehr, K. J, McDuffie, A. R., Aguirre, J. M, Bartell, T, G, & Drake, C. (2016) Learning to leverage children's multiple mathematical knowledge bases in mathematics instruction. *Journal of Urban Mathematics Education. 9(1),* 48-78.

CHALLENGING MATH-MUSIC INTEGRATION

Sandra J Hall
University of Saskatchewan
sandra.hall@usask.ca

Katryne Dubeau
University of Saskatchewan
katryne.dubeau@usask.ca

Egan J Chernoff
University of Saskatchewan
egan.chernoff@usask.ca

Cross-curricular integration is an effective way to enhance student understanding and create real-life connections. Research has shown that the math-music connection increases students' conceptual knowledge, spatial-temporal reasoning and improves motivation. Through the lens of the cognitive-affective model of conceptual change, we will identify reasons non-musical teachers struggle to integrate math and music and offer an approach that addresses this disconnect. We argue that the expectation for non-musical mathematics teachers to integrate music into their lessons is discouraging and ineffective due to their lack of musical literacy.

Keywords: Integrated STEM / STEAM, Affect, Emotion, Beliefs, and Attitudes, Teacher Knowledge

The current educational landscape is such that teachers face daily challenges that affect their instructional methodologies in the classroom. Educators encounter such "widely varying intellectual, social, and affective differences" (Gregoire, 2003, p. 149) in modern classes that even the most effective educator can struggle. The results of the 2018 Organisation for Economic Co-operation and Development Programme of International Student Assessment found that just under 30% of American students and 15% of Canadian students scored under the minimum level of mathematical proficiency (PISA, 2018). These innumerate students will struggle in a range of personal and professional settings where mathematical literacy is essential. To rectify these shortcomings, mathematics educators find themselves under additional pressure to address these weaknesses to ensure their students meet or exceed international standards (Pelletier et al., 2002). Unfortunately, when teachers are required by school administrators to meet these expectations, they become more controlling and critical, offering students minimal choice and autonomy, as well as giving them nominal time to work alone; this can lead to higher anxiety and less growth in mathematical understanding (Deci et al., 1982; Pelletier et al., 2002).

Since the inception of curricular mathematics, resourceful educators have continually searched for innovative and creative ways to enhance student learning (Beckmann et al., 2012). Cross-curricular integration is not a new topic and there are an incredible number of resources available for educators to access around the globe. However, based on the above teacher challenges, many math teachers do not feel they have time to integrate subjects they are unfamiliar with, such as music. Based on ample research (Catterall et al., 1999; Chrysostomou, 2004; Civil, 2007; Harney, 2020; Jones & Pearson, 2013), it is now widely known that integrating music into mathematics instruction increases student motivation, creativity and engages "learners through self-reflection and active inquiry" (Parsons, 2004). Yet, many math teachers, pre- and in-service, do "not feel qualified to teach music […]. [C]lassroom observations revealed that their hesitancy and resultant lack of confidence were truly well founded" (Wiggins & Wiggins, 2004, p. 18). It is reasonable that teachers would struggle as proficiency in music literacy requires instruction and practice. Additionally, many teachers have pre-conceived notions of their (lack of) musical ability built on never having received instruction or recalling poor experiences with music when they were young (Henessy, 2000). This assumed lack of ability and/or unfortunate previous experience, paired with lack of preparation time, often prevents math teachers from integrating music fully into their teaching (Battersby & Cave, 2014). Our proposal demonstrates that expecting teachers to assimilate music into their mathematics lesson is demoralizing for those that do not have the skills necessary to do so.

Theoretical Model

Through the lens of the cognitive-affective model of conceptual change (Gregoire, 2003), we will argue that it is beyond the scope of many in-service math teachers to integrate music into their lessons, even though current research has proven this is an excellent way to improve student motivation, "intellectual performance, spatial-temporal reasoning and other skills advantageous for learning" (Holmes, 2016, p. 4; also see Hallam, 2015; Rauscher & Zupan, 2000; Schellenberg, 2004). This report will discuss the reasons why most mathematics teachers avoid cross-curricular integration, particularly with music, and offer a practical solution.

The Challenge

After several years of experience, most teachers find a groove they settle into. Researchers have found that mathematics teaching strategies in typical American classrooms demonstrate or review a concept or procedure, "provide students with step-by-step instructions, then assign students problems on which to practice the procedure" (Stipek et al., 2001, p. 214). These traditional beliefs regarding math education put the teacher in full control of what and how students learn, emphasize speed and performance, and highlight their "value of extrinsic, teacher-controlled motivational strategies, such as giving praise, rewards, or punishment" (Stipek et al., 2001, p. 215). Integrating music into this traditionalist teaching approach (or method?) would overwhelm teachers as their "pre-existing subject-matter beliefs constrain them from adopting practices that conflict with those beliefs" (Gregoire, 2003, p. 148). Those with "a low sense of efficacy are [even] less likely to try new methods for meeting their students' needs" (Gregoire, 2003, p. 171). A math teacher's belief about their subject and their practice is a complex relationship. If change in these subject beliefs cannot occur, "maintaining radically new ways of instruction is almost impossible" (Gregoire, 2003, p. 149).

How could asking math teachers to start integrating music be considered demoralizing? Wiggins and Wiggins (2008) explain:

> We would not allow someone who had stopped studying mathematics at the fifth[-]grade level to teach mathematics. We would be appalled at the idea that someone could teach language arts if he or she had not read a book or written a word since the age of eleven. Yet we expect that […] teachers can teach music when their last formal musical instruction, if any, may have occurred at that age or earlier. (p. 4)

From this perspective, it seems irrational to expect educators to assimilate technical music theory into their mathematics lessons. However, student learning experiences are enhanced when integration occurs. To assist educators in making these connections, proper training and sufficient resources are essential so "they are not left having to reinvent the curriculum on their own" (Gregoire, 2003, p. 172).

Non-musical math educators that have embraced the "reform" approach or whose beliefs have them situated in the middle ground (Schoenfeld, 2004) will not experience as many difficulties in integrating music as traditionalist teachers but will still require appropriate training. The constructivist approach provides students a chance to become intrinsically motivated with lessons and “activities that require reasoning and creativity, gathering and applying information, discovering, and communicating ideas” (Stipek et al., 2001, p. 214). Incorporating music (where failure is expected and turned into growth) as a natural part of learning mathematics “encourage[s] students to think critically and creatively, engag[ing] in inquiry-based problem solving, collaborat[ion] with peers, and make meaningful, real-life connections between music and math” (Harney, 2020, p. 205). When students develop autonomy and focus on process instead of performance, the classroom becomes a safe place to take risks, accept being wrong and find the correct answers in alternative ways (Stipek et al., 2001).

A Different Approach and Conclusions

We theorize that the only way non-musical mathematics teachers from traditional to constructivist methodologies would feel comfortable integrating music into their math classroom is if a qualified music teacher and math teacher work together for at least one full class of each grade level/class. Providing authentic experiences to increase teacher efficacy and self-confidence will create a successful environment for music/math integration. Teachers working together creating integrated lesson plans, formulating instructional methodologies and experimenting with new projects and activities will build competence in both educators. Based on the research, administrators who support math/music integration “take advantage of the rich connections between music and mathematics” (An et al., 2014, p. 168) leading to improved motivation, deeper conceptual understanding and greater achievement of national standards (Wentworth, 2019). While further research is necessary, we aim to provide an explanation as to why the expectation for non-musical math teachers to integrate music in their instructional approach will be ineffective unless proper supports and resources are readily available.

References

An, S. A., Tillman, D., Shaheen, A., & Boren, R. (2014). Preservice teachers’ perceptions about teaching mathematics through music. Interdisciplinary Journal of Teaching and Learning, 4(3), 150–171.

Battersby, S. L., & Cave, A. (2014). Preservice classroom teachers’ preconceived attitudes, confidence, beliefs, and self-efficacy toward integrating music in the elementary curriculum. Applications of Research in Music Education, 32(2), 52–59. https://doi.org/10.1177/8755123314521033

Beckmann, S., McCallum, W., Spangler, D. A., & Tucker, A. (2012). (rep.). (C. Kessel, Ed.) The Mathematical Education of Teachers II. American Mathematical Society & the Mathematical Association of America. Retrieved February 22, 2023, from chrome-extension://efaidnbmnnnibpcajpcglclefindmkaj/https://www.cbmsweb.org/archive/MET2/met2.pdf.

Catterall, J., Chapleau, R., & Iwanaga, J. (1999). Involvement in the arts and human development: General involvement and intensive involvement in music and theatre arts. In E. B. Fiske (Ed.), Champions of change: The impact of the arts on learning (pp. 1-18). Washington, DC: The Arts Education Partnership.

Chrysostomou, S. (2004). Interdisciplinary approaches in the new curriculum in Greece: A focus on music education. Arts Education Policy Review, 105(5), 23-30.

Civil, M. D. (2007). Using music to improve learning in Mathematics (thesis). New York.

Deci, E. L., Spiegel, N. H., Ryan, R. M., Koestner, R., & Kauffman, M. (1982). Effects of performance standards on teaching styles: Behavior of controlling teachers. Journal of Educational Psychology, 74(6), 852–859. https://doi.org/10.1037/0022-0663.74.6.852

Gregoire, M. (2003). Is it a challenge or a threat? A dual-process model of teachers’ cognition and appraisal processes during conceptual change. Educational Psychology Review, 15(2), 147–179.

Hallam, S., Cross, I. and Thaut, M. (Eds) (2011). Oxford Handbook of Music Psychology. Oxford: Oxford University Press.

Harney, K. (2020). Music and mathematics. In Integrating Music Across the Elementary Curriculum (pp. 205 244). Essay, Oxford University Press.
Hennessy, S. (2000). Overcoming the red-feeling: The development of confidence to teach music in primary school amongst student teachers. British Journal of Music Education, 17, 183–196.
Holmes, S. (2016). The impact of participation in music on learning mathematics (dissertation).
Jones, S. M., & Pearson, D. (2013). Music: Highly Engaged Students Connect Music to Math. General Music Today, 27(1), 18–23. https://doi.org/10.1177/1048371313486478
OECD (2019), "What can students do in mathematics?", in PISA 2018 Results (Volume I): What Students Know and Can Do, OECD Publishing, Paris. DOI: https://doi.org/10.1787/f649d3c2-en
Parsons, M. (2004). Art and integrated curriculum. In E. Eisner and M. Day (Eds.), Handbook of research and policy in arts education (pp. 775-794). Mahwah, NJ: Lawrence Erlbaum.
Pelletier, L. G., Séguin-Lévesque, C., & Legault, L. (2002). Pressure from above and pressure from below as determinants of teachers' motivation and teaching behaviors. Journal of Educational Psychology, 94(1), 186–196. https://doi.org/10.1037/0022-0663.94.1.186
Rauscher, F. H. and Zupan, M. A. (2000). 'Classroom keyboard instruction improves kindergarten children's spatial-temporal performance: A field experiment'. Early Childhood Research Quarterly, 15(2),215-228.
Schellenberg, E. G. (2004). 'Music lessons enhance IQ'. Psychological Science, 15(8),511- 514.
Schoenfeld, A. H. (2004). The math wars. Educational Policy, 18(1), 253–286. https://doi.org/10.1177/0895904803260042
Stipek, D. J., Givvin, K. B., Salmon, J. M., & MacGyvers, V. L. (2001). Teachers' beliefs and practices related to mathematics instruction. Teaching and Teacher Education, 17(2), 213–226. https://doi.org/10.1016/s0742 051x(00)00052-4
Wentworth, E. R. (2019). Mathematics and music: The effects of an integrated approach on student achievement and affect (dissertation). ProQuest LLC, Ann Arbor, MI.
Wiggins, R. A. & Wiggins, J. (2008). Primary music education in the absence of specialists. International Journal of Education & the Arts, 9(12). Retrieved [February 5, 2023] from http://www.ijea.org/v9n12/

EXPLOITATION COLLECTIVE DE TÂCHES ROUTINIÈRES ET ÉMERGENCE D'INCERTITUDES MATHÉMATIQUES

COLLECTIVE INVESTIGATION OF ROUTINE TASKS AND MATHEMATICAL UNCERTAINTIES

Geneviève Barabé
Université de Montréal
Genevieve.Barabe@UMontreal.ca

Cette recherche, ancrée dans la théorie de l'enaction, étudie l'évolution de tâches routinières à travers l'activité collective de résolution de ces tâches. Une telle étude permet de s'intéresser au potentiel de l'exploitation de ce type de tâche en classe de mathématiques. L'analyse de vidéos de séances de classe montre que la résolution collective de ce type de tâche peut faire émerger des incertitudes et générer une activité authentique de résolution de problèmes en classe.

Mots-clés: Résolution de problèmes, Incertitude, Enaction, Collectivité.

Problématique

La formulation et la résolution de problèmes mathématiques sont, depuis longtemps, des champs de recherche centraux en didactique des mathématiques. Un tel intérêt peut s'expliquer par le fait que plusieurs voient en celles-ci l'essence même de ce qu'implique faire des mathématiques (Halmos, 1980; Arsac & Germain et Mantes,1988). Au fil des années, une emphase a été mise sur l'usage de tâches non routinières, soit des tâches qui nécessitent de la personne qui tente de les résoudre, de surmonter un défi ou de faire face à une incertitude (Polya, 1945). À l'opposé, une *tâche routinière* ne représenterait pas de défi puisque la personne saurait quoi mettre en œuvre ou appliquer pour la résoudre aussitôt la tâche lue ou énoncée (*Ibid*.).

Les travaux de recherche ont largement montré le potentiel des tâches non routinières à réussir de faire vivre aux élèves une activité mathématique riche et authentique. Ce constat est d'ailleurs mis en évidence dans la synthèse de Liljedahl et Cai (2021) qui soulève que les récentes études montrent que la nature des tâches proposées et le contexte de classe influencent grandement l'activité mathématique des élèves. Cette influence du contexte sur l'activité mathématique des élèves est également relevée par d'autres chercheurs qui soutiennent, toutefois, que tout ne reposerait peut-être pas nécessairement sur la tâche elle-même. Mason (2019) affirme en ce sens être de plus en plus convaincu que ce n'est pas la tâche en soi qui est riche, mais plutôt la manière dont elle est utilisée en classe. Small (2022) propose également aux milieux scolaires une présentation intitulée « It's not the tasks! : It's the follow-up! » affirmant que « many of us would argue that it's less about the task itself and more about the questions teacher asks as and after students work on it ». Agre, en 1982, avançait d'ailleurs déjà en ce sens que le contexte pourrait contribuer à l'émergence de problèmes mathématiques chez les élèves :

> If a time-tested procedure exists for bringing about desired state of affairs, there may not exist a problem, because carrying out the procedure may be easy. But if extra effort is required because the income is in doubt, the situation may qualify as a problem. » (p. 131)

Cette idée d'incertitude qui peut émerger de la situation est importante pour permettre à une tâche, quelle qu'elle soit, de devenir un problème. La notion de « problème mathématique » se définissant d'ailleurs par cette notion d'incertitude : « […] problem solving is the process of moving toward a goal when the path to that goal is uncertain. We solve problem every time we achieve something without having known beforehand how to do so » (Martinez, 1998, p. 605).

Dans cet ordre d'idée, Beghetto (2017) propose aux enseignants d'insérer de l'incertitude en classe en exploitant les tâches routinières dont ils disposent dans leurs cahiers scolaires. Pour ce faire, il suggère de créer des ouvertures en classe en remplaçant les caractéristiques qui semblent prédéterminées des tâches routinières par des aspects à être déterminés *durant* la résolution, permettant d'introduire de l'incertitude. Autrement dit, malgré l'apparence peu porteuse des tâches routinières, certains chercheurs soutiennent qu'elles pourraient être exploitées de sorte à générer une activité authentique de résolution de problèmes mathématiques. Cette perspective est au cœur de la recherche présentée ici, qui a pour objectif d'étudier, de manière empirique, ce qui peut se produire lorsque des tâches routinières sont exploitées en classe, notamment au regard de l'incertitude et de son rôle sur l'activité mathématique des élèves.

Cadrage Théorique

Cette recherche prend appui sur la théorie de l'enaction (e.g. Maturana & Varela, 1992), une théorie biologique de la connaissance qui offre une manière de comprendre les actions des individus à travers leurs inter-actions avec l'environnement. L'insistance sur les mots « inter » et « action » par le trait d'union a pour objectif d'accentuer le fait que, du point de vue de l'enaction, la connaissance s'observe dans les actions de chacun et pointe vers le rôle fondamental de l'environnement dans ces actions (Kieren, 1995). Cette théorie explique les mécanismes en jeu lors des inter-actions entre une personne et l'environnement, offrant une manière de comprendre les actions de chacun en contexte d'exploitation de tâches routinières. L'enaction amène à s'intéresser, d'une part, à la classe en tant qu'entité, que collectivité, qui ensemble met en avant une activité mathématique. D'autre part, elle invite à étudier l'évolution, qui, dans cette étude, porte sur l'évolution de tâche routinière, soit sur ses transformations, à travers les inter-actions avec la collectivité que forme la classe.

Un regard centré sur la classe en tant que collectivité

Pour Maturana et Varela (1992), les actions d'une personne sont orientées par sa structure qui lui permet d'inter-agir avec l'environnement et de l'aborder selon cette structure. Ce phénomène est appelé le *déterminisme structurel*, soit le fait que la structure d'une personne détermine les changements qui se produisent chez elle, et donc les actions qu'elle pose dans ses inter-actions avec l'environnement. L'environnement et l'individu sont toujours en inter-actions si bien qu'il y a une nécessaire compatibilité entre eux permettant leur fonctionnalité mutuelle. Tant que cette compatibilité demeure, chacun agit en tant que source de perturbations de la structure de l'autre, ce qu'ils nomment le *couplage structurel*. L'environnement agit comme un déclencheur de changements chez une personne et, réciproquement, celle-ci agit comme un déclencheur de changements pour l'environnement. Ces inter-actions et changements se mobilisent dans une boucle d'influence mutuelle où chacun influence l'autre dans son évolution.

La constitution d'un système social comme la classe entraine la *co-ontogénie* entre les personnes qui expérimentent alors une histoire partagée de changements structurels réciproques. La vie sociale de la classe leur permet de coordonner leurs activités à travers cette co-ontogénie. Les inter-actions, qui découlent de cette coordination, donnent lieu à de nouvelles possibilités d'action pour tous en classe. Sous une telle perspective, l'enseignement est vu comme une activité réciproque où les actions des uns génèrent des possibilités d'actions pour les autres :

> The children's mathematical actions were occasioned by the teacher acts but those teacher actions were in turn occasioned by the children's mathematical activity. This reciprocity was not simply a "back and forth" activity. This reciprocal teaching/learning occurred in a complex web with each person's actions resulting in new possibilities for all of the others. (Kieren, 1995, p. 10)

Pour l'enaction, l'individu n'est pas isolé de son environnement, si bien que ses actions sont partagées et créées à travers et par l'environnement. Le terme « partagé » ne signifie pas que des actions mathématiques se chevauchent, comme dans un diagramme de Venn, ni que des individus réalisent des actions qu'ils connaissent déjà, mais plutôt que ces actions émergent et sont constituées de manière inter-active (Towers & Martin, 2015). En classe, les actions mathématiques déployées sont vues comme étant un produit des inter-actions des personnes et de l'environnement. Ainsi, dans chaque action faite en classe, il y a une contribution de l'environnement dans cette même action. L'action est tout autant attribuable à l'inter-action qu'à l'individu (*Ibid.*). Comme Martin, Towers et Pirie (2006) le soulignent, l'idée n'est pas d'enlever de l'importance aux actions déployées par un individu, mais plutôt de se centrer sur le niveau auquel elles se constituent et prennent place, soit au niveau de la collectivité. Un tel intérêt pour la prise en compte de la collectivité au sein de la classe de mathématiques s'est d'ailleurs développé dans les travaux de recherche en didactique des mathématiques dans les vingt dernières années (voir e.g. Thom & al., 2020). C'est sous un tel angle collectif que les données de cette étude ont été étudiées.

Une étude de l'évolution de la tâche à travers les inter-actions en classe

Lorsqu'une tâche mathématique est proposée à résoudre en classe, elle agit en tant que déclencheur d'une réaction mathématique de la part de la collectivité. La tâche peut être vue comme une invitation lancée aux individus à faire émerger un problème mathématique pour eux (Simmt, 2000). Pour Varela (1996), la codétermination entre une personne et son environnement implique que les problèmes qu'elle rencontre proviennent du sens qu'elle donne au monde dans lequel elle vit. Les problèmes ne sont pas situés dans la nature à attendre qu'une personne les trouve et les résolve. Ils sont mis en avant par l'individu à travers ses inter-actions avec l'environnement. En classe, un élève aborde ainsi une tâche mathématique avec ses expériences et selon les stimuli qu'il perçoit de son environnement. En réaction à ces stimuli, il se pose un problème qui, pour lui, est significatif à adresser à ce moment précis. Proulx et Maheux (2017) invitent à voir ce moment de *pose de problème* comme l'entrée de chacun dans la résolution du problème. Lorsque ce premier pas de poser le problème est fait, la tâche se modifie en retour puisqu'elle prend une couleur particulière en fonction de ce premier pas. Ils argumentent ainsi que chaque étape de résolution de problèmes peut être vue comme constituant à la fois la pose d'un problème et un pas dans la résolution :

> This gives the sense that each act of solving is followed by a reposing, which calls for a new solving move, and so forth. But there is more. The conceptualisation of mathematical problem-posing that we derive from our reading of Varela leads us to consider the dialectical nature of posing and solving. That is, not only does the posing and the solving continually follow and precede each other like two feet moving in/for walking, the two dimensions emerge at the very same time, each being conditional on and for the other. (p. 163)

Ils proposent en ce sens d'utiliser l'expression *pose|résolution de problèmes* à l'instar de résolution de problèmes pour mettre l'accent sur cette relation dialectique entre la pose et la résolution d'un problème. Lorsqu'une tâche mathématique est proposée en classe, elle agit comme un déclencheur d'une réaction chez la collectivité qui alors fait émerger différents problèmes mathématiques à résoudre en entrant dans une telle activité de pose|résolution de problèmes. Les travaux sur l'enaction permettent ainsi de s'intéresser à l'évolution d'une tâche mathématique; une évolution qui se produit à même l'activité de pose|résolution de problèmes déployée pour la solutionner et à même les inter-actions entre la collectivité et la tâche proposée.

Méthodologie

Les données de cette étude proviennent d'un projet de recherche centré sur l'activité mathématique des élèves en contexte de résolution de problèmes (voir e.g. Proulx, 2018). Ce projet a été mené dans des classes de 5^e^ année (10-11 ans), 6^e^ année (11-12 ans) et de 2^e^ secondaire (13-14 ans). Un total de 56 séances de classe qui ont été réalisées et captées sur vidéo.

D'une manière générale, les séances de classe suivent une structure qui s'inspire de celle proposée par Douady (1994) : le chercheur-enseignant (CE) propose une tâche oralement ou par écrit; un certain temps est donné aux élèves pour sa résolution; ils sont invités à proposer leurs idées et stratégies en plénière; le CE les note au tableau, ou encore, les élèves viennent eux-mêmes le faire; d'autres stratégies pouvant permettre de résoudre la tâche sont demandées; puis les différentes stratégies sont comparées entre elles. À tout moment, le CE et les élèves peuvent apporter, toujours en plénière, de nouvelles idées ou solutions, questionner ou commenter les différentes propositions. Ces séances donnent ainsi lieu à une exploitation collective des tâches routinières proposées à résoudre.

Les tâches utilisées dans ce projet de recherche peuvent être qualifiées de routinières, car les notions, procédures ou stratégies mathématiques pour les résoudre avaient déjà été enseignées. Les tâches étaient également sélectionnées par les enseignants eux-mêmes, dans leurs cahiers scolaires, et suivaient leur planification annuelle. Voici quelques exemples de tâches routinières travaillées en classe dans ce projet : « Estime la somme de : 152 496 + 608 947. *5^e^ année*. », « Combien donne $\frac{1}{2} + \frac{3}{4}$? *6^e^ année* », et « Résoudre 2x + 3 = 5. *2^e^ secondaire* ».

À partir d'une adaptation du modèle d'analyse de vidéo de Powell, Francisco et Maher (2003) et d'une grille d'analyse reposant sur les ancrages théoriques développés, une analyse globale des 56 séances de classe a été réalisée. Puis, 7 séances ont été choisies afin de bien représenter l'ensemble des données pour en faire une analyse détaillée. L'analyse de l'évolution des tâches routinières de ces séances a été réalisée, l'unité d'analyse étant la collectivité, tel que discuté ci-haut. Cette analyse a permis de relever la présence d'incertitudes qui ont joué un rôle dans l'évolution des tâches routinières et donc dans l'activité mathématique en classe.

Résultats

Dans ce qui suit, un exemple synthétique d'analyse, celle d'une tâche routinière portant sur la divisibilité par 4, est proposé en guise d'illustration des analyses conduites.

L'analyse synthétique de l'évolution de la tâche divisibilité par 4

La tâche routinière « Est-ce que 498 est divisible par 4? » est proposée à résoudre à une classe de 6^e^ année. Voici ce qui s'est produit dans cette séance qui a duré une cinquantaine de minutes.

Une fois la tâche énoncée, les élèves ont eu une quinzaine de secondes de réflexion (mode calcul mental), puis le CE leur a demandé de partager leurs stratégies. Différentes stratégies sont proposées pour la résoudre. Notamment, pour débuter, une première affirmation portant sur la parité du dividende et du diviseur est donnée: « Si je ne me trompe pas, tous les nombres pairs peuvent se diviser par un nombre pair. » Le CE demande des explications sur la stratégie, et une référence à l'enseignante de l'an dernier qui disait cela est donnée en guise de réponse. Par la suite, une autre stratégie s'appuyant sur la décomposition du nombre et la vérification de la divisibilité par 4 de chaque partie de la décomposition est expliquée. Ceci mène la collectivité à investiguer la vraisemblance de cette stratégie même de décomposition du nombre, faisant émerger une première sous-tâche mathématique, issue de la résolution de la tâche initiale, à savoir s'il est possible de décomposer un nombre en plusieurs parties et de vérifier si chaque partie est divisible pour savoir si le nombre complet l'est.

La collectivité se penche, pendant un certain temps, sur cette première sous-tâche proposant deux stratégies pour y répondre. L'une valide la stratégie en affirmant qu'elle aide à mieux comprendre le nombre. L'autre la valide aussi, prenant appui sur le nombre 496, et sur une décomposition judicieusement bien choisit, pour exemplifier et réexpliquer la stratégie proposée. Les traces suivantes sont laissées au tableau :

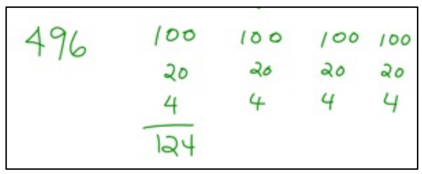


Figure 1 : Traces de la stratégie de décomposition du nombre à partir du nombre 496

À ce moment, la collectivité met en avant d'autres stratégies pour résoudre la tâche routinière initiale. En particulier, elle propose de vérifier la divisibilité par 4 de chaque chiffre qui compose le nombre. Plusieurs des stratégies avancées jusqu'à présent ne sont pas expliquées ni justifiées de manière générale (par exemple, qu'arriverait-il avec la stratégie de décomposition de 496, représentée à la Figure 1, si la décomposition en 490 et 6 avait plutôt été proposée?). Ceci mène la collectivité à se questionner sur des manières d'y arriver, à partir d'un nouveau nombre, 589, amenant la collectivité à résoudre une seconde sous-tâche mathématique.

Après environ cinq minutes de réflexions individuelles ou en petites équipes, le CE ramène le groupe en plénière. Il propose, pour débuter, de tenter d'expliquer la première stratégie : « Peut-on savoir si 589 se divise par 4 à partir de la stratégie que tout nombre pair se divise par un nombre pair? Comment on y arriverait? ». Le travail de cette sous-tâche amène la collectivité à mettre en avant de nouvelles stratégies. L'une d'entre elles est investiguée de manière soutenue. Celle-ci repose sur la formulation d'une conjecture portant sur la parité du produit d'une multiplication par 4, qui est mise en route en classe comme suit :

> Maria : Quand je prends un nombre et que je le multiplie par 4, ça donne un nombre pair.
> CE : Ok. Tu nous dis que quand tu as un nombre et que tu le multiplies par 4, ça te donne un nombre pair. Comment tu sais ça?
> Maria : Bien, parce que 4 x 4 = 16, 4 x 2 = 8, 4 x 3 = 12.

Cette conjecture devient, en elle-même, le nouvel objet de résolution de la collectivité qui se penche sur sa vraisemblance. Une troisième sous-tâche mathématique à savoir s'il existe des contre-exemples à celle-ci émerge alors. Après avoir donné quelques exemples qui semblent la valider, la collectivité tente de la justifier au plan mathématique amenant la troisième sous-tâche à se préciser. Différentes stratégies visant à la résoudre sont présentées et discutées par la collectivité jusqu'à ce que la cloche sonne et mette fin à la séance.

Les différentes stratégies mises en oeuvre dans l'activité qui a pris place en classe peuvent être vues comme une activité de pose|résolution de problèmes mathématiques, comme discuté précédemment. Dans l'action en classe, certaines stratégies ont mené la collectivité à investiguer les stratégies elles-mêmes, c'est-à-dire que certaines sont devenues l'objet d'attention, soit un nouvel objet de résolution, de la collectivité. Ceci a permis de faire émerger des sous-tâches mathématiques sur lesquelles la collectivité s'est penchée pendant un certain temps. Ces sous-tâches proviennent de l'activité collective de pose|résolution de problèmes déployée pour résoudre la tâche routinière initiale, mais ont fait bifurquer sa résolution, pendant un moment, pour se pencher sur celles-ci. La Figure 2 ci-bas offre une synthèse permettant d'illustrer l'évolution de cette tâche routinière à travers les inter-actions dans cette classe de 6^{e} année.

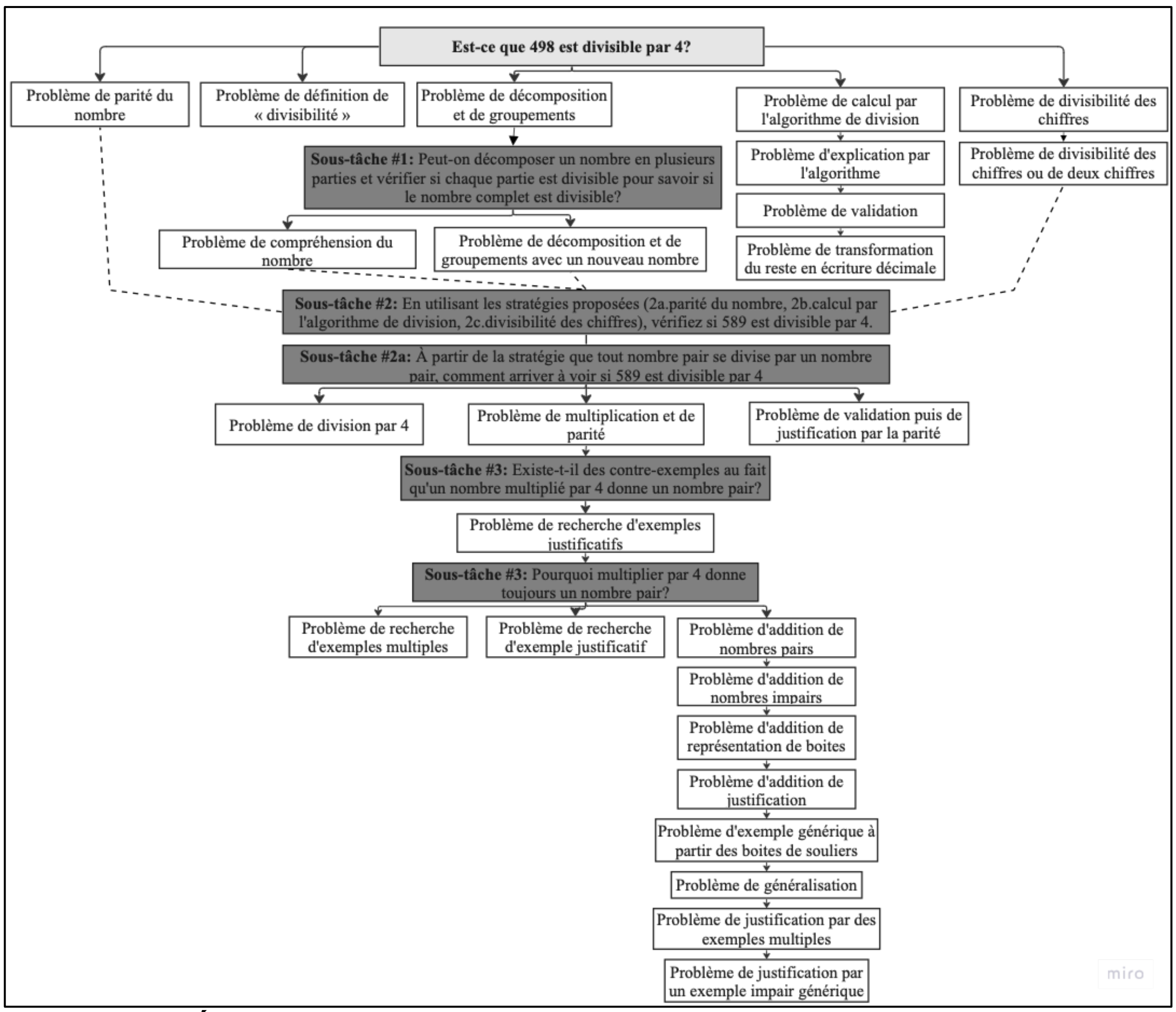


Figure 2 : Évolution de la tâche de divisibilité par 4 dans une classe de 6^{e} année

Des apports à une conceptualisation de l'incertitude en classe de mathématiques

Plusieurs incertitudes, relevées lors de l'analyse des données, apparaissent porteuses au regard de l'activité mathématique vécue par la collectivité. L'analyse montre que l'émergence d’incertitudes a souvent permis l’investigation de nouvelles avenues et un engagement dans un travail mathématique profond sur des enjeux conceptuels importants. Beghetto (2020) parle en ce sens des *actionnables uncertainties,* soit des incertitudes productives qui sont reconnues pour être un moteur de nouvelles investigations et pour ouvrir sur de nouvelles possibilités de penser et d’agir. Il soutient que lorsqu’elles surviennent en classe, elles augmentent le niveau d’attention de la personne qui se trouve dans cet état, et l’entraine dans une impasse qui éveille chez elle le besoin d’explorer et de mettre en œuvre de nouvelles actions. Beghetto (2020) propose une conceptualisation de l’incertitude en trois niveaux, pouvant être représentée comme ceci :

Figure 3: Continuum des niveaux d’incertitude

Les *mundane uncertainties* font référence à des incertitudes qui sont d’un bas niveau d’intensité. Celles-ci passent souvent inaperçues ou n’entrainent que peu ou pas de réaction chez une personne. Ce type d’incertitude se vit de manière plus implicite, comme la personne n’en est que peu ou pas consciente. Dans les séances analysées, ce type d’incertitude pourrait être représenté lorsqu’une demande d’explication, de justification ou de validation est énoncée, mais qu’elle n’est pas adressée puisqu’elle n’engendre pas de réaction chez la collectivité. En effet, l’analyse des données montre que de telles demandes laissent parfois planer un doute au niveau mathématique, mais qui ne semble pas résonner chez la collectivité qui ne semble pas exprimer le besoin de le surmonter. Par exemple, lorsque l’affirmation que tout nombre pair se divise par un nombre pair est énoncée en classe, une demande d’expliquer celle-ci est lancée, mais n’est pas répondue par la collectivité dans le sens que celle-ci ne s’engage pas mathématiquement envers elle. La collectivité se contente de la réponse de ne pas savoir et de l’argument d’autorité qu’il s’agit des propos de l’enseignante de l’an dernier. À travers cette réponse, il est possible de déceler une incertitude quant à la manière de mettre en avant une justification mathématique satisfaisante en lien avec cette affirmation. Toutefois, à cet instant, la collectivité ne semble pas embêtée par le fait de ne pas savoir comment la justifier mathématiquement. Cette incertitude n’est donc pas vraiment perçue par la collectivité qui ne cherche pas à la surmonter ni à s’y engager. De leur côté, les *profound uncertainties,* qui se situent de l’autre côté du continuum dans la Figure 3, font référence à des incertitudes qui sont extrêmement intenses et qui apparaissent comme étant inaccessibles; ne pouvant être vraiment connues, comprises, ni résolues par la personne qui se trouve face à une telle incertitude. Ce type d’incertitude n’a toutefois pas semblé être présent dans les données. Entre ces deux pôles, se trouve, pour Beghetto, une grande variété d’expériences d’intensité modérée d’incertitude, soit les incertitudes productives. Dans cette séance sur la divisibilité par 4, lorsque la demande de s’intéresser à la stratégie que tout nombre pair se divise par un nombre pair est plus tard réitérée à partir du nombre 589, ce qui était de l’ordre d’une *mundane uncertainty* est devenue une *actionable uncertainty*. En effet, une invitation à s’engager avec l’incertitude de la justification mathématique de la stratégie est lancée puis répondue. La collectivité s’y engage, prenant le temps d’explorer de nouvelles possibilités et de mettre en avant de nouvelles actions visant à surmonter cette incertitude. L'analyse des données révèle la présence fréquente de telles *actionables uncertainties* dans les séances de classe.

Ainsi, bien que les tâches proposées dans le cadre de cette étude étaient de nature routinière, des incertitudes sur la validité de certains éléments mathématiques mis en avant en classe ou encore sur la manière de les expliquer ou de les justifier mathématiquement ont émergées en classe. Ces incertitudes naissent du travail collectif fait dans l'action de les résoudre. Elles sont le fruit des inter-actions entre la collectivité et la tâche initiale, émergeant de leur coévolution. Ces incertitudes ne sont, en ce sens, pas entièrement prévisibles. En effet, il n'est pas possible d'être certain, *a priori*, qu'une incertitude précise émergera des inter-actions, qu'elle sera abordée par la collectivité à un moment prédéterminé ou encore qu'elle occupera un temps précis de l'activité collective afin de la surpasser. Il est toutefois possible d'identifier certains éléments qui pourraient potentiellement devenir une source d'incertitude en classe, par exemple, en faisant une analyse *a priori* de la tâche routinière initiale. Toutefois, ce qui, dans l'activité collective, devient la source d'une incertitude, et en particulier d'une incertitude productive dans ce qu'elle permet de produire au plan mathématique en classe, ne peut être entièrement prédit ou contrôlé. C'est d'ailleurs sous les angles d'interdépendance entre la collectivité et l'environnement, et du manque de prédictibilité, de contrôle, de prédétermination ou encore de certitude que Beghetto (2020) propose de concevoir la notion même d'incertitude :

> Uncertainty refers to a state of doubt. *It connotes a lack of determinateness, sureness, stability, control, and predictability*. Although uncertainty typically refers to one's own experience of doubt in the present moment, *it also includes past- and future-oriented doubts about others, features of the environment, and the interrelationship among self, others, and context* (Jordan & McDaniel, 2014). (p. 2, c'est moi qui met l'emphase)

Cette définition met en lumière que l'incertitude peut être relative à une activité qui se veut collective qui ne peut être entièrement prédéterminée et où des doutes surgissent à travers les relations avec les autres et l'environnement. À cet égard, il semble raisonnable que l'incertitude soit présente dans les données. En tournant le regard vers la collectivité, il apparait que les tâches initialement vues comme étant routinières ne le sont plus tellement lors de l'activité collective qui se déploie pour les résoudre. Ceci mène à penser que la résolution collective d'une tâche routinière implique davantage que de simplement la résoudre pour soi-même. Elle implique une résolution *collective* qui s'intéresse aux stratégies, bonnes ou erronées, mises en avant par la collectivité. Cette résolution collective peut engendrer des incertitudes productives qui sont à investiguer, permettant à des problèmes mathématiques d'émerger au cœur même de l'activité collective de résolution de tâches d'apparence routinière.

Conclusion

Cette recherche met en lumière l'exploitation collective de tâches routinières et ce qu'elle peut engendrer comme activité authentique de résolution de problèmes. La recherche met ainsi en exergue l'importance du contexte offert en classe, de son rôle qui apparait prédominant; pouvant aller au-delà de la tâche proposée en classe puisqu'à travers ses inter-actions avec la collectivité, celle-ci est appelée à évoluer. D'autres études semblent toutefois nécessaires pour mieux comprendre et circonscrire les conditions, les possibilités et les contraintes qu'offre l'exploitation collective de tâches routinières auprès des élèves. Alors qu'un engagement mathématique soutenu a été observé lors de l'analyse des données, il convient aussi de se demander si la nature routinière des tâches, jouant un rôle dans le sentiment de contrôlabilité de la tâche, pourrait agir sur l'engagement mathématique, et ce, auprès de différentes clientèles d'élèves. De même, des enjeux mathématiques importants ont été travaillés dans les séances, conduisant à se demander si l'exploitation collective de tâches routinières pourrait être un levier au développement d'une compréhension conceptuelle chez la collectivité. En somme, alors qu'il a souvent été question d'une dichotomie entre les tâches routinières et non routinières, cette recherche invite plutôt à penser en termes de rapprochement; mettant l'emphase sur l'activité mathématique vécue en classe davantage que sur la tâche en elle-même.

Références

Agre, G. P. (1982). The concept of problem. *Educational Studies, 13*(2), 121-142.

Arsac, G., Germain, G. & Mante, M. (1988). *Problème ouvert et situation-problème.* Lyon: IREM de Lyon.

Beghetto, R. A. (2017). Lesson unplanning: toward transforming routine tasks into non-routine problems. *ZDM Mathematics Education, 49*, 987-993.

Beghetto R.A. (2020) Uncertainty. In: Glăveanu V. (dir.) The Palgrave Encyclopedia of the Possible. Palgrave Macmillan, Cham.

Douady, R. (1994). Ingénierie didactique et évolution du rapport au savoir. *Reperes-IREM, 15*, 37-61.

Halmos, P. R. (1980). The art of mathematics. *The American Mathematical Monthly, 87*(7), 519-524.

Kieren, T. (1995). Mathematics Teaching (In-the-middle): Enactivist view on learning and teaching mathematics. *Communication présentée au Canadian National Mathematics Leadership conference*. Kingston, Ontario: Université Queens.

Liljedahl, P. & Cai, J. (2021). Empirical research on problem solving and problem posing: a look of the state of the art. *ZDM- Mathematics Education, 53,* 723-735.

Martin, L., Towers, J. & Pirie, S. (2006). Collective mathematical understanding as improvisation. *Mathematical Thinking and Learning, 8*(2), 149-183.

Martinez, M. E. (1998). What is problem solving? *The Phi Delta Kappan, 79*(8), 605-609.

Mason, J. (2019). Evolution of a task domain. *Digital Experience in Mathematics Education, 5,* 145-165.

Maturana, H. & Varela, F. J. (1992). *The tree of knowledge*. Boston, MA: Shambhala.

Polya, G. (1945). *How to solve it*. New-York: Doubleday.

Powell, A. B., Francisco, J. M. & Maher, C. A. (2003). An analytical model for studying the development of learners' mathematical ideas and reasoning using videotape data. *Journal of Mathematical Behavior, 22*(4), 405-435.

Proulx, J. (2018). On teaching actions in mathematical problem-solving contexts. *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Pyschology of Mathematics Education.* Greenville, Sc: University of South Carolina & Clemson University.

Proulx. J. & Maheux, J.-F. (2017). From problem solving to problem posing and from strategies to laying down a path in solving: Taking Varela's ideas to mathematics education research. *Constructivist Foundation, 13*(1), 160-190.

Simmt, E. (2000). *Mathematics knowing in action: A fully embodied interpretation*. Thèse de doctorat. Université d'Alberta, Edmonton, Canada.

Small (2022, February 22nd). *It's not the task it's the follow-up.* https://www.onetwoinfinity.ca/its-not-the-task-its-the-follow-up/

Thom, J. S., Glanfield, F., Mgombelo, J., Proulx, J., McGarvey, L. & Towers, J. (2020). Research tool for collectivity: tracking mathematics classes. In Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Dir.). (2020). *Mathematics Education Across Cultures: Actes de la 42e conférence de North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico.

Towers, J. & Martin, L. (2015). Enactivism and the study of collectivity. *ZDM Mathematics Education, 47,* 247-256.

Varela, F. J. (1996). *Invitation aux sciences cognitives*. Paris: Éditions du Seuil.

DETECTIVES, NINJAS, AND THE TASTE OF MATH: ELEMENTARY TEACHERS' REASONING ABOUT MATHEMATICS CURRICULAR RESOURCES

Doris Fulwider, M.S.Ed.
Purdue University
dfulwide@purdue.edu

Amy M. Olson, Ph.D.
Duquesne University
olsona@duq.edu

Stephany Panza, M.S.Ed.
Duquesne University
panzas1@duq.edu

Teachers hold complex goals when making decisions about their curriculum. In particular, the proliferation of supplemental resources means teachers negotiate both how to use the adopted curriculum and how to supplement it with additional resources. Through a study of the language teachers use, the decision-making of three elementary classrooms is brought to life. This study highlights the unique curricular decisions each teacher makes within the context of their own classroom and reveals curricular decision-making themes that emerge across the teachers' classrooms.

Keywords: Teacher Beliefs, Curriculum, Elementary School Education

Objectives or Purposes of the Study

Language provides insight into how "we base our actions…on what we take to be true" (Lakoff & Johnson, 1980, p. 160) and teachers' stories provide insight into the multiple professional, personal, and practical experiences they bring to their classroom decision-making (Clandinin & Connelly, 1996). In this study, and in celebrating the theme of *Engaging All Learners*, we examine teachers' language and reasoning about the decisions they make when using multiple curricular resources to better understand their practice as mathematics teachers. These stories highlight the intention teachers have to deliver instruction that targets the learning needs and interests of their students and to bring a sense of joy into the mathematics classroom.

Research Question

What key words, lines of reasoning, and figurative language do elementary mathematics teachers use when explaining the decisions they make about using multiple curricular resources?

Perspective(s) or Theoretical Framework

Curricular Resources

Recently, online curricular resources have proliferated and there is growing evidence that teachers are making use of these resources widely (Pepin, et al., 2013; Sawyer et al., 2020). While many teachers continue to rely on textbooks to implement content and instructional strategies, districts vary in the quality and quantity of textbooks that are selected and the degree to which they are supported or mandated (Polikoff et al., 2020). Consequently, teachers have varying needs and supports for supplementing their curriculum. Rather than think about curriculum implementation in terms of fidelity to a textbook or singular resource (Brown et al., 2009; Superfine et al., 2015), it becomes important to investigate both the degree of teacher's curricular control, here defined as the degree to which teachers feel empowered to make decisions to meet their students' needs (Dampson et al., 2019), and the context of their decision making with respect to exploring their curricular options.

Silver (2022) systematically reviewed the literature surrounding the decisions teachers make as they supplement any officially adopted curriculum with additional curricular resources. One of the major findings of his review suggests that most teachers supplement their district- and school-provided resources to meet goals specific to their instructional approach to the content, school context, and students' needs. Thus, most teachers have become curriculum "curators" for their classrooms (Hu et al., 2020), and their choices influence the mathematics experience and learning opportunities for their students. Given the pervasiveness of this trend, it is increasingly important to understand how teachers reason about their use of curricular resources.

Context of Ongoing Impacts from the COVID-19 Pandemic

An additional rationale for the importance of exploring teachers' decisions around curricular supplementation is that content, context, and student needs are likely to be impacted by teachers' need to be responsive to the impacts of the COVID-19 pandemic. It has been well documented that learning opportunities during COVID-19 emergency remote teaching and the subsequent return to classrooms varied widely, raising and exacerbating equity concerns and discussions of learning loss (Aguilera & Nightingale, 2020; Authors, 2021; Gross & Opalka, 2020; Grossman et al., 2021; Huck & Zhang, 2021; Reilly & Ball, 2020; Valant, 2020).

Methods or Modes of Inquiry

Participants

The data presented here focus on three teachers, whose self-chosen pseudonyms are Audrey, Kasie, and Jamie. Audrey is a second-grade teacher with six years of experience. Kasie is a third-grade teacher with 13 years of experience. Jamie is a fourth-grade teacher, also with 13 years of experience. All three teachers are generalists and teach across subject areas, but they all feel most confident and interested in teaching mathematics.

Context

All three teachers work in the same Midwestern elementary school. The school enrolls over 200 students. The population is racially diverse (37% White, 29% Black, 22% Hispanic, 11% multiracial) and economically disadvantaged (78%). Standardized test data suggest that the mathematics proficiency for elementary students (66%) is above the state average (48%).

The district provides a mathematics pacing guide with three-week windows focused on standards. Students take benchmark assessments at the end of each three weeks using an online application. Teachers reported having between 45 and 75 minutes to spend on mathematics each day, depending on the schedule for "specials" (e.g., music, art) with most days close to 60 and some days allowing up to an extra 30 minutes for mathematics intervention based on the previous benchmark assessment scores. The teachers reported using between 12 and 14 curricular resources for daily mathematics lesson planning.

At this school, face-to-face instruction resumed in Fall 2020, but learning was still disrupted throughout the past two years by shifts in best practice for COVID-19 mitigation (e.g., for spacing and orienting desks). These interviews took place in Spring 2022 and the teachers referenced and were responsive to challenges due to the COVID-19 pandemic, including some "unfinished learning" where students are not proficient at content from previous grade levels.

Data Collection

Data presented here are drawn from the two individual interviews. The first interview focused on identifying the mathematics curricular resources teachers used and the degree of autonomy teachers had in selecting and adapting their curricular resources. The second interview focused on the lesson planning process and examined how teachers created cohesive learning from the variety of resources they used. Consistent with Clandinin (2013), the interview design was intentional in that it allowed for us to explore teachers' lived experiences through telling, retelling, and reliving across interactions.

Analysis

The data presented here are based on audio recordings and transcriptions of the interviews. The transcriptions were de-identified through use of pseudonyms prior to being subjected to the descriptive/interpretative analysis by the researchers. Rather than analyze response by interview question, the two transcriptions from individual interviews with each teacher were analyzed holistically as a rich source of language and reasoning. Using Quinn's (2005) approach, three coders examined interview data to explore: (a) key words (e.g., repeated phrases, surprising word choices); (b) lines of reasoning (e.g., how teachers explain what they are doing); and (c) figurative language (e.g., metaphors) teachers used in explaining their decision making about use of curricular resources. No a priori codes were defined. Three researchers independently coded the interviews. All coding disagreements were resolved in discussions between the researchers.

Findings

Audrey

Key Words. For Audrey, the most repeated words/phrases had emotional valence (e.g., "love/don't really love/hate" [n = 26]). Other commonly repeated words and phrases focused on her student-centered approach (e.g., "meet their needs" and "where they are" [n = 23]) and her responsiveness to student ability and achievement levels (e.g., "high-ability"/ "high-achieving" and "lower groups"/ "lowest groups"/ "low-achieving" [n=16]).

Lines of Reasoning. Audrey depends upon a variety of supplemental curricular resources to "meet the needs" of her students. Her explanations of how and why she incorporates supplemental resources revolve around three major themes: accommodating wide variation in student learning levels, timing of her math block, and love for teaching and her students.

First, when discussing the desire to "meet the needs" of a range of achievement levels, Audrey describes her biggest challenge as having the "lowest" in the grade level and the "highest in the exact same room." In her words:

> I have nine IEPs this year, and I have three high ability students. And so, the majority of my IEPs are for a learning disability that ... the content is … well above where they are. And then, for my high ability students, the content is well below where they are … I have 26 students in my classroom, and that's a lot of students, all at a different level.

Audrey uses "the bare minimum" of her district's mandated curriculum, because it "is very hard to do independently" and not "grade-level appropriate." She reflects that "it would be easier if, maybe, I had access to the grade before, and their content, because then I could really help my kiddos that are struggling." She expresses the following frustration:

> It's supposed to meet them where they are, and it can meet some of my students who are above average pretty well, but any student that's on grade level or lower, it does not meet them.

She purposefully chooses supplemental materials that allow her to "access pre-k through 12th grade" content. She values the opportunity to differentiate discreetly and describes the following scenario:

> I will have a student that is high ability, that's very gifted at multiplication, and I can find that resource for them to push them farther. And I can do the exact same thing with my lowest

> intervention group. I can put them on skip counting by twos. And, the thing I like about that is that none of the students know who's on what.

Second, Audrey also considers the timing of her math block. She describes the structure of her math block as "wonky" and "sporadic," with "those weird six or seven minutes at the beginning or at the end" due to the special area class schedules and feels "envious" that her co-teacher "has a wonderful, beautiful math block ... every day with no interruption for at least an hour." Audrey values supplemental electronic apps for those inconvenient spots of time (e.g., when there are "six minutes until the bell rings" and she gives students "the option of doing this, this or this" on a device).

Finally, Audrey makes the choice to manage multiple resources "honestly, for the comfort of" her students. Additionally, it is clear through a spontaneous self-reflection that experience has helped her navigate the complexities surrounding that management: "I have gained a lot of confidence in my teaching over the last few years. I'm very passionate about my kids. I'm there late. I'm there early. I love teaching."

Figurative Language. Figuratively, Audrey is resolved to sweeten the "taste" of math for students because she believes they are "burnt out" from textbook-heavy mathematics instruction in previous grades. This metaphor comes to life as she describes teaching graphing skills during a "Peep-themed" day:

> … yesterday, my entire day was Peep-themed because it's getting ready to be Easter. And so, I wanted to find graphing that related to Peeps, without me having to go and make it all, and there was perfect content. So, we had a taste test that we could graph, with all of the students, whose favorite Peep they got to eat... They were hyped up for it. They loved it.

Audrey reasons that "math could get very boring" if instruction is limited to "worksheets from the book" and reports that many of her students "hate math by the time they get to" her. She reasons that "if I want success, and I want my students to learn, I need to find a different way to do that." The idea of changing how mathematics tastes and making it "kid friendly" is repeated throughout her descriptions of teacher-created resources she finds online or creates herself. In addition to her "Peeps" day, her students collect data in an "Oreo" themed unit and participate in regular "parties" where math is paired with food and fun.

Kasie

Key Words. Similarly to Audrey, the most repeated words/phrases for Kasie were affective, with a focus on how students respond positively to curricular activities (e.g., "interest," "enjoy," "fun" [n = 42]). She also spoke often about curricular expectations and mandates (e.g., "standards" [n = 22]). Like Audrey, she worried about meeting the needs of her students, especially those who are the extremes of achievement in her classroom (e.g., students who "struggle" or "are frustrated" [n = 12] and those "higher" students who are "early finishers" [n = 6]).

Lines of Reasoning. Kasie explains her lesson planning process as first referring to the standards presented in the pacing guide, next accessing the curriculum provided by the district to teach them, and finally supplementing with other curricular resources only "as needed." The lines of reasoning she uses to rationalize incorporating supplemental resources revolve around two major themes. First, like Audrey, she discusses the need to engage "struggling" students while at the same time providing independent practice for "early finishers." Second, she talks about the importance of consistency, and discusses the need to establish consistent "patterns" and "routines" for all students in her mathematics lessons.

In terms of meeting the needs of students at the extremes of achievement in her classroom, Kasie focuses mostly on engaging the students who "struggle" in mathematics. She has "a lot of students that have ADHD, or just struggle, in general are lower in math." Her strategy for supporting these students is to use an online educational marketplace to get materials for a "fun way to incorporate math" so students will "engage" and not feel "frustrated." Kasie knows, from teaching experience, the concepts that low-achieving "kids kind of zone out on." For example, with elapsed time, Kasie has found a supplemental detective-themed unit, in which elapsed time problems were integrated as clues for solving crimes around the world:

> But the fact that they were solving a crime and I made them detective badges and gave them magnifying glasses, and just those little extra things that the book doesn't do, or the other stuff that we have access to doesn't give them, made it more interesting for them. And I know that elapsed time is something that our students struggle with, so that was one that I wanted to definitely find something more engaging.

A second major reason Kasie sources supplemental curricular resources is to differentiate the independent practice her students do following whole class instruction. She works to maintain a balance between assigning things that are "too low" for students who would "breeze through it too quickly" and not "too high" or "too difficult," causing students to disengage. Since Kasie has access to different grade levels within an online program purchased by her district, she can either "push" her higher students or "go back a grade level or two" for students who need it. Kasie explains that the application provides "extra practice filler" for her "early finishers" and for students who "don't get to it," it "never counts against their grade."

Finally, Kasie recognizes that beyond learning the mathematics, students can struggle with interacting with multiple materials that present content differently than the textbook (e.g., "... it's more them struggling with figuring out how the program is working, not struggling to figure out the math"). She explains that:

> … it usually is harder the more and more you bring in because the kids get used to one kind of material, and then you throw other stuff at them. Sometimes they're more thrown off by the process of the material, instead of the actual process of the standard.

Because of this, she highlights the importance of establishing a "pattern" or "routine" when planning whole-class math lessons. Kasie describes how and why she chooses to "start with" the adopted curriculum for the "main teaching" of her lessons and then "branch out from there," when needed:

> … we usually follow the same kind of pattern. In the book, it starts with, ... they do a problem solving, and then we watch a video, and then we do some group problems together, then they do independent problems, and we check them. So, kind of following that same pattern. The kids know what to expect, then. Because that's how the material follows in the book.

Figurative Language. Figuratively, Kasie conceptualizes doing mathematics as slow and methodical problem-solving "detective" work. This metaphor comes to life when she describes what she enjoys most about teaching math: "... seeing the kids figure out those different ways of getting answers ... and their thought process through problem solving with it." Kasie values building in time to explore strategies, pause videos for discussion, and encourage her students to collaborate while solving problems. She appreciates the problem-solving opportunities the adopted curriculum provides her students, saying “the kids seem to grasp on and be able to verbalize and talk about how they're getting their answers...” Finally, Kasie nurtures students’ detective skills through engaging them in supplemental “puzzles,” “riddles,” and “brain teasers.”

Jamie

Key Words. Because Jamie is an experienced interventionist, it is not surprising that “standards” is the word she used most frequently (n = 32) when talking about her curricular decision making. Like Audrey and Kasie, Jamie’s transcripts are also full of affective and emotional word choices that describe both her own and her students’ responses to curricular resources. Jamie's affective word choices (e.g., “enjoy”, “like”, “fun”, “interest”, “exciting” [n = 29]) are wholly positive and are especially prevalent when she describes her decision-making for the 30 minutes she has allotted for addressing remediation needs each day. Jamie also makes several references to curricular content in ways different than her peers (e.g., “vocabulary”, "key concept” [n = 15]).

Lines of Reasoning. Jamie pays close attention to the pacing guide windows and standards she is required to cover in organizing her lessons by the week. She introduces a new concept on Monday, engages students in “small-group learning … centers” Tuesday through Thursday, and assesses student progress on Friday. The lines of reasoning she uses to explain how she incorporates supplemental resources revolve around three major themes: boosting student engagement, adapting instruction for differing levels of learning, and deepening her own understanding and experience with the standards.

First, Jamie is concerned with ensuring her students are engaged. For example, she uses a fall-themed project from an online educational marketplace to “review a bunch of different concepts” during intervention time and reports that “the kids enjoy it, because they're sitting there thinking about costumes and things like that, but you know, they're also doing math at the same time.” When introducing new concepts, Jamie creates a chart or poster with students to support her instruction (e.g., drawing “a factor ninja” and talking “about how the factor ninja breaks up the number.”) She does this because “it's something that helps the kids really learn or remember that concept; whereas, if you read it in the textbook, it's just, it's not as engaging … so that's why I tend to go more towards those things.” She also elicits engagement by getting kids “up and moving,” incorporating “current events type of things,” and integrating the use of technology (e.g., using QR codes to “check their answers” on task cards that are spread around the room).

Second, Jamie’s impetus for selecting supplemental resources is to adapt instruction for diverse learners in her classroom. Like Audrey and Kasie, she perceives her students to range widely in terms of achievement. Jamie identifies the students working above grade level as “high-flyers,” those working on grade level as “gen-ed kids,” and those with IEPs as having “different needs.” Like Audrey and Kasie, Jamie values the online program purchased by her district, because she finds it very “adaptable.” She loves that it “breaks standards down into little parts, and it also allows you to go back grade levels or go ahead grade levels. So, it really helps with differentiation, and adapting for every kid that you have in your room.”

Finally, Jamie, like Audrey, is more likely to source alternative resources than use the provided curriculum because “when I look at all of these different places that I’m getting materials from, it helps me to broaden my look at the standard understanding.” When using the district’s materials, she thinks about the cost (e.g., feeling “guilty” for not using them) and efficiency (e.g., using materials because “they’re there”). However, she also believes that her experience across resources makes her a better teacher and more able to support her students:

> I actually think that it's beneficial for the teacher, especially, to be able to see the different materials and make the connections across those, because it gives you a broader view of what deeper understanding for the student looks like, and I see that as being beneficial … I like using different materials for that reason, that I can make sure that they truly understand and it's not just a surface-level understanding.

Figurative Language. Figuratively for Jamie, mathematics is envisioned as quick and efficient “ninja” work. She values “quick, concise videos … that just don’t take a whole lot of time,” anchor charts, and task cards sourced online. She implements speedy formative assessment routines (e.g., exit tickets and asking students to “jot down their answer really quick” on sticky notes). Finally, Jamie values the time-saving practice of following producers of high-quality materials on a social media platform over taking time for her own searches.

Discussion and/or Conclusions

Consistent with Silver (2022) and Hu et al. (2020), all three teachers in this study are engaged in extensive levels of curriculum supplementation and are acting as curriculum curators for their classrooms. Although it is clear that the three teachers featured in this study bring different professional, personal, and practical experiences to their curricular decision-making, several similarities emerge when looking across their reasoning for *why* they source supplemental curricular resources. Silver (2022) argued teachers make these decisions to meet goals specific to their instructional approach to the content, school context, and students’ needs. This study provides support for all three sets of goals.

Most prevalent in the interviews, these three teachers endorse goals related to meeting student needs. They all used affective and emotional language throughout their interviews, especially when speaking about student engagement as well as student struggles and experiences of frustration. These teachers strongly emphasized student engagement as an important strategy for meeting the needs of underperforming students (Kasie, Jamie) and students who do not enjoy the content (Audrey).

Moreover, all three teachers speak about the unprecedentedly wide range of achievement they are currently experiencing in their school context and all three use supplemental resources to provide opportunities outside the range of what is expected by their adopted textbook. The teachers attributed this range in part as an effect of COVID-19 and “unfinished learning” at previous grade levels. Although not explicitly a focus of this study, it has emerged as a significant and interesting topic for future consideration.

In this study, goals associated with the school context included approaches for using the adopted curriculum versus when to supplement. For example, Kasie feels it is important to rely heavily on the adopted curriculum to establish routines while Jamie feels “guilty” for not using the adopted curriculum more, but also feels that she has become a better teacher by seeing the standards taught in different ways across resources. In contrast, Audrey is unapologetic about using “the bare minimum” of the adopted curriculum, believing that it does not do a good job of meeting her students’ needs.

Finally, these teachers brought different supplementation goals related to the content, and specifically to how they envisioned mathematics and their role as mathematics teachers. These especially become clear in the figurative language used by each teacher with Audrey wanting to change the "taste" of math for her students, Kasie prioritizing problem solving "detective" skills, and Jamie looking for efficiencies and to build her student's "quickness" with math.

In conclusion, these findings bring teacher voice to researchers' understandings of the reasoning teachers bring to making decisions across multiple curricular resources. Additionally, they demonstrate that teachers working in the same school context can bring both similar and different goals to their use of curricular resources and teaching mathematics more generally. These findings emphasize the importance of closely attending to teachers' language and reasoning and the value of teacher voice in research.

References

Aguliera, E., & Nightingale-Lee, B. (2020). Emergency remote teaching across urban and rural contexts: Perspectives on educational equity. Information and Learning Sciences, 121(5-6), 471-478. https://doi.org/10.1108/ILS-04-2020-0100

Clandinin, D. J. (2013). Engaging in narrative inquiry. Routledge Taylor & Francis Group.

Brown, S. A., Pitvorec, K., Ditto, C., & Kelso, C. R. (2009). Reconceiving fidelity of implementation: An investigation of elementary whole-number lessons. Journal for Research in Mathematics Education, 40(4), 363-395.

Clandinin, D. J., & Connelly, F. M. (1996). Teachers' professional knowledge landscapes: Teacher stories. Stories of teachers. School stories. Stories of schools. Educational Researcher, 25(3), 24–30. https://doi.org/10.2307/1176665

Dampson, D. G., Apau, S. K., & Amuah, U. (2019). Freedom to choose within limits: Teacher autonomy from the perspectives of basic school teachers in Ghana. *European Journal of Educational Management, 2*(1), 35-44. https://doi.org/10.12973/eujem.2.1.35

Gross, B., & Opalka, A. (2020). Too many schools leave learning to chance during the pandemic. Center for Reinventing Public Education. https://www.crpe.org/sites/ default/files/final_national_sample_brief_2020.pdf

Grossmann, M., Reckhow, S., Strunk, K. O., & Turner, M. (2021). All states close but red districts reopen: The politics of in-person schooling during the COVID-19 pandemic. Educational Researcher, 50(9), https://doi.org/637-648.10.3102/0013189X211048840

Hu, S., Trophy, K. T., Evert, K., Lane, J. L. (2020). From cloud to classroom: Mathematics teachers' planning and enactment of resources accessed within virtual spaces. *Teachers College Record, 112*(6), 1-33. https://doi-org.ezproxy.lib.purdue.edu/10.1177/016146812012200606

Huck, C., & Zhang, J. (2021). Effects of the COVID-19 pandemic on K-12 education: A systematic literature review. New Waves-Educational Research and Development Journal, 24(1), 53-84.

Ormond, B. M. (2017). Curriculum decisions - The challenges of teacher autonomy over knowledge selection for history. Journal of Curriculum Studies, 49(5), 599-619. https://doi.org/10.1080/00220272.2016.1149225

Lakoff, G., & Johnson, M. (1980). Metaphors we live by. University of Chicago Press.

Quinn, N. (2005). Finding culture in talk: A collection of methods. Palgrave Macmillan.

Pepin, B., Gueudet, G., & Trouche, L. (2013). Re-sourcing teachers' work and interactions: A collective perspective on resources, their use and transformation. *ZDM Mathematics Education*, 45, 929–943. doi:10.1007/s11858-013-0534-2

Polikoff, M. S., Campbell, S. R., Koedel, C., Le, Q. T., Hardaway, T., & Gasparian, H. (2020). The formalized processes districts use to evaluate mathematics textbooks, Journal of Curriculum Studies, 52(4), 451-477. https://doi.org/10.1080/00220272.2020.1747116

Reilly, K., & Ball, M. (2020). As the school year approaches, education may become the pandemic's latest casualty, Time. Available online at: https://time.com/5870132/schools-Coronavirus/

Sawyer, A. G., Dick, L. K., & Sutherland, P. (2020). Online mathematics teacherpreneurs developers on Teachers Pay Teachers: Who are they and why are they popular? Education Sciences, 10(9), 248- 268. https://doi.org/10.3390/educsci10090248

Silver, D. (2022). A theoretical framework for studying teachers' curriculum supplementation. *Review of Educational Research, 92*(3), 455-489. https://doi.org/10.3102/00346543211063930
Superfine, A., Marshall, A., & Kelso, C. (2015). Fidelity of implementation: Bringing written curriculum materials into the equation. *Curriculum Journal*, *26*(1), 164–191. https://doi.org/10.1080/09585176.2014.990910
Valant, J. (2020). School reopening plans linked to politics rather than public health. Brown Center Chalkboard. https://www.brookings.edu/blog/brown-center-chalkboard/2020/07/29/school-reopening-plans-linked-to-politics-rather-than-public-health/

DEVELOPING A QUALITATIVE ANALYSIS PROCESS WITH A MULTI-RESEARCHER TEAM

Tara Heikila
Washington State University
tara.heikila@wsu.edu

Nicole Stripling
University of Arkansas
ngstripl@uark.edu

Kate Webster
Brigham Young University
kwebste4@byu.edu

Dawn Teuscher
Brigham Young University
dawn.teuscher@byu.edu

Amy Roth McDuffie
Washington State University
mcduffie@wsu.edu

Shannon Dingman
University of Arkansas
sdingman@uark.edu

Keywords: Curriculum, Instructional Vision, Research Methods.

Teachers use curricular reasoning (CR) as they design and enact instruction with their students, curriculum materials, and standards in mind (Roth McDuffie & Mather, 2009). Teachers' CR has not been measured to the extent of other critical practices: professional noticing (cf., Schack et al., 2017) and facilitating mathematical discussions (cf., Smith & Sherin, 2019). As part of a larger project, we aim to develop and validate a questionnaire and an observation protocol to formatively measure middle school teachers' mathematical CR (Dingman et al., 2021).

Purpose and Relationship to PME-NA's Goals

This poster relates to PME-NA's conference theme of "engaging all learners" in the specific areas of investigating curricula design features by considering student engagement, and interest in supporting learning. This poster presents our research team's qualitative data analysis process. Three subgroups, each with an experienced researcher and a graduate student, applied iterative approaches to identify data patterns for ways middle school mathematics teachers use CR to engage learners. This work illuminates the creativity in data analysis: using established methods for coding data, writing analytic memos, and creating data matrices, we applied these methods in unique ways consistent with the data each subgroup analyzed (Saldaña, 2021).

Methods, Results, and Implications

Our team analyzed multiple pre- and post-interviews for eight teachers to identify the ways teachers used different CR aspects as they made decisions while planning and enacting lessons. This poster will present ways each subgroup analyzed the three CR aspects: *analyzing curricular materials*, v*iewing mathematics from the learner perspective*, and *considering mathematical meaning*. Our approaches shared the common goal of leveraging existing data to capture characteristics of teachers' CR, while maintaining the teachers' perspectives and voices. The subgroups facilitated graduate students' learning about analysis methods through legitimate peripheral participation (Lave & Wenger, 1991) alongside faculty researchers. In turn, faculty learned from graduate students as they questioned why and how we might use different methods. We will illustrate how we created space for dialogue about data analysis, wove six researchers' perspectives together, and discussed different approaches to analyzing data. Our process has implications for other researchers as they consider various approaches to analyzing complex data sets.

Acknowledgements

This material is based upon work supported by the National Science Foundation under Grant DRL-2201167. Any opinions, findings, and conclusions are recommendations expressed in this material are those of the authors and do not necessarily reflect the view of the National Science Foundation.

References

Dingman, S., Teuscher, D., Olson, T. A., & Kasmer, L. A. (2021). Conceptualizing Curricular Reasoning: A Framework for Examining Mathematics Teachers' Curricular Decisions. Investigations in Mathematics Learning, 13(4), 267-286. https://doi.org/10.1080/19477503.2021.1981742. (2021).

Lave, J., & Wenger, E. (1991). Situated learning: Legitimate peripheral participation. Cambridge University Press.

Roth McDuffie, A., & Mather, M. (2009). Middle school mathematics teachers' use of curricular reasoning in a collaborative professional development project. In J. T. Remillard, B. A. Herbal-Eisenmann, & G. M. Lloyd (Eds.), Mathematics teachers at work: Connecting curriculum materials and classroom instruction (pp. 302-320). Routledge.

Saldaña, J. (2021). The Coding Manual for Qualitative Researchers (4th ed.). Sage.

Schack, E. O., Fisher, M. H., & Wilhelm, J. A. (2017). Teacher noticing: Bridging and broadening perspectives, contexts, and frameworks. Springer.

Smith, M., & Sherin, M. G. (2019). The 5 Practices in Practice: Successfully Orchestrating Mathematical Discussion in Your Middle School Classroom. ERIC.

EMBODIMENT AS EVIDENCE FOR STUDENT ENGAGEMENT IN AN INQUIRY-ORIENTED MATHEMATICS CLASSROOM

Kaylee Fantin-Hardesty
Colorado State University
Kaylee.fantinhardesty@colostate.edu

Rachel Tremaine
Colorado State University
rachel.tremaine@colostate.edu

Jocelyn Rios
Colorado State University
Jocelyn.rios@colostate.edu

Hortensia Soto
Colorado State University
Hortensia.Soto@colostate.edu

Student engagement is an impactful component of student experience in mathematics classrooms and can shape academic and affective outcomes. The measurement of engagement in classroom settings has been limited to self-report measures or observational frameworks which privilege verbal participation. By conducting a microanalysis of two students' engagement with a meaningful mathematics task in an inquiry-oriented discrete mathematics classroom, we provide evidence for embodiment as a lens through which to effectively observe behavioral, affective, and cognitive engagement. We emphasize embodiment's potential to (a) illuminate engagement which may have gone unnoticed under observational scales which privilege verbalization, (b) account for the multimodal nature of utterances, and (c) consider the broader classroom context in which students engage with mathematical tasks.

Keywords: Embodied Cognition, Research Methods, Undergraduate Education

Student engagement is an impactful component of students' experiences and outcomes in mathematics classrooms. Increased levels of student engagement support students' academic achievement (Carini et al., 2006; Pilotti et al., 2017) and mediate against negative shifts in mathematics identity (Voigt et al., 2022). Alongside the study of the consequences of student engagement comes the predicament of how to measure it; Fredricks et al. (2016) write that "developing valid and reliable measures [for student engagement] is especially important in math and science because engagement in these subjects is so critical to academic achievement and career choices related to STEM" (p. 14). In this paper, we propose embodied cognition as a lens through which to observe students' engagement with a meaningful mathematics task. We provide evidence for this proposal via a microanalysis of two students' engagement with a meaningful mathematics task and discuss our findings.

We draw upon Fredricks et al.'s (2004) definition of student engagement which comprises of three domains: *behavioral* engagement, *affective* engagement, and *cognitive* engagement. *Behavioral* engagement consists of on-task, participatory classroom actions that indicate a student is performing classroom tasks in the way that the cultural and institutional authority in that classroom expects. *Affective* engagement encompasses emotional responses to and emotional investment in the task at hand and its associated discoveries, and *cognitive* engagement involves students' self-regulation of their learning and thoughts, and their perseverance in the face of cognitive challenge. While additional domains of engagement have been proposed and utilized (Joshi et al., 2022; Quintero et al., 2022; Veiga, 2016), we utilize Fredricks et al.'s (2004) domains due to their specific prevalence in mathematics engagement literature, and the ways in which they provide a bases for other methods of observing engagement in the classroom.

Comprehensive pictures of student engagement are best painted when they account for the interplay between the three domains. Fredricks et al. (2004) note that the three domains are necessarily “dynamically interrelated” (p. 61), and students’ experiences of engagement in one domain are correlated with experiencing engagement in others (Böheim et al., 2020; Joshi et al., 2022). Measuring engagement as a multifaceted construct in the mathematics classroom most often utilizes self-reported data (Fredricks & McColskey, 2012), which is valuable to studies of engagement (Appleton et al., 2006; Garcia & Pintrich, 1991), and is also limited. Survey items are often broad enough to be interpreted in several ways and may not be answered with fidelity to the researchers’ interpretation. This limits what can be gleaned from self-reports regarding students’ engagement in the classroom (Fredricks & McColskey, 2012; Hodgson et al., 2017).

Consequently, scholars have investigated how one might observe student engagement in the classroom. For example, hand-raising or designated on/off-task behaviors have been proposed as observable measures of behavioral engagement (Böheim et al., 2020; Hodgson et al., 2017). While these do provide methods through which to observe student engagement, verbal participation is often privileged in such scales. Women, students of color, and individuals of lower academic status may have less opportunities to engage in verbal participation (Black & Radovic, 2018; Civil, 2014; Fink, 2022). For example, Böheim et al.’s (2020) notion of hand-raising as an observable indicator of student engagement is rooted in the assumption that *all* students in a classroom are equally comfortable contributing vocally to a whole-class discussion. If we are to study student engagement in the classroom, there exists a need to develop, refine, and advocate for the use of observational tools which meaningfully incorporate engagement beyond verbalization in the classroom. This need for mechanisms to document classroom engagement that do not privilege but incorporate verbalizations led us to explore embodiment as a lens through which student engagement could be effectively observed.

Theoretical Framing: Embodied Cognition

In this paper, we use the terms “embodiment” and “embodied cognition” interchangeably and draw from Radford’s (2009) definition of embodied cognition as the notion that “thinking does not occur solely in the head but also in and through a sophisticated semiotic coordination of speech, body, gestures, symbols, and tools” (p. 111). This notion insinuates that evidence of students’ internal learning processes—including their engagement—goes beyond inner cognitive dialogue and expresses itself in a myriad of observable ways in the classroom. Within an embodied cognition paradigm, we focus on the concept of “utterances” as a categorical lens through which to conceptualize embodiment, as presented by Nemirovsky and Ferrara (2009). Utterances are body activities and actions involved in a conversation. They include gesture, gaze, body poise, body motion, facial expressions, eye motion, and more. Further, when a speaker includes an object in the conversation, the object becomes part of an utterance.

An embodied cognition lens has been used in the classroom for a variety of purposes, including to study how students collectively discover mathematics (Nemirovsky et al., 2012), and to understand the role that tools and materials have on students’ cognition (Radford, 2014). Embodied cognition is a lens that pays mind to verbal participation but it also, by its very nature, incorporates many other forms of classroom participation. While some scholars study how incorporating embodiment into task design can enhance engagement with those tasks (Georgiou & Iannou, 2020; Lindgren et al., 2016), we intentionally studied embodiment in the context of an activity not explicitly designed to be embodied to effectively assess whether embodiment can provide evidence for student engagement in the mathematics classroom. Our investigation centers around the research question: *In what ways can embodiment provide evidence for students' engagement with meaningful mathematical tasks?*

Methodology

Setting and Data

This study took place in Fall 2022 at a large research university in the Rocky Mountain region of the United States. The class selected for this observation is an undergraduate Discrete Mathematics course, within which an experienced instructor (Dr. A) follows an Inquiry-Based Mathematics Education (IBME) teaching paradigm. In IBME, instructors encourage students to participate in the classroom in ways which reflect actions of expert mathematicians such as exploring patterns, generating conjectures, proving theorems, (re)inventing definitions, and comparing solutions (Laursen & Rasmussen, 2019). Such classrooms are an optimal setting to investigate student engagement, as one of its four pillars is student engagement with meaningful mathematics tasks as one (Laursen & Rasmussen, 2019).

Over the course of two class periods, we videotaped a total of 2.5 hours of data. Early in the research process, we narrowed our focus to a particular task Dr. A set forth for the students: the "Locker Problem." We refined our unit of analysis by focusing on students who engaged with the task using two-color counters, provided by Dr. A. When asked to split into groups, four students from two groups– Rebecca, Lauren, Carly, and Jason– all chose to use the two-color counters to tackle the problem. While Carly and Jason immediately began working together, Rebecca and Lauren worked individually for several minutes before coming together to enhance their understanding of the problem. It is for this reason– the presence of both individual and social interactions with the task– that we narrowed our unit of analysis to be the videotaped data of Rebecca and Lauren.

Analysis

Video data of Lauren and Rebecca engaging in the Locker Problem task totaled approximately 13 minutes of our collected data. We transcribed Lauren and Rebecca's verbal utterances and performed three phases of coding. The initial phase entailed descriptive writing through an *ethnographic microanalysis of interaction* (microanalysis) (Garcez, 1997). Garcez's work details this style of analysis as one in which communication necessarily "involves conversationalists contained in physical bodies, occupying space in simultaneously constraining and enabling social situations, who must reflexively make sense of each other's actions as they act" (p. 187). Microanalysis has a rich history in embodied cognition literature (Alibali et al., 2019; Nemirovsky et al., 2020; Kelton & Ma, 2020; Walkington et al., 2019) and entails transcribing participants' second-by-second utterances for the entirety of the 13 minutes of data.

Once we comprehensively described the data in this way, we selected smaller snippets of 1-3 minutes, and within those episodes, smaller excerpts of 10-25 seconds (as modeled by Nemirovsky et al., 2020). These clips exemplified instances in which Lauren and Rebecca needed to "reflexively make sense of each other's actions as they act" (Garcez, 1997, p. 187). The interactions chosen were not exclusively between Lauren and Rebecca; Episode 1 centers around Rebecca making sense of Dr. A's verbalizations, Episode 2 centers around Rebecca and Lauren reaching a shared understanding, and Episode 3 involves Rebecca and Lauren grappling with another group's verbalized conclusion. In each of these chosen episodes, we position Rebecca and Lauren as agentic bodies in a wider classroom space, one rich with embodiment in interactional ways.

Upon selecting these episodes, we engaged in two rounds of thematic analysis (Braun & Clarke, 2006) using *a priori* codes from the embodiment and engagement literature. The first round entailed coding the specific types of utterances as aligned with those provided in Nemirovsky and Ferrara (2009): sound production, eye motion, facial expression, gaze, body motion, body poise, tone of voice, and hand gesture. As we knew Lauren and Rebecca worked with the counters as necessary materials in their learning process, we also included "materials" as an *a priori* code. Because Nemirovsky and Ferrara explicitly acknowledge that additional types of utterances other than those they specify exist, we remained open to acknowledging and describing other forms of embodiment. During our second round of coding, we coded the type of engagement evidenced by the embodiment of the participants: behavioral, affective, or cognitive. This was done with careful consultation of Fredricks et al.'s (2004) definitions and descriptions. Organized into episodes, below we provide context for each selected episode, include details of our microanalysis, and provide screenshots from the episode which illustrate the students' engagement in embodied ways.

Results

Just prior to the beginning of our first selected episode, Lauren and Rebecca worked individually on the locker problem. They both used the two-colored counters and quietly worked through their worksheets. The two students did not talk with each other, but Rebecca leaned in towards Lauren, indicating that she seemingly wanted to engage with Lauren. Dr. A walked over to the two students and began conversing with Lauren.

Episode 1: Responsive Resetting (04:58 – 05:25)

When Dr. A walked over to Lauren and Rebecca, she prompted them to explain their thinking. Lauren verbally discussed what she had written on her worksheet, and she and Dr. A engaged in a verbal conversation about Lauren's thought process. In this episode, Dr. A attempted to reorient Lauren toward the problem, but simultaneously elicited non-verbal engagement from Rebecca.

Rebecca was leaning in towards Lauren and Dr. A, apparently listening to their conversation and thus behaviorally engaging. After several seconds, at time 05:04, Rebecca leaned back and reached for the counters in front of her. As Dr. A said, "If I rephrase that question as, we have the lockers of ten students" (Timestamp 05:10 – *Figure 1*), Rebecca began resetting her counters to their original "closed" position. This counter-resetting action appeared to be in direct response to Dr. A's verbalized notion of "rephrase[ing] the question." While Dr. A's verbalization was directed toward Lauren, Rebecca clearly thought about Dr. A's remarks in an on-task and participatory way and evaluated her own prior notions of the problem. Rebecca was behaviorally engaged in her attention and response to Dr. A's verbalization as well as cognitively engaged as evidenced by her resetting of the counters.

After she reset her counters, Rebecca laid her hand flat with fingers spread and started rearranging her counters (Timestamp 05:15 – *Figure 1*). Her use of gesture and her continued interaction with materials indicated her ongoing behavioral engagement. She then used her right pointer finger to point to the counters one-by-one, and in doing so extended her material environment to a line of 20 "closed" counters. After Rebecca had rearranged her counters and added more in front of her, she pulled away from her counters and looked over at Lauren's worksheet, where Dr. A was pointing (Timestamp 05:24 – *Figure 1*). Her body position and eye gaze were demonstrative of further, continued behavioral engagement.

Figure 1: Selected Screenshots from Episode 1

Between Episodes 1 and 2

Fourteen seconds after the conclusion of Episode 1, Dr. A left the students. For twelve seconds, the two sat in silence, until Lauren turned to Rebecca and verbally speculated on pattern existence present in their counters. Lauren listed off the "open" counters in front of her– counters one, four, and nine– while Rebecca pointed to those "open" counters using the counters in front of her. This did not seem to lead them toward any conclusive thoughts, so Rebecca began counting something else: the number of "open" counters between the "closed" ones.

Episode 2: Collaborative Conjecturing (01:33-01:48)

This episode begins when Rebecca started counting the "closed" counters in between the "open" counters, both verbally and by using spread-out fingers of her left hand. She drew Lauren's attention to the numeric value of these gaps, and in doing so, they came to a shared understanding of the pattern they saw without verbally expressing it. At 01:35, Lauren's abrupt change in body posture, and deliberate reaching toward Rebecca (*Figure 2)* indicated that Lauren had made a relevant realization. Without verbalizing the realization, she gained confirmation from Rebecca that they were on the same page cognitively from Rebecca's clapping and pointing toward Lauren (Timestamp 01:37 – *Figure 2*). The ways in which they both positioned themselves and reached toward each other with their gestures indicated to the other collaborator that they arrived at a similar conclusion on the pattern they saw (namely, that each "open" counter was separated by an increasing multiple of 2 "closed" counters). Rebecca's reiteration of the counts confirmed their shared understanding of the pattern at timestamp 01:41. Lauren's verbalization of "Dude, that's so cool!" and her brief eye contact with Rebecca seemed to confirm that their understanding of this pattern was a shared understanding as developed by Rebecca's verbal counting and gestures and a source of shared excitement.

Rebecca's gestures and associated verbalizations during her initial verbal and gestural counting was evidence of on-task participation and thus of behavioral engagement. The point at which Lauren abruptly shifted her body posture and extended her arm indicates a realization; we infer that this was the point when Lauren identified the pattern and mentally made a conjecture about how future "closed" lockers will be separated. Her extended arm gesture, body position, and facial expressions demonstrated behavioral and cognitive engagement associated with coming to a conjecture. Both students' smiling and laughing indicated that they were affectively engaged with the activity.

Episode 2 - 01:35

Episode 2 - 01:37

Figure 2: Selected Screenshots from Episode 2

Between Episodes 2 and 3

After making the conjecture detailed in Episode 2, the students sat in silence for approximately six seconds (1:49 - 1:55). Then Lauren proposed that they try the problem using thirty counters, and she and Rebecca pushed their desks together, end-to-end, and started combining their counters into a continuous line spanning both of their desks. This resulted in one long material environment with 30 counters. However, they did not reset their counters, leading them to neglect to account for integers 11-20. This left them with the conclusion that 1, 4, 9, 16, 22, 24, 25, 26, 27, and 30 are "open," breaking their earlier conjecture. They grew quiet, seemingly recognizing that their conjecture did not work and stared silently at Lauren's worksheet for 13 seconds. We continue with the embodied narrative in Episode 3.

Episode 3: Second-guessing Squares (00:44-00:55)

In this episode, Lauren and Rebecca attempted to reconcile the materials in front of them with what they heard from the classroom context around them. This episode centers on the embodiment they illustrated as they listened to a group that was positioned in front and to the right of where they were seated in the classroom. A student member of that group, Anthony, told Dr. A that he believed that any locker with a number that is a square should be closed. This contradicted the counters on display in front of Lauren and Rebecca.

Lauren and Rebecca's observable embodiment was influenced by their classroom surroundings. In front of them, counters 1, 4, 9, 16, 22, 24, 25, 26, 27, and 30 were "closed," making Anthony's claim that all squares should be closed contradictory to the cues they received from their counters. Their acknowledgement of this contradiction started at the beginning of the episode (Timestamp 00:49), where they both expressed confusion and focused on Dr. A and Anthony's conversation. At 00:50, they shared this confusion with each other via expressive eye contact. Both embodied uncertainty of what to make of Anthony's claim and Dr. A's validation of it via their facial expressions. At 00:52, they turned back to their respective materials (the worksheet for Lauren and the counters for Rebecca), possibly attempting to reconcile the notion of squares being "open" with what they had observed in their materials or had written down. *Figure 3* showcases this sequence of events. While Anthony in the front group began formalizing his group's conjecture, Lauren and Rebecca continued to express confusion. Lauren looked at the counters and shook her head and Rebecca brought her left hand to her forehead in an apparent show of deep thought about the counters in front of them.

In this episode, Lauren and Rebecca's embodiment provided evidence for cognitive and affective engagement. The affective engagement was well-evidenced by the facial expressions they individually adopted as they listened to Anthony's conjecture and the facial expressions they wore as they surveyed their own counters which contradicted Anthony's assertion. They also illustrated their individualized affective engagement in their body movements and positionings. At 00:52 Lauren shook her head as she looked at her worksheet; this movement indicative of "no" seemed to apply to the mismatch between Anthony's words and the pair's observations. At the end of the episode, Rebecca brought her left hand to her forehead, indicating that she was deep in thought and thus cognitively engaged with the material. The students' cognitive engagement was also evidenced by their facial expressions as their confusion indicated that they grappled seriously with Anthony's conjecture, which led to the cognitive dissonance associated with their materials illustrating something else.

Episode 3 - 00:49 Episode 3 - 00:50 Episode 3 - 00:52

Figure 3: Selected Screenshots from Episode 3

In each episode, a multitude of utterances were observed. Further, no domain of engagement existed purely on its own; embodiment evidenced cognitive and behavioral engagement in Episode 1, all three domains in Episode 2, and affective and cognitive engagement in Episode 3. *Table 2* below provides documentation of which utterances were coded as evidencing which forms of engagement throughout the three episodes.

Table 1: Types of Utterances which Evidenced Domains of Engagement

Behavioral	Affective	Cognitive
Gaze	Gaze	Gaze
Gesture	Gesture	Gesture
Body Poise	Facial Expression	Body Poise
Materials	Body Movement	Facial Expression
Verbalization	Sound Production	Materials
Environment alteration	Verbalization	

Discussion

To begin the discussion, we restate our research question: *In what ways can embodiment provide evidence for students' engagement with meaningful mathematical tasks?* We saw that various types of utterances provided evidence for Lauren's and Rebecca's collaborative (Episode 2 and Episode 3) and individual (Episode 1) engagement with the Locker Problem task. *Table 1* is a summary of the types of utterances exhibited by the participants for each of Fredricks et al.'s (2004) domains of engagement. Through our microanalysis, we assert that embodiment provides evidence for students' engagement in ways which (a) illuminate engagement which may have gone unnoticed under observation scales which privilege verbalization, (b) account for the multimodal nature of both utterances and engagement as a construct, and (c) account for the broader classroom context in which the students engage with the task.

In each episode, studying embodiment as evidence for student engagement revealed domains of engagement which may have gone unnoticed had we purely looked at the students' verbalizations. This is particularly clear in Episode 1 via Rebecca's embodied responses to Dr.

A's verbal prompting. In addition, the cognitive engagement associated with conjecturing in Episode 2 may have gone unnoticed without an embodied lens, as Lauren and Rebecca *never* directly verbalized or wrote down their shared conjecture, but rather communicated and verified its existence in an embodied way. Further, the only verbalization made in Episode 3 was Lauren's rhetorical "What?" at timestamp 00:50, but additional realms of embodiment (e.g., gaze and facial expression) indicate that Lauren and Rebecca were continually engaged with the task and attempted to reconcile Anthony's assertion with their own materials.

Further, the utterances which provided evidence for the three domains of engagement enabled us to describe the students' engagement in a way reflective of their interrelated nature. Much of the embodiment that we observed did not fall squarely into one category of engagement or utterance, but rather provided evidence for multiple domains and types respectively. For example, in Episode 2, Rebecca laughed and clapped her hands, and then pointed to Lauren as she cognitively constructed her conjecture. This was coded as Facial Expression, Sound Production, and Gesture (types of utterances), and Affective and Cognitive (domains of engagement). The multiple assignment of *a priori* codes is unsurprising, given what Nemirovsky and Ferrara (2009) refer to as the "multimodality" of any given utterance, as well as the interactional nature of Fredricks et al.'s (2004) three types of engagement. Much of the embodiment we observed was built from several simultaneous utterance types, and illustrated multiple different domains of engagement, as is reflective of the nature of these constructs.

Embodiment, as an evidential lens for student engagement, enabled consideration of the classroom context. This claim has been supported by literature indicating that the social, physical, and technological classroom context influences student engagement (Hodgson et al., 2017; Kahn, 2014; Keith, 2016). In Episode 3, the cognitive and affective engagement the students experienced as they tried to reconcile their counters with the conclusion of the group in front of them was evidenced by the embodied ways in which they responded to the other groups' verbalizations. Their confused facial expressions, eye gaze between their materials and the other group, and eye contact between each other served as evidence of their attempts to make sense of this finding, and thus be cognitively and affectively engaged in the continuation of the task. Our microanalysis of embodiment allowed for this by mediating the students' responsive utterances with the broader classroom context in which they were situated.

Both triangulation of our data with other observational and self-report measures, as well as any comparative analysis with other mathematics courses, other majors, or other institutional settings, is an important topic for future work. We also note that embodiment, as is the case with other observational scales for student engagement, may be limited in regards of what we are seeing. For example, the embodiment Lauren and Rebecca exemplified in Episode 3 illustrated affective and cognitive engagement. If we consider the linkage between cognitive and behavioral engagement established by Böheim et al. (2020), it is likely that they were also behaviorally engaged, although that wasn't evidenced by their utterances. We further recognize the need to triangulate these data with other measures of engagement, such as self-reports. This is particularly relevant for cognitive engagement, as it foundationally relies on the notion of students self-regulating their own learning (Fredricks et al., 2004), something which an observer can only speculate on without concrete knowledge of that students' thought process.

Conclusion

This paper has set forth an evidence-based argument for the potential of embodiment, particularly Nemirovsky and Ferrara's (2009) notion of utterances, as a novel way through which to observe behavioral, affective, and cognitive engagement in a mathematics classroom. In proposing the lens of embodiment as an observation tool, we broaden who is seen as engaging in classroom mathematics tasks, as we assert that engagement happens in embodied ways which are *not* purely verbal. We know that students experience high levels of engagement within embodied tasks (Georgiou & Iannou, 2020; Lindgren et al., 2016); our work builds upon this notion by asserting that embodiment in and of itself is indicative of student engagement. This study demonstrates its potential to provide researchers and instructors a novel way to view their students' engagement with meaningful mathematical tasks in the classroom.

References

Alibali, M. W., Nathan, M. J., Boncoddo, R., & Pier, E. (2019). Managing common ground in the classroom: Teachers use gestures to support students' contributions to classroom discourse. *ZDM*, *51*(2), 347-360. http://dx.doi.org/10.1007/s11858-019-01043-x

Appleton, J. J., Christenson, S. L., Kim, D., & Reschly, A. L., (2006). Measuring cognitive and psychological engagement: Validation of the Student Engagement Instrument. *Journal of School Psychology, 44*(5), 427-445. http://dx.doi.org/10.1016/j.jsp.2006.04.002

Black, L., & Radovic, D. (2018). Gendered positions and participation in whole class discussions in the mathematics classroom. In U. Gellert (Ed.), *Inside the Mathematics Class* (pp. 269-289). Springer. http://dx.doi.org/10.1007/978-3-319-79045-9_13

Böheim, R., Urdan, T., Knogler, M., & Seidel, T. (2020). Student hand-raising as an indicator of behavioral engagement and its role in classroom learning. *Contemporary Educational Psychology, 62*. https://doi.org/10.1016/j.cedpsych.2020.101894

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology, 3*(2), 77-101. http://dx.doi.org/10.1191/1478088706qp063oa

Carini, R.M., Kuh, G. D., & Klein, S. P. (2006). Student engagement and student learning: Testing the linkages. *Research in Higher Education, 47*(1). http://dx.doi.org/10.1007/s11162-005-8150-9

Civil, M. (2014). Guest editorial: Musings around participation in the mathematics classroom. *The Mathematics Education, 23*(2), 3-22.

Fink, H. (2022). Obscuring racial inequities through gender-focused discourse: An analysis of student participation in a calculus class. In Lischka, A. E., Dyer, E. B., Jones, R. S., Lovett, J. N., Strayer, J., & Drown, S. (Eds.), *Proceedings of the 44th Annual Meeting of the North American Chapter of the International Group for the Psychology in Mathematics Education* (pp. 387-395). Nashville, Tennessee, USA.

Fredricks, J. A., Blumenfeld, P. C., & Paris, A. H. (2004). School engagement: Potential of the concept, state of the evidence. *Review of Educational Research, 74*(1), 59–109. http://dx.doi.org/10.3102/00346543074001059

Fredricks, J. A., & McColskey, W. (2012). The measurement of student engagement: A comparative analysis of various methods and student self-report instruments. In S.L. Christenson, A. L. Reschly, & C. Wylie (Eds.), *Handbook of Research on Student Engagement* (pp. 763-782). Springer. http://dx.doi.org/10.1007/978-1-4614-2018-7_37

Fredricks, J. A., Wang, M., Shall Linn, J., Hofkens, T. L., Sung, H., Parr, A., & Allerton, J. (2016). Using qualitative methods to develop a survey measure of math and science engagement. *Learning and Instruction, 43*, 5-15. http://dx.doi.org/10.1016/j.learninstruc.2016.01.009

Garcez, P. M. (1997). Microethnography. In N.H. Hornberger & D. Corson (Eds.), *Encyclopedia of Language and Education* (pp. 187-196). Springer. http://dx.doi.org/10.1007/978-94-011-4535-0_18

Garcia, T., & Pintrich, P. R. (1991). Student motivation and self-regulated learning: A LISREL model. Office of Educational Research and Improvement, United States Government.

Georgiou, Y., & Ioannou, A. (2020). Developing, enacting and evaluating a learning experience design for technology-enhanced embodied learning in math classrooms. *TechTrends, 65*(1), 38– 50. http://dx.doi.org/10.1007/s11528-020-00543-y

Hodgson, T. R., Cunningham, A., McGee, D., Kinne, L. J., & Murphy, T. J. (2017). Assessing behavioral engagement in flipped and non-flipped mathematics classrooms: Teacher abilities and other potential factors. *International Journal of Education in Mathematics, Science, and Technology, 5*(4), 248-261. http://dx.doi.org/10.18404/ijemst.296538

Joshi, D. R., Adhikari, K. P., Khanal, B., Khadka, J., & Belbase, S. (2022). Behavioral, cognitive, emotional, and social engagement in mathematics learning during COVID-19 pandemic. *Plos one, 17*(11). http://dx.doi.org/10.1371/journal.pone.0278052

Kahn, P. E. (2014). Theorising student engagement in higher education. *British Educational Research Journal, 40*(6), 1005-1018. http://dx.doi.org/10.1002/berj.3121

Keith, H. (2016). *The effects of community-building on achievement, motivation, and engagement in undergraduate mathematics* [Thesis, Western Kentucky State University].

Kelton, M. L., & Ma, J. Y. (2020). Assembling a torus: Family mobilities in an immersive mathematics exhibition. *Cognition and Instruction, 38*(3), 318-347. http://dx.doi.org/10.1080/07370008.2020.1725013

Laursen, S. L., & Rasmussen, C. (2019). I on the Prize: Inquiry approaches in undergraduate mathematics. *International Journal of Research in Undergraduate Mathematics Education, 5*(1), 129–146. https://doi.org/10.1007/s40753-019-00085-6

Lindgren, R., Tscholl, M., Wang, S., & Johnson, E. (2016). Enhancing learning and engagement through embodied interaction within a mixed reality simulation. *Computers & Education, 95*, p. 174-187. http://dx.doi.org/10.1016/j.compedu.2016.01.001

Nemirovsky, R., & Ferrara, F. (2009). Mathematical imagination and embodied cognition. *Educational Studies in Mathematics, 70*(2), 159-174. http://dx.doi.org/10.1007/s10649-008-9150-4

Nemirovsky, R., Rasmussen, C., Sweeney, G., & Wawro, M. (2012). When the classroom floor becomes the complex plane: Addition and multiplication as ways of bodily navigation. *Journal of the Learning Sciences, 21*(2), 287–323. http://dx.doi.org/10.1080/10508406.2011.611445

Nemirovsky, R., Ferrari, G., Rasmussen, C., & Voigt, M. (2020). Conversations with materials and diagrams about some of the intricacies of oscillatory motion. *Digital Experiences in Mathematics Education, 7*(1), 167-191. http://dx.doi.org/10.1007/s40751-020-00073-5

Pilotti, M., Anderson, S., Hardy, P., Murphy, P., & Vincent, P. (2017). Factors related to cognitive, emotional, and behavioral engagement in the online asynchronous classroom. *International Journal of Teaching and Learning in Higher Education, 29*(1), p. 145-153.

Quintero, M., Hasty, L., Li, T., Song, S., & Wang, Z. (2022). A multidimensional examination of math anxiety and engagement on math achievement. *British Journal of Educational Psychology, 92*(3), 955-973. http://dx.doi.org/10.1111/bjep.12482

Radford, L. (2009). Why do gestures matter? Sensuous cognition and the palpability of mathematical meanings. *Educational Studies in Mathematics, 70*(2), 111–126. http://dx.doi.org/10.1007/s10649-008-9127-3

Radford, L. (2014). Towards an embodied, cultural, and material conception of mathematics cognition. *ZDM, 46*(3), 349–361. https://doi.org/10.1007/s11858-014-0591-1

Veiga, F. H. (2016). Assessing student engagement in school: development and validation of a four dimensional scale. *Procedia: Social and Behavioral Sciences, 217*, 813-819. http://dx.doi.org/10.1016/j.sbspro.2016.02.153

Voigt, M., Hagman, J. E., Street, C., Guglielmo, J., Martinez, A., & Tremaine, R. (2022). A quantitative critical analysis of instructional practices and math identity. In A. E. Lischka, E. B. Dyer, R. S. Jones, J. N. Lovett, J. Strayer, & S. Drown (Eds.), *Proceedings of the 24th Annual Conference on Research in Undergraduate Mathematics Education*, p. 666-674. Boston, Massachusetts, USA.

Walkington, C., Chelule, G., Woods, D., and Nathan, M. J. (2019). Collaborative gesture as a case of extended mathematical cognition. *Journal of Mathematical Behavior, 55*. http://dx.doi.org/10.1016/j.jmathb.2018.12.002

EXPANDING ON COMPLEX INSTRUCTION THROUGH LEARNER AGENCY

Larissa Peru
University of Arizona
larissap@arizona.edu

Kasie Betten
Sunnyside Unified School District
kasieb@susd12.org

Since the social turn in mathematics education pushed mathematics educators and researchers to "see meaning, thinking, and reasoning as products of social activity" (Lerman, 200, pg. 8), there has been a growing field of research regarding the design and implementation of collaboration in mathematics instruction. From a secondary teacher perspective, the application of this pedagogical philosophy often falls short without critical self-reflection. Students cannot reap the benefits of mathematics from a social lens if educators do not provide the classroom structures to allow for agentic learning. This poster will present the theoretical framework, methods, and implications of an expansion on Complex Instruction (Cohen & Lotan, 2014) as a means to address the following research question: How does designing instruction around learner agency impact students' mathematical identity?

Cohen and Lotan (2014) offer Complex Instruction (CI) as a means to create classroom communities where all students have access and opportunity; all students feel capable of doing mathematics; and all students' mathematical competencies are recognized, honored, valued, and leveraged. This curricular innovation offers a marriage of this philosophy with Black and William's (2009) conceptualization of Formative Assessment (FA). Under this philosophy, lessons: clarify the learning target and success criteria; facilitate classroom discussion that elicits evidence of student understanding; provide opportunities for feedback that moves learning forward; allow students to see their peers as resources; and activate students' ownership of their learning. This curricular innovation focuses on expanding the structures and guidelines for collaboration, peer feedback, and self-assessment from CI, to allow students to demonstrate learner agency.

Over the course of one academic school year,12th-grade mathematics students were engaged in instructional structures to build their capacity for collaboration, peer feedback, self-assessment, and ultimately, learner agency. Table 1 outlines the implemented structures, as well as their association with elements of CI and FA.

Complex Instruction	Structure	Formative Assessment
Establish and Maintain Norms	Consistent (nearly daily) reflection on the lesson's learning target and success criteria establishes the norm for personal and community accountability for learning	Clarity around the learning target and success criteria
Multiple Abilities Treatment	Multiple opportunities to elicit varied forms of evidence from students are intentionally planned for both the teacher and student-to-student	Eliciting evidence of student understanding and using peers as a resource
Assigning Competence	Specific and intellectually meaningful feedback is provided throughout a lesson to build a sense of confidence for students	Feedback that moves learning forward

Table 1. CI and FA Structures

Ultimately, students were able to articulate what they discovered about themselves as learners. They found value in collaborating with their peers as they recognized the opportunity for unique perspectives to extend their own understanding. Rather than viewing the teacher as the keeper of knowledge who grants a summative grade, the teacher was viewed as a partner in learning. Because instructional decisions were made clear to students, learning was demystified, and students developed a stronger sense of self in relation to the content. Without intentional classroom structures, social learning in a mathematics classroom fails to make a significant impact on students' mathematical identity. The goal of this poster is to encourage educators to think about how CI and FA can cooperate to forge agentic learners who are empowered to take charge of their learning.

References

Lerman, S. (2000). The social turn in mathematics education research. *Multiple perspectives on mathematics teaching and learning*, *1*, 19-44.

Black, P., & Wiliam, D. (2009). Developing the theory of formative assessment. *Educational Assessment, Evaluation and Accountability (formerly: Journal of personnel evaluation in education)*, *21*, 5-31.

Cohen, E. G., & Lotan, R. A. (2014). *Designing groupwork: Strategies for the heterogeneous classroom* (3rd ed.). New York, NY: Teachers College Press.

EXPLORING INTELLECTUAL AUTHORITY IN WORK-SHARING INTERACTIONS IN ONE SIXTH-GRADE MATHEMATICS CLASSROOM

Miriam Simone Leshin
Stanford University
mLeshin@stanford.edu

Opportunities for students to share their thinking with the class—which I refer to as "work-sharing practices—require a profound shift in who is positioned with intellectual authority in mathematics classrooms. This study explores work-sharing practices in one sixth grade mathematics classroom through an interactional lens. Video analysis revealed three types of work-sharing interactions along a continuum of distributions of intellectual authority, ranging from the presenting student holding authority to the teacher holding authority. Notably, in the center of the continuum were instances in which the student was initially positioned with authority, but that authority shifted to the teacher in the next moment, largely based on correctness of the student's work. Findings suggest the need to deepen our understanding of authority dynamics in work-sharing interactions, as the field works to center students' thinking.

Keywords: Instructional activities; equity; classroom discourse

Purpose of study

For the past several decades, mathematics education researchers and policymakers have called for increased opportunities for students to generate their own mathematical strategies, to share that thinking with their peers, and to critique their peers' reasoning (Common Core State Standards, 2013; National Council of Teachers of Mathematics, 2000; California Department of Education, 2021). Research has shown that teachers can implement specific pedagogies to elicit, attend to, and build on students' mathematical ideas (Carpenter et al., 1996; Fennema et al., 1993), leveraging students' novel thinking as an instructional resource for their peers' learning (Singer-Gabella et al, 2016) and supporting students to engage with each other's thinking (Franke et al 2015). These instructional strategies have been shown to be both effective and equitable in terms of increasing students' opportunities to learn (Bartell et al, 2017).

Opportunities for students to share their thinking with the class—which I refer to as "work-sharing practices"—require a profound shift in who is positioned as holding intellectual authority in mathematics classrooms. Whereas teachers and students typically position the teacher, textbook, and/or mathematics itself as the authority in traditional math classrooms (Amit & Fried, 2005), inviting students to share their thinking with their peers reconfigures the classroom such that students come to be viewed as active constructors—rather than passive receivers—of knowledge who can author their own ideas (Boaler & Greeno, 2000). Work-sharing practices have the potential to shift the social negotiation of authority in classrooms (Langer-Osuna, 2016), to redefine and expand who is seen as competent (Gresalfi et al, 2009), and to subvert narrow and racialized notions of mathematical ability (Louie, 2017; Shah, 2017).

Like any pedagogy, work-sharing practices can be implemented in a range of ways, such that the extent to which students are positioned with intellectual authority and their thinking is physically and intellectually centered may vary across classrooms. Many studies have examined work-sharing in terms of teachers' instructional moves (Smith et al, 2011; Franke et al, 2015), viewing the teacher as the main determinant of how students share their work and how they come to be viewed by their classmates. Given the co-constructed and dynamic nature of classroom interaction (Philip et al, 2020), however, there is much to be learned about how teachers and students construct these practices in and through interaction. Within work-sharing interactions, some students may be positioned with authority while others may not be (Wood, 2013; Langer-Osuna, 2016) and some students' ideas may be privileged while others may be silenced (Leander, 2002). Unpacking the ways in which intellectual authority is distributed across and shifts within work-sharing interactions is essential to the implementation of pedagogies that robustly and equitably center students' thinking. This case study of one sixth-grade math classroom explores work-sharing through an interactional lens to investigate:

6. How is intellectual authority distributed during work-sharing practices?
7. When and under what conditions does intellectual authority shift in the moment, if at all?

Theoretical framework

I draw on literature on interactional positioning and intellectual authority within mathematics classrooms. Cobb and colleagues (1996, 2001) defined "classroom mathematical practices" as mathematical activities that become established over time and are locally and jointly constructed by students and teachers through interaction. In this study, I use the term "practices" to refer to co-constructed interactional routines in which students share their mathematical work publicly during whole-class discussions. Viewing work-sharing practices as constructed by teachers and students through interaction affords lenses on how teachers and students position themselves and each other, which may be consequential for how authority is distributed (Cobb et al, 2009).

To consider how some participants in a practice come to be seen as holding intellectual authority in particular moments, I draw on the construct of positioning. Davies and Harré (1990) define moment-to-moment positioning as, "the discursive process whereby selves are located in conversations as observably and subjectively coherent participants in jointly produced story lines" (p. 48). The authors define two types of positioning: interactive, in which one person positions another, and reflexive, in which one positions oneself. As an iterative process, interactional positioning enables us to consider the moment-to-moment locations that students may hold within an interaction, as positioned by themselves and by others through verbal and nonverbal communication. In-the-moment positions within learning environments are shaped by a range of factors, from local co-constructed notions of what it means to be mathematically competent (Gresalfi et al, 2009) to racial and gender categories (Gholson & Martin, 2014).

The extent to which students are positioned as having intellectual authority within mathematics classrooms is of particular importance, in light of current recommendations for students to author their own strategies and to share that thinking with their peers (CCSS, 2013), which depart from dominant forms of instruction that position mathematics as a discipline as the authority (Amit & Fried, 2005). Building on Cobb et al (2009) and Engle et al (2014), I consider intellectual authority in work-sharing to be the extent to which students position themselves reflexively and/or are positioned interactively by their peers and their teacher as a credible source of information in relation to their mathematical contributions. Further, I draw on Engle and colleagues' (2014) influence framework to consider two linked components that relate to the negotiation of students' authority: their access to the conversational floor (i.e. the degree to which they can initiate and complete turns of talk when desired) and to the interactional space (i.e. the degree to which they are visually and physically attended to by others when speaking).

The extent to which students position themselves and get positioned by their peers and teacher with intellectual authority can have material consequences, decreasing opportunities to contribute to groupwork (Esmonde, 2009) and to identify with mathematics (Langer-Osuna, 2011, Cobb et al, 2009). Much of this work focuses on interactional positioning and authority in collaborative contexts (Wood, 2013; Langer-Osuna, 2016); however, Turner and colleagues (2013) note that whole-class discussions are distinctive in that they "afford opportunities for students to take on roles publicly" (p. 203). Sharing work in the whole-class space thus affords students opportunities to position themselves or to be positioned publicly with authority, which may be particularly consequential for their engagement in and identification with mathematics.

Methods

Study Setting & Participants

I use ethnographic approaches (Emerson et al, 2011) to characterize these practices at the classroom level and interaction analysis (Jordan & Henderson, 1995) to uncover how students are positioned in moments of sharing. The sixth-grade mathematics teacher in this study, Ms. L, [6] was selected because she engages students in sharing their thinking with the class. This was determined through initial conversations and then verified during observations in the fall of 2021. Change Academy, where Ms. L teaches, is a public charter school in a small city in Northern California that serves a majority working-class student population. There were 30 students in Ms. L's class during the study, most of whom identified as Latinx, with the remaining students identifying as Filipinx, East Asian, and/or Black. Ms. L identifies as Vietnamese-American. Four of the focal students identify as Latinx and the other two identify as Filipinx. Three are female-identifying and three are male-identifying. Catalina, the focal student whose work is featured in this paper, identifies as a Mexican American female.

Data Sources & Analysis

The primary data source consisted of 13 hours of classroom video, recorded daily during a geometry unit in spring 2022. Additional data sources were used to triangulate video analysis: fieldnotes from throughout the unit and semi-structured interviews (Glesne, 2005) with six focal students at the end of the unit. Focal students were selected to be representative of the class, in terms of race, gender, achievement, and participation. Five focal students had shared work during the video recorded unit and were shown a picture of them sharing in the interview.

To begin data reduction, a time-indexed content log (Derry et al, 2007) of the video recording was created that outlined work-sharing activities across the 13 hours. Work-sharing activities (n = 15) were defined as instances in which: 1.) a representation of the student's work was made publicly visible to the class, 2.) that student stood at the front of the classroom speaking about it. Following Angelillo and colleagues' (2007) approach to examining video interactions, each instance was viewed multiple times across several rounds of analysis. In the first round, analytic memos were constructed about each instance, in relation to reflexive and interactive positioning of the presenting student and their ideas (Davies & Harré, 1990), the extent to which correctness was determined (as noted in Esmonde, 2009), and teacher framing and class engagement more broadly. These memos revealed patterns in how intellectual authority was distributed across instances and seemed to shift between the presenting student and the teacher in particular moments. This revelation led to additional memos in conversation with the literature on intellectual authority in classrooms (Cobb et al, 2009; Engle et al, 2014; Langer-Osuna, 2016; Amit & Fried, 2005). In the second round of analysis, each work-sharing instance was broken into several "interactional states" (Langer-Osuna et al, 2020), between which

[6] All teacher, student, and school names are pseudonyms.

authority seemed to shift. Using Engle and colleagues' (2014) influence framework as a guide, analytic memos were written about each interactional state, in relation to who had access to the interactional space and the conversational floor, who was positioned as having intellectual authority, when positioning around authority seemed to shift, and what interactive and reflexive positioning acts made these shifts possible. Following this round of analysis, additional memos were written about each activity, which noted under what conditions authority seemed to shift between states and from which the continuum of three distributions emerged. Finally, student interviews were transcribed and open coded for emergent themes, after which analytic memos were written to connect students' reflections on the moments in which they shared their work (in the interview) to the interaction analysis of those same moments (in the video).

Results

Video analysis revealed that intellectual authority was distributed in distinct ways across the set of work-sharing interactions and that it shifted within some work-sharing interactions. Of the 15 instances of work-sharing that occurred during the video recorded unit, three types of interactions emerged along a continuum of distributions of intellectual authority. On one side of the continuum were "expert student" interactions (n = 6) in which students were positioned as having intellectual authority for the majority of the interaction. On the other side of the continuum were "teacher mediated interactions (n = 3) in which the teacher positioned herself as the intellectual authority, despite the student occasionally having access to the conversational floor. In the center of the continuum were "student to teacher" interactions (n = 6) in which students were initially positioned as having intellectual authority on their work but at some point during the interaction the teacher undermined their authority and positioned herself as the authority on the mathematics. These types are defined in table 1 and explored further below.

Table 1: Continuum of authority distributions across work-sharing interactions

Expert student	Student to teacher	Teacher mediated
Student positioned as authority for majority of clip	Student initially gets positioned as authority, but during work-sharing the teacher withdraws that authority and positions herself instead as mathematical authority.	Teacher positions self as authority, despite students having access to the floor

Student ⟷ Teacher

RQ1: Distributions of Intellectual Authority

In "expert student" instances (n = 6), the presenting student was positioned as an intellectual authority on their work by Ms. L, their peers, and in some cases themselves. These instances typically began with Ms. L telling the class to "follow along" and "have eyes and ears up here so we can pay attention to our classmate" and telling the presenting student to "show us how you got…" This language positioned the presenting student as someone to learn from and to pay attention to, as well as someone capable of actively "showing" the class their thinking. The presenting student then walked up to the whiteboard in the front and stood in the center, as Ms. L moved to the side of the whiteboard and told them to "go ahead." This physical movement, combined with the verbal signal from Ms. L to "go ahead," effectively granted the student access to the interactional space and the conversational floor. The presenting student stood at the front, pointing to their work and explaining their reasoning, sometimes annotating it with a whiteboard marker at the same time. In these ways, many of these students reflexively positioned themselves with confidence in their thinking. After the student finished explaining, Ms. L asked the class if they agreed with their classmate and simultaneously motioned an agreement signal herself, which many students immediately mirrored back or gave a thumbs up. Ms. L then asked the class if they had questions for the presenting student "especially if you got a different value," positioning that student as a credible source of information and positioning their work as an answer key that other students could use to check their work against. In two of these six instances, a student asked a question directly to the presenting student (e.g. "Why did **you** draw…?"), positioning them as capable of explaining their own thinking. These instances ended with Ms. L thanking the presenting student, the class clapping for them, and them sitting down. This co-constructed closure seemed to signal the end of the work-sharing interaction and of several minutes of that student being positioned as an intellectual authority.

One example illustrates the variation of positioning in these interactions. When Ms. L called on Abigail to share, she was initially reluctant to come up. Ms. L encouraged her to share, saying "You can do this!" and her classmates cheered "Go Abigail!" as she eventually walked up to the board. Once at the front, Abigail drew out and explained her method for decomposing and rearranging a parallelogram into two rectangles to find its area. After she finished explaining, her classmates applauded unprompted, despite this applause usually happening right after Ms. L thanked the student for sharing. Ms. L then asked the class if they had questions for Abigail and one student asked her to explain why she had decomposed the shape that way. In this example, Abigail did not initially position herself as an intellectual authority, however, her classmates and teacher did through encouragement, unprompted applause, and a question about her work.

In "teacher mediated" instances (n = 3) at the other end of the continuum, the presenting student was not positioned as an intellectual authority, despite standing in front of the class. In these instances, students were called up to annotate a shape projected to the whiteboard, rather than having a photograph of their work projected or writing out their work on the whiteboard in real-time. These students did not explain their reasoning, but instead stated their answer at the front, after which Ms. L began asking Initiate-Response-Evaluation (Mehan, 1979) questions to them or the class. In these instances, the work that was shared was initially incorrect or incomplete. Throughout Ms. L's IRE questioning, Ms. L either modified the projected work or another student came up to add to it. In cases of the latter, despite the initial presenting student and the student who came up standing at the front together, they did not talk directly to each other, but instead the conversation was mediated through Ms. L. Unlike the expert student interactions, these instances did not have a consistent form of closure. In one case, a student sat down while Ms. L spoke (without being thanked or clapped for) and in the other two cases, Ms. L thanked the presenting student but there was no applause. These instances resembled less of a performance that centered one student's thinking and more of a moment in which students were briefly physically, but not intellectually, centered.

In the center of the continuum were "student to teacher" instances (n=6), which began much like the expert student clips but ended more similarly to the teacher mediated instances. Because these instances involved distinct shifts in terms of who was positioned with authority, I explore them in more detail in the following section.

RQ2: Shifts in Authority

In student to teacher interactions, the presenting students were initially positioned with authority as they stood at the front, explained their thinking, and pointed to their work. After their explanations, however, that student's access to the interactional space and the conversational floor, as well as their positioning as an authority, were undermined. An example is shown below during which Catalina came up to show how to find the area of a triangle. The problem was projected on the white board and Catalina drew on the projected triangle with a whiteboard marker (see figure 1). Catalina explained her thinking while drawing:

Ms. L	Go ahead, Catalina. [moves to the side]
Catalina	[picks up whiteboard marker and begins writing with it] Okay um, so what I did um…what I did is…I tried to draw a line right here [draws dotted line segment] to make it a lot more bigger [draws another dotted line segment], expanding it. And then what I did, I took the piece, I took the little index card [picks up index card] and lined it up right here [puts it in the corner of one vertex of the triangle], so it's right, so I chose this to be my base [writes "b" above side of triangle]. Um, I moved it right, I moved it right here, and I put it right here [moves index card along the base], moved it over here, to reach this corner, which is the vertex, which is my opposite vertex. Then I tried to also expand this right here [draws another dotted line segment], so it's easier for me to count. And this base turned into [points to base and moves finger along length of base], turned into 12 [moves hand to write above the base, then takes it off], this turned into [counts each unit square in the base and dotted line extension] 15, turned into 15 [writes 15 above the base]. And this was [points to dotted line height] 8 [writes 8 next to dotted line]. And, uh, I did 15 times 8 [writes out 15 x 8 next to triangle], which is 120 [writes = 120]. Uh, so, my total, is 120 square units [writes square units]. [caps marker and turns to face Ms. L]
Ms. L	What is the formula to find the area of a triangle? What did we say earlier? Compared to a parallelogram?
Catalina	Um…
Ms. L	Do you remember?
Catalina	Not really.
Ms. L	Can someone help out? [turns body away from Catalina toward the class]

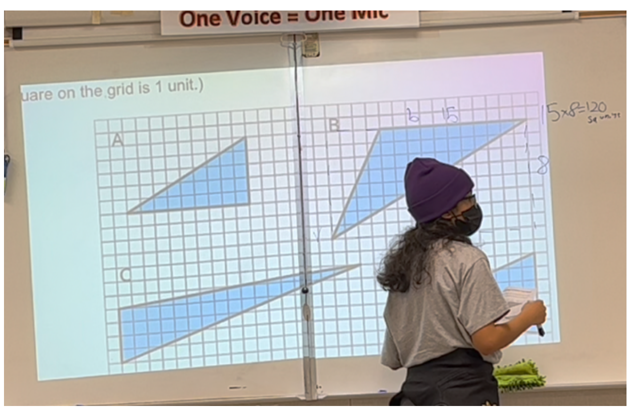


Figure 1: Catalina sharing her work finding the area of a triangle

After this exchange, Ms. L called on another student to share the formula for the area of a triangle and asked several follow-up questions related to the formula. This shifting of the conversation away from the presenting student's strategy to the formula for finding area happened in nearly all of the student to teacher instances. Moreover, in many of these instances Ms. L wrote out the formula next to the presenting student's work and solved the problem that way, though that did not happen in the Catalina example, perhaps because class ended shortly after the excerpt shown above. In shifting the conversation away from the students' thinking to the formula and in authoring her own representation of the problem, Ms. L undermined Catalina and other presenting students' authority. Interestingly, neither Ms. L nor other students in the class explicitly noted that Catalina's answer was incorrect. Instead, it seemed to be implied that she was incorrect based on Ms. L offering another way to solve the problem and arriving at a different answer. Notably, Ms. L did not ask the class if they agreed with Catalina's work, as was done in the expert instances.

All students in these instances presented work that was incorrect, though some were partially correct and/or employed creative strategies that could have been revised to find the area accurately. Like the Catalina example, neither Ms. L nor the other students explicitly shared that the work was incorrect in these instances, yet it seemed to be implicitly communicated through the offering of another strategy with another answer. In many instances, the presenting student stood up at the front while Ms. L authored another strategy on the whiteboard. Typically, Ms. L gradually moved to the center as she wrote and the student gradually backed up to the side. In this way, the student lost access to the physical interactional space. In most of these instances, the presenting student never regained access to the conversational floor after this moment, though in one case the student had a brief opportunity to explain their work again. These interactions typically ended with Ms. L thanking the student and the class applauding, which might be read as an affirmation of the presenting student's effort.

Correctness emerged as the condition under which authority shifted within a work-sharing interaction, as all students in the "expert student" interactions presented work that was deemed to be correct by the teacher and the class through agreement signals, whereas all students in the other two instances had authored work that was partially or fully incorrect or incomplete. It should be noted that the identity of the presenter did not seem to be a condition under which authority shifted within an instance, as some students participated in interactions of all types. In other words, there was one student who shared work in both expert student instances and teacher-mediated instances and another who shared work in all three types of instances.

Although correctness determined authority shifts, it was not a prerequisite for sharing. While in some cases students volunteered to share and in others Ms. L selected students to share, this variation did not correlate with the type of instance. That is, it was not the case, as might be expected, that all students who presented in expert instances volunteered to share or that all students who presented in teacher-mediated instances were selected by Ms. L to share. Additionally, in cases in which Ms. L did select a student to share, this selection was often based on completion rather than correctness, as she noted at the beginning of some instances that she had seen that the presenting student had solved the problem based on scanning the room. In this way, students who had written out their thinking were prioritized in work-sharing, rather than those who may have arrived at the correct answer without documenting their thinking.

Analysis of student interviews confirmed that students were aware of this emphasis on answers and completion. Several students cited the same handful of peers as frequent work-sharers in their classroom, referred to some classmates as smart and others as not, and shared anxiety about being incorrect and getting corrected while presenting. Catalina shared:

> What I felt [when sharing my work] was kind of nervous and anxious that some people didn't really understand…I really felt kind of paranoid, thinking people will probably use what I

> said there against me…like correcting me about like, almost everything that—being paranoid that I got almost everything wrong.

In this quote, Catalina noted her fear of being wrong and of being corrected. She then named two students in her class who she worried might correct her, "because they're both really smart."

At the same time, Catalina and others mentioned feeling pride in sharing their work with the class, as she explained: "I felt pretty excited to share my work. I felt really happy since I really—I knew a lot more about it, I was able to solve the problem and get like, an actual answer." Interestingly, students seemed to both relish the opportunity to be positioned with authority and to fear the withdrawal of that authority from Ms. L and their peers through correction.

Discussion & Conclusion

Findings revealed that Ms. L and her students constructed work-sharing practices in which intellectual authority was distributed to presenting students in some instances, but which shifted to Ms. L in other instances. The expert interactions in this study demonstrate that intellectual authority can be distributed to students during work-sharing, reconfiguring traditional authority dynamics in the classroom (Boaler & Greeno, 2000) and creating opportunities for students to share their thinking and engage with their peers' ideas, as has been recommended (CCSS, 2013). In these instances, students wrote and explained their thinking, were physically attended to by their peers and their teacher, and were applauded for and asked questions about their work. We might even think of the expert instances as one way to assign competence to presenting students with lower status (Cohen & Lotan, 2014), as was the case with Abigail who did not reflexively position herself with authority but was positioned this way by her teacher and her peers.

However, the ways in which presenting students' authority was undermined in the student to teacher instances should caution us to look more closely at work-sharing practices. These interactions reveal that who is positioned with authority can abruptly shift in the moment and that these shifts are co-constructed. As Catalina and other students shared in their interviews, they feared being corrected while sharing work not only by their teacher but also by a subgroup of their peers. Students and teachers alike are immersed in rigid notions of mathematical activity as answer-finding (Louie, 2017), which may explain the central role that correctness played in shifting authority. The co-constructed and situated nature of classroom interaction (Philip et al, 2020) demands that we look beyond teacher moves (Smith et al, 2011) to consider how students contribute to classroom practices, as well as the ways in which pacing pressures may incentivize teachers to withdraw authority from an incorrect student, rather than to explore the mistake.

This emphasis on correctness has concerning implications for students. If those with correct answers are positioned with authority and those with incorrect answers have authority withdrawn from them, this pattern not only reinforces rigid notions of mathematical activity (Louie, 2017) but may also hinder students' development of positive mathematics identities (Langer-Osuna, 2011; Cobb et al, 2009), especially for those who often arrive at incorrect answers. Further, this pattern has the potential to be particularly damaging for marginalized students like Catalina, who already contend with racialized and gendered discourses about mathematical ability (Shah, 2017) that seep into classrooms and shape in-the-moment positions (Gholson & Martin, 2014).

Future research might seek to understand not only when authority shifts in these moments, but how those shifts are made possible through discursive moves, such as types of positioning around authority (Wagner & Herbel-Eisenmann, 2014). Unpacking how these shifts happen is essential given the ways in which the distribution of intellectual authority can shape students' mathematics learning and identities (Langer-Osuna, 2017; Cobb et al, 2009). Researchers and practitioners alike might more closely examine work-sharing interactions to better understand how intellectual authority may be distributed to more students, how it might be undermined and shifted, and, most importantly, how positioning students with authority might be sustained.

References

Amit, M., & Fried, M.N. "Authority and authority relations in mathematics education: A view from an 8th grade classroom." *Educational studies in Mathematics* 58 (2005): 145-168.

Angelillo, C., Rogoff, B. & Chavajay, P. (2007). Examining shared endeavors by abstracting video coding schemes with fidelity to cases. In R. Goldman, R. Pea, B. J. Barron & S. Derry (Eds.), *Video research in the learning sciences* (pp. 189-206). Mahwah, NJ: Erlbaum.

Bartell, T., Wager, A., Edwards, A., Battey, D., Foote, M., & Spencer, J. (2017). Toward a framework for research linking equitable teaching with the standards for mathematical practice. *Journal for Research in Mathematics Education*, *48*(1), 7-21.

Boaler, J., & Greeno, J. G. (2000). Identity, agency, and knowing in mathematics worlds. *Multiple perspectives on mathematics teaching and learning*, *1*, 171-200.

California Department of Education. (2021, October 22). California Digital Learning Integration and Standards Guidance. Retrieved February 17, 2023, from https://www.cadlsg.com/guidance/

Carpenter, T. P., Fennema, E., Franke, M. L. (1996). Cognitively Guided Instruction: A Knowledge Base for Reform in Primary Mathematics Instruction. *The Elementary School Journal, 97*(1), 3-20.

Cobb, P., Gresalfi, M., & Hodge, L. L. (2009). An interpretive scheme for analyzing the identities that students develop in mathematics classrooms. *Journal for Research in Mathematics Education*, *40*(1), 40-68.

Cobb, P., Stephan, M., McClain, K., & Gravemeijer, K. (2001). Participating in classroom mathematical practices. *The journal of the Learning Sciences*, *10*(1-2), 113-163.

Cobb, P. & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational psychologist*, *31*(3-4), 175-190.

Cohen, E. G., & Lotan, R. A. (2014). *Designing groupwork: strategies for the heterogeneous classroom third edition*. Teachers College Press.

Common Core State Standards. (2013). Retrieved February 17, 2023, from https://www.cde.ca.gov/be/st/ss//documents/ccssmathstandardaug2013.pdf

Davies, B., & Harré, R. (1990). Positioning: The discursive production of selves. *Journal for the theory of social behaviour*, *20*(1), 43-63.

Derry, S. J., Pea, R. D., Barron, B., Engle, R. A., Erickson, F., Goldman, R., Hall, R., Koschmann, T., Lemke, J.L., Sherin, M.G., & Sherin, B. L. (2010). Conducting video research in the learning sciences: Guidance on selection, analysis, technology, and ethics. *The Journal of the Learning Sciences*, *19*(1), 3-53.

Emerson, R. M., Fretz, R. I., & Shaw, L. L. (2011). *Writing ethnographic fieldnotes* (2nd ed.) Chicago, IL: University of Chicago Press.

Engle, R. A., Langer-Osuna, J. M., & McKinney de Royston, M. (2014). Toward a model of influence in persuasive discussions: Negotiating quality, authority, privilege, and access within a student-led argument. *Journal of the Learning Sciences*, *23*(2), 245-268.

Esmonde, I. (2009). Mathematics learning in groups: Analyzing equity in two cooperative activity structures. *The Journal of the Learning Sciences*, *18*(2), 247-284.

Fennema, E., Franke, M. L., Carpenter, T. P., & Carey, D. A. (1993). Knowledge in Instruction. *American Educational Research Journal*, *30*(3), 555-583.

Franke, M. L., Turrou, A. C., Webb, N. M., Ing, M., Wong, J., Shin, N., & Fernandez, C. (2015). Student engagement with others' mathematical ideas: The role of teacher invitation and support moves. *The Elementary School Journal*, *116*(1), 126-148.

Gholson, M., & Martin, D. B. (2014). Smart girls, Black girls, mean girls, and bullies: At the intersection of identities and the mediating role of young girls' social network in mathematical communities of practice. *Journal of Education*, *194*(1), 19-33.

Glesne, C. (2005). Making Words Fly: Developing understanding through interviewing. In *Becoming Qualitative Researchers: An Introduction*. Boston, MA: Pearson Publishing.

Gresalfi, M., Martin, T., Hand, V., & Greeno, J. (2009). Constructing competence: An analysis of student participation in the activity systems of mathematics classrooms. *Educational Studies in Mathematics*, *70*(1), 49-70.

Jordan, B., & Henderson, A. (1995). Interaction analysis: Foundations and practice. *Journal of the Learning Sciences*, *4*(1), 39-103.

Langer-Osuna, J., Munson, J., Gargroetzi, E., Williams, I., & Chavez, R. (2020). “So what are we working on?”: How student authority relations shift during collaborative mathematics activity. *Educational Studies in Mathematics*, *104*, 333-349.

Langer-Osuna, J. M. (2017). Authority, identity, and collaborative mathematics. *Journal for Research Mathematics Education*, *48*(3), 237-247.

Langer-Osuna, J. M. (2016). The social construction of authority among peers and its implications for collaborative mathematics problem solving. *Mathematical Thinking and Learning*, *18*(2),107-124.

Langer-Osuna, J. M. (2011). How Brianna became bossy and Kofi came out smart: Understanding the trajectories of identity and engagement for two group leaders in a project-based mathematics classroom. *Canadian Journal of Science, Mathematics & Technology Education*, *11*(3), 207-225.

Leander, K. M. (2002). Silencing in classroom interaction: Producing and relating social spaces. *Discourse processes*, *34*(2), 193-235.

Louie, N. L. (2018). Culture and ideology in mathematics teacher noticing. *Educational Studies in Mathematics*, *97*(1), 55–69.

National Council of Teachers of Mathematics. (2000). Principles and standards for school mathematics. Reston, VA.

Mehan (1979). Learning Lessons: Social Organization in the Classroom. Cambridge, MA: Harvard University Press.

Philip, T. M., & Gupta, A. (2020). Emerging perspectives on the co-construction of power and learning in the learning sciences, mathematics education, and science education. *Review of Research in Education*, *44*(1), 195-217.

Shah, N. (2017). Race, ideology, and academic ability: A relational analysis of racial narratives in mathematics. *Teachers College Record*, *119*(7), 1-42.

Singer-Gabella, M., Stengel, B., Shahan, E., & Kim, M. J. (2016). Learning to leverage student thinking: What novice approximations teach us about ambitious practice. *The Elementary School Journal*, *116*(3), 411-436.

Smith, M. S., & Stein, M. K. (2011). *Five practices for orchestrating productive mathematics discussions*. Reston, VA: National Council of Teachers of Mathematics, Inc.

Turner, E., Dominguez, H., Maldonado, L., & Empson, S. (2013). English learners' participation in mathematical discussion: Shifting positionings and dynamic identities. *Journal for Research in Mathematics Education*, *44*(1), 199-234.

Wagner, D., & Herbel-Eisenmann, B. (2014). Identifying authority structures in mathematics classroom discourse: A case of a teacher’s early experience in a new context. *ZDM*, *46*, 871-882.

Wood, M. B. (2013). Mathematical micro-identities: Moment-to-moment positioning and learning in a fourth-grade classroom. *Journal for Research in Mathematics Education*, *44*(5), 775-808.

EXPLORING THE AFFORDANCES AND LIMITATIONS OF BETWEEN-GROUP MOVEMENT IN ELEMENTARY MATHEMATICS CLASSROOMS

Tye Campbell
Utah State University
tgcamp22@gmail.com

Sheunghyun Yeo
Daegu National University of Education
shyeo@dnue.ac.kr

Erin Rich
The University of Alabama
eerich@crimson.ua.edu

Mindy Green
Revere School
mindy.green@revereschool.com

In this exploratory study, we examine how between-group movement, as an autonomy-promoting practice, might incentivize or disincentivize sixth-grade students' engagement in two mathematical practices: (1) making sense of problems and persevering in solving them; and (2) constructing viable arguments and critiquing the reasoning of others. Exploring both the affordances and limitations of between-group movement, we found that between-group movement supported groups to construct viable justifications, among other sense-making mathematical practices. However, we also found that some groups over-scaffolded during between-group conversations which disincentivized meaningful engagement in mathematical practices.

Keywords: Instructional Activities and Practices, Classroom Discourse, Communication

In standard collaborative classrooms, students generally remain seated in their small groups without opportunities to interact with other groups. In contrast to traditional collaborative classrooms, recent literature reveals a new classroom movement structure that allows groups to physically move to discuss their ideas between groups (Pruner & Liljedahl, 2021; Liljedahl, 2020). We refer to this movement structure as "between-group movement."

This study took place in a rural K-12 school in the Midwestern United States. The aim was to explore how between-group movement, as an autonomy-promoting practice, might incentivize or disincentivize sixth-grade students' engagement in two mathematical practices: (1) making sense of problems and persevering in solving them; and (2) constructing viable arguments and critiquing the reasoning of others. We examined three critical events of students engaging in between-group conversations to uncover the affordances and limitations of between-group movement.

We found six primary affordances of between-group movement and four primary limitations of between-group movement. Some affordances included: (1) Between-group movement can support groups to construct mathematically valid justifications and solutions; (2) Between-group movement can support groups to rely on other groups as a resource when they get "stuck". Some limitations included: (1) Some students might over-scaffold during a between-group conversation and decrease the challenge of the task, which removes opportunities for groups to productively struggle through problem-solving; (2) Between-group movement may limit groups' capacities to consider multiple solution approaches.

The findings suggest that between-group movement can be leveraged as a constructive autonomy-promoting practice under certain conditions.

References

Liljedahl, P. (2020). *Building thinking classrooms in mathematics, Grades K-12: 14 teaching practces for enhancing learning*. Corwin.

Pruner, M., & Liljedahl, P. (2021). Collaborative problem solving in a choice-affluent environment. *ZDM–Mathematics Education*, *53*, 753-770.

Schoenfeld, A. H. (2020). Mathematical practices, in theory and practice. *ZDM*, *52*(6), 1163-1175.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. New York: Cambridge University Press.

Yin, R. K. (2009). *Case study research: Design and methods* (4th ed.). Washington, DC: Sage.

FRACTION ADDITION THROUGH THE MUSIC

Maria T. Sanz
Universidad de Valencia
m.teresa.sanz@uv.es

Carlos Valenzuela
Universidad de Guadalajara
carlos.valenzuela@academicos.udg.mx

Emilia López-Iñesta
Universidad de Valencia
emilia.lopez@uv.es

Guillermo Luengo
Universidad de Valencia
guiluena@alumni.uv.es

This study examined the effects of an academic intervention, associated with music, on the conceptual understanding of musical notation and arithmetic of fractions of first-year students of high school from a mixed Spanish multicultural and socioeconomic public school. The students (N = 12) had previous concepts about musical instruction, as well as operations with fractions, particularly addition. This is an observational study in which a battery of four tasks was administered before and after an instruction based on a musical environment, music being a semiotic function. The instruction included 9 sessions of 50 minutes each. The results prior to the intervention show deficiencies in a concept that was not new to the students, however, after the intervention the students were competent in addition with fractions.

Keywords: addition of fractions, semiotic function, music, multidisciplinary intervention.

Introduction

Although there is the idea that music and mathematics are very different subjects, the reality is very different. In ancient Greece, Pythagoras (ca. 580-500 BC) realized the relationship between different sounds and the proportion of the strings that emitted them. This led to a description of music using mathematical symbols (Henning, 2009; Weber, 1991, cited in Mall et al., 2016). But the relationship between mathematics and music does not occur because they are close subjects, but "because musical parameters can be transformed and organized using mathematical techniques, and vice versa" (Mall et al., 2016, p. 12). Thus, these are two disciplines in which two systems of compatible signs follow one another: musical notation and numbers (natural and rational). Currently, these systems interact creating a semiotic set (Arzarello, 2006). With all this, interdisciplinary projects acquire great relevance, in particular, projects in which music and mathematics converge.

This context defines the need to continue research on the incorporation of music and its characteristics in the teaching and learning process of mathematics, and vice versa. In particular, with the present work the objective is to evaluate the influence of music, regarding the value of musical notes, in the learning process of the addition of fractions in first grade students of high school (ages between 12 and 14 years old).

Background and frame of reference

According to Avdulova (2018, cited in Veraksa et al., 2022), symbolic play represents the child's ability to link concrete experience with abstract thought by connecting the meaning with the signifier (Piaget, 1962). In this process, the child develops the semiotic function, that is, an ability to represent an absent object or an event that is not directly perceived through symbols or signs (Veraksa et al., 2022).

In the modern teaching of mathematics, the interaction between perception and action also becomes increasingly important. Different levels of communication work together in semiotic sets (Arzarello, 2006) and students use the cycles of action and perception to develop mathematical understanding. The identification of patterns and their interpretation as a system of signs constitutes a fundamental mathematical task. The repetitions, and therefore the regularities, can be found by observing written symbols. These regularities are the basis of mathematical understanding (Mall et al., 2016).

In this context, the use of musical notation as a tool to acquire the concept of fraction and, in particular, the addition of these numbers, is within what is called addition using discrete models. The school tradition advocates a teaching-learning process of fractions based on continuous models (Rico, 1997), but separating them from a discrete context (Kieren, 1993) means that students are not able to function in contexts in which other meanings or subconstructs of the fraction underlie, for example, as measure, ratio, or operator.

With respect to adding fractions, research has shown that many students learn operations with fractions through memory-based, procedural-oriented instruction, attaching little meaning to such operations and thus making serious errors over time (Behr et al., 1992; Cramer and Henry, 2002; Cramer et al., 2002; Kennedy and Steve, 1997). The Rational Number Project Curriculum is committed to learning fractions through everyday life contexts that are reflected in word problems. Others like Braithwaite et al. (2017) bet on contexts such as the number line; or there are those who see musical arithmetic as a basis for the teaching-learning process of fractions (Oshanova et al., 2022).

Continuing with the last idea, and given that the literature is extensive on the use of music as a multimodal approach for the instruction of fractions (Azaryahu et al., 2020; Courey et al., 2012; Lovemore et al., 2022) in the present work seeks to answer, how does the value of musical notes improve competence in the addition of fractions of students in the first year of high school?

Methodology

This work is part of an experimental and observational study. Within observational studies, this is defined as a short-term longitudinal study. We have worked on the same sample, students belonging to the first grade of high school of the Spanish educational system, during a period of 2 months, obtaining data at the beginning and at the end of an intervention.

This section gives rise to three subsections in which the sample studied, the method used, and finally, the instrument and the research variables are presented.

Sample

Regarding the study participants, it is a small group of 12 students belonging to a public high school in Valencia, Spain. Among the students there is a high number of immigrants and students at risk of school absenteeism. In turn, there is great diversity with respect to the economic and cultural levels of the families of the students that make up the classroom. The ages of the students are between 12 and 14 years old.

Method

The study has three different sections. First, a battery of four questions was asked, prior to the intervention (pre-test). After this, an intervention is carried out with an approximate duration of two months (9 sessions of 50 minutes each). Finally, a second battery of four questions (post-test) is carried out. Both batteries of questions contain questions related to the contents of fractions that the students have already seen in the previous course (6th of primary education) and that will be extended in their present course.

This paper is part of a broader study that covers the concepts of fractions, representation, ordering, equivalence, and fraction arithmetic. Due to the extension allowed, only results related to the arithmetic of fractions, in particular addition, are presented here. In the proposed intervention, it is the third, fifth and sixth sessions that focus their content on this specific concept; which are detailed below.

Session 3. The main purpose of this session is to explain the addition and subtraction of fractions with the same denominator using musical elements. In his case, the manual (Adiel, 2021), is limited to refreshing the algorithmic process of this operation. In addition, it introduces a graphic example using a continuous model based on the area model (Figure 1).

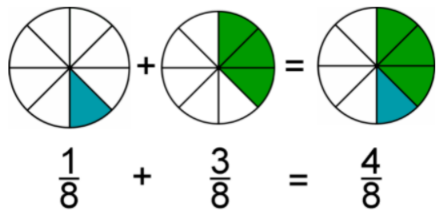


Figure 1: Example of graphic sum (Adiel, 2021)

To this aspect, in the intervention carried out, two graphic ways of carrying out or verifying these operations from a musical field are exposed: a linear one and a discrete one. To exemplify the linear model, the strings of a cello are used (Figure 2).

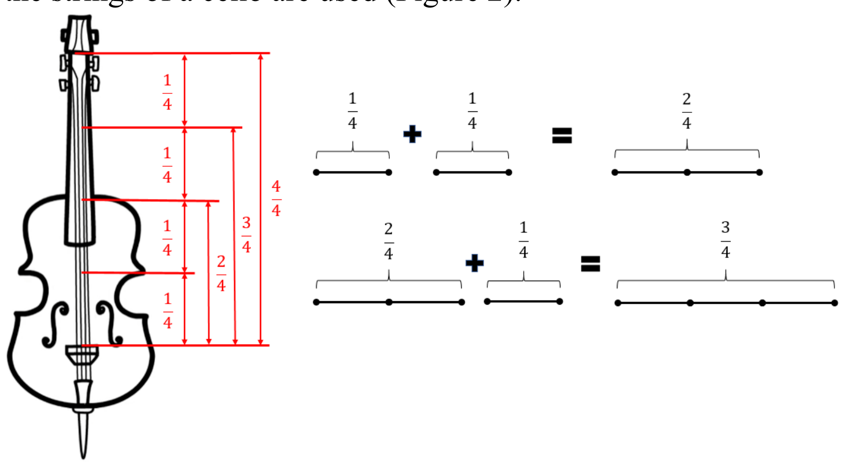


Figure 2: Addition of fractions linear model, cello strings

Instead, to exemplify the discrete model, musical notes are used (Figure 3).

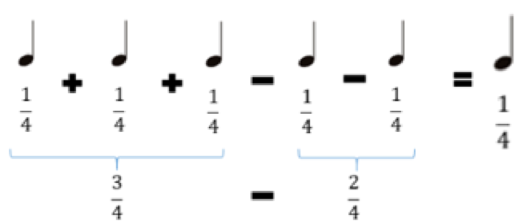


Figure 3: Addition and subtraction of fractions with discrete model

Session 5. In this session the addition and subtraction of fractions is worked on again, but with the particularity that the denominator differs. As in the previous session, the graphic support provided by the musical representation continues to be the element that promotes the achievement of the main objective stated at the beginning of the document.

As a practical example, the students are presented with the one shown in Figure 4. The aforementioned figure shows the process followed to add two fractions with different denominators and, at the same time, the graphic representation from the musical perspective that is has been using in previous sessions, specifically with the use of the linear model.

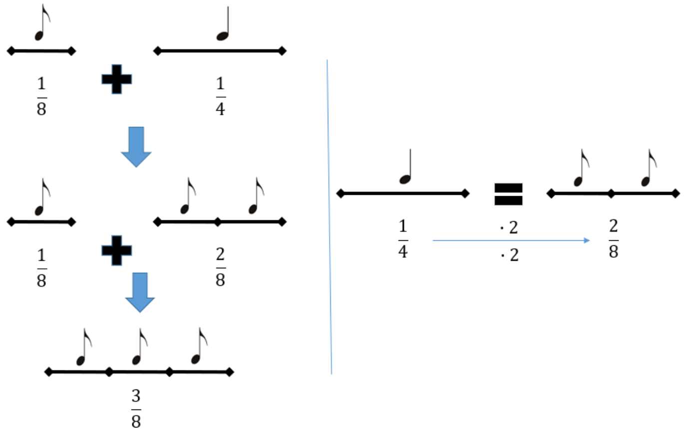


Figure 4: Addition of fractions with different denominators

Session 6. The main objective of this session is to propose to the students a more realistic situation, in the musical field, in which to make direct use of the addition of fractions.

For this purpose, an extract of four bars from Symphony No. 9, the Ode of Joy (van Beethoven, 1824), commonly called Hymn of Joy (Figure 5), is used to show how the discrete representation model exposed in the previous sessions is not an artificial construction. The students verify that this codification that has been presented to them previously is, in reality, used by composers and performers in their day to day. The students will observe that the compass is nothing more than the sum of the values of the notes located between the dividing lines.

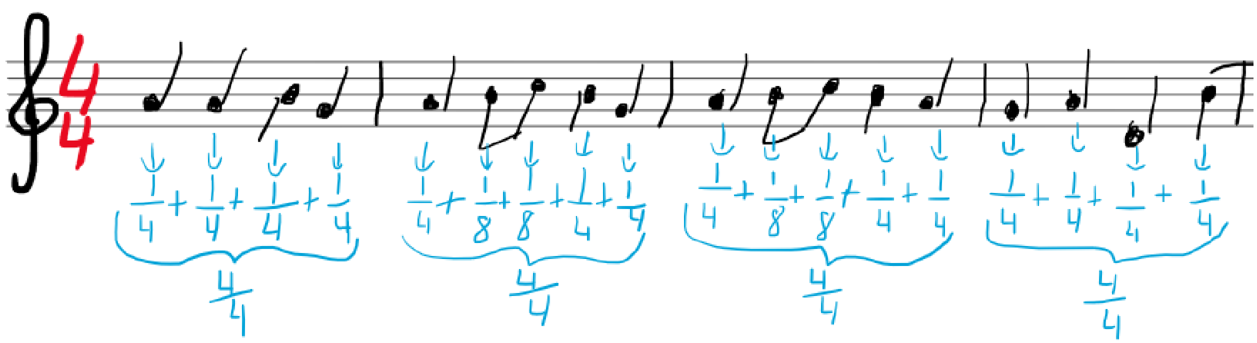


Figure 5: Hymn of Joy extract, addition of fractions

Likewise, they will realize that they themselves make use of this encoding during their music classes at the center. It is hoped, then, that at this point they will see a manifest utility of fractions. Utility of which they themselves were users without knowing it, thus creating that contextualization that arouses their attention and interest.

Instrument and Research Variables

To obtain research data, two pencil and paper batteries with 4 questions each are used. Although some data changes in the questions between the pre-test and the post-test, the isomorphism between the two is preserved to make possible an evaluation of the knowledge evolution of the students.

Q1. Represent, in as many ways as possible, the following fraction: ¼

Q2. Order, from smallest to largest:

Pre-test: 2/2,1/4, 7/8, 5/4. Post-test: 1/2, 3/4, 2/8, 2/2
Q3. Write and represent two equivalent fractions.
Q4. Calculate, you can rely on representations:
Pre-test: a) 1/2+3/8; b) 2/2-1/4. Post-test: a) 2/4+1/2, b) 6/8-1/4.

Following the objective and question of this report, the focus is on the addition of fractions, so only the analysis and research variables of Q4 will be determined.

The question Q4 allows to check the abilities in the addition of fractions of the students. In particular, it allows analyzing whether they use alternative methods, such as representation, which are suggested to them, or only apply the algorithm of said operation. On the other hand, possible errors made in carrying out this process will also be highlighted, if any.

In order to carry out a descriptive analysis of this issue, the research variables defined are three:

- Success: relationship between students who have answered the question satisfactorily with respect to the total.
- Process: this field introduces the method used to solve the question: operate with the decimal expression (ED), use the graphical representation (RP) or apply the algorithm (AL).
- Mistakes: use of the multiplication algorithm (ALmult), use of the division algorithm (ALdiv), common denominator (CD).

Results and discussion

First, a descriptive analysis of the results obtained in the pre-test is carried out, followed by the analysis of the results obtained in the post-test. Subsequently, the evolution of the students globally is presented and ends with the individual evolution.

Q4 Results

As can be seen in the pre-test column, in Table 1, only one student out of 12 has managed to successfully add fractions.

This is in line with what is obtained in previous studies (Braithwaite, 2017) in which it is determined that students between the ages of 13 and 14 continue to have difficulties when performing this operation.

In the same column it can also be verified that the method used in the pre-test, by the student who has solved correctly (ED, 1/12) has not involved the manipulation of the fractions, since the student has obtained the expression decimal and has operated with these expressions (Carpenter et al., 1976) (see Figure 6).

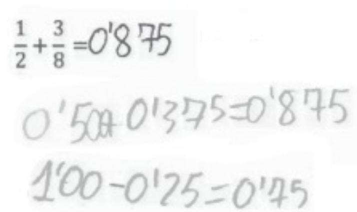


Figure 6: Decimal Expression Method

Although this is a perfectly valid strategy, it also confirms that at the time of the pre-test the students did not remember the tools to answer the question by operating on the fractions, a fact that is evident in the errors made.

Table 1: Q4 Results

	Pretest	Post Test
Success	1/12	10/12

Method	ED	1/12	-
	RP	-	1/12
	AL	-	9/12
Null		3/12	-
Mistake	ALmult	7/12	-
	ALdiv	1/12	1/12
	CD	0	1/12

It is determined in the pre-test a general tendency on the part of the students to commit the same type of error, confusion with the rule of the algorithm of multiplication of fractions (7/12). The students apply the Multiplication algorithm, but adapted to addition, thus adding numerator with numerator and denominator with denominator (Figure 7) (Howard, 1991; Keller et al., 1940; NAEP, 2013).

$$\frac{1}{2}+\frac{3}{8}=\frac{4}{10}$$

Figure 7: Errors when applying the addition algorithm, use of the Multiplication Algorithm

Another error marked by previous studies (Mata and Porcel, 2006) is the adaptation of the use of the Division Algorithm. Thus, the student who uses it in the pre-test performs a cross addition (Figure 8).

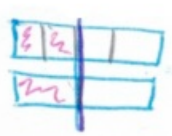
$$\frac{1}{2}+\frac{3}{8}=\frac{5}{9}$$

Figure 8: Errors when applying the addition algorithm, use of the Division Algorithm

This confusion in the rules of the algorithms verifies the fact that the rule is rote learning, which implies that over time it is forgotten or confused with similar rules (Hart, 1981).

Analyzing now the results obtained after the intervention -post-test-, a significant increase in success can be observed (10/12), which leads to determine that the use of music as a methodology has favored the teaching-learning process. (Azaryahu et al., 2020; Courey et al., 2012; Lovemore et al., 2022). This statement is highlighted, since the students who have participated in this study, had previous knowledge in the sum of fractions, because it is a knowledge, which in the Spanish curriculum begins in 4th grade of primary pducation and is repeated in 5th and 6th grade. of this primary level, which would indicate that their knowledge should be consolidated in the first year of secondary education, which, as has been verified, the results in this study provide other information.

If the methods used to solve the activity are analyzed, it can be observed that the application of the fraction addition algorithm is the process most used by the students (9/12), being a student the one who has chosen to solve the operations, exceptionally graphically (Figure 9) (Rau et al., 2015).



$$\frac{2}{4}+\frac{1}{2}=\frac{4}{4}$$

Figure 9: Use of the Graphical Representation in the addition of fractions

Finally, regarding the errors in the post-test, it should be noted that a student presented confusion in the use of the division algorithm. This student, as can be seen in Table 2 of the following section, had made an error by applying the multiplication algorithm. In addition, an error appears related to making the common denominator (Vinner et al., 1981) (Figure 10).

$$\frac{2}{4}+\frac{1}{2}=\frac{3}{8}$$

Figure 10: Errors when applying the addition algorithm, Common Denominator

Evolution Q4

Regarding the evolution (Table 2), first of all, the significant increase in students who have performed both operations after the intervention stands out, going from one to ten students.

Table 2: Individual evolution Q4

Student	Pretest			Post Test		
	Success	Metho d	Mistake	Success	Method	Mistake
S1,S2,S3, S4,S7,S12	0	AL	ALmult	1	AL	-
S9	0	AL	ALmult	1	RG	-
S10	0	AL	ALmult	0	AL	ALdiv
S16	0	AL	null	0	AL	CD
S15	0	AL	null	1	AL	-
S6	0	AL	ALdiv	1	AL	-
S13	1	ED	-	1	AL	-

Regarding the methods used by the students in the resolution, in the pre-test, the only student who carried out the operations did so using the decimal expression of the proposed fractions, as explained in detail above. On the other hand, in the post-test it is observed that no student makes use of this strategy and, instead, all of them operate directly with the fractions and there is even a student who performs the operations graphically (Figure 9).

Another positive aspect is that the blank answers in the post-test disappear, the two students (S15 and S16) go to an algorithmic resolution, one correct (S15) and the other with an error in obtaining the common denominator.

Continuing with the errors, the use of the division algorithm persists, although it is not maintained by S6 who does so in the pre-test, but it is the student S10 who goes from using the multiplication algorithm to that of division.

Conclusion

The present observational study shows that students between 13 and 14 years of age have significant deficiencies in the addition of fractions, despite being a mathematical concept that has been present in the Spanish curriculum since the age of 10 (LOMLOE, 2022). However, an interdisciplinary intervention, in which a system of musical signs is used as a base as a semiotic function to try to improve skills with fractional numbers, particularly with the addition of fractions, allows reversing a dramatic initial situation.

The results presented here are part of an investigation in which, in addition to the addition of fractions, the difficulties that students have in graphic representation, ordering and equivalence of fractions are analyzed, and how music allows solving part of the problems. initial deficiencies. Despite the fact that, in the addition of fractions, a direct consequence of the intervention carried out is not observed, it is obtained in the rest of the concepts evaluated, in particular, in the equivalence of fractions, where the students make use of the system of musical signs, to solve the situations raised. This situation is highlighted, since as indicated by Vinner et al. (1981) for the sum of fractions there are two ideas that must be present, a common denominator and equivalent fractions, the second idea being assumed by the students who have participated in this research.

References

Adiel, M. (2021). Agrega - Suma y resta de fracciones no homogéneas [[Web; accedido el 10-06-2021]]. http://agrega.educacion.es/visualizar/es/es_2013120913_9122920/false

Arzarello, F., (2006). Semiosis as a Multimodal Process. Revista Latinoamericana de Investigación en Matemática Educativa, RELIME, (Esp), 267-299. https://www.redalyc.org/comocitar.oa?id=33509913

Azaryahu L, Courey SJ, Elkoshi R, & Adi-Japha E. (2020). 'MusiMath' and 'Academic Music' – Two music-based intervention programs for fractions learning in fourth grade students. Development Science, 23(4), 1-17. https://doi.org/10.1111/desc.12882

Carpenter, T. P., Coburn, T. G., Reys, R. E., & Wilson, J. W. (1976). Using research in teaching: Notes from National Assessment: addition and multiplication with fractions, The Arithmetic Teacher, 23(2), 137-142. Retrieved Feb 22, 2023, from https://pubs.nctm.org/view/journals/at/23/2/article-p137.xml

Behr, M. J., Harel, G., Post, T., & Lesh, R. (1992). Rational number, ratio and proportion. In D. Grouws (Ed.), Handbook of research on mathematics teaching and learning (pp. 296–333). New York: Macmillan.

Braithwaite, DW., Tian, J., & Siegler, RS. (2017) Do children understand fraction addition? Developmental Science, 21(4), 1-9. https://doi.org/10.1111/desc.12601

Courey, SJ, Balogh, E., Siker, JR & Paik, J. (2012) Academic music: music instruction to engage third-grade students in learning basic fraction concepts. Educational Studies in Mathematics, 81, 251–278. https://doi.org/10.1007/s10649-012-9395-9

Cramer, K., & Henry, A. (2002). Using Manipulative Models to Build Number Sense for Addition of Fractions. In B. Litwiller & G. Bright (Eds.), Making Sense of Fractions, Ratios, and Proportions: 2002 Yearbook (pp. 41-48). Reston, VA: National Council of Teachers of Mathematics.

Cramer, K., Post, T. & delMas, R. (2002). Initial Fraction Learning by Fourth- and Fifth-Grade Students: A Comparison of the Effects of Using Commercial Curricula with the Effects of Using the Rational Number Project Curriculum. Journal for Research in Mathematics Education, 33(2),111-144. https://doi.org/10.2307/749646

Hart, K. (1981). Fractions. Mathematics in School, 10(2), 13-15.

Howard, A. C. (1991). Addition of Fractions—the Unrecognized Problem, The Mathematics Teacher, 84(9), 710-713. https://pubs.nctm.org/view/journals/mt/84/9/article-p710.xml

Keller, M. W., Shreve, D. R. & Remmers, H. H. (1940). Diagnostic Testing Program in Purdue University. The American Mathematical Monthly, 47(8), 544-548.

Kennedy, L. M., & Steve, T. (1997). Guiding children's learning of mathematics (8th ed.). Belmont, CA: Wadsworth/Thomson.

Kieren, T. E. (1993). Rational and fractional numbers: From quotient fields to recursive understanding. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), Rational numbers: An integration of research (pp. 49–84). Lawrence Erlbaum Associates, Inc.

LOMLOE. (2022). Ley Orgánica 3/2020, de 29 de diciembre, por la que se modifica la Ley Orgánica 2/2006, de 3 de mayo, de Educación.

Lovemore, T., Robertson, SA, & Graven, M. (2022). Task design grapplings in integrating music and fraction representations. In KR Langenhoven & CH Stevenson-Milln. Book of Proceedings of the 30th Annual Conference of the Southern African Association for Research in Mathematics, Science and Technology Education (pp. 49-61). Western Cape: SAARMSTE.

Mall, P., Spychiger, M., Vogel, R. & Zerlik, J. (2016). European Music Portfolio (EMP)- Maths:'Sounding Ways into Mathematics': Manual para el profesorado. Frankfurt am Main: Frankfurt University of Music and Performing Arts. http://maths.emportfolio.eu/images/deliverables/Teacher_Handbook_Spanish_Version.pdf

Mata, L. E. & Porcel, E. A. (2006). Análisis de los errores cometidos en el algoritmo de la Suma de Fracciones por ingresantes a la Fa.C.E.N.A. Universidad Nacional del Nordeste.

NAEP] (2013). National Assessment of Educational Progress. Retrieved March 1, 2020, from https://nces.ed.gov/NationsReportCard/nqt/Search

Oshanova, N. T., Shekerbekova, S. T., Sagimbaeva, A. E., Arynova, G. C., & Kazhiakparova, Z. S. (2022). Methods and techniques of formation of arithmetic musical competence in students. International Journal of Learning and Change, 14(1), 46-56.

Piaget, J. (1962). The stages of the intellectual development of the child. Bulletin of the Menninger Clinic, 26(3), 120–128.

Rau, M. A., Aleven, V., & Rummel, N. (2015). Successful learning with multiple graphical representations and self-explanation prompts. Journal of Educational Psychology, 107(1), 30-46. doi:https://doi.org/10.1037/a0037211

Rico, L. (1997). La educación matemática en la enseñanza secundaria. Horsori van Beethoven, L. (1824). Sinfonía n. ° 9, la Oda a la alegría. Viena

Veraksa, N., Colliver, Y. & Sukhikh, V. (2022). Piaget and Vygotsky's Play Theories: The Profile of Twenty-First-Century Evidence. In Nikolay, V. and Ingrid P. S., Piaget and Vygotsky in XXI century. Discourse in early childhood education (pp. 165-190). Springer

Vinner, S., Hershkowitz, R., & Bruckheimer, M. (1981). A Forum For Researchers: Some Cognitive Factors as Causes of Mistakes in the Addition of Fractions. Journal for Research in Mathematics Education, 12(1), 70-76. https://doi.org/10.5951/jresematheduc.12.1.0070

HOW A TEACHING PRACTICE THAT BUILDS ON STUDENT THINKING HELPS TEACHERS DRAW OUT CONCEPTUAL CONNECTIONS

Joshua M. Ruk
University of Wisconsin-Whitewater
rukj@uww.edu

Laura R. Van Zoest
Western Michigan University
laura.vanzoest@wmich.edu

Past research has identified factors that help maintain the cognitive demand of tasks, including drawing conceptual connections*. We investigated whether teachers who were engaging in the teaching practice of* building*—and thus focusing the class on collaboratively making sense of their peers' high-leverage mathematical contributions—drew conceptual connections at a higher rate than has been found in previous work. The rate was notably higher (54% compared to 14%). By comparing multiple enactments of the same task, we found that this higher rate of drawing conceptual connections seemed to be supported by (1) eliciting student utterances that delve more deeply into the underlying mathematics, (2) giving students more time to explore the underlying math, and (3) using previously learned abstractions to help move the class toward understanding the new abstract concepts underlying a task.*

Keywords: Classroom Discourse, Instructional Activities and Practices

The important role that high-cognitive-demand tasks play in student learning is ubiquitous in mathematics education (e.g., NCTM, 2014). Unfortunately, the high cognitive demand of tasks is often not maintained as those tasks are enacted in classrooms (e.g., Henningsen & Stein, 1997). As a result, much work has been done to understand the complexity of maintaining high levels of cognitive demand during task enactments. For example, Stein et al. (1996) identified factors that maintain and lower cognitive demand. These factors have been utilized by other studies (e.g., Sullivan, 2019) to better understand the maintenance of cognitive demand. One of these factors that has not received much attention is *drawing conceptual connections*. This is despite the fact that developing conceptual connections is at the core of the type of student learning envisioned by NCTM (1989, 2000, 2014).

The *teaching practice of building* (e.g., Leatham et al., 2022; hereafter referred to as *building*) is a teaching practice designed to take full advantage of MOSTs (Mathematical Opportunities in Student Thinking)—high-leverage student mathematical contributions that "provide an in-the-moment opportunity to engage the class in joint sense making about that contribution to better understand the important mathematics within it" (Van Zoest et al., in press). This important mathematics that students can come to better understand—the *mathematical point* (MP) of the contribution (e.g., Van Zoest et al., 2016) is central to building. The teacher identifies the MP when deciding if a student contribution is a MOST, keeps it in mind throughout the joint sense making discussion, and finally, ensures that the MP—the mathematics that the students had the opportunity to learn as a result of the discussion—is made explicit as the discussion concludes (e.g., Leatham et al., 2022).

It seems that the emphasis that building places on MPs and student thinking may support teachers to draw out conceptual conceptions, and thus to better maintain the high cognitive demand of tasks as they are enacted in their classrooms. To better understand the phenomenon of conceptual connections and how building might support drawing them out, this study investigated how teachers who were attempting to build on MOSTs in their classroom engaged in drawing out conceptual connections from their students.

Literature Review

Cognitively demanding tasks are challenging problems, or sets of problems, that require students to use their existing knowledge, sometimes in new and unique ways, along solution pathways that are not immediately clear (Stein et al., 1996). The use of such tasks can lead to student learning gains (Stein & Lane, 1996). As mentioned above, the high cognitive demand of these tasks is often not maintained as they are enacted in classrooms (Henningsen & Stein, 1997), and as a result, much work has been done to understand the complexity of maintaining high levels of cognitive demand during task enactment. The foundation of that work is Stein et al.'s (1996) study of 520 task enactments by teachers utilizing reform-based teaching practices, which led to identifying seven factors that maintain, and six factors that lower, cognitive demand. *Drawing conceptual connections* is one of the factors that help maintain cognitive demand.

Drawing conceptual connections occurs when a teacher or student explicitly makes connections between a task and its underlying mathematical concepts. Stein et al.'s (1996) study found drawing conceptual connections in only 13% of tasks where cognitive demand was maintained during enactment. Henningsen and Stein (1997) looked specifically at tasks that began at the highest level of cognitive demand—doing math—and similarly found that drawing conceptual connections occurred in only 14% of such tasks for which the cognitive demand was maintained. In another study, Sullivan (2019) provided a group of 16 teachers with over 300 hours of professional development (PD) to help them maintain high levels of cognitive demand and share mathematical authority through high-quality classroom discourse. They analyzed the teacher-identified "best" enactments using the Instructional Quality Assessment (IQA; Boston, 2012), a toolkit that assesses elements of ambitious instruction in mathematics. On the IQA Mathematical Residue Rubric, which considers the extent to which conceptual connections are drawn during whole-group discussion, the teachers in Sullivan's study had an average score of 2.08 (out of 4). A rating of 2 reflects "the teacher is telling students what connections should have been made" or "students make superficial contributions that are taken over by the teacher" (Boston, 2017, Mathematics Residue Rubric). Smith and Stein (2018) created *5 practices for orchestrating productive mathematics discussions* as a way to help teachers maintain the cognitive demand of task enactments, and described the 5th practice "connecting different students' responses and connecting the responses to key mathematical ideas" (Smith & Stein, 2018, p. 10) as "the most challenging of the five practices because it calls on the teacher to craft questions that will make the mathematics visible and understandable" (p. 70). Perhaps because drawing conceptual connections is the least prominent factor that helps maintain cognitive demand, less research has been done to understand it than many of the other factors, and consequently less is known about it as well.

We can infer some things about drawing conceptual connections from Hiebert and Wearne (1993), who looked at classroom discourse to understand the maintenance of cognitive demand. They found that some types of questions teachers ask, such as "Why does this procedure work?" or "What's going on with this strategy?", helped maintain cognitive demand. Similarly, increased length of student utterances during whole class work also positively correlated with increased cognitive demand. Essentially, a teacher asking probing questions that require students to give explanations, rather than one-word answers, is an indicator of maintaining cognitive demand. Although Hiebert and Wearne did not discuss drawing conceptual connections specifically, when teachers ask students to explain the nature of a problem using descriptive answers, for example, they are creating a path that can lead to students making conceptual connections.

To better understand how teachers draw conceptual connections, we investigate these questions: (1) Do teachers who are attempting to build on MOSTs in the context of enacting a high cognitive demand task rate higher on drawing conceptual connections than those who are not? (2) What can we learn about drawing conceptual connections from analyzing their instruction?

Theoretical Framework

We approach this work from a *Knowledge in Pieces* epistemological perspective (e.g., diSessa & Sherin, 1998; diSesssa, Sherin, & Levin, 2016). Knowledge in Pieces (KiP) models knowledge as "a complex system of many local abstractions of experience" (Walkoe & Levin, 2020, p. 28). Harlow & Blanchini (2020) describe key ways in which the KiP perspective influences teaching, including teachers: (1) recognizing students' initial ideas as "useful and productive for building understanding that is consistent with canonical knowledge" (p. 397) and (2) designing their instruction around their students' ideas, often in the midst of their instruction. MOSTs are often incomplete or non-canonical thinking (Van Zoest et al., 2017) and building is a teaching practice that supports teachers to act in ways consistent with a KiP perspective because it centers student thinking and provides a pathway for facilitating collaborative sense-making about that thinking (Leatham et al., in press). Thus building on MOSTs is a teaching practice consistent with the KiP epistemology.

Building on MOSTs is comprised of four elements (Leatham et al., 2021, p. 1393):

1. *establish* the student mathematics of the MOST as the object to be discussed;
2. *grapple toss* that object in a way that positions the class to make sense of it;
3. *conduct* a whole-class discussion that supports the students in making sense of the student mathematics of the MOST; and
4. *make explicit* the important mathematical idea from the discussion [the mathematical point (MP) of the MOST].

Although the centrality of the MP to building heightens opportunities to draw conceptual connections throughout the practice, the focus of the fourth element, make explicit, is drawing a conceptual connection between the whole-class discussion and the MP of the MOST. For this reason, the instruction of teachers who are attempting to build on MOSTs seems a fruitful context for investigating the way in which teachers draw conceptual connections.

Mode of Inquiry

Six middle school teacher-researchers (TRs) in the larger MOST project, which was focused on conceptualizing the teaching practice of building (for more details, see Leatham et al., 2022), provided 24 videotaped classroom task enactments (each teacher enacted two tasks twice; see Figure 1 below for the tasks). The enactments were analyzed using The Instructional Quality Assessment (IQA; Boston, 2012) by researchers trained in using the IQA who had no connection to the project. We also analyzed these enactments with the **R**eorganized Factors that **U**ndermine or **K**eep Cognitive Demand (RUK; Ruk, 2020). The RUK is a succinct tool designed to measure the factors that maintain and lower cognitive demand (as identified by Stein et al., 1996) so that such measurements can be compared across studies. The Conceptual Connections category of the RUK looks at the extent to which conceptual connections to underlying concepts were made and seemed to be understood by students. Our analysis of these results was compared to what is known about this factor and used to create hypotheses about how the teachers drew conceptual connections during the task enactments. Based on these hypotheses, questions for an online survey and teacher interviews were created. For example, teachers were asked "When enacting this task, what would you ideally like to hear students say to show you that they understood the underlying mathematics of this task?" These questions were given to the five teachers who were able to participate in this part of the data collection. Their responses allowed us to verify or disprove the hypotheses. For more details about the larger study on the maintenance of high cognitive demand, see Ruk (2021).

Results & Discussion

The teachers in our study (TRs), who were attempting to build on MOSTs in the context of enacting a high-cognitive demand task, rated higher on drawing conceptual connections than those in past research studies. Specifically, the TRs drew conceptual connections during 54% of the enactments, compared to 13% and 14% in Stein et al. (1996) and Henningson and Stein's (1997) studies, respectively. Furthermore, on the IQA Mathematical Residue Rubric (similar to the RUK Conceptual Connections category), the TRs had an average score of 2.83, compared to 2.07 in Sullivan's (2019) study. It is important to note that both of these groups were middle school mathematics teachers who were committed to NCTM-Standards-based teaching, thus this difference is quite striking. The higher ratings for drawing conceptual connections when attempting to build on MOSTs are likely due to the structure of the building practice and the focus on drawing out student contributions related to the underlying mathematics of the MOST that is being built on. This specificity may have done more to draw out conceptual connections than a general focus on reform-based teaching practices (Henningsen & Stein, 1997), or high-quality classroom discourse (Sullivan, 2019).

We now turn to what we can learn about drawing conceptual connections from analyzing these teachers' enactments. When teachers attempt to draw conceptual connections, they attempt to surface connections between in-class work and the mathematical concepts underlying this work. The RUK looks not only at this attempt but also at how well these connections appear to be understood by students. Figure 1 shows each teacher's score on the RUK Conceptual Connections category for each of their enactments. The variability across enactments both within and between teachers may reflect the fact that these teachers were attempting a new practice. This variation, however, allowed us to analyze differences in teacher actions between the highest-rated (those rated 4) and lowest-rated (those rated 1) enactments. In the interest of space, we draw our examples from the Variables task (Figure 1a).

Task	Enact-ment	RUK Conceptual Connections Rating by TR					
		TR1	TR2	TR3	TR4	TR5	TR6
a. Variables: Which is larger, x or $x + x$? Explain your reasoning.	1st	1	4	4	3	3	2
	2nd	3	1	3	3	4	1
b. Percent Discount: The price of a necklace was first increased 50% and later decreased 50%. Is the final price the same as the original price? Why or why not?	1st	3	1	4	1	4	1
	2nd	1	2	4	3	2	1

Figure 1: Conceptual Connections Rating by Teacher-Researcher for Each Enactment

Comparing task enactments rated highest and lowest on the RUK Conceptual Connections category led to the identification of three main differences. First, during whole-class discussion for the highest-rated enactments students themselves made utterances that delved more deeply into the underlying mathematics, even without the teacher needing to push them. In contrast, during the lowest-rated enactments, teachers tried to push students to explore the underlying mathematics of the task more deeply but were less successful. Second, students in the lowest-rated enactments were not given time to explore the underlying mathematics after it surfaced. In highest-rated enactments, students were given additional time to do this. Third, the exploration of previously learned abstractions seems to be a needed precursor for understanding a task's underlying mathematics. For the highest-rated enactments, this exploration was used mainly to help move the class toward understanding the new abstract concepts underlying a task. For the lowest-rated enactments, this exploration was used partially for this purpose, but mostly for solidifying abstract concepts needed to engage with the task in the first place. In the following, we describe each of these three patterns. Before doing so, it is important to note that there was no indication that students in highest-rated enactments were different from students in the lowest-rated enactments in ways that would affect the study.

Utterances of the Underlying Mathematics

All of the task enactments analyzed contained student utterances that helped move the class toward surfacing connections to the underlying mathematical concept, but only those in the highest-rated enactments actually surfaced the concept. Figure 2 shows the underlying mathematics concept for the Variables task and two examples of student utterances that surfaced the underlying concept. The highest-rated task enactments also contained teacher comments that elicited student utterances that delved more deeply into the underlying mathematics by specifically asking them to make connections between the discussion and the underlying mathematics. Here are two examples:

- Okay, because some of you initially just said that x+x is bigger 'cause it's double the size but now you guys are saying it really depends on the value of the variable. So, if you go back to the original statement, what did we figure out? Can somebody summarize for us what we just learned?
- It depends on the number that x is? Can somebody restate what we just learned in this one problem combining everyone's thinking? What should we consider when we are comparing two expressions?

Variables task	Underlying math concept	Examples of student utterances that surfaced the underlying concept

Which is larger, x or $x + x$? Explain your reasoning.	The domain of the variable must be considered to determine relative values of variable expressions.	"If the x is a positive then x+x is bigger. If x is a negative, then x is bigger and then if x is zero then they're equal." "It just depends on what x is, if the x is negative, then x is negative, if positive then x+x is positive."

Figure 2: Task, Underlying Concept, and the Utterances that Surfaced this Concept

In contrast, the lowest-rated enactments were not as effective at eliciting student utterances that delved more deeply into the underlying mathematics. For example, in one lowest-rated enactment students had not mentioned the case of x=0 (which is not needed to answer the question), so the teacher asked, "Is there anything else we should look at, are there any other numbers out there besides positives and negatives?" After a student said zero, the teacher led them to expand their statement to say "[if x is zero] none of them are any bigger than the other," and then the teacher extrapolated this to mean that x and x+x are equal. The teacher then tried to surface the underlying mathematics by asking students to recite all of the cases (x positive, negative, and zero). Rather than saying anything clearly related to domain, students made vague utterances such as "they're both right and they're both wrong" or "try with both negative and positive numbers if it's a variable." Students never uttered an encompassing statement of the underlying math, and the teacher made an attempt themself by saying, "be more general, just think [of] different numbers for x. You might have to try other values. Not just positives and negatives." It seems that although the teacher could see the connection between zero and the problem, the students did not. Thus, when the teacher focused on zero without reorienting the students, they were no longer in a sense-making position and appeared to lose the sense that they had already made. This points to the importance of teachers being explicit about grounding their questions in the discussion—something we saw examples of teachers in the highest-rated enactments doing in the bulleted questions above.

Time to Explore

Directly after the teacher's utterance of the underlying mathematics, the teacher ended the task by saying, "Ok, make sure your name is on your paper." This abrupt ending gave the students no time to discuss the underlying mathematics. This was problematic because these particular cases (positive, negative, and zero) do not generalize to all variable comparisons. Similar patterns were present in other lowest-rated enactments as well. For example, in one enactment it seemed students could have uncovered the underlying mathematics themselves, as one student said, "[I]t depends on the value of x." However, the teacher did not allow time to consider this idea further. Had the teacher allowed the class to consider the student's idea, they likely would have connected it to the context of the problem, since they had discussed positive, negative, and zero earlier. However, across the lowest-rated enactments, students were not given time to discuss the underlying mathematics. Thus, students again had to make the connections between the contextual examples from this problem and the underlying mathematics on their own. To contrast this, we return to the highest-rated task enactments. During the whole-class discussion for these enactments students said things like, "[I]t will depend on x. If it is a positive, x+x will be greater, but if it is a negative x, x it will be greater. Or if it was a 0, they'd be equal," and "[It] depends on the x-value, if it's positive, negative, or zero." These utterances discussed positive, negative, and zero—the three components needed to fully consider the domain in the context of this task. Additionally, these utterances occurred earlier during whole-class discussion than any utterances considering zero during the lowest-rated enactments. After these initial utterances considering negative values and zero, students in the highest-rated enactments were allowed additional discussion time, and several other students made similar utterances, so ultimately multiple students uttered complete examples of the underlying mathematical concept. Again, this is in comparison to the lowest-rated enactments where only the teacher uttered such complete examples, and the whole-class discussion ended shortly thereafter. Overall, teachers in the highest-rated enactments supported students to provide robust utterances of the underlying math, as opposed to the lowest-rated enactments where teachers allowed for only superficial utterances.

Previously Learned Abstractions

For all 12 of the Variables task enactments, students gave a concrete example of x having a negative value. Students said things like: "-9 plus -9 is -18, and -18 would be less than -9." For the highest-rated enactments, such examples emerged *after* more abstract utterances that if x is negative, then x plus x is less than x, and for the lowest-rated enactments at least one concrete example came *before* an abstract utterance. This shows a different pattern for whole-class discussion in the highest- and lowest-rated enactments. Overall, the lowest-rated enactments started with misconceptions, moved to concrete examples, then to an abstraction of those concrete examples, and concluded with a brief statement related to the underlying mathematics that was not discussed further by the class. For these enactments, concrete examples were, at least partially, used to understand the mathematical concepts needed to engage with the task. For example, if a student said, "Isn't negative 3 plus negative 3 equal to negative 6," they were not saying this strictly as an example of the more abstract concept that if x is negative, then x plus x is less than x, but rather as a way to verify their understanding of adding negative numbers. Conversely, the highest-rated task enactments started with an abstraction of the mathematics needed to find a solution, moved to concrete examples of this abstraction, and ended with discussion of, a new abstraction of the tasks' underlying mathematics, which seemed to be understood by the majority of the class. For these enactments, concrete examples were in service of understanding the underlying mathematics of the task, as opposed to focusing on understanding prerequisite concepts.

These observations suggest that exploration of previously learned abstractions may be a needed prerequisite to understanding a task's underlying mathematics. However, for the highest-rated enactments, this exploration was predominantly used to move toward understanding new abstract mathematical concepts underlying the task—the underlying mathematics of the task. But for the lowest-rated enactments, this exploration primarily focused on solidifying an abstract concept needed to engage with the task—such as adding negative numbers.

Implications

This study found higher rates of drawing conceptual connections than has been found in previous work. The difference is likely due to the building practice calling for explicit utterances of the underlying mathematics. Thus, even though building was not specifically developed to support the maintenance of high cognitive demand tasks, because aspects of the practice aligned with factors that support the maintenance of cognitive demand, teachers who attempted to build in the context of using a high-cognitive demand task by default increased their ability to maintain the high-cognitive demand of the task. This finding suggests that developing teaching practices that support factors that have been found to maintain high cognitive demand may be an important way to increase the maintenance of high cognitive demand tasks during enactments. Our results also showed that conceptual connections were drawn out at higher levels when multiple students made complete utterances of the underlying mathematics and the class was given sufficient time to explore the ideas that surfaced. Furthermore, the exploration of previously learned abstractions seems a necessary prerequisite to understanding a task's underlying mathematics, and this exploration is more productive if it is used to move towards understanding new abstract mathematical concepts underlying the task rather than to solidify abstract concepts needed to engage with the task.

Overall, to draw conceptual connections, teachers need to explicitly ask for them and allow students enough time to discuss the underlying mathematics so that it is uttered by numerous students, and make sure students are taking what they learn from engaging with the task and using it to move towards new abstractions, rather than simply rehashing previously learned abstractions. Most importantly, the results of this study show that teachers' actions are needed to help students draw conceptual connections. If students are allowed to explore underlying mathematics, as opposed to a teacher directly telling them what it is, it appears that all students have the ability to draw conceptual connections.

Acknowledgements

This research report is supported by a Western Michigan University Dissertation Completion Fellowship and by the U.S. National Science Foundation (NSF) under Grant Nos. DRL-1720410, DRL-1720566, and DRL-1720613. Any opinions, findings, or conclusions, expressed in this material are those of the authors and do not necessarily reflect the views of the NSF.

References

Boston, M. (2012). Assessing instructional quality in mathematics. *The Elementary School Journal, 113*, 76-104. doi:10.1086/666387

diSessa, A. A., & Sherin, B. L. (1998). What changes in conceptual change? *International Journal of Science Education, 20*(10), 1155-1191. https://doi.org/10.1080/095006998020100

diSessa, A. A., Sherin, B., & Levin, M. (2016). Knowledge analysis: An introduction. In A. diSessa, M. Levin, & N. Brown (Eds.), *Knowledge and interaction: A synthetic agenda for the learning sciences* (30-71). Routledge.

Harlow, D.B., Bianchini, J.A. (2020). Knowledge-in-Pieces—Andrea A. diSessa, David Hammer. In B. Akpan, & T. J. Kennedy (Eds.), *Science Education in Theory and Practice*. Springer. https://doi.org/10.1007/978-3-030-43620-9_26

Henningsen, M., & Stein, M. K. (1997). Mathematical tasks and student cognition: Classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education, 28*, 524-549.

Hiebert, J., & Wearne, D. (1993). Instructional tasks, classroom discourse, and students' learning in second-grade arithmetic. *American Educational Research Journal, 30*, 393-425.

Leatham, K. R., Van Zoest, L. R., Freeburn, B., Peterson, B. E., & Stockero, S. L. (2021). Establishing student mathematical thinking as an object of class discussion. In D. Olanoff, K. Johnson, & S. M. Spitzer (Eds.), *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1392-1400). PME-NA.

Leatham, K. R., Van Zoest, L. R., Peterson, B. E., & Stockero, S. L. (in press). A Decomposition of the Teaching Practice of Building. In *Proceedings of the NCTM 2022 Research Conference*.

National Council of Teachers of Mathematics. (1989). Curriculum evaluation standards for school mathematics. Author.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all*. Author..

Ruk, J. M. (2020). A conceptualized framework for assessing factors that maintain and lower cognitive demand during task enactment. In M. Askew, B. J. Dougherty, & J. T. Zilliox (Eds.), *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 2351-2352). PME-NA.

Ruk, J. M. (2021). *Maintenance of cognitive demand during repeated task enactments using a teaching practice that builds on student thinking* (Publication No. 29150738) [Doctoral dissertation, Western Michigan University]. ProQuest Dissertations & Theses Global.
Smith, M. S., & Stein, M. K. (2018). 5 practices for orchestrating productive mathematics discussions (2nd ed.). Reston, VA: National Council of Teachers of Mathematics.
Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal, 33*, 455-488. doi:10.2307/1163292
Stein, M. K. & Lane, S. (1996). Instructional Tasks and the development of student capacity to think and reason: An analysis of the relationship between teaching and learning in a reform mathematics project. *Educational Research and Evaluation, 2*(1), 50-80.
Sullivan, P. (2019). Examining the tensions between high quality discourse and sharing mathematical authority with students. In S. Otten, A.G. Candela, Z. de Araujo, C. Haines, & C. (Eds.), *Proceedings of the 41st Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1576–1584). PME-NA.
Van Zoest, L. R., Stockero, S. L., Leatham, K. R., Peterson, B. E., Atanga, N. A., & Ochieng, M. A. (2017). Attributes of instances of student mathematical thinking that are worth building on in whole-class discussion. *Mathematical Thinking and Learning*, *19*(1), 33-54.
Van Zoest, L. R., Stockero, S. L., Leatham, K. R., & Peterson, B. E. (2016). Theorizing the mathematical point of building on student mathematical thinking. In C. Csíkos, A. Rausch, & J. Szitányi (Eds.), *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 323–330). PME.
Van Zoest, L. R., Stockero, S. L., Peterson, B. E., & Leatham, K. R. (in press). (Counter)Productive Practices for Using Student Thinking. *Mathematics Teacher: Learning & Teaching PK-12*.
Walkoe, J., & Levin, M. (2020). Seeds of algebraic thinking: Towards a research agenda. *For the learning of mathematics, 40*(2), 27-31.

IT'S NOT ABOUT THE MATH: ENACTING CARING INSTRUCTION IN A COMMUNITY COLLEGE MATHEMATICS CLASSROOM

Megan Selbach-Allen
Stanford University
mselbach@stanford.edu

Abstract: *Math remains a major barrier for students interested in pursuing STEM careers even as policy makers decry the need for more graduates with these majors. One important place students from diverse backgrounds begin their studies is in community colleges even as they remain understudied. This work is an in-depth case study of one 'successful' classroom that is positioning students for future success in STEM majors. Building on prior work around the importance of care in community college classrooms, the study reveals how an instructor uses 'non-math' talk regularly in his class to connect students with the broader importance of what they are learning and express caring. This 'non-math' talk appears to be supplemented with deliberate community building and positioning students as intellectual authorities. Future work will elicit student perspectives and build a framework around this care.*

Keywords: Instructional Activities and Practices; Undergraduate Education

Introduction

For years American policy makers have decreed the need for a more educated workforce especially in STEM fields as we advance into a 21st Century filled with increasingly complicated technology and hyper-connected by the internet. At the same time many advocates in education point to the continuing underrepresentation of certain groups, primarily women and Black and Latinx people, in science, technology, engineering and math (STEM) fields and the need to address this underrepresentation (Chen, 2013). For these and other students who come from families with few socioeconomic resources or who are the first in their family to attend college the barriers to entering STEM fields are magnified. While STEM is composed of science, technology, engineering and mathematics, math is the gate through which students enter or are kept out of STEM. In order to pursue majors and careers in these fields, students often have to excel in early math courses like calculus.

Work in determining how to teach math better and engage all learners is a large ongoing problem that is central to addressing the larger societal issues of increasing the number of students pursuing STEM fields and improving representation in those fields. Community colleges are a particularly understudied area in math education research and an important setting for working to address issues of underrepresentation in STEM fields. Roughly 50% of students who go on to complete 4-year degrees pass through community colleges in the US and they disproportionally serve students with fewer resources and who are the first in their family to attend college (National Student Clearinghouse Research Center, 2017).

In a 2012 paper Ann Sitomer and colleagues (2012) noted that while there is much work and overlap in K-12 and undergraduate math education research, one area that distinctly lacks research is community college math instruction. Furthermore, the majority of the research that does exist follows larger picture trends such as course completion rates and student trajectories. Research examining what is happening in classrooms themselves has been done but is very limited (Mesa et al. 2014). To close this gap instructors have done work studying their own practice and researchers have looked to adapt work done in K-12 or university settings.

A 2019 study sought to do just this in examining how the framework of mathematical knowledge for teaching (Ball et al., 2008) was enacted by experienced community college faculty (Nabb & Murawska, 2019). For their study they asked instructors what is unique about teaching 'developmental math' at community colleges. The interviews with teachers revealed a somewhat surprising finding. A major theme of their results was the importance of caring and trust. Their work does align with prior work by Weston & McAlpine (1998) which revealed caring as a primary theme in a deep analysis of the practice of six outstanding math professors at a university. They also found that professors mentioned content, strategies, evaluation, learning outcomes, passion, and research as important aspects of teaching.

Theoretical Framework

This work draws upon a definition of success from the work of Vilma Mesa and colleagues in their examination of research problems in community college math classroom (Mesa et al., 2014). They take a two-pronged definition of success with two related factors: students learning the content of the class and making progress towards their personal, academic goals. This more expansive definition is important because it goes beyond the more common metrics of course and degree completion to consider that students' personal values and goals. Ultimately if students were completing classes but not able to finish their degrees or meet the goals they have for obtaining their education then the educational environment they are in is not 'successful'.

With this more expansive view of success then the role of the teacher is about more than delivering content but helping to ensure students are making progress towards their personal goals. Considering the broader role of the teacher and Nabb & Murawska's (2019) work which revealed that community college instructors themselves identified the important role of 'care' in their work, I look to define what is 'caring' in the context of a community college math classroom. Nel Noddings (1992) places the care of teachers in the 'caring relation' and distinguishes between teachers work to meet students assumed needs versus their expressed needs. She associates assumed needs primarily with teachers' beliefs about what they think their students need such as to learn the content of a courses. Assumed needs are setup in contrast to expressed needs such as a student's dislike of school or the subject of study. Teachers are expected to be attentive listeners and balance students expressed and assumed needs. This framework offers a good starting point, and it does mention that caring relations are not always equal, but it does not explicitly address issues around power and equity and how caring relations in a classroom are impacted by broader societal structures. These are important considerations for caring in the context of diverse classrooms. From this framework comes the research question for this study: How does a successful community college mathematics instructor enact 'caring' in the classroom?

Methods

Study Setting and Participants

This study is occurring in a community college mathematics classroom in the Bay Area of California. The instructor of this class identifies as a white male and also attended community college before moving on to a 4-year university and ultimately obtaining his master's degree in mathematics. He has many years of experience with over 20 years of teaching experience and has been teaching this course sequence of compressed pre-calculus and calculus for 8 years.

The 45 students in this classroom are typical of community college students. They are racially, ethnically, and socio-economically diverse. Some students have prior exposure to calculus content in high school, while others have not. The students vary in age with many coming straight from high school but others having been in the work force for a number of years. The majority of students continue to work while attending classes. All students in the class aspire to major in either engineering or computer science.

In the vein of wisdom of practice studies (Shulman & Wilson, 2004), this teacher and class context were deliberately chosen and are defined as successful. With this case study I am looking to understand what might be considered 'best practice' from an exemplar case and this example is not meant to be representative. The goal of this study is to understand from deep examination of a single case what might be some mechanisms driving the success of this classroom environment.

Evidence of the success of this classroom comes from examination of course pass rates which are significantly higher than the typical rate for the college. This evidence, which while it could be viewed as evidence of students learning content, alone does not meet the broader criteria of success defined above, was supplemented with data from a pilot study of interviews with former students and classroom observations to ascertain how this environment was helping students meet their academic goals. The pilot study interviews with former students revealed the unique nature of this classroom and the support students felt in the environment.

Data Collection and Sources

The primary sources of data for this study are classroom observations and teacher interviews. Additionally, the in-person class observations are supplemented with access to the electronic platforms used for class facilitation and communication. In addition to the teacher focused data, an initial survey was given to students to gage their feelings about and engagement with math and the same survey will be given again at the end of the academic year. Students completed a math autobiography essay which will also provide some information about each students' background before entering the class and interviews with students will be conducted during the spring semester to learn about individual students' experiences of the classroom environment.

Data Analysis

This study is still in the initial phases of analysis with the focus primarily on the classroom observation fieldnotes. The fieldnotes were reviewed and open-coded. This coding revealed the common presence of 'off-topic' conversations. These instances were then reviewed in further detail to look for common themes.

Preliminary Findings

Initial Results showed the high degree to which the teacher engaged in discussion and content that did not fall within the formal curriculum of pre-calculus or calculus but could be seen as relevant to the students broader goals. Early review of field notes also reveals routines around community building and positioning of students as authorities on mathematics and sources of knowledge for one another.

'Non-math' Talk

During the Fall semester the instructor invited guest speakers into the class on four different occasions and took the students on one field trip. During week three of the semester he invited two representatives from the school's transfer office to come share with the students information about pathways for transfer and the role of their office. The speakers shared for approximately 15 minutes when the instructor shared that the class would be visiting the transfer day fair the following week and he had created an assignment for the students to talk with at least three different schools. The following week he gave up around 90 minutes of instructional time to take his students to the school transfer day. In interviewing him about why he did this he shared about how he had seen these interactions change students' motivations in the classroom. He shared how after talking with a school at the transfer day one student understood the importance of getting good grades since the school was competitive for transfer. Other students shared about opportunities they were not previously aware of. When framing this visit to the class he emphasized about how the ultimate goal should not be just passing this class but about their broader academic goals and where do they want to go in their lives.

In addition to the transfer day, the instructor gave up around 60 minutes of instruction during week 10 to a workshop on writing resumes. Prior to the workshop the instructor shared how internships can be important pathways to jobs in STEM fields and learning how to write a good resume was an important part of that. He had previously assigned students work on a draft of their resume prior to the workshop.

In addition to these instances of outside speakers the instructor also regularly shared messages about mindset and learning to students. After an exam where some students had struggled he shared a TED talk about growth mindset and encouraged students to watch it before working on their exam corrections assignment. This example illustrates how the non-math talk around ideas like growth mindset was also reinforced by structures in the class such as exam corrections assignments that allowed students to get points back by reviewing their work and learning from their mistakes.

'Community Building

Community building appears to be another important aspect of this class. Everyday before class starts the instructor moves through the class fist-bumping each student and thanking them for coming to class. Early on he made an effort to know every student's name and greet them by such. He also made an assignment out of having each student schedule a one-to-one meeting with him so he could be sure to have a conversation with each student. However, beyond his interactions with the students he encourages the students to work with one another and to see the class as a learning community. Every day also started with students being asked to check-in with one another and he generally had students turn and talk to a neighbor 8-12 times per a class. These directions to talk with a neighbor were framed in a way that positioned the student as an authority with wording like, "I am confident you understand but I want you to check with your neighbor." The instructor would also make comments to the class, "no one is left behind" as he directed them to work together on a problem.

Discussion, Implications and Continued Work

This study is a single case study and thus cannot speak to broad situations but serves as a basis for further study. It also builds on other studies that have illustrated the importance of issues that go beyond content delivery in community college mathematics classrooms. This class illustrates potential mechanisms for how caring is enacted by an instructor and could serve as an initial point for building a framework for what caring looks like in these contexts. Further work will also elicit student perspectives to understand how they experience the classroom environment. While learning the math is important, it appears from this example that engaging all learners is not about the math but about care and community that goes beyond it.

Acknowledgments

The author would like to thank the professor and students who participated in this study and acknowledge the important work they are engaged in. They would also like to acknowledge the BLINDED University for a Dissertation Support Grant to support the conduct of this research.

References

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389-407.

Chen, X. (2013). STEM Attrition: College Students' Paths into and out of STEM Fields. Statistical Analysis Report. NCES 2014-001. *National Center for Education Statistics*.

Mesa, V., Wladis, C., & Watkins, L. (2014a). Research problems in community college mathematics education: Testing the boundaries of K-12 research. *Journal for Research in Mathematics Education*, *45*(2), 173-192.

Nabb, K., & Murawska, J. (2019). Mathematical knowledge for teaching developmental courses at the community college: An unexplored terrain. *Community College Journal of Research and Practice*.

National Student Clearing House Research Center. (Spring, 2017). *Contribution of Two-Year Public Institutions to Bachelor's Completions at Four-Year Institutions*. https://nscresearchcenter.org/wp-content/uploads/SnapshotReport26.pdf

Noddings, N. (2012). The caring relation in teaching. *Oxford review of education*, *38*(6), 771-781.

Sitomer, A., Strom, A., Mesa, V., Duranczyk, I. M., Nabb, K., Smith, J., & Yannotta, M. (2012). Moving from Anecdote to Evidence: A Proposed Research Agenda in Community College Mathematics Education. *MathAMATYC Educator*, *4*(1), 35-40.

Shulman, L. S., & Wilson, S. M. (2004). *The wisdom of practice : essays on teaching, learning, and learning to teach* (1st ed.). Jossey-Bass.

Weston, C., & McAlpine, L. (1998). How six outstanding math professors view teaching and learning: The importance of caring. *The International Journal for Academic Development*, *3*(2), 146-155.

"I UNDERSTAND THAT THEIR MINDS MAY BE ELSEWHERE": TOWARDS A CULTURALLY RESPONSIVE MATH PEDAGOGY

Tarah Michelle Donoghue
University of Colorado - Boulder
tarah.donoghue@colorado.edu

Victoria Hand
University of Colorado - Boulder
victoria.hand@colorado.edu

Brenda A Aguirre Ortega
University of Colorado - Boulder
brag2087@colorado.edu

Victor Baldemar Leos
University of Colorado - Boulder
victor.leos@colorado.edu

This brief research report examines the discursive shifts of a secondary mathematics teacher participating in a collaborative learning community centered on culturally responsive mathematics teaching. We draw on two frameworks to analyze the teacher's discursive moves. The first framework comes from Lefstein et al. (2020) on generative discourse practices in learning communities. The second framework — FAIR (Louie et al., 2021) — offers noticing practices for deficit versus anti-oppressive mathematics teaching. Through these lenses, we found that the teacher's initial discourse practices were marked by deficit framing and noticing. The teacher's discourse practices begin to shift towards a culturally responsive pedagogy in response to a particular artifact that captured student noticing and reframed the teacher's problem of practice.

Keywords: Professional Development; Culturally Relevant Pedagogy; Discourse; Teacher Noticing.

Introduction

Professional development initiatives for secondary mathematics teachers are increasingly focused on learning to teach mathematics in a culturally responsive way. Yet, learning to become a culturally responsive mathematics teacher has been complicated by teachers' participation in dominant discourses practices around deficit ideologies of students of color, their communities, and families. These dominant discourse practices are stitched into the fabric of mathematics teaching and learning and are often invisible to mathematics teachers from dominant backgrounds.

Recently, teacher professional development has taken place in learning communities, where teachers come together to share artifacts and dilemmas from their instructional practice. Teacher learning communities are more likely to be more deeply grounded in teachers' everyday work and to support teacher collaborative conversation and reflection. Teacher learning communities focused on culturally responsive mathematics teaching may be more likely to support teachers in rejecting deficit ideologies in moments of everyday classroom life.

This study examined the discursive practices of one secondary mathematics teacher as they participated in a collaborative learning community that brought teachers and researchers together to learn about and build culturally responsive teaching practices. The teacher's discursive practices in this collaborative learning community were of particular interest since they appeared to shift over time in relation to the goals of the professional development to help mathematics teachers learn to see and respond to issues of equity and justice in their schools. We sought to explore the relation of these shifts to specific aspects of the collaborative learning community and in relation to anti-deficit ideologies.

Theoretical Framework

Lefstein et al., (2020) define discourse as “an essential medium through which members of a community of practice coordinate their work, make meaning, and negotiate roles and identities,” while also being “constitutive of practitioners’ professional vision: how they gaze upon and make sense of phenomena in their domain of work” (p. 2). In their review of empirical research on teacher collaborative discourse in professional learning activities, Lefstein et al., (2021) identified five discourse practices that were essential to productive shifts in practice. The first discourse practice, *revealing and probing problems of practice*, what we will refer to as Revealing, relates to the ways that teachers take risks with others in sharing aspects of their teaching with which they are grappling. The second practice, *providing reasoning and evidence*, or what we call Reasoning, refers to the ways that teachers support their claims about their practices, classroom, or other aspects of their teaching. In this case, providing concrete evidence was shown to hone teacher discourse toward problem-solving. The third practice, *making connections to general principles*, or what we call Generalizing, relates to moving beyond particular events and situations to general patterns and principles. The fourth practice, *building on others’ ideas*, or what we call Building, refers to the ways that teachers’ discourse reflects dialogue and elaboration. The final practice, *offering different perspectives*, or what we call Contrasting, reflects the ways that perspectives in the discourse provide points of contrast and challenge. Taken together, these five aspects of teachers’ collaborative discourse practices, Revealing, Reasoning, Generalizing, Building, and Contrasting provide markers of generative opportunities to learn.

Research on how these generative discourse practices support secondary mathematics teachers in learning to teach mathematics in culturally responsive ways is less prevalent. One framework that seems particularly promising for understanding these shifts is the FAIR framework (Louie et al., 2021). The FAIR framework is conceptualized as a sociopolitical perspective on mathematics teacher noticing, or the social and political lens through which teachers frame, attend to, interpret, and respond to everyday moments of classroom life. Research on mathematics teacher noticing has revealed that teachers often attend to moments of classroom life through interactional frames guided by deficit or colorblind perspectives (Author, 2022; Louie, 2018). Louie et al. (2021) argue that anti-deficit noticing is marked by: (1) framing students as human beings instead of simply math learners, (2) framing mathematics learning as a vibrant and creative process instead of a fixed way of knowing, and (3) framing both informational and interpersonal aspects of interaction as central to mathematics learning, instead ignoring the latter. Studies of teachers’ development of critical consciousness reveal similarly that teachers’ discourse remains persistently deficit in nature. As a result, supporting shifts in teachers’ discourse practices towards ones that are anti-deficit, asset-based, and relational has been challenging.

One of the ways that teachers can become aware of deficit ideologies through which they perceive of students is through the perspectives of the students themselves. Students’ perspectives are often ignored in classroom teaching (Dominguez, 2019), apart from what teachers perceive in ongoing social interaction. While soliciting students’ perspectives on teaching is not new, it is often through formative assessment around the subject matter or classroom practices. In contrast, students can also be invited to share their perceptions of what their teacher seems to be noticing about them. Teachers might believe they are noticing and responding to students in caring ways, for example, while students might perceive this noticing as deficit or oppressive. This disconnect can spark deep reflection for teachers around their assumptions about students.

This study examines how one secondary mathematics teacher responded to the noticing of students in her classroom in the context of a professional development on culturally responsive mathematics teaching. Our research questions include:

1. In what ways did the teacher's initial discourse practices in relation to students, mathematics teaching, and broader sociopolitical systems reflect deficit ideologies?
2. What was the nature of the shifts in teachers' collaborative discourse around the noticing prompt?
   a. How did the teacher begin to frame, attend to, interpret and respond to students' noticings in ways that were aligned with culturally responsive mathematics teaching?

Methods

This qualitative case study was conducted on a one-year collaborative learning community focused on culturally responsive mathematics teaching and anti-racist noticing practice. The learning community was guided by the authors of this study, who met with secondary mathematics teachers on a bimonthly over Zoom. Data collection included a baseline and final interview, recordings of the Zoom meetings, teacher artifacts, and two noticing interviews. The participants of the PD each created their own action research plan in which they applied their learning from the PD in a meaningful way for their own school and teaching practice.

Subjects

The case study focuses on Carrie, a mathematics teacher at a middle school in a metropolitan area. Carrie identified as a white woman who had been teaching mathematics for approximately eight years. The demographics of the high school at the time were: 64% white, 14% Hispanic, 11% Black, 8% two or more races, 2% Asian or Pacific Islander, and %1 Native American. Carrie was chosen as the focal teacher for this study due to significant shifts in discourse revealed during initial open coding.

Positionality

The authors of this paper designed and facilitated the teacher learning community. The first author identifies herself as a white, woman. She taught middle school mathematics for three years and is a fifth-year PhD candidate. She studies culturally sustaining pedagogies in math classrooms using an Indigenous epistemological framework. The second author also identifies as a white woman and is the advisor for the three other authors. She studies issues of equity in mathematics education and has found it challenging to untangle from dominant deficit discourses. The third author identifies as Latina, international graduate student. She is a third-year PhD student who for the last two years has been working as a facilitator and researcher of learning spaces that promote Political Conocimiento for mathematics teachers. The fourth author identifies as a first-generation Latino graduate student. He is a second-year PhD student who has spent the last two years working with math and science teachers in professional learning communities around equitable teaching practices and inclusive pedagogies.

Data Collection

In a preliminary analysis, we identified one artifact in particular, the student noticing survey (SNS), to be particularly supportive of shifts in teachers' deficit discourses. The SNS was designed by the authors, who then presented it to the teachers for their review and editing. It was designed to prompt students in the teachers' classes to describe what think their teacher pays attention to about them (the student), and to other students in the class.[7] The SNS was then turned into an online format (found here). The teachers distributed the SNS to one or two mathematics classes of their choice. Following the collection of completed surveys, the teachers and authors engaged in a discussion about the survey results. Carrie continued to refer to the SNS in her discursive practices throughout and beyond the professional learning community.

To analyze Carrie's discourse practices before, around, and after the SNS, we examined transcripts of her initial interview, her utterances during the professional development meeting, transcripts and notes from her action research plan meeting, the action research plan, and her final interview.

Analysis

To analyze the data, we first transcribed recordings of all PD sessions and participant interviews. Following the transcription, we reviewed our field notes, teachers' artifacts, and the transcriptions for themes in teachers' discourse practices. We proceeded to develop hypotheses about these shifts. Data was thematically coded in Dedoose by the first author based on key elements of the PD. The initial coding process generated ten codes (e.g.,positioning, tools, collaboration, culturally responsive, and deficit). The second author re-coded the data to seek interrater consensus. Following the initial coding, we honed our analysis on one particular teacher, Carrie, due to her significant discursive changes throughout the PD, and her strong response to the SNSs responses from her students.

The second round of coding focused on Carrie's discursive moves over time and employed codes based on the two frameworks from Lefstein et al. (2020) and Louie et al. (2021) (e.g., generalizing, contrasting, deficit, anti-deficit, framing, building, reasoning, revealing, interpreting, and reflection).The analysis revealed particular patterns in Carrie's discursive moves over time. For example, deficit discourse was coded 7% across the data at the start of the PD, and 3% across the data towards the end. Similarly, anti-deficit discourse was coded 0 %at the start, and 21% at the end. Themes emerged based on these patterns in Carrie's discourse and around aspects of the collaborative learning community

Findings

This study explored the extent to which Carrie engaged in productive discourse practices in the professional development, and how this engagement related to the SNS, and to shifts in her noticing of themselves, students, mathematics, and society. Our results show that as Carrie interacted with the results of the survey in a collaborative way, Carrie also worked through the practices of generative discourse, and in doing so experienced shifts away from her initial discourse practices and towards discourse practices aligned with culturally responsive teaching. Carrie also took up the discourse practices in circular ways, meaning that her noticings and discourse practices continually evolved as they generated questions and problems of practice through new orientations. Here, we describe their discursive activities around the survey in detail. Students were not participants in the research; therefore we focus on Carrie's comments and reflections about the survey rather than direct student responses.

---

[7] Students were not participants in the research study, so researchers did not analyze the survey results.

Carrie's perspective of her students at the beginning of the professional development indicated a deficit perspective. At the beginning of the school year, she framed students as being non-compliant with her requests and described them as "lacking stamina and lacking grit" when it came to mathematical tasks. In this framing, Carrie employed deficit discourses that locate educational problems with students and their families. She complained that she was experiencing many disruptions in class, saying, "…[students were] constantly talking and over each other and over me and if one thing happens that's interesting everything gets lost". In this utterance, we note that Carrie was providing evidence about what was occurring in her classroom to position the students as "non-compliant." It is interesting to note what Carrie is attending to, that students are talking over each other and that they get distracted, which again stems from a frame of deficit ideologies and behavior management, rather than one of relationality or engagement.

Carrie also remarked about her students' perceived lack of care about their education, stating, "I feel frustrated when students don't try. I tell them all the time that I care about their education but I don't understand why some don't." In this utterance, Carrie is interpreting students' behavior as 'not trying', and that lack of effort as 'not caring' about her class. From a fair perspective, reasonable responses to these interpretations often revolve around trying to motivate or punish students toward new behavior.

PD sessions began with mindful moments or reflective journaling to encourage and invite all attendees to come into the learning space intentionally. In a session prior to distributing the SNS, Carrie journaled: "I make the assumption that they don't care about their education when it could be something else entirely. … I should ask what their actions really mean." This statement indicates early recognition of the disconnect between students and the teacher, preempting the SNS. This disconnect was troublesome for Carrie, and she showed in this statement awareness that she must attend to and respond to different issues than those she had prioritized. As Carrie reflected on her perception of students' lack of care for education during and between PD sessions, she decided to give the SNS to this "challenging" class, mindful of the fact that she knew she needed to change something in order for the disconnect to become a connection, for her pedagogy to be more culturally responsive, and focused on improving the dynamic with this class. Before distributing the survey, she had tried other ways of connecting with and celebrating her students in her "challenging" class, including taking notes of what students were doing well and sending those to the students' home adults. After several sessions of the PD, she decided to distribute the SNS to this class to glean further information about what she could do to improve.

When Carrie first viewed the results of the SNS, she noticed a disconnect between what she believed she was noticing about her students, and how the students perceived what she noticed about them. Her initial response was "I am sorry about the results." Aligned with prior discourse about how challenging she felt this class was for her, she continued, "I felt like I've made improvements, but their survey results say otherwise." In collaborative discussions, Carrie began to wonder about this disconnect, turning it into an informal problem of practice. She reasoned, "my students largely felt like I didn't care about them as people. They felt that I cared only about their mathematical understanding and improvements. In reflection, this is probably true." In this discursive move, Carrie has shifted her framing. She acknowledges that her framing (and noticing) has largely been guided by a focus on her students' mathematical development, which Louie et al. (2021) describe as a deficit, dominant framing in which teachers attend to students' mathematical progress, interpret students as either on track, gifted or behind, and respond by labeling them as accordingly. Here, Carrie is beginning to examine this framing, as her students tell her in the survey that she is not noticing them as human beings. The generative discursive contrast pushes Carrie to reconsider the way she is making sense of her instruction.

This discursive turn is further reflected in another utterance Carrie made during the PD, in which she stated, "What is most important to them [the students] in the moment may not be the math that I am trying to teach them." While this discursive move continues to be framed from a colorblind perspective, it indicates that Carrie is becoming curious about what is going on for her students, apart from her interpretations of them. Carrie also began working on a topic for a culminating action research project at this time, including who her community partner would be. The requirements of the PD included partnering with someone outside of the participant's own classroom for initial conversations about her action research plan involved bringing in support from outside the school, from a local youth action and advocacy group. This proved to be complicated for various reasons, including a strict requirement around adherence to the school-designed and provided curriculum.

Ultimately, Carrie arrived at a focus for her action research project, which we view as discursively revealing a new problem of practice: "What do I need to notice about myself as a white woman, my mathematics instruction, and my largely BIPOC students to be able to develop relationships of trust and a supportive learning environment?" In this question, Carrie's discourse has shifted dramatically. Instead of asking why BIPOC students in her class do not care about learning mathematics, she is probing herself as a noticer. The new framing here is of mathematics learning as a relational and racialized activity. As a white woman working with largely BIPOC youth, Carrie interprets this kind of learning as only taking place in a space of trust. Thus, she needs to attend to her students as humans, not just math learners.

Carrie herself attributed this new framing as a result of the disconnect between her perception of her teaching and how students received it. She described how this new framing shifted how she noticed and responded to her students:

> I made a big change in how I see my students. Instead of just seeing the mathematicians inside of them, I learned to focus on who they are as individuals. I began to notice when students weren't themselves for a day and let me know that I see them, that I understand that their minds may be elsewhere. This compassion turned a whole classroom culture from one that was very negative to one that was much nicer to be a part of.

The SNS provided a tool for Carrie to learn from her students, so she could attend to issues that her students saw and felt, not just ones that she observed, as she leaned into culturally responsive mathematics teaching. Through generative discourse during PD sessions and reflective practices on her own, Carrie shifted her teaching practice toward a culturally responsive pedagogy.

Discussion

Supporting shifts in the discourse practices of secondary mathematics teachers towards ones that are critical, asset-based, and relational has been challenging for the field of mathematics education. In this study, we focused heavily on a specific artifact, the student noticing survey, as a tool for generative discourse to promote a shift in one teacher's discursive practices. This study combined the FAIR framework and discourse practices that research indicates are generative in teacher learning communities to provide insights into how one teacher's interactions around a student noticing survey within a learning community shifted her discursive practices, from colorblind and deficit towards relational and asset-based.

At the beginning of the PD, Carrie was upholding deficit perspectives, consistent with framing as described in the FAIR framework (Louie et al., 2021). We found that reframing and shifting attention were supported by the generative discourse practice of revealing. Carrie revealed multiple problems of practice throughout the PD, which shifted as she began to see herself and her students differently. The shifts were also supported by building and contrasting, in which she brought evidence of her problem of practice to the learning community, which was interpreted differently by different members. This enabled Carrie to attend to different aspects of the survey and to generate new questions. The most prevalent generative discourse principles were generalizing and contrasting. Generalizing helped Carrie continuously attend to, interpret, and respond to her problem of practice as it evolved, while contrasting was supported by the discourse of the learning community, as previously mentioned. Generalizing and contrasting discourse practices supported the cycle of interpretation and responding as Carrie shifted her own discourse around her teaching and her students.

Conclusion

This study investigated how a specific tool can be used to facilitate generative discourse among teacher learning communities and support a shift in one teacher's classroom practices toward culturally responsive pedagogy. This work confirms extant research on mathematics teaching around the persistence of dominant, deficit ideologies. This study contributes to the literature on culturally responsive teaching by providing deeper insight into how directly attending to students' perspectives of teachers' attention can result in a shift in the teacher's classroom practices and ability to create and sustain a culturally responsive learning environment. In practice, this study informs professional development for mathematics teachers, focusing on the significance of generative discourse as a tool for supporting teachers in framing and re-framing problems of practice consistent with anti-deficit and anti-racist ideologies.

References

Author. (2022).

Dominguez, H. (2019). Theorizing reciprocal noticing with non-dominant students in mathematics. *Educational Studies in Mathematics*, *102*(1), 75–89. https://doi.org/10.1007/s10649-019-09896-5

Lefstein, A., Louie, N., Segal, A., & Becher, A. (2020). Taking stock of research on teacher collaborative discourse: Theory and method in a nascent field. *Teaching and Teacher Education*, *88*, 102954. https://doi.org/10.1016/j.tate.2019.102954

Louie, N., Adiredja, A. P., & Jessup, N. (2021). Teacher noticing from a sociopolitical perspective: The FAIR framework for anti-deficit noticing. *ZDM - Mathematics Education*, *53*(1), 95–107. https://doi.org/10.1007/s11858-021-01229-2

Louie, N. L. (2018). Culture and ideology in mathematics teacher noticing. *Educational Studies in Mathematics*, *97*(1), 55–69. https://doi.org/10.1007/s10649-017-9775-2

LATINX BILINGUALS TRANSLANGUAGING WITHIN THE UNOFFICIAL SPACE OF A MATHEMATICS CLASSROOM

Hector Morales Jr.
Northeastern Illinois University
H-morales3@neiu.edu

Keywords: Equity, Inclusion and Diversity, Classroom Discourse, High School Education

This paper examines the meaning-making practices of Latinx bilingual students with a challenging mathematics task. My aim is to draw a better understanding of how learning takes place among a group of students who participate in multiple social spaces in the classroom but who also form a unique unofficial space or "underlife" within the same classroom. There are many times when students' knowledge and language are displaced or not valued in the official space of the classroom, so a counterscript emerges motivated by a resistance to the teacher's dominate script in the official space of the classroom. I will be using Gutierrez, Rymes, and Larson's (1995) notion of script as a way of talking about the normal or routine discourse practices of students and teachers in various social spaces in a mathematics classroom. We draw on translanguaging to reconceptualize bilingualism as a liberating and empowering communicative practice, and a resource capable of transforming learning that goes beyond the transition to a dominant school language. A translanguaging (See Garcia, 2017) lens allows us to better understand holistically how bilingual students use their linguistic, multimodal, and mathematical repertoire to make meaning as they persevere with challenging ideas.

Methods

I observed a group of 12^{th}-grade Latinx bilingual students and their teacher, Ms. Davidson (Morales, 2004), a monolingual English-speaking teacher working on a multi-day task. The task referenced *Alice in Wonderland,* in which Alice's height is doubled or halved by consuming certain potions. All but one student were born in the U.S., yet all students still use home language (Spanish) as part of their linguistic repertoire. I focus on these episodes because it presents opportunities to study how students and the teacher interact within various social spaces of the classroom and how they engaged in translanguaging to make meaning of a complex mathematical situation involving exponential growth.

Findings

Two themes emerged from the data: 1) In the official space where the teacher facilitated discussions within the small group students operated within a limited mediational means where the teacher dominated discussions and asked a series of short answer questions primarily trying to help students remember the mathematical ideas of doubling and exponential functions. The students tried to understand what they were reading in the textbook to what the teacher was telling them. A counterscript developed and continued during the next few days while working without Ms. Davidson. 2) These students also participated in the unofficial space of the classroom where they used cultural resources, constructed mathematical meanings, and developed the mathematical ideas of doubling and exponential functions through their participation in activity. This group of students' activity and actions in the unofficial space were in fact complementary to the larger classroom activities but not perceived as such by the teacher at the end.

References

García, O. (2017). Translanguaging in schools: Subiendo y bajando, bajando y subiendo as afterword. Journal of Language, Identity & Education, 16(4), 256-263. doi:10.1080/15348458.2017.1329657

Gutierrez, K. D. (1994). How Talk, Context, and Script Shape contexts for Learning: A Cross-Case Comparison of Journal Sharing. Linguistics and Education, 5, 335-365.

Gutierrez, K. D., Larson, J., & Kreuter, B. (1995). Cultural Tensions in the Scripted Classroom: The Value of the Subjugated Perspective. Urban Education, 29(4), 410-442.

Gutierrez, K. D., Rymes, B., & Larson, J. (1995). Script, Counterscript, and Underlife in the Classroom: James Brown versus Brown v. Board of Education. Harvard Educational Review, 65(3), 445-471.

Gutierrez, K. D., & Stone, L. D. (2000). Synchronic and Diachronic Dimensions of Social Practice. In C. D. Lee & P. Smagorinsky (Eds.), Vygotskian perspectives on literacy research: Constructing meaning through collaborative inquiry (pp. 150-164). Cambridge [England] ; New York: Cambridge University Press.

Hiebert, J. & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. Second Handbook of Research on Mathematics Teaching and Learning, 1, 371-404.

Irwin, V., De La Rosa, J., Wang, K., Hein, S., Zhang, J., Burr, R., Roberts, A., Barmer, A., Bullock Mann, F., Dilig, R., and Parker, S. (2022). Report on the Condition of Education 2022 (NCES 2022-144). U.S. Department of Education. Washington, DC: National Center for Education Statistics. Retrieved [date] from https://nces.ed.gov/pubsearch/ pubsinfo.asp?pubid=2022144.

Langer-Osuna, J. M., Moschkovich, J., Norén, E., Powell, A. B., & Vazquez, S. (2016). Student agency and counter-naratives in diverse multilingual mathematics classrooms: Challenging deficit perspectives. In R. Barwell et al. (Eds.), Mathematics Education and Language Diversity (pp. 163-173): Springer.

Morales Jr., H. (2004). A Naturalistic Study of Mathematical Meaning-Making by High School Latino Students. (Unpublished doctoral dissertation). University of Illinois at Chicago.

Morales Jr., H. (2012). Case 4: Twelfth-grade English language learners and the making of mathematical meanings. In S. Celedon-Pattichis & Ramirez, N.G. (Eds.), Beyond Good Teaching: Advancing Mathematics Education for ELLs (pp. 109-18). National Council of Teachers of Mathematics.

Morales, Jr., H. & DiNapoli, J. (2018). Latinx bilingual students' perseverance on a mathematical task: A rehumanizing perspective. REDIMAT: Journal of Research in Mathematics Education, 7(3), 226-250.

National Council for Teachers of Mathematics. (2014). Principles to actions: Ensuring mathematical success for all. Reston, VA.

MAKING TIME FOR DISCOURSE: EFFECTS OF TIME SCAFFOLDS ON STUDENTS' ON-TASK MATHEMATICS ENGAGEMENT

Kristy Litster
Valdosta State University
klitster@valdosta.edu

This inquiry looks at differences in mathematics engagement for 34 groups of Grade 5 students with and without time scaffolds for individual and group reflections, discussions, and problem solving. Results showed no significant differences in the quantity of time engaged in individual work versus group discourse. Results showed that students without time scaffolds were more likely to run out of time to compete or discuss tasks at the end of a large task-set. In contrast, students with time scaffolds were more likely to complete tasks, but less likely to discuss them.

Keywords: Instructional Activities and Practices; Classroom Discourse; Cognition

Time is a precious commodity for teachers, and how they use that time to engage students with mathematical ideas and concepts in ways that can be transferable to their authentic and daily uses, as opposed to rote abstract procedures, is important (Vural & Vural, 2020). Discourse around real-world tasks can be a great strategy to encourage student engagement with mathematical ideas and concepts and increase the depth of student thinking (Bennet, 2014; Blackham, 2022; Litster et al., 2020). Discourse can also support skills students need such as solving problems in daily life, problem solving, critical thinking, and communication (Vural & Vural, 2020). However, research is divided on whether the best use of time to support thinking is for students to explore math tasks as a collaborative group or whether there should be time scaffolds in place that provide time for students to work both individually and collaboratively.

Thus, the purpose of this inquiry was to explore the differences in mathematics engagement when students are provided with open versus segmented time scaffolds for individual and group reflections, discussions, and problem solving when engaging with tasks at four levels of cognitive demand. Specifically, engagement was measured in two ways a) quantity of time engaged in verbal discussions of the mathematics tasks and b) quantity of tasks students completed and discussed within the assigned task-sets.

Theoretical Framework

Teachers can use discourse in whole-group or small group settings. One disadvantage of whole-class discourse is that only a few students have the opportunity to share their thinking; in contrast, having students work in small groups or with partners allows multiple students to share their thinking at the same time, increasing engagement (Charalambous & Litke, 2018; Coakley, 2018; Hunt et al., 2018; Litster et al., 2020). There are two ways teachers may choose to organize the timing for small-group discourse: collaboratively or with time scaffolds for individual and group work.

When students work collaboratively, they are engaged in mathematical thinking to deconstruct problems, identify strategies, and overcome misconceptions to build a shared understanding of the mathematics (Blackham, 2022; Hwang & Hu, 2013; NCTM, 2014). This has the potential for students to use time effectively and dismiss inaccurate or less efficient strategies early. However, without a positive class culture and discussion norms to support equitable participation, it can also lead to imbalances in power where one student takes over and other students are “free-riders” or excluded from the discourse or problem solving (Bennet, 2014; Cohen et al., 1999; Horn, 2012; Litster, 2020).

To overcome problems with positions of power or equity, other researchers recommend using time scaffolds to allow time for individual reflection, internal discourse, and problem solving before coming together to discuss the problem together, reflect on strategies, and/or work together (Kalamar, 2018; Walter, 2018). Scaffolding time for individual preparation can increase equitable participation and support more in-depth responses and reflections when students ultimately come together to discuss the tasks (Kalamar, 2018; Rowe, 1986),

Methods

This inquiry utilized verbal and written data from a larger study with 96 Grade 5 students working in 34 small groups to solve two task-sets relating to operations with fractions and decimals. Using a crossover design, half the groups completed Task-Set 1 (H) with a Harry Potter theme without time scaffolds for individual or group reflections, discussion, and problem solving, while the other half completed the same task-set with time scaffolds. Then, a week later, the first half completed Task-Set 2 (D) with a Diary of Wimpy Kid theme with time scaffolds while the other half did not have any time scaffolds.

Groups were given 70 minutes to complete each task-set. Groups without time scaffolds were allowed to set their own pace for discussion, reflection, and problem solving to complete 13 tasks within the task-set. Groups with time scaffolds had alternating times for individual or group reflection, discussion, and/or problem solving by breaking up the task-set into three parts: Part 1 and 2 (DOK1 & DOK 2 focused tasks) had 10 minutes for individual reflection and problem solving followed by 10 minutes of group reflective discussion and problem solving. Part 3 (DOK 3 & 4 focused tasks) had 15 minutes for individual reflection and problems solving following by 15 minutes of group reflective discussion and problem solving.

The researcher quantized the data sources in three ways for the analyses (Saldana, 2016). First, video data was analyzed to identify the time (to the second) students were engaged on-task discussing the mathematics, on-task working silently to solve problems, or off-task (e.g., talking about other topics beyond the scope of the task-set, playing with items in desk). Second, the frequency of discourse contributions across all groups were counted based on intended DOK levels of the tasks and discourse timing. A change in topic, idea, or speaker indicated a new discourse contribution. Third, written and verbal data was analyzed to identify the number of tasks that were not discussed or answered, by DOK level, task-set, and discourse timing.

Results

Quantity of Time Engaged in Mathematics Discourse

Results relating to the amount of time engaged in verbal discussions of the mathematics tasks show that there was an average difference of less than one minute in the time that groups spent discussing the mathematics tasks regardless of the presence of time scaffolds. Table 1 illustrates the time differences for groups that were on-task the entire time as well as groups that were off-task at least once during the task-set. The Ns in the table indicate the number of groups who were on- or off-task for Time Scaffolds and No Time Scaffolds, respectively.

Table 1: Frequency of Time Spent Discussion Math in Minutes

Heading	Time Scaffolds Mean	Time Scaffolds Min-max	No Time Scaffolds Mean	No Time Scaffolds Min-max	Difference (Ref – Col)
On-Task (N = 16, 17)	14.75	11.10-17.80	15.41	11.67-20.10	-0.66
Task-Set 1 (H)	14.65	13.06-17.80	15.81	11.67-20.10	-1.16
Task-Set 2 (D)	14.81	11.10-17.23	14.47	13.08-18.43	0.34
Off-Task (N = 18, 17)	8.83	3.05-13.15	8.45	2.70-12.60	0.38
Task-Set 1 (H)	7.75	3.05-11.48	9.65	8.80-11.12	-1.90
Task-Set 2 (D)	9.78	4.32-13.15	7.90	2.70-12.60	1.88
Overall (N = 34)	11.17	03.05-17.80	12.03	02.70-20.10	-0.86

As seen in Table 1, groups of students who were on-task during the entire task-set spent an average of 15 minutes actively engaging in group mathematical discourse and the remaining 55 minutes engaging in individual reflection, computation, or writing – regardless of discourse type. Groups of students who were off-task at least once during the task-set spent an average of 8 or 9 minutes engaged in group mathematical discourse. Overall, students spent an average of 11-12 minutes engaging in mathematical discourse. This indicated that, students engaged in discussions a similar amount of time regardless of the presence of time scaffolds for individuals and groups.

Quantity of Tasks Engaged With

Results for quantity of tasks engaged with showed that more tasks were completed for Task-Set 1 when students did not have time scaffolds and more tasks were completed for Task-Set 2 when there were time scaffolds. This is most likely due to class effects in the crossover design (same groups who has Task-Set 1 without scaffolds had Task-Set 2 with scaffolds). Results also show that there were differences in the percentage of tasks completed and discussed. Table 2 shows the percentage of completed tasks in the written work, while Table 3 shows the percentage of tasks discussed (regardless of whether they were completed in the written work).

Table 2: Percentage of Completed Tasks by Task-Set and DOK

Scaffold and Task-Set	Levels of Cognitive Demand for Tasks DOK1	DOK2	DOK3	DOK4
No Time Scaffolds				
Task-Set 1(H) (Group 1)	100%	100%	99%	100%
Task-Set 2(D) (Group 2)	100%	97%	88%	94%
Time Scaffolds				
Task-Set 1(H) (Group 2)	99%	99%	99%	94%
Task-Set 2(D) (Group 1)	100%	97%	95%	100%

Table 3: Percentage of Tasks Discussed by Task-Set and DOK

Scaffold and Task-Set	Levels of Cognitive Demand for Tasks			
	DOK1	DOK2	DOK3	DOK4
No Time Scaffolds				
Task-Set 1(H) (Group 1)	100%	99%	94%	94%
Task-Set 2(D) (Group 2)	98%	88%	80%	81%
Time Scaffolds				
Task-Set 1(H) (Group 2)	88%	95%	63%	94%
Task-Set 2(D) (Group 1)	66%	61%	55%	100%

Groups without time scaffolds had the discretion to move to the next task or spend extra time discussing a particular task. However, this may have limited the time for some groups to discuss or complete the higher DOK level tasks at the end of the task-set. For example, two groups spent a lot of time on Task-Set 2 calculations for Part 1 questions due to a misconception (e.g., calculated 44.2 x 44.2 x 44.2 instead of 4200 + 44.2). This meant that they only completed tasks D1-D8 before they were out of time, with one group jumping from D8 to D13 during the last 5 minutes (e.g., "We're almost out of time. This one is funner so let's jump to this."). This accounts for the high incompletion percentage for Task-Set 2 DOK3 and DOK4 tasks. For groups that completed all tasks, but did not discuss all tasks, one student in the group completed the task independently.

In contrast, groups with time scaffolds prompted students to discuss tasks across DOK levels; However, it also created conflicts for some groups who either completed their group discussions, reflections, and problem solving early (and then proceeded to off-topic discussions) or other groups that wanted extra time to reflect on the tasks, but were forced to move on to the next set before they had completed their discussion. This made it less likely that students would run out of time before reaching the higher DOK tasks at the end of the task-set, and more likely that students would run out of time before the last task in each level. For example, during the 10 minutes provided for group work in Part 1 (Tasks H1-3). One group that had no differences in their responses for Part 1 were able to discuss all three tasks in 5 minutes and then spent the remaining 5 minutes justifying and checking their responses. In contrast, a different group where each student had a different response for H1, spent the entire 10 minutes discussing their solutions and strategies for H1 and never discussed H2 or H3, even though they each had a response written for these tasks.

Discussion and Implications

These results show that regardless of how teachers organize the timing of classroom discussions, students spend about the same amount of time discussing tasks, and students still need time for personal reflection and problem solving. When working with multiple tasks, students without time scaffolds are more likely to discuss every task as they dictate their own timings for individual and group discourse of the tasks. Students with time scaffolds are less likely to talk about every task, spending more time focusing on one or more questions. However, the time scaffolds did ensure they were more likely to talk about a task in each part of the task-set. Educators should be aware that neither method is more efficient for use of time – some groups were more efficient with time scaffolds while others were more efficient without; some groups needed more time to talk about a problem whereas others needed less. Educators should monitor their students' engagement to move the discussion and work time along at a pace that meets their students' needs.

References

Bennett, C. A. (2014) Creating Cultures of Participation to Promote Mathematical Discourse, *Middle School Journal, 46*(2), 20-25. DOI: 10.1080/00940771.2014.11461906

Blackham, V. (2022) *Eliciting mathematical justifications in the classroom: An investigation of tools for success.* Doctoral Dissertations (University of Idaho). https://www.lib.uidaho.edu/digital/etd/items/blackham_idaho_0089e_12410.html

Charalambous, C. Y., & Litke, E. (2018). Studying instructional quality by using a content-specific lens: the case of the Mathematical Quality of Instruction framework. *ZDM,* 1-16.Coakley, 2018;

Cohen, E. G., Lotan, R. A., Scarloss, B. A., & Arellano, A. R. (1999). Complex instruction: Equity in cooperative learning classrooms. *Theory into practice, 38*(2), 80-86.

Horn, I. (2012). *Strength in numbers: Collaborative learning in secondary mathematics.* NCTM, Reston, VA

Hunt, J. H., MacDonald, B., Lambert, R., Sugita, T., & Silva, J. (2018). Think-Pair-Show-Share to Increase Classroom Discourse. Teaching Children Mathematics, 25(2), 78-84.

Hwang, W. Y., & Hu, S. S. (2013). Analysis of peer learning behaviors using multiple representations in virtual reality and their impacts on geometry problem solving. Computers & Education, 62, 308-319.NCTM, 2014

Kalamar, C. (2018). Questioning techniques that increase student engagement during the mathematics lesson (Master's thesis). Moravian College, Bethlehem, PA. Retrieved from https://www.moravian.edu/static/education/2018/thesis/Kelsey.pdf

Litster, K. (2020). Small-group discourse: Supporting the inclusion of elementary students that typically struggle with mathematics in cognitively demanding discourse. In A. I. Sacristán, J. C. Cortés-Zavala, & P. M. Ruiz-Arias (Eds.), Proceedings of the 42nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (p. 2009-2016). Mazatlán, México.

Litster, K., MacDonald, B. L., & Shumway, J. F. (2020). Experiencing active mathematics learning: Meeting the expectations for teaching and learning in mathematics classrooms. In A. Apposa, R. M. Welder, & Z. Feldman (Eds.) Special Issue: Supporting Mathematics Teacher Educators' Knowledge and Practices for Teaching Content to Prospective (Grades K-8) Teachers. The Math Enthusiast 17(1 & 2, June), 615-640.

National Council of Teachers of Mathematics (NCTM). (2014). *Principles to Action: Ensuring Mathematical Success for All.* Reston, VA: NCTM.

Rowe, M. B. (1986). Wait time: Slowing down may be a way of speeding up! *Journal of Teacher Education, 37*(1), 43-50. https://doi.org/10.1177/002248718603700110

Vural, Ö. F., & Vural, S. (2020). An Examination of 5th Grade Mathematics Curriculum in Terms of 21st Century Skills. International Journal of Educational Research Review, 6(2), 82-92.

Walter, H. A. (2018). Beyond turn and talk: Creating discourse. *Teaching Children Mathematics, 25*(3), 180-185.

MATHEMATICS TEACHERS' CONCEPTIONS OF THE THEORY OF MULTIPLE INTELLIGENCES IN GHANA

Sumaila Yakubu
Montana State University-Bozeman
sumaila.yakubu@student.montana.edu

Michael Johnson Nabie
University of Education Winneba
Nabizie@yahoo.com

Fenqjen Luo
Montana State University-Bozeman
fluo@montana.edu

Keywords: Equity, inclusion and diversity, Teacher Knowledge.

The Ghanaian mathematics classroom is typically heterogeneously diverse but the government's inclusive and Free Senior High School policies have worsened the situation. Teaching in heterogeneously diverse classrooms requires theoretical conceptions that recognize the distinct nature of learners, address individual differences, and ensure equity and inclusion (Nabie, 2016). The theory of Multiple Intelligences (MIs) provides space for diverse contexts and experiences to maximize student learning and achievement (Wang, 2018). The National Teaching Council (2017) and National Council for Curriculum and Assessment (2020) advocate mathematics teachers' employment of creative pedagogical strategies to ensure inclusion and address diversity in Ghanaian classrooms. Despite these advocacies, little or nothing is seen in the Ghanaian mathematics classroom as providing for diversity. Literature on the theory of MIs that has the potential to address the challenge of diversity in the Ghanaian mathematics classroom setting is also limited. Mathematics teachers still stick to their traditional methods of teaching described as "one-size-fits all" (Ako et al., 2019, p. 49), which favor only a few learners whose unique intelligences and abilities are met and leave behind the majority.

The study was designed to identify the Ghanaian mathematics teachers' conceptions of the theory of MIs for implementation. This study seeks to deepen our understanding of how mathematics teachers conceive the theory of MIs for implementation in the classroom as well as inform the development of teacher training programs and professional development opportunities to support teachers in their implementation of MIs.

Methods

The study collected data from 12 Senior High School mathematics teachers using the Mathematics Teacher's Questionnaire (MTQ) and a semi-structured interview guide. The MTQ that was pilot-tested, and test-retesting conducted to obtain a correlation co-efficient of 0.79 as a measure of its reliability. The semi-structured interview data was subjected to member checking for credibility. Data was analyzed thematically in relation to the research question and codes were grouped into categories.

Findings

Four (4) mathematics teachers were identified from the MTQ to be familiar with the theory of MIs. Interview data from the four (4) mathematics teachers indicate that teachers' conceptions of the theory of MIs could be put into three themes, namely theory of multiple intelligences: (a) as multiple learning styles, (b) as multiple strategies for teaching, and (c) for diverse student needs. The findings show that despite mathematics teachers' prior knowledge of the theory of MIs, much is still expected of them with regards its understanding and implementation in the mathematics classroom.

References

Ako, M., Kwame, S. O., Asare, M., & Amihere, A. K. (2019). Knowledge and usage of differentiated instructional strategies in junior high schools in Kwadaso Municipal, Kumasi – Ghana. *International Journal of Education and Research*, *7*(8), 49-62.

Nabie, M. J. (2016). *Fundamentals of the psychology of learning mathematics* (3rd ed.). KAB Impression.

National Council for Curriculum and Assessment (NaCCA). (2020). *Mathematics Common core programme (CCP) curriculum*. National Council for Curriculum and Assessment (NaCCA).

National Teaching Council. (2017). *National teachers' standards for Ghana: Guidelines*.

Wang, N. (2018). The application of multiple intelligences theory in mathematics teaching. *Proceedings of the 2018 2nd International Conference on Economic Development and Education Management (ICEDEM 2018)*. https://doi.org/10.2991/icedem-18.2018.4

MATH TEACHERS LEARNING TO BE CULTURALLY RESPONSIVE

John Eston Brown
University of Arizona
jebrown@arizona.edu

Keywords: Culturally Relevant Pedagogy; Equity, Inclusion, and Diversity; Social Justice; K-12

Although research has shown many ways teachers have successfully implemented culturally responsive math teaching (CRMT) practices, we need to know more about how teachers take up these practices or why they decide not to. Some teachers sometimes do not take up this work because they fear losing their job. However, some teachers want to see themselves and be seen by others as culturally responsive but still feel they have work to do. To understand how and why teachers engage in CRMT practice, we need to know more about how they identify with, make sense of, and own cultural responsiveness in math class. Specifically, the study will attempt to answer the following research questions:

1. How do teachers identify as culturally responsive?
    a. How do teachers *engage* in cultural responsiveness?
    b. How do teachers *imagine* themselves and others as culturally responsive?
    c. How do teachers *align* their practices with other culturally responsive educators?
2. How do teachers *negotiate* culturally responsive math teaching (CRMT) practices?

Wenger (1998) argues that there are three modes of belonging: engagement, imagination, and alignment. *Engagement* entails how one participates (or not) in a community of practice, like attending professional development. On the other hand, *imagination* refers to one's ability to see oneself and others as a part of the community of practice. Lastly, *alignment* involves acting in a certain way to contribute to the community of practice's common interest. While forming an identity within a community of practice consists of developing and understanding one's membership, it also includes how members negotiate and take ownership of meanings. Wenger (1998) refers to *negotiability* as the member's ability to legitimately contribute to how a community of practice defines and develops its practice. Like Crisan and Rodd's (2017) translation of Wenger's framework into a math teacher identity framework, I have translated what it means to identify with cultural responsiveness and negotiate CRMT practices.

I decided to use an exploratory case study (Yin, 2009) to help CRMT research further explore such a complex phenomenon. I recruited four teachers (two elementary, two secondary) from a southwestern city who have participated in different professional development settings about culturally responsive practices. I gathered demographic information on an initial survey before the first observation. I observed each teacher twice and followed up with an interview after each observation. While the first interview focused on how teachers view their CRMT practices, the second interview provided space for teachers to reflect on how their backgrounds influence their practices. After transcribing and de-identifying the data, I used two rounds of thematic coding in my analysis. The first round assigned each turn in the transcript one or multiple mode(s) of belonging (i.e., engagement, imagination, alignment). The second round of coding assigned one of Aguirre and Zavala's (2021) nine areas of culturally responsive mathematics teaching to each code from the previous round. After I analyzed the interviews, I sent a version of my analysis to the participants to gather feedback. Since data analysis is still incomplete, results and implications are still developing.

References

Crisan, C. & Rodd, M. (2017). Learning mathematics for teaching mathematics: Non-specialist teachers' mathematics teacher identity. *Mathematics Teacher Education and Development*, *19*(2), 104-122.

Aguirre, J.M. & Zavala, M. (2021). Evolution of a culturally responsive mathematics teaching lesson analysis tool. In H. Kang (Chair), *Articulating theoretical and methodological approaches for studying mathematics and science teacher learning for equity and justice*. Presentation as part of a research symposium at the annual meeting of the American Education Research Association, Virtual.

Wegner, E. (1998). *Communities of practice: Learning, meaning and identity*. New York: Cambridge.

Yin, R.K. (2009). *Case study research: Design and methods* (4th Edition). Thousand Oaks, CA: Sage.

MULTILINGUAL CLASSROOMS: DEVELOPMENT OF AN OBSERVATIONAL ANALYTIC TOOL TO EXAMINE MATHEMATICS INSTRUCTION

AULAS MULTILINGÜES: DESARROLLO DE UNA HERRAMIENTA ANALÍTICA DE OBSERVACIÓN PARA EXAMINAR LA ENSEÑANZA DE LAS MATEMÁTICAS

Michael W. Krell	Abigayle Dirdak	Beatriz Quintos
University of Maryland	University of Arizona	University of Maryland
mkrell@umd.ued	adirdak@math.arizona.edu	bquintos@umd.edu
Jonee Wilson	M. Alejandra Sorto	Claudia Galindo
University of Virginia	Texas State University	University of Maryland
vma8ze@virginia.edu	sorto@txstate.edu	galindo@umd.edu

This brief research report describes the refinement and testing of an observational rubric designed to identify and assess elements of classroom mathematics instruction that research has found to support multilingual student learning. The aim of this process is to (a) combine existing rubrics that capture teaching strategies and positioning construct protocol, (b) test the combined rubric in multiple elementary classroom settings, (c) revise the rubric in light of testing to create a more consistent version, and (d) retest with a larger sample of classrooms. Initial results include revised instrument category rubrics and level descriptors, and the creation of a new conjectural code category.

Keywords: Social Justice; Instructional Activities and Practices; Equity, Inclusion, and Diversity; Elementary School Education

Introduction

Education inequities persist across the United States, significantly affecting multilingual students. Multilingual students are children learning the language(s) spoken at home in addition to English (García & Kleifgen, 2010). Many multilingual students attend schools with limited resources that are less well-equipped to support their specific learning needs in general (NASEM, 2017) or to draw on their cultural and linguistic resources in the ways necessary for an equitable mathematics education in particular (Nasir & Hand, 2006).

To support the improvement of learning opportunities for multilingual students, an interdisciplinary team of researchers across three universities in the Mid-Atlantic, Southwest, and Midwest United States received a collaborative National Science Foundation grant, "Together/Juntos," to *develop and research an innovative mathematical partnership that engages teachers, parents, and multilingual students in elementary grades in underserved communities.* One of the main goals of this project focuses on improving the quality of mathematics instruction in linguistically diverse classrooms. To analyze progress towards this goal, we needed a classroom tool that would (a) measure "high-quality" mathematics instruction that is relevant for multilingual students, (b) validate student differences, and (c) be culturally responsive.

Purpose

The primary purpose of this study is to refine and test an observational tool that responds to the mathematics-instructional and positioning practices of teachers of multilingual students in elementary grades. In this paper, we describe the adaptation process of the Quality of Linguistically Diverse Teaching (QLDT) protocol (Sorto et al., 2018) and the Equity and Access Rubrics for Mathematics Instruction (EAR-MI) (Wilson et al., 2019) to a more heterogeneous context with respect to languages present in the classroom and geographical region. Ultimately this adapted rubric will serve as a primary analytic tool answering broader research questions about the impact of professional development on teachers' quality of mathematics instruction for, and the classroom positionality of, multilingual students.

Theoretical framework

Elevating multilingual students' mathematics and linguistic knowledge requires teachers with asset-based dispositions and specific pedagogical skills. Chval and Chávez (2011) argue mathematics instruction for multilingual students requires teachers who support students' mathematics learning, develop students' language or languages, incorporate students' life experiences into instruction, and facilitate productive interactions. Aligned with these recommendations, Moschkovich (2002) suggests classroom instruction must be grounded in a situated and sociocultural perspective of mathematics learning that rejects a deficit view of multilingual students and focuses on the discourse embedded within and constituent of mathematics practices. While some aspects of classroom mathematical discourse, such as the use of mathematical vocabulary or multiple meanings of words, are easier to identify and assess, there is a need for tools that assess classroom mathematical discourse practices more broadly in a way that is tailored to multilingual students and that integrate attention to student identities and classroom positioning (Davies & Harré, 1999; Holland et al., 2001). Our approach to assessing quality mathematics instruction, and the tool we are developing, incorporates the above ideas to help researchers understand what successful teacher support of multilingual mathematics students looks and sounds like, and to characterize the sorts of interactions that take place in such an environment.

Methods

Phase 1: Initial Development

The development of the observational tool started with a review of existing observational rubrics from studies that investigate mathematics classroom interactions. These included (1) the Mathematical Quality of Instruction (MQI) tool (Hill, 2014), (2) the QLDT protocol, which augments the MQI with dimensions that focus on teaching moves or routines specifically tailored for linguistically diverse classrooms (Sorto & Bower, 2017; Sorto et al., 2018), and (3) the EAR-MI (Wilson et al., 2019; Wilson, 2022), which was designed to complement the Instructional Quality Assessment (IQA) rubrics (Boston, 2012). Sorto and colleagues (2018) developed the QLDT to leverage the elements of Chval and Chávez's (2011) research-based instructional practices to support multilingual students in mathematics. It contains six instructional strategy categories: "Connections of mathematics with students' life experiences," "Connections of mathematics with language," "Meaning and multiple meanings of words," "Use of visual aids or support," "Record of written essential ideas and concepts on board," and "Discussion of students' mathematical writing." Each category can be coded as "Not Present", "Low", "Mid", or "High." The EAR-MI is a set of rubrics that attends to practices observed in classrooms where there was evidence of conceptually oriented instruction and of more equitable access for students who are typically pushed to the margins (Wilson & Smith, 2022). Among the diverse practices that the EAR-MI highlights, we focused on one in particular: positioning students as competent by framing students' actions and statements as intellectually valuable. The EAR-MI positioning rubric highlights moments when the teacher explicitly states that a student is capable of participating in rigorous mathematical activity and/or is making important contributions to the learning community. This positioning rubric was added to the QLDT at the onset of Phase 2.

Phase 2: Testing and Revising

During the Together/Juntos project's first year, 22 classroom lessons from nine teachers were videotaped at two school sites. The first site was a dual language P-8 school in a large city in the Southwest U.S. This school serves roughly 580 students, of whom 80% are eligible for free or reduced lunch and 92% are Latinx. Students at this school learn to read and write in both English and Spanish, although mathematics classes are primarily taught in English. All teachers observed are bilingual, as are the majority of the students. Five teachers were recorded teaching mathematics lessons in their classrooms: three fifth grade teachers, one third grade teacher, and one second grade teacher. The second site was a suburban K-5 elementary school in a diverse community outside of a large Mid-Atlantic city in the U.S. The school has roughly 550 students, of whom about half are eligible for free or reduced lunch. More than half of the students (56%) are Black, 19% are White, and 13% are Latinx. Students at this school receive classroom instruction in English. Four teachers were recorded teaching mathematics in their classrooms: two fourth grade teachers, one fifth grade teacher, and one third grade teacher. Half of these teachers speak and understand a little Spanish, the rest do not.

Three trained doctoral students ("the coders") coded 22 lessons following the QLDT protocol, which is modeled after the MQI. For every 7.5-minute video segment, each coder independently assigned a code (Not Present, 0; Low, 1; Mid, 2; or High, 3) to each QLDT instructional strategy element, as well as to the EAR-MI positioning element. After coding an entire lesson, coders would then compare and discuss their coding and arrive at consensus where necessary. After coding all 22 videos, coders compiled their notes and email exchanges to begin revising the instrument rubrics. The overarching goals during this revision process were to remove ambiguities remaining from Phase 1, improve clarity, and brighten the distinction between the different rubric categories and coding levels. For example, the original description of the "Use of visual aids or support" category did not foreground the idea of dynamism, although the QLDT creator highlighted this idea as central during initial instrument training sessions (see Table 1 below). Furthermore, the original description includes examples that could be considered static (e.g. "formula charts . . . times tables"), and the term "dynamic" itself is not defined nor does it appear in the category title.

Initial Results

We have revised all six QLDT rubric categories and further integrated the EAR-MI positioning rubric into a tool we currently call the QLDT+P. Table 1 presents an example of a revised code (level descriptors have been omitted for reasons of space). Overall, the revised codes offer more thorough and explicit guidance based on coders' experiences wrestling with applying the original descriptors to various classroom circumstances, including specific language to help deal consistently with recurring ambiguous cases and examples drawn from actual lessons to further illustrate and clarify this guidance. The revisions provide more direction as well regarding when a situation should be classed as Not Present, Low, Mid, or High in at least three ways: (1) by including more decision rules for when something should be classed as one level or the other, (2) by providing extended examples within the descriptors of when something would count as one level or another, and (3) by overhauling ambiguous terminology.

Table 1: Use of dynamic visual aids or support

Original description	Revised description
For example, concrete objects or manipulatives, videos, and illustrations in classroom conversations. Concrete objects may include times tables, formula charts, protractors, 2D models, or dynamic foldables. This code does not include the visual support of the blackboard. Any visual support needs to add to the static nature of the blackboard, anchor charts, or other visually displayed information.	The lesson or activity uses dynamic aids that support mathematical sense/meaning making or reasoning to some extent. For example, using concrete objects or manipulatives, videos, or illustrations that are animated or dynamic in some other way during classroom conversations. “Concrete objects” may include protractors, rulers, 2D models, or dynamic foldables as well as objects from everyday life. The key word here is “dynamic” in the sense of movement, change, or energy. Static supports such as black- or whiteboards, written materials or notes, PowerPoint slides, etc., are not captured by this code. Dynamic visual support needs to add to the static nature of the blackboard, anchor charts, or other visually displayed information. It is possible for ordinarily static supports like black-/whiteboard writing, formula/anchor charts, number lines, or times tables to be made dynamic through teacher or student actions such as annotation with arrows or circles accompanied by movement, physically acting out scenarios or ideas related to the supports, etc. However, simply including arrows or other static indicators of motion on a handout or chart does not count.

We are also developing and testing a new category aimed at capturing multilingual students' use of what Schleppegrell (2007) and others (e.g., Herbel-Eisenmann & Otten, 2011; Pimm, 1987) call the “classroom mathematics [linguistic] register.” Although elements of the classroom mathematics register are already captured in the QLDT+P code category “Meaning and multiple meanings of words,” this category is oriented towards students *learning about* differences in the mathematical and non-mathematical meanings of words, rather than *employing* such meanings. We feel that capturing moments when multilingual students capitalize on opportunities to use the classroom mathematics register is a potential gap in the current version of the QLDT+P. Hence, we have designed a new code to pilot during the next iteration of QLDT+P video analysis.

Discussion

The original combination of QLDT and EAR-MI rubrics captures many aspects of classroom mathematics instruction that are needed for multilingual students' mathematics learning, including key sociocultural elements related to identity and positioning. We believe that our revised tool will allow for more nuanced and consistent assessment of quality mathematics instruction for linguistically diverse students. To test whether this is the case, we are training coders on the revised QLDT+P tool to analyze a new round of classroom videos generated by our larger, National Science Foundation-funded research project. These analyses, in turn, will generate insights not only about the reliability of the tool itself, but also about our broader research questions regarding the impact of a professional development on teachers' quality of mathematics instruction for multilingual students. As such, our enterprise fits the 2023 PME-NA theme of engaging all learners, addressing in particular the question of how to design learning environments that take students and learning into account.

Acknowledgments

This research was supported by the National Science Foundation grants (DRL-1055067, DRL-2010417, DRL-2010260, DRL-2010230). The authors wish to thank the administrators, teachers, and students who allowed us to visit their classrooms.

References

Boston, M. (2012). Assessing instructional quality in mathematics. The Elementary School Journal, 113(1), 76–104.

Hill, H. C. (2014). Mathematical quality of instruction (MQI) [Coding tool]. Harvard Graduate School of Education.

Chval, K. B., & Chávez, O. (2011). Designing math lessons for English language learners. Mathematics Teaching in the Middle School, 17(5), 261 – 265.

Davies, B., & Harré, R. (1999). Positioning and personhood. In R. Harré & L. van Langenhove (Eds.), Positioning theory: The moral contexts of intentional action (pp. 32-52). Blackwell.

García, O., & Kleifgen, J. A. (2010). Educating emergent bilinguals: Policies, programs, and practices for English Language Learners. Teachers College Press.

Herbel-Eisenmann, B. A., & Otten, S. (2011). Mapping Mathematics in Classroom Discourse. Journal for Research in Mathematics Education, 42(5), 451–485.

Holland, D., Lachicotte, W., Jr., Skinner, D., & Cain, C. (2001). Identity and agency in cultural worlds. Harvard University Press.

Moschkovich, J. (2002). A situated and sociocultural perspective on bilingual mathematics learners. Mathematical Thinking & Learning, 4(2 & 3), 189–212.

Nasir, N. S., & Hand, V. M. (2006). Exploring sociocultural perspectives on race, culture, and learning. Review of Educational Research, 76(4), 449–475.

National Academies of Sciences, Engineering, and Medicine [NASEM]. 2017. Promoting the educational success of children and youth learning English: Promising futures. The National Academies Press.

Pimm, D. (1987). Speaking mathematically: Communication in mathematics classrooms. Routledge & Kegan Paul.

Schleppegrell, M. J. (2007). The linguistic challenges of mathematics teaching and learning: A research review. Reading & Writing Quarterly, 23(2), 139–159.

Sorto, M. A., & Bower, R. S. G. (2017). Quality of instruction in linguistically diverse classrooms: It matters! In A. Fernandes, S. Crespo, & M. Civil (Eds.), Access and equity: Promoting high quality mathematics in grades 6-8 (pp. 27 – 40). NCTM.

Sorto, M. A., Wilson, A., & White, A. (2018). Teacher knowledge and teaching practices in linguistically diverse classrooms. In J. Moschkovich, D. Wagner, A. Bose, J. Rodrigues, & M. Schütte (Eds.) Language and communication in mathematics education: International perspectives (pp. 219-231). Springer.

Wilson, J. (2022). Initial steps in developing classroom observation rubrics designed around instructional practices that support equity and access in classrooms with potential for "success". Teachers College Record, 124(11), 179-217.

Wilson, J., Nazemi, M., Jackson, K., & Wilhelm, A. G. (2019). Investigating teaching in conceptually oriented mathematics classrooms characterized by African American student success. Journal for Research in Mathematics Education, 50(4), 362-400.

Wilson, J., & Smith, E. (2022). Increasing access for multilingual learners in mathematics classrooms. Mathematics Teacher Learning and Teaching PK-12, 115(2), 104-10.

PICKING AND ADAPTING A PREEXISTING TEACHING MATH FOR SOCIAL JUSTICE LESSON

Kathryn E. Roman
Portland State University
kroman@pdx.edu

Christopher Orlando Roman
Portland State University
roman7@pdx.edu

Manqing Gao
Portland State University
manqgao@pdx.edu

Keywords: Equity, Inclusion, and Diversity; Instructional Activities and Practices; Social Justice

Teaching math for social justice (TMfSJ) is a pedagogical practice that teachers can use to help students see that mathematics is relevant to their lives and understand injustices and inequities in the world around them (Gutstein, 2003). While much of the literature involving TMfSJ lessons (TMfSJLs) has emphasized implementation of these lessons (e.g. Esmonde, 2014; Kokka, 2020), there has been less emphasis on picking *and* adapting preexisting TMfSJLs (see Berry III et al., 2020; Conway IV et al., 2022; for examples of preexisting lessons and see Felton-Koestler, 2020; Harper et al. 2021; Thanheiser & Sugimoto, 2020 for exceptions to picking these lessons). Thus, the research question we focus on is: How can teachers pick and adapt a preexisting TMfSJL?

Methods

The data for this study stems from a lesson that we implemented in an online synchronous capstone course for undergraduate math majors that focused on uncovering racism with mathematics. The data included instructional notes from each author and student work from the 12 undergraduate mathematics majors. To analyze the data, we compared the instructional notes and reached agreement on the elements that were central to picking and adapting the lesson–(1) student interests, (2) student math knowledge, and (3) knowledge of preexisting lessons.

Preliminary Results

We picked and adapted a preexisting TMfSJL letting students' interests drive our selection. We first investigated the students' interests by looking through the students' introduction slides, the current topics the students were interested in exploring with math, and their significant circles (Esteban-Guitart & Moll, 2014). From these slides, we were able to create a joint list of topics: healthcare, education, income, and the environment. We then searched the lessons in Berry et al. (2020) to pick specific TMfSJLs that matched these topics while also ensuring the mathematics would challenge the undergraduate math majors. After narrowing our search down to five lessons, we looked at concepts the students struggled to understand throughout the class. We noticed that the students relied on Google Sheets in a previous lesson to construct a line of best fit. While Google Sheets was useful in helping students generate a quick response to the task, it was not as useful in helping students understand the underlying concept behind constructing a line of best fit. Thus, we picked *Lesson 6.3: Culturally Relevant Income Inequality* (Berry III et al, 2020, p. 127) as it connected to one of the students' interests (income) and its math goal could be adapted to creating and analyzing lines of best fit by hand.

The original goal of Lesson 6.3 involved analyzing and comparing the rate of change of the median annual income of whites, Blacks, Hispanics, and Asians in 1967, 1967, 1970, and 1987, respectively, to their median annual income in 2014. We adapted this math goal by having students create (and later analyze) a line of best fit *by hand* of the difference between two races' median annual incomes–Asians and whites, whites and Blacks, or Blacks and Hispanics–with data points ranging from 1970 to 2020.

References

Berry III, R. Q., Conway IV, B. M., Lawler, B. R., & Staley, J. W. (2020). *High school mathematics lessons to explore, understand, and respond to social injustice*. Corwin Press.

Conway IV, B. M., Id-Deen, L., Raygoza, M. C., Ruiz, A., Staley, J. W., & Thanheiser, E. (2022). *Middle school mathematics lessons to explore, understand, and respond to social injustice* (B. R. Lawler, Ed.). Corwin Press.

Esmonde, I. (2014). "Nobody's rich and nobody's poor … It sounds good, but it's actually not": Affluent students learning mathematics and social justice. *Journal of the Learning Sciences*, *23*(3), 348–391. https://doi.org/10.1080/10508406.2013.847371

Esteban-Guitart, M., & Moll, L. C. (2014). Funds of identity: A new concept based on the funds of knowledge approach. *Culture & Psychology*, *20*(1), 31–48. https://doi.org/10.1177/1354067X13515934

Felton-Koestler, M. (2020). Teaching sociopolitical issues in mathematics teacher preparation: What do mathematics teacher educators need to know? *The Mathematics Enthusiast*, *17*(2), 435–468. https://doi.org/10.54870/1551-3440.1494

Gutstein, E. (2003). Teaching and learning mathematics for social justice in an urban, Latino school. *Journal for Research in Mathematics Education*, *34*(1), 37–73.

Harper, F. K., Maher, E. M., & Jung, H. (2021). Whiteness as a stumbling block in learning to teach mathematics for social justice. *Investigations in Mathematics Learning*, *13*(1), 5–17.

Kokka, K. (2020). Social justice pedagogy for whom? Developing privileged students' critical mathematics consciousness. *The Urban Review*, *52*(4), 778–803. https://doi.org/10.1007/s11256-020-00578-8

Raygoza, M. C. (2020). Counting the experiences and beliefs of secondary teachers striving to teach mathematics for social justice in urban schools. *Urban Education*, *55*(8–9), 1142–1171. https://doi.org/10.1177/0042085916672289

Thanheiser, E., & Sugimoto, A. (2020). Mathematics to understand and critique the world: Reconceiving mathematics in a mathematics content course for elementary school teachers. *Investigations in Mathematics Learning*, *12*(3), 179–193. https://doi.org/10.1080/19477503.2020.1768761

PIQUING STUDENT INTERESTS: USING VOTING TO ENGAGE STUDENTS IN MATHEMATICS

Lino Guajardo
Texas State University
lrg74@txstate.edu

Holly Zolt
Texas State University
hmz10@txstate.edu

Hiroko Warshauer
Texas State University
hiroko@txstate.edu

Max Warshauer
Texas State University
max@txstate.edu

This study examines the design and implementation of a problem-solving task intended to motivate students to explore mathematics in unlikely places and investigate the students' perception of the task. The task given to 19 middle school students centered on Voting. Four voting methods provided context for determining outcomes using mathematics involving proportional reasoning, fractions, ratios, and logic. The task design provided clues for how 60 children voted for 4 choices of candy. Student work and surveys were analyzed qualitatively. Student work elicited multiple strategies using both ratios and algebraic thinking. Student perceptions were examined for mathematical, non-mathematical, and real-world perspectives. Findings indicate students see mathematics embedded in counting votes and that fairness in voting is not easily achievable. They then considered if satisfaction was optimizable.

Keywords: Instructional Activities and Practices, Middle School Education

Students often view mathematics as something that occurs only in school mathematics classrooms and removed from our daily lives or general interests (Boaler & Dweck, 2015). Mathematics is "often siloed to its own specific time and format" (Smith & Oslick, 2023, p. 14) and in many instances focused on procedures, calculations, and answers. However, teachers can use tasks that generate an exciting environment for exploring mathematical ideas creatively through connections and integration of other fields of study (Aljarrah & Tower, 2023; Darling-Hammond, 2000). Our research considers the design and implementation of a collection of tasks and activities that enable and motivate students to explore mathematics in unlikely places. We also investigate students' perceptions of the tasks.

Relevant Literature and Theoretical Perspectives

Engaging students in their mathematical activity, particularly those activities that have an air of originality and novelty (Aljarrah & Tower, 2023) and set in a creative environment "enables learners to generate and expand ideas, suggest hypotheses, apply imagination, and look for alternative, not-yet imagined approaches" (Craft, 2001 p. 14). We are guided by instructional practices that are beneficial to engaging group work such as the use of complex instruction and designs of tasks with active learning through cooperative interaction (Cohen, 1994). As Esmonde (2009) notes, there are possible pitfalls of these types of activities. One must be aware of group dynamics when enacting group activities within the classroom, specifically when particular students take on an "expert" role. Esmonde (2009) discusses a negative effect on some student's learning when one (or multiple) vocal students take control of the conversations. One way we believe to ensure multiple students interacting and partaking in the discussions is through enacting activities that are designed to pique student hobbies and non-mathematical interests.

We take a sociocultural perspective in designing the tasks (Vygotsky, 1962) that students construct new knowledge supported by the social nature of their interactions with their peers. We also encompass aspects of the culturally relevant pedagogy framework (Ladson-Billings, 2020) in using students' experiences and backgrounds to help create activities and engage students in the mathematical content contained within.

Literature on using student interest in mathematics is limited. Poston & Vandenkieboom (2019) examined the use of chess to increase test scores while Ke (2019) examined how integrating mathematical problem solving with architectural design using a 3D epistemic stimulating game resulted in better performance in a math context problem solving test over a control group. An activity for middle school students discussed by Hunt and colleagues (2014) incorporated voting with the study of ratios and found that by developing the concepts from concrete, semiconcrete, to abstract representations helped build students' conceptual understanding of the mathematics. Ainley (2012) discusses two metaphors that can describe the relationship between interest and engagement: hook and switch. A "switch connects students' existing personal interests with opportunities to express those interests." (pg. 286)

Hence, our report focuses on two research questions:

- In what ways can topics that are not inherently associated with mathematics be leveraged in mathematical task design to engage middle-school students in a productive manner?
- How do the students perceive their experiences in enacting the activity?

Methods

The results presented in this paper were collected at the end of a multiple summer research project aimed at exploring how mathematics could be related to other areas of study that are not necessarily seen as mathematical. During the summer of 2020, six graduate research assistants from a large research institution in the southern U. S. each designed six activities relating mathematics to other fields of study. In the summer of 2021, three of the graduate assistants in turn refined and implemented a subset of the activities with middle school students from a summer math camp that was held online due to the Covid pandemic. Out of 36 activities designed in 2020, 12 were implemented in 2021. Some of the topics included were origami, billiards, medicine, voting, juggling, and astronomy.

To implement these activities in the camp setting, each graduate assistant prepared their lesson for the week and discussed their plans with the research team. Given feedback from the research team, adjustments were made, and the following day the graduate assistants spent one hour training two camp counselors on how to implement this activity with the students. This cycle was repeated over the course of two weeks of the camp session as each graduate student implemented a total of four activities. After each activity, feedback was collected from both the counselors and students via a survey. The activity discussed in this paper is from one of those implementations. 19 students' written work was submitted via the classroom management system *Canvas*, additionally 17 student surveys were collected.

Activity

The activity that we analyzed centered on voting methods. Voting is a central part of American society, but more broadly, a central part of democracy. It is a topic that students tend to have first-hand knowledge of as they have typically experienced some type of voting be it for electing a class president or voting for their favorite foods.

Voting can be leveraged in mathematics through its relationship with fractions and ratios. The activity designed promotes understanding of both concepts as well as logic. It additionally promotes the development of soft skills used such as creative thinking and communication.

The activity is a combination of a reading task, a PowerPoint presentation, and a logic puzzle. Prior to the presentation and puzzle, students were given a short article about four different types of voting methods that are used (Jones, 2002). Through the reading, students are exposed to basic definitions of each voting method as well as examples of how each method works (*First Past the Post, Run-off, Elimination,* and *Ranking*). Following the reading, students were given a short presentation on the article and a logic puzzle. The students were prompted: Suppose 60 people are voting on the best candy to use as prizes in their classroom. The four choices up for a vote are Reese's Peanut Butter Cups (R), Jolly Ranchers (J), Smarties (S), and Kit Kats (K). Using the clues below, determine how many people voted for each candy. Students were also given clues and table as seen in Figure 1.

Clues: (Hint: The clues are in the order you need them in.)

1. In method 1, Reese's peanut butter cups wins.
2. In method 2, Jolly Ranchers wins with 2/3 of the total votes.
3. In method 3, Smarties wins with an 8:7 vote ratio, beating Reese's, which comes in second.
4. One-tenth of the total people like Jolly Ranchers least of all the options.
5. One person more than 1/5 of all the people like Smarties second best.
6. There is a ratio of 11:20 in terms of people who like Jolly Ranchers third best to total people.
7. Of the people who like Smarties best, there is a 1:6 ratio of people who preferred Jolly Ranchers to Kit Kats.
8. In method 4, Kit Kats wins with the following point totals: Kit Kats 171, Jolly Ranchers 154, Smarties 139, and Reese's 136. (Use this to check your answer).

The table below shows all the different ranking orders: where **R** is for Reese's, **J** for Jolly Ranchers, **S** for Smarties, and **K** for Kit Kats.

Number of people	*First choice*	*Second Choice*	*Third Choice*	*Fourth Choice*
	R	**K**	**J**	**S**
	R	**S**	**K**	**J**
	J	**R**	**K**	**S**
	J	**K**	**S**	**R**
	S	**K**	**J**	**R**
	S	**J**	**K**	**R**
	K	**S**	**J**	**R**

Figure 1: Voting Clues and Table

Analysis

For this paper, we present our preliminary analysis of the Voting Activity. We used the written student work to qualitatively analyze the ways in which students attempted to solve the given logic problem and the mathematical strategies they employed by open coding (i.e., did they use ratios, proportions, system of equations, other) (Miles, Huberman & Saldana, 2019). For the student survey responses, again we open coded for what was the overall perception of the activity mathematically and non-mathematically, and how the students saw this as beneficial or connected to the real world.

Results

In this paper we discuss results related to the ways the topic of voting was used to leverage mathematical activity for middle grade students. Additionally, we present student survey responses on their thoughts of the activity and the non-mathematical idea that was present within the activity.

Mathematical Activity

Within the voting activity, students were presented with the four different voting methods. With these four different voting methods, students discussed how proportional reasoning, fractions, and ratios can be used to discuss the ways in which people are "happy" or "not happy" with the outcome.

Each of these voting methods can elicit ratio and proportional reasoning in different ways. Students used ratios to help further their reasoning of how to deduce the answers to this puzzle. For example, when considering Clue 7, a student wrote the following on their worksheet "1:6, SJKR:SKJR= 1:6 = a:6a." Yet, we did not just see students using proportional reasoning or ratios to make sense of their logic puzzles. Frequently, we saw students create multiple systems of equations. The same student referenced above later created a system of equations to help deduce other portions of the puzzle. Another example can be seen by considering the student work in Figure 2 below. Here, we see the student set up a system of equations to deduce the values of *x* and *y*, where *x* and *y* represent the number of people who voted for Jolly Ranchers as their first choice in two different ways.

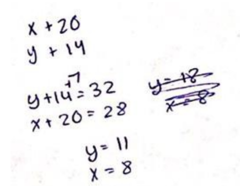


Figure 2: Student work of system of equations

Student Perceptions

In regard to student perceptions, we focus on the responses to the following survey questions:

1. Describe what you learned from the activity (mathematical and non-mathematical) and
2. What would you like to learn more about related to the activity?

In terms of the mathematical ideas learned, all responses from the students were within the context of voting. Of the responses, multiple students discussed the ways in which mathematics is used in areas such as voting with the comment, "I learned how you can use math to see how voting works and see how the people get their favored thing in a ratio." In response to the non-mathematical ideas learned, we saw multiple students focus on the idea of fairness, "any voting method will never be 100% fair or correct," "no voting methods are 100% fair." These responses also transcend into the second question above on what they would like to learn more about regarding the activity. Out of the responses, many students focused on knowing more about voting, either as in "more methods of voting" or methods used in the real world such as "real-life politics" or "voting systems in different countries." Additionally, we saw students interested in optimizing satisfaction, "most optimal voting method" or "which way is fairest under certain circumstances." Notice that both comments can be seen as ways in which mathematicians begin to conjecture in their own research.

Discussion

Throughout this report, our goal was to showcase how nonmathematical ideas could be incorporated into mathematics classrooms to engage students' interests. The activity showcased closely aligns with common core standards in grades six and seven, making it an activity suitable for classrooms settings. Moreover, this activity presents students with the opportunity to reason with multiple solutions. While we acknowledge that there are limitations in online implementations, we consider this implementation informative as online classes become a more normal part of society. As designed, our activities can be adapted for both in-person and online environments. Regarding students' perceptions, we highlight how students' interests in voting were piqued and mathematical understanding deepened (self-reported) through engagement with this activity. While limited to a report of one task design, it can inform designs of other mathematical tasks that capitalize on other nonmathematical ideas. A broader question that we see in relation to this activity and further research include exploring how this activity and other nonmathematical ideas could be situated within the realm of culturally relevant pedagogy and deliberately designed towards collaborative and collective mathematical problem solving.

References

Ainley, M. (2012). Students' interest and engagement in classroom activities. Handbook of research on student engagement, 283-302

Aljarrah, A., & Towers, J. (2023). The Emergence of Collective Mathematical Creativity Through Students' Productive Struggle. Canadian Journal of Science, Mathematics and Technology Education, 1–17.

Boaler, J., & Dweck, C. S. (2015). Mathematical Mindsets: Unleashing Students' Potential Through Creative Math, Inspiring Messages and Innovative Teaching. John Wiley & Sons, Incorporated.

Cohen, E. G. (1994). Designing groupwork: Strategies for the heterogeneous classroom (2nd ed.). New York: Teachers College Press.

Craft, A. (2001). An analysis of research and literature on creativity in education. Qualifications and Curriculum Authority.

Darling-Hammond, L. (2000). Teacher quality and student achievement. Education Policy Analysis Archives, 8, 1. Esmonde, I. (2009). Ideas and identities: Supporting equity in cooperative mathematics learning. Review of Educational Research, 79(2), 1008-1043.

Hunt, A., Bush, S. B., & Karp, K. (2014). Systematic Interventions for Teaching ratios. Mathematics Teaching in the Middle School, 19(6), 360-367.

Jones, K. (2002). The mathematics of voting. Math Explorer, 5(1), 4-5

Ke, F. (2019). Mathematical problem solving and learning in an architecture-themed epistemic game. Association for Educational Communications and Technology, 67, 1085-1104. https://doi.org/10.1007/s11423-018-09643-2

Ladson-Billings, G. (1995). Toward a theory of culturally relevant pedagogy. American Educational Research Journal, 32(3), 465-491.

Miles, M., Huberman, A. M., & Saldana, J. (2019). Qualitative Data Analysis: A Methods Sourcebook (4th ed.) Sage Publishing.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). Common Core State Standards for Mathematics. Washington, DC: Authors.

Poston, D., & Vandenkieboom, K. (2019). The effect of chess on standardized test score gains. Sage Open, 9(3), 1-22.

Smith, A., & Oslick, M. E. (2023). MATH AND LITERACY: Looking at how picture books can help students understand math concepts. Literacy Today, 40(3), 74–75.

Vygotsky, L. S. (1962). Thought and language. Cambridge MA: MIT Press.

RACE, ABILITY, AND MATHEMATICAL IDENTITY

Camille Griffin
University of Illinois, Urbana-Champaign
cgriffi5@illinois.edu

Erica N. Mason
University of Illinois, Urbana-Champaign
enmason@illinois.edu

Keywords: Equity, Inclusion, and Diversity, Student with Disabilities, Classroom Discourse, Middle School Education

Black middle-school students with disabilities often face barriers to accessing rigorous mathematics as well as opportunities to develop a positive mathematical identity (Lambert, 2017; Padilla & Tan, 2019). This critical academic identity consists of beliefs about an individual's "ability to do mathematics, the relevancy of mathematical knowledge, the opportunities and barriers to enter mathematics fields, and the motivation and persistence needed to obtain mathematics knowledge" (Martin, 2009, pp. 19). Mathematical identity is an important educational outcome that contributes to mathematics academic success (Gonzalez et al., 2020). Yet, Black students with disabilities may experience Discourses that separate them from notions of normality and being a knower and doer of mathematics.

Constraining students' development of a mathematical identity is problematic because we know math serves as a gatekeeper (Cribbs et al., 2015). Students with strong mathematics experience and skills have access to more advanced courses, entrance to more selective colleges, and pathways to math-dependent careers that tend to provide material wealth (Rogers, 2020). These avenues are often not available to students tracked into remedial, special education, or carceral educational learning environments (Hines et al., 2021). Students in lower mathematics tracks are less likely to receive the content they need to access college preparatory math classes (Gholson & Robinson, 2019). To help students positively construct their mathematical identity there is a need for research that considers the multifaceted, contextual messages students receive about their race and ability (McGee & Martin, 2011). This research project supplies a preliminary perspective of these messages for one middle school classroom community and the possible implications of classroom discourse related to mathematical equity and inclusion for Black students with disabilities.

This study examines the classroom community interactions among sixth-grade students from a predominantly Black and Latiné, low-income middle school. Inclusion criteria required that the selected classroom community comprised students with and without disabilities and at least one Black student. Using audio recordings, field observation notes, and reflexive memos, our research approach and procedures identified the semantic relationships among spoken words or phrases (little "d" discourse) to interpret the semiotic meanings of mathematical identity conveyed through classroom conversation (big "D" discourse). Our objective was to understand what Discourses about race and ability exist in a middle school mathematics classroom. We used intersectional qualitative research methodology (Esposito & Evans-Winters, 2022) and Positive Discourse Analysis (Rogers, 2018) tools to examine broad messages of mathematical power, agency, and solidarity. We conducted four coding phases and present preliminary results from this analysis. Our findings may invite researchers and teachers to recognize the Discourses related to mathematics identity construction for Black middle school students with disabilities and the contours of meaning assembled through classroom conversation.

References

Cribbs, J. D., Hazari, Z., Sonnert, G., & Sadler, P. M. (2015). Establishing an explanatory model for mathematics identity. Child Development, 86(4), 1048–1062. https://doi.org/10.1111/cdev.12363

Esposito, & Evans-Winters, V. E. (2022). Introduction to intersectional qualitative research. SAGE Publications, Inc.

Gholson, M. L., & Robinson, D. D. (2019). Restoring mathematics identities of Black learners: A curricular approach. Theory into Practice, 58(4), 347–358. https://doi.org/10.1080/00405841.2019.1626620

Gonzalez, L., Chapman, S., & Battle, J. (2020). Mathematics identity and achievement among Black students. School Science and Mathematics, 120(8), 456–466. https://doi.org/10.1111/ssm.12436

Hines, D. E., Boveda, M., & Lindo, E. J. (2021). Racism by another name: Black students, overrepresentation, and the carceral state of special education. Information Age Publishing, Incorporated.

Lambert, R. (2017). 'When I am being rushed it slows down my brain': constructing self-understandings as a mathematics learner. International Journal of Inclusive Education, 21(5), 521–531. https://doi.org/10.1080/13603116.2016.1251978

Love, H. R. & Beneke, M. R. (2021). Pursuing justice-driven inclusive education research: Disability Critical Race Theory (DisCrit) in early childhood. Topics in Early Childhood Special Education, 41(1), 31-44–44. https://doi-org.proxy2.library.illinois.edu/10.1177/0271121421990833

Martin, D. B. (2009). Liberating the production of knowledge about African American children and mathematics. Mathematics teaching, learning, and liberation in African American contexts (pp. 3–36). London, UK: Routledge.

Martin, D. B. (2019). Equity, inclusion, and antiblackness in mathematics education. Race Ethnicity and Education, 22(4), 459–478. https://doi.org/10.1080/13613324.2019.1592833

Martin, J. R., & Rose, D. (2007). Working with discourse: Meaning beyond the clause (2nd ed.). London: Continuum.

McGee, E. O., & Martin, D. B. (2011). "You would not believe what I have to go through to prove my intellectual value!" Stereotype management among academically successful Black mathematics and engineering students. American Educational Research Journal, 48(6), 1347–1389. https://doi.org/10.3102/0002831211423972

Padilla, A., & Tan, P. (2019). Toward inclusive mathematics education: a metatheoretical reflection about countering ableism in mathematics standards and curriculum. International Journal of Qualitative Studies in Education, 32(3), 299–322. https://doi.org/10.1080/09518398.2019.1576941

Rogers, K. D. (2020). Centering the "M" in STEM: A review of Black students' math experiences. The Negro Educational Review, 71, 7–52.

Rogers R. (2018). Reclaiming powerful literacies: new horizons for critical discourse analysis. Routledge Taylor & Francis Group.

REIMAGINING PARENTAL INVOLVEMENT: USING FAMILIES' FUNDS OF KNOWLEDGE TO ENGAGE ALL LEARNERS IN MATHEMATICS

Enyinnaya Onyeukwu
Cobb County School District
Enyinnaya.onyeukwu@cobbk12.org

Belinda P. Edwards
Kennesaw State University
bedwards@kennesaw.edu

Keywords: *Teaching Practice and Classroom Activity, Equity and Justice*

Parental involvement in education has been traditionally defined in the literature by what schools can do to attract parents into the building to assist in educating their children (Abrams & Gibbs, 2000). This traditional school conception of parental involvement in mathematics education does not align with culturally different parenting traditions found in Black and Latinx households (Cunningham, 2021) and represents a deficit view centering student achievement around parents' lack of school participation, parenting skills, or a disorganized homelife (Nieto, 2001). Gonzalez, Moll, and Amanti (2005) noted that there is a reservoir of mathematics knowledge and usage in households that are not always evident to household family members and teachers.

The purpose of this qualitative study was to reimagine parental involvement during their children's middle school experience by exploring parental, family, and community Funds of Knowledge (FoK) and how such knowledge can be used by a middle school mathematics teacher to promote student mathematics learning (Gonzalez et al., 2005). The research questions guiding the study were: (a) What are some FoK of Black and Latinx middle school parents and families? (b) How did a middle school mathematics teacher capture and use parents' and families' FoK to create lessons that connect students' lived experiences to mathematics learning?

As it relates to mathematics teaching and learning, FoK seeks to build a bridge between the mathematics parents or families understand and use informally at home and students engage with in their classroom environments. The concept of FoK provided an analytic framework for the qualitative study that acknowledges the strengths and assets of Black and Latinx parents and families, recognizing that their voices and experiences might be used as an intellectual resource when developing mathematics education learning opportunities (Gonzalez et al., 1993). This theoretical lens supports an exploration of bridging the lived experiences and perspectives of Black and Latinx parents of middle school students to mathematics teaching and learning in the classroom environment (Glesne, 2016).

The current study takes a critical perspective of the deficit portrayal of parent and family engagement by listening to the stories and experiences of parents and families via a questionnaire, semi-structured interview, focus group conversations, and exploring the community in which students live; then using what is learned about the use of mathematics in the home and community to design cognitive demanding mathematics learning opportunities in the classroom (Stein & Smith, 1998).

Results show FoK can be captured by participating in church/church events located within the school's community. Those FoK centered on food, gardening, music, and cultural traditions and celebration discourse. These led to the creation of Number & Operation, Measurement, Algebra, and Geometry middle school mathematics tasks. Building reciprocal and mutual relationships with parents/families through participation in community events enables teachers to create learning activities that enrich students' lived experiences in the classroom environment.

References

Abrams, L. S., & Gibbs, J. T. (2002). Disrupting the logic of home-school relations: Parent involvement strategies and practices of inclusion and exclusion. Urban Education, 37(3), 384-407.

Cunningham, J. (2021). “We made math!”: Black parents as a guide for supporting black children's mathematics identities. Journal of Urban Mathematics, 14(1), 22-44.

Epstein, J.L. (1995). School/family/community partnerships: Caring for the children we share. Phi Delta Kappan, 92(3), 701-712.

Glesne, C. (2016). Becoming qualitative researchers: An introduction. Pearson.

González, N., Moll, L.C., & Amanti, C. (2005). Funds of knowledge: Theorizing practices in households, communities, and classrooms. Routledge.

Gonzalez, N., Moll, L., Floyd-Tenery, M., Rivera, A., Rendon, P., Gonzales, R., & Amanti, C. (1993). Teacher Research on Funds of Knowledge: Learning from Households (Report 6). National Center for Research On Cultural Diversity And Second Language Learning. https://escholarship.org/uc/item/5tm6x7cm

Nieto, S. (2001). Language, culture, and teaching: Critical perspectives. Routledge.

Stein, M. K., & Smith, M. S. (1998). Mathematical Tasks as a Framework for Reflection: From Research to Practice, Mathematics Teaching in the Middle School, 3(4), 268-275.

REIMAGINING THE EMERGENT PERSPECTIVE WITHIN RESEARCH-PRACTICE LEARNING COMMUNITIES

Beth L. MacDonald	Amanda L. Cullen	Allison M. Kroesch
Illinois State University	Illinois State University	Illinois State University
blmacd1@ilstu.edu	almille@ilstu.edu	amkroes@ilstu.edu

Neet Priya Bajwa	Edward Mooney	Julien Corven
Illinois State University	Illinois State University	Illinois State University
nbajwa@ilstu.edu	mooney@ilstu.edu	jccorve@ilstu.edu

To approach Cai et al.'s (2017) call to blur the lines between research and practice, we examine how early childhood, special, and mathematics educators form a research-practice partnership with the shared focus of supporting young children's development of equitable problem-solving activity. To situate this work, we draw from Cobb and Yackel's (1996) Emergent Perspective, considering both the child's psychological construction of knowledge and the adaptations made by community members in a partnership framework. Thus, in this theoretical brief, we describe an Emergent Research-Practice Learning Community Framework and unpack its three main constructs: shared focus, shared work, and shared vision.

Keywords: Early Childhood Education, Problem Solving, Professional Development, Teacher Educators

A gap between research and practice is well known and has been lamented for years (Arbaugh et al., 2010; Heid et al., 2006; National Association for the Education of Young Children [NAEYC] & National Council of Teachers of Mathematics [NCTM], 2010). Recently, Cai et al. (2017) called for a "radically different" approach to linking research and practice (p. 468). They envisioned "a world in which research and practice in mathematics education are tightly intertwined, a world in which the boundaries between research and practice become

blurred" (Cai et al., 2017, p. 472). This initiative critically presses researchers in mathematics education to take up partnership research design models between teachers and researchers in hopes to blur the boundaries between practice and research worlds. In such a partnership, teaching and research partners share the research and teaching activities jointly. The authors of this brief reflect on what this research practice partnership looks like theoretically.

Theoretical Foci

In this theoretical brief, we unpack three main aspects of a research-partnership framework (shared focus, shared work, and shared vision), while focusing on how young children become sense makers in their own microcultures (e.g., small group activity) and how this can extend to classrooms and school cultures. Broadly, this collaboration is set within a University-School partnership in seven Prekindergarten, Kindergarten, and Grade 1 classrooms. In this brief report, we reflect on the theoretical aspects of this work and how children may engage in sensemaking in equitable ways from the organizational features of this partnership.

By taking up a research-practice partnership design, we posit there is a critical need to develop theoretical frames to explain phenomena under examination and better contribute to the field in meaningful ways. We would be remiss to not root research-practice partnerships with equitable practices at the center while explaining the theoretical stance for these varying educational cultures. In this brief, we consider how theories around children's problem-solving activity, professional learning, and reflexive research-teaching practices can mediate equitable practices for the sake of research-practice partnership development. To examine such a complex system in this theoretical brief, we take up the Emergent Perspective theoretical lens (Cobb & Yackel, 1996) and develop an Emergent Research-Practice Learning Community framework.

Theoretical Framework

Early childhood mathematics education is fundamentally important, setting the stage for rich learning opportunities for young children well beyond their early childhood classroom experiences and predicting later learning, achievement, and social development (McWayne et al., 2009; National Mathematics Advisory Panel, 2008; Schmitt et al., 2017). Moreover, problem solving in early childhood education receives the least amount of attention in the classroom (Karoly et al., 2008) and in the research field (Fox & Diezmann, 2007), leaving early childhood educators with a lack of curricula, resources, and professional development focused on problem-solving instruction. Coupling these issues with the fact that equitable teaching practices (NCTM, 2018) prior to entering kindergarten also varies dramatically (Rudd et al., 2008; Stipek et al., 2012), children often experience dramatic differences in their mathematics performance prior to entering Kindergarten (Puma et al., 2010). This framework essentially states that when research studies begin at the student level (utilizing Psychological Constructivist approaches), efforts remain unfinished until an interactionalist perspective is taken up to examine the emergent perspectives from research-practice partnership members in the classroom (see Figure 1).

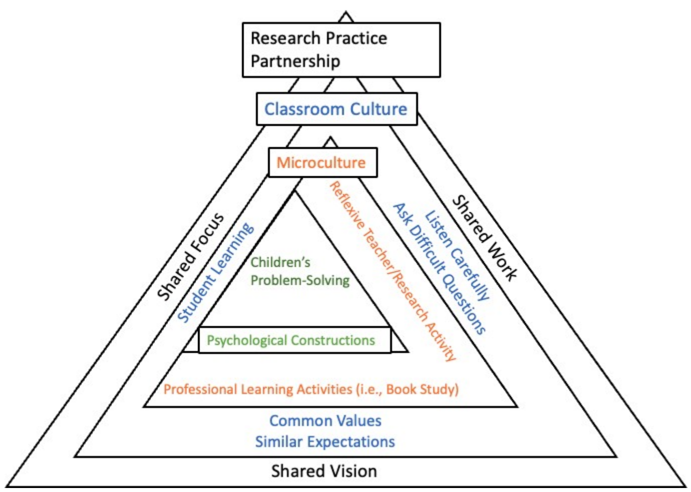


Figure 1. Emergent Research-Practice Learning Community Framework

To serve these issues, we begin at the center of this framework, a *shared focus,* on the development of children's problem-solving activity (see green text in Figure 1). Another aspect of this theoretical framework is how reflexive research-teacher collaborations are formed to better provide resources, time, and support when implementing high-quality and equitable mathematics instruction in teachers' classrooms. Following the COVID-19 pandemic, the US has experienced high levels of educator attrition rates (Steiner & Woo, 2021). Wu-Pope (2022) posits the solution to retaining quality educators is to develop research-practice partnerships with more intentionality. Essentially, we echo this claim and aim to increase teacher (and researcher) learning through ongoing collaboration in classrooms. Thus, the Emergent Research-Practice Learning Community framework includes a second level, activities in classroom microcultures (see orange text in Figure 1). The learning communities explicate *shared work* between educators and researchers in a reflexive manner and through the development of a *shared vision* of equity-oriented learning environments (see Figure 1).

Cobb and Yackel (1996) stated that adaptation of activity from culture (e.g., microculture) to culture (classroom culture), allows meaningful examination of how features of an organizational system are adapted for children's problem solving. In particular, at the student level, Cobb and Yackel explained that the "qualities of students' thinking are *generated by* or *derived from* the organizational features of the social activities in which they participate" (p.220). In the remaining brief, we consider the constructs and activity at the microculture level; we examine how professional learning activities, reflexive teaching, and research activity can provide equitable organizational features for children's problem-solving activity.

Shared Focus: Children's Problem-Solving Activity

The first component in our framework is shared focus: children's problem-solving activity.

In this framework, we broadly define *problem solving* as children's autonomous engagement with their unfinished mathematics learning through their sensemaking of contextualized tasks. By having a shared focus, Louis et al. (1996) posited the importance of having "an undeviating concentration on student learning" (p. 760). To better articulate the shared focus, we consider Cognitively Guided Instruction [CGI] problem types (Carpenter et al., 1999, 2015) with the intersection of how children construct units (Steffe & Cobb, 1988) and develop a solution path with particular tools (e.g., number line, rekenrek).

We utilize Carpenter et al.'s (1999, 2015) CGI problem taxonomies. Within their taxonomy of additive word problems, Carpenter et al. identified four types based on the underlying structure of the problem: join, separate, part-part-whole, and compare. Each join or separate problem type can have a missing start, change, or result, providing opportunities for reversible thinking (Simon et al., 2016). When children engage with these problem types, Carpenter and colleagues (1999, 2015) found students early strategies included direct modeling of the solution, or varying degrees of counting activity.

To explain nuanced development of children's direct modeling and counting, we also examine children's problem solving through their units construction and units coordination. A *unit* is a measured out set of items or length and to *coordinate units* requires a child to develop abstract relationships between units. For instance, a child may first solve a problem by counting on (i.e., groups five bears, and says, "five, six, seven), and then by double counting to keep track of a total counted and the number of skip counts (i.e., "two, four, six, that's three twos"). We posit, that by engaging in reversible activity (through varying problem types) with pre-numerical units (perceptual units, figurative units, etc.), children are better positioned to develop abstract units (internalized and interiorized units) at earlier ages. When children are first relying on pre-numerical units that are not yet coordinated, their physical activity with varying tools (i.e., number line, bears) often evidences mental activity developed later (Steffe & Cobb, 1988). This work is critical in establishing sensemaking opportunities for young children in early childhood mathematics education, but it is not sustainable without understanding how educators and researchers can partner to enact this shared work and examine their shared vision.

Shared Work: Reflexive Teaching and Research Activities

To begin adapting problem-solving strategies in a microculture, we theorize the importance of the second component in our framework is shared work through reflexive teacher-researcher engagement in classroom practices and research activities. By team teaching alongside varying members of the research-practice partnership, microcultures are created (Webb et al., 2009. In the Emergent Research-Practice Learning Community, these microcultures include collaborative teaching experiences and station teaching experiences (cf. team teaching as a specific form of co-teaching; Cook & Friend, 1995). These microcultures impact how children's mathematics discourse is orchestrated (Smith et al., 2019) and how children are grouped (Webb et al.). All partnership members share in the work of anticipating children's solution paths before lessons as well as monitoring children in small groups as they engage in reasoning, sensemaking, and productive struggle to solve problem types with the solution path or tool of their choice. After small group work, partnership members share in the work of selecting and sequencing children to share their solution paths with the whole class. During the whole class discussion, partnership members team teach by co-facilitating the discussion of children's solution paths. Every child is encouraged to share, critique, and connect ideas and solution paths. These five practices allow for meaningful adaptations within classroom microcultures for all children to be positioned as knowers and doers of mathematics by enhancing their mathematical identity and agency.

When engaging in station teaching, children's small group problem-solving activities become contextualized and reflexive for partnership members (Webb et al., 2009). For instance, in this shared work, partnership members have opportunities to engage in equitable instructional practices through how they group their students for small-group problem solving. By creating heterogeneous groups, community members are able to "try out" varying ways to engage children with problem-solving tasks and talk about their activity and solution paths (Webb et al., 2009). Thus, reflexive teaching and research partnerships, which members take up to learn more about equitable grouping structures, are formed through their shared work.

Shared Vision: Professional Learning Activities

The third component of this framework, shared vision, explains how equitable teaching practices (NCTM, 2014; 2018) are promoted through professional learning experience. Essentially, through professional learning opportunities (e.g.., professional book study, lesson study), partnership members begin noticing patterns in their students' activity (Pak, 2022) and reflecting on their teaching practices (Muir & Beswick, 2007), which largely take up the Equitable Teaching practices (NCTM, 2018). For instance, when designing and implementing tasks that promote problem solving and reasoning, partnership members reflect on tasks, anticipate student strategies and develop questions about multiple pathways children may develop. Moreover, partnership members could also engage in contexts where problems are designed around the varying funds of knowledge of children and families (NCTM, 2018).

Through discussions around professional materials and action research activity, partnership members may connect equitable teaching practices and equitable learning opportunities, while reflecting on their instruction and their students' learning opportunities.

Conclusion and Implications for Research in Practice Partnerships

We described how a focus on children's problem solving could be leveraged equitably through community members shared work around their orchestration of discourse, and their shared vision around their equitable task development. In short, by taking up such a theoretical framework, partnership members are better positioned to examine how varying pedagogical cultures afford development of a shared focus, shared work, and shared vision. Given the impact this framework could have with research studies, we have questions as to how constructs in this model (e.g., shared focus, shared vision, shared work) may explain young children's equitable access to their own sensemaking in mathematics classrooms over time.

References

Arbaugh, F., Herbel-Eisenmann, B., Ramirez, N., Knuth, E., Kranendonk, H., & Quander, J. (2010). Linking research and practice: The NCTM research agenda conference report. Retrieved from https://www.nctm.org/uploadedFiles/Research_and_Advocacy/Linking_Research_and_Practice/Linking_Research_20100511.pdf

Cai, J., Morris, A., Hohensee, C., Hwang, S., Robison, V., & Hiebert, J. (2017). A future vision of mathematics education research: Blurring the boundaries of research and practice to address teachers' problems [Editorial]. Journal for Research in Mathematics Education, 48(5), 466–473. doi:10.5951/jresematheduc.48.5.0466

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). Children's mathematics: Cognitively guided instruction. Heinemann.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (2015). Children's mathematics: Cognitively guided instruction (2nd ed.). Heinemann.

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. Educational Psychologist, 31, 175–190. doi:10.1080/00461520.1996.9653265

Cook, L., & Friend, M. (1995). Co-teaching: Guidelines for creating effective practices. Focus on Exceptional Children, 28(3), 1–16. doi:10.17161/fec.v28i3.6852

Fox, J. L., & Diezmann, C. M. (2007). What counts in research? A survey of early years' mathematical research, 2000–2005. Contemporary Issues in Early Childhood, 8(4), 301¬–312. doi:10.2304/ciec.2007.8.4.301

Heid, M. K., Middleton, J. A., Larson, M., Gutstein, E., Fey, J. T., King, K., Strutchens, M. E., & Tunis, H. (2006). The challenge of linking research and practice. Journal for Research in Mathematics Education, 37(2), 76–86.

Karoly, L., Ghosh-Dastidar, B., Zellman, G. L., Perlman, M., & Fernyhough, L. (2008). Prepared to learn: The nature and quality of early care and education for preschool-age children in California. Santa Monica, CA: RAND.

Louis, K. S., Marks, H. M., & Kruse, S. (1996). Teachers' professional community in restructuring schools. American Educational Research Journal, 33(4), 757–798. doi:10.3102/00028312033004757

McWayne, C. M., Green, L. E., & Fantuzzo, J. W. (2009). A variable-and person-oriented investigation of preschool competencies and Head Start children's transition to kindergarten and first grade. Applied Developmental Science, 13(1), 1–15. doi:10.1080/10888690802606719

Muir, T., & Beswick, K. (2007). Stimulating reflection on practice: using the supportive classroom reflection process. Mathematics Teacher Education and Development, 8, 74-93.

National Association for the Education of Young Children & National Council of Teachers of Mathematics. (2010). Early childhood mathematics: Promoting good beginnings. Authors. https://www.naeyc.org/sites/default/files/globally-shared/downloads/PDFs/resources/position-statements/psmath.pdf

National Council of Teachers of Mathematics. (2014). Principles to actions: Ensuring mathematical success for all. Author.

National Council of Teachers of Mathematics. (2018). Catalyzing change in high school mathematics: Initiating critical conversations. Author.

National Mathematics Advisory Panel. (2008). Foundations for success: The final report of the National Mathematics Advisory Panel. U.S. Department of Education. http://www.statlit.org/pdf/2008NMAP.pdf

Pak, B. (2022). Expanding professional noticing to examine teachers' decision making during intervention in small groups. In Lischka, A. E., Dyer, E. B., Jones, R. S., Lovett, J., Strayer, J., & Drown, S. (Eds.), Proceedings of the forty-fourth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (pp. 1837–1841), Nashville, TN.

Puma, M., Bell, S., Cook, R., & Heid, C. (2010). Head Start impact study. Washington, DC: U.S. Department of Health and Human Services.

Rudd, L. C., Lambert, M., Satterwhite, M., & Zaier, A. (2008). Mathematical language in early childhood settings: What really counts? Early Childhood Education Journal, 36, 75–80.

Schmitt, S. A., Geldhof, G. J., Purpura, D. J., Duncan, R., & McClelland, M. M. (2017). Examining the relations between executive function, math, and literacy during the transition to kindergarten: A multi-analytic approach. Journal of Educational Psychology, 109(8), 1120–1140. doi:10.1037/edu0000193

Simon, M. A., Kara, M., Placa, N., & Sandir, H. (2016). Categorizing and promoting reversibility of mathematical concepts. Educational Studies in Mathematics, 93, 137-153.

Smith, M.S., Bill, V.L., & Sherin, M. G. (2019). The 5 practices in practice: Successfully orchestrating mathematics discussions in your elementary classroom. Thousand Oaks, CA: Corwin.

Steffe, L. P., & Cobb, P. (1988). *Construction of arithmetical meanings and strategies.* Springer-Verlag. doi:10.1007/978-1-4612-3844-7

Steiner, E. D., & Woo, A. (2021). *Job-related stress threatens the teacher supply.* RAND. Retrieved February 17, 2023 https://www.rand.org/content/dam/rand/pubs/research_reports/RRA1100/RRA1108-1/RAND_RRA1108- 1.pdf

Stipek, D., Schoenfeld, A., & Gomby, D. (2012). Math matters, even for little kids. *Education Week, 31*(26), 27–29.

Webb, N. M., Franke, M. L., De, T., Chan, A. G., Freund, D., Shein, P., & Melkonian, D. K. (2009). 'Explain to your partner': teachers' instructional practices and students' dialogue in small groups. *Cambridge Journal of Education, 39*(1), 49-70.

Wu-Pope, J. (2022). How research-practice partnerships meet high-performing teachers' needs. *Phi Delta Kappan, 104*(1), 28-32.

RESPONSIVE EVALUATION OF STUDENT WORK: A THEORETICAL PROCESS

Julien Corven
Illinois State University
jccorve@ilstu.edu

To base teaching on student thinking requires analyzing and interpreting students' thinking, key components of the construct of professional noticing (e.g., Jacobs et al., 2010). Although substantial research has been conducted using this construct, less attention has been paid to teachers' evaluations of student work based on these analyses and interpretations. In this theoretical report, I argue that evaluation of student thinking is a key prerequisite for effective instructional decision making for responsive teaching. I then present a synthesized framework of evaluation criteria and an elaborated process for effective and responsive evaluation of student written work using these criteria. Finally, I analyze excerpts from two interviews with elementary prospective teachers to demonstrate the utility of these products.

Keywords: Teacher Noticing, Preservice Teacher Education, Assessment

Effective mathematics teaching requires basing instruction on students' thinking (e.g., National Council of Teachers of Mathematics, 2014). To engage in such responsive teaching, teachers need the skills of eliciting (e.g., Shaughnessy & Boerst, 2018b), understanding (e.g., Association of Mathematics Teacher Educators, 2017), evaluating (e.g., Dyer & Sherin, 2016), and productively responding to (e.g., Jacobs et al., 2010) student thinking. Therefore, teacher preparation programs must provide multiple opportunities for prospective teachers (PTs) to learn how to enact these skills. This theoretical report contemplates PTs' ability to evaluate student mathematical thinking as reflected in written mathematical work, a construct which is underdeveloped in the literature. I argue that thoughtfully evaluating student work and thinking is an essential aspect of responsive teaching because it shapes how teachers can productively respond to students' mathematical understandings and ideas (Dyer & Sherin, 2016; Jacobs et al., 2010; van Es & Sherin, 2021). In this report, I address the following research questions: *What criteria for evaluation of student work and/or thinking have been documented in the literature?* and *Can these criteria be synthesized into a process that can be used to evaluate students' written mathematical work in a systematic way supportive of responsive teaching?*

Evaluating Students' Thinking in Responsive Teaching

One essential meaning of the verb 'evaluate' is "to determine the significance, worth, or condition of… usually by careful appraisal and study" (Merriam-Webster, n.d.). Thus, evaluation is necessarily judgmental, but it should be thoughtful (i.e., based on evidence and/or reasoning). I define "evaluation of student thinking/work" as the process teachers use to make reasoned judgments about students' mathematics using relevant criteria and against relevant standards to inform their next steps for teaching, including providing feedback and responding to students.

Robertson et al. (2015) defined responsive teaching as teaching that embodies three characteristics: (1) foregrounding the substance of students' ideas, (2) connecting students' ideas to key ideas within the discipline, and (3) taking up and pursuing students' ideas. Evaluation of student thinking plays a critical role in applying these tenets. For example, to connect a student's idea to a key idea within the discipline, a teacher must evaluate how closely the student's idea aligns with it. Ideally, this evaluation would help the teacher consider how to help move the student’s thinking towards that key idea. Also, during typical classroom instruction, due to time constraints, teachers must choose which ideas will be pursued by the whole class (Stein et al., 2008; Stockero et al., 2017). To make such decisions, teachers must evaluate students’ ideas in reference to their relevance to the learning goal (Stein et al., 2008).

Potential Criteria for Evaluating Students’ Written Work

From a synthesis of the literature, I define six criteria that teachers could use to evaluate students’ written mathematical work to support instruction aligned with the tenets of responsive teaching and a seventh criterion, personal preference, that could be detrimental to responsive teaching if used by a teacher to evaluate students’ written work (Table 1).

Table 1: Definitions of Potential Criteria for Evaluating Students’ Written Work

Criterion	Definition
Strategic competence	The extent to which a student’s strategy was appropriate for solving the problem, including discussions of mathematical limitations of the strategy for solving this or similar problems.
Logical validity	The extent to which a student’s reasoning forms a logically valid mathematical argument.
Conceptual understanding	The extent to which a student’s work demonstrates understandings of mathematical concepts related to the topic of the problem.
Efficiency/ sophistication	The extent to which a student’s strategy and/or work for the problem is efficient or mathematically sophisticated, including judgments related to the placement of the strategy within a developmental progression.
Clarity	The extent to which a student’s thinking, reasoning, and work are communicated in ways others can easily understand.
Accuracy/procedural understanding	The extent to which a student obtains a correct answer, correctly completes computations, and/or knows/follows steps of a procedure.
Personal preference	The extent to which the evaluator prefers a student’s strategy for solving the problem due to their own relationship with the strategy.

In the next sections, I will describe and synthesize literature that implicates the first six of the criteria in Table 1 and comment on how teachers’ evaluations of student work or thinking on each criterion could contribute to responsive teaching.

Strategic Competence

Building from one of the strands for mathematical proficiency from the *Adding it up* report (National Research Council [NRC], 2001), Copur-Gencturk and Doleck (2021) defined three aspects of strategic competence. First, the student must devise a valid solution strategy for solving the problem. Second, the student must translate the problem and/or strategy into a useful mathematical representation. Finally, the student must execute their chosen solution strategy accurately. However, when evaluating student work for a non-computational problem, I consider accuracy to be a separate criterion for evaluation. An appropriate strategy that contains a minor calculation error in its application in a written solution could be seen by a teacher as strategically competent despite its inaccuracy.

Monitoring students as they work on a mathematical task is one of the five practices for orchestrating a productive mathematical discussion (e.g., Stein et al., 2008). Notably, the goal of monitoring is “to identify the mathematical learning potential of particular strategies or representations used by the students, thereby honing in on which student responses would be important to share with the class as a whole" (Stein et al., 2008, p. 326). Thus, as teachers circulate, they should be evaluating the student work they see for its strategic competence, which will then inform their decision for which strategies may be best to share with the class. Such decisions align with tenet (3) of responsive teaching: taking up and pursuing students' ideas.

Logical Validity

Although logical validity could also be viewed as an aspect of strategic competence, I consider strategic competence to be a criterion for evaluating the reasonableness of a student's strategy selection. Logical validity is more fine-grained, as it considers the validity of each step of reasoning (implicit arguments) in the student's work. As Özdemir and Pape (2012) noted, teachers can support students by asking them to explain their reasoning for selecting a strategy (strategic competence) and justifying why it works (logical validity).

As part of the process for selecting student solutions to present to the class (Stein et al., 2008), a teacher may decide to have a student share work that illustrates a common error in reasoning to help the class understand why it is not valid (Ayalon & Rubel, 2022). Doing so helps students connect their ideas to key disciplinary ideas, tenet (2), through contrast. However, Morris (2007) suggested that PTs may have difficulty applying this criterion, as they will often fill in logical gaps in students' arguments with their own reasoning.

Conceptual Understanding

Teachers often have a particular learning goal in mind when they plan a lesson and the tasks or activities contained within it. Hiebert et al. (2018) and Morris et al. (2009) described the process of decomposing the learning goal into component mathematical subconcepts. These subconcepts are specific conceptual understandings that students may have an opportunity to demonstrate as they work on a task. Evaluating student work to determine which conceptual understandings the student has demonstrated is therefore important for considering next steps.

Cohen and Benton (1988) recommended that teachers analyze students' thinking for evidence of specific conceptual understandings as they circulate while monitoring student work. As Stein et al. (2008) noted, one key goal of monitoring student responses is to ensure that approaches to the task aligned with the learning goal (i.e., that demonstrate specific conceptual understandings) will be available for class discussion. Diagnosing these specific conceptual understandings foregrounds the substance of students' ideas, tenet (1).

Efficiency/Sophistication

To orchestrate an effective mathematical discussion, teachers need to the sequence of solutions presented to the class. Stein et al. (2008) suggested one possible effective sequence is to present a strategy that is more accessible to students before presenting a more complex strategy. Thus, it would be important for teachers to evaluate students' solutions on an axis of efficiency, sophistication, or placement on a developmental progression. A study by Ayalon and Rubel (2022) confirmed that PTs often sequence strategies in increasing order of complexity, and PTs justified this decision by discussing students' ability to access more strategies. Additionally, several studies (e.g., Clements & Sarama, 2021; Moreno et al., 2021; Schack et al., 2013) have examined how instruction on learning trajectories succeeded in supporting PTs and teachers to learn to notice students' mathematical thinking, a prerequisite to tenets (1) and (2). Therefore, evaluation on this criterion may support responsive teaching by improving teachers' noticing.

Clarity

During a classroom discussion, students may make mathematical contributions, either verbally or through presenting their written work. Van Zoest et al. (2020) provided a framework that teachers can use to think about such contributions. According to the framework, a student's contribution can be non-mathematical, clarification-needed, inference-needed, or standalone (Van Zoest et al., 2020). A teacher's evaluation of a mathematical contribution on this framework could help them determine their next teaching move. For example, if the teacher evaluates the contribution as "inference-needed," they may ask a student to restate the contribution with additional context and precision before asking the class to engage with it. In this process, the student is given an opportunity to clarify their meaning if an incorrect inference was made. By making an evaluation of the clarity of a student's contribution and responding appropriately, the teacher can support the class in understanding the connection of the idea shared to key conceptual understandings or disciplinary concepts, tenet (2).

Accuracy/Procedural Understanding

For some procedural tasks, the learning goal concerns students' ability to demonstrate accurate computation and ability to follow the steps of a procedure. However, for other tasks, although teachers may wish to take note of computational or procedural errors, evaluations on this criterion may be of a lower priority in terms of responding productively to students' ideas. Nevertheless, as I explain below, it has been an evaluation criterion observed in prior research.

Prior Studies on Considering Students' Written Work

Several studies have found that PTs and teachers often attend to the accuracy of students' written work. For example, in a study by As'ari et al. (2019), most algebra teachers' ideal solutions for understanding student thinking employed accurate symbol manipulation after translating the given information into algebraic equations. As'ari et al. (2019) concluded the teachers in this study valued accuracy and procedural understanding highly. However, because the teachers also valued correctly mathematizing the given information into an algebraic representation, these teachers may also have valued solutions that demonstrated strategic competence (NRC, 2001). Additionally, PTs often assume a correct answer in student work indicates conceptual understanding (e.g., Bartell et al., 2013; Shaughnessy & Boerst, 2018a; Spitzer et al., 2011). Studies by Son (2013) and Lee (2021) concluded that, although PTs often successfully attended to errors in students' work, few PTs developed clear interpretations of the student's underlying mathematical difficulty. These studies strongly suggest that procedural understanding and accuracy are criteria that PTs and teachers use to evaluate students' work.

Prior qualitative research has provided some additional nuance to these findings. For example, Michael (2005) claimed that elementary PTs value solutions where all information and steps are shown, and their evaluations of solutions appeared to be positively influenced by features that made the work easy to follow. Corven (2021) also found that elementary PTs often preferred work that was clear to them, regardless of the solution's accuracy or validity.

Personal preferences may also affect PTs' evaluations of student work. For example, Van Dooren et al. (2002) observed that PTs generally prefer student solutions that match the one they would have used to solve a problem when asked to choose from among several correct examples of student work. A replication and expansion of this study by Michael (2005) noted a statistically significant, positive correlation between PTs' frequency of using a particular strategy to solve word problems and their evaluation scores for student work using a similar strategy.

Processes for Evaluating Student Work

Philipp's (2018) study on diagnostic competence provided insights into how mathematics teachers think about students' written solutions to mathematics problems. According to Philipp, after reconstructing a student's solution from their perspective, a teacher evaluates the student's work in terms of its strengths and deficits. However, Philipp's process is problematic for several reasons. First, although "deficits" may be an accurate characterization of how teachers in Philipp's study viewed aspects of students' work or thinking, a growing body of research (e.g., Louie et al., 2021) encourages the use of anti-deficit approaches to such analyses. Second, Philipp (2018) specifically defined deficits as "errors" (p. 121), but other aspects of the work could also raise concerns for a teacher (e.g., gaps in logical reasoning, unclear parts of the work, or not demonstrating an important subconcept of the learning goal). Finally, Philipp's process ceases after strengths are identified, but interactions with a student after teachers attend to and interpret their mathematics can help develop the student's thinking (e.g., van Es & Sherin, 2021), even when no errors or concerns are identified. To address these concerns, I created Figure 1, which is a refinement and elaboration of Philipp's (2018) idealized diagnostic process.

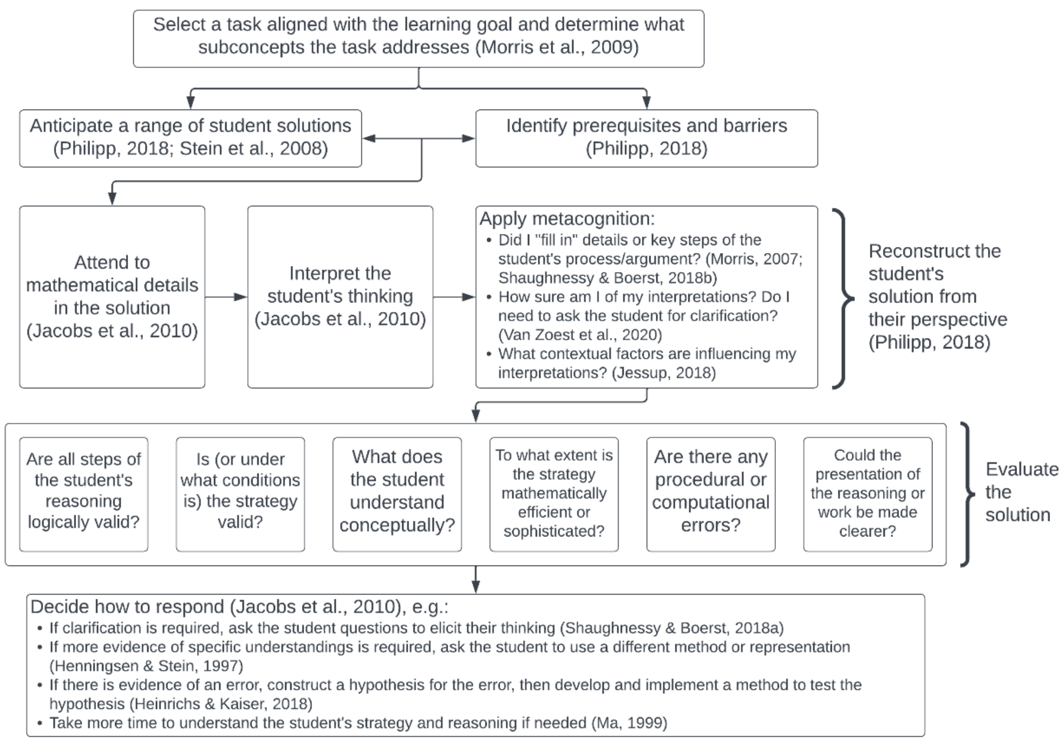


Figure 1: An Elaborated Process for Evaluating Students' Written Mathematical Work

The row labeled "Evaluate the solution" in Figure 1 poses questions aligned with the six productive criteria for evaluation in Table 1. However, some solutions may not contain sufficient evidence to answer all the questions in the "Evaluate the solution" row. For example, Spitzer et al. (2011) state that, for a procedural solution, an appropriate answer to the question "What does the student understand conceptually?" would be that there is not enough evidence to substantiate any such claim. The "Evaluate the solution" row in Figure 1 is also not meant to be followed strictly from left to right. Expert teachers will apply the evaluation criteria implicated by those questions flexibly, strategically, and purposefully depending on the details in a student's work.

Application of the Elaborated Process

I will now demonstrate an application of the framework in Figure 1. Below is an excerpt from an interview with Charlotte (pseudonym), an elementary PT who had just finished her first mathematics content course. Charlotte was asked to reconstruct the student's thinking shown by the solution in Figure 2, which was a response to the story problem, "A licorice rope was 36 inches long. Tammy cut the entire rope into pieces 4.5 inches long to share with her friends. How many pieces did she make?" (adapted from Moore et al., 2020).

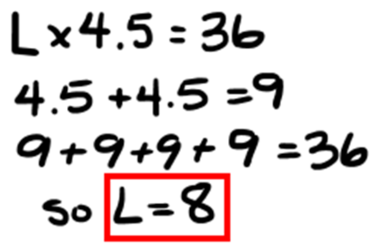


Figure 2: A Missing Factor Approach to Solving a Division Story Problem

Charlotte: So, I think they understand, just by starting out, I think they understand the correlation between division and multiplication. So, they switched the 36 divided by 4.5 to a multiplication problem. So, they understood that 36 inches [*trails off*]…. [T]hey don't know yet how many groups of 4.5, so they switched it to the problem L groups of 4.5 inches equals 36. So, to make it easier for them, they did 4.5 plus 4.5 to get a whole number, to get 9, and they understand that 9 goes into 36 four times. So, 9 plus 9 plus 9 plus 9 equals 36…. [B]y doing the 9 plus 9 plus 9 plus 9 is just saying like parentheses, 4.5 plus 4.5, close parentheses four times, so they can see that L equals 8….

Interviewer: So, they had the idea that each of these 9's represents two copies of 4.5?

Charlotte: Yes, and it looks like they could have made their work clearer by doing, like under that maybe, parentheses 4.5 plus 4.5, and do that across the line…. But it seems like they're able to understand it [so] that they don't need the work. If they were struggling and not getting the right answer, as a teacher, I might have been like, well, show your idea behind the 9. But, since they have the work above it, it seems like they understand what they're trying to say…. Maybe when they were first learning this, they did put that line under it. But maybe they're further along in the process where they understand now that, with the work above it, that 4.5 and 4.5 will give the 8 instead of 4….

Interviewer: In your opinion, how would you evaluate the reasoning shown in this solution?

Charlotte: I think that it's good that they understand the concept that division and multiplication relate to each other with switching the problem up like that. And then, with the 4.5 plus 4.5 equaling 9, it's good that they made it a whole number so they're not messing around with the decimals and everything. So, I think they showed the work they needed for this problem. I understood it, and it seems like they understood it as a student. So, overall, I think they understand the concept of division.

Charlotte's engagement with the student work was generally aligned with Figure 1. Charlotte interpreted the student's work as showing that "they understand the correlation between division and multiplication," and she cited mathematical details from the student's work ("L groups of 4.5 inches equals 36") to support her inference. Thus, Charlotte both attended to and interpreted mathematical details in the student's work (Jacobs et al., 2010). Further, she recognized where she had filled in a step in the student's reasoning ("they could have made their work clearer by doing, like under that maybe, parentheses 4.5 plus 4.5") that was implicit in the work (metacognition; Morris, 2007). Charlotte did not indicate a need to elicit the student's thinking for confirmation, suggesting she considered the contribution as standalone (Van Zoest et al., 2020). Charlotte also compared the work to (hypothesized) past performance (Jessup, 2018) when she said, "Maybe when they were first learning this, they did put that line under it. But maybe they're further along in the process [now]" (contextual factor). Thus, Charlotte was generally able to reconstruct the solution from the student's perspective to prepare for evaluation.

Charlotte considered and evaluated the logical validity of the student's reasoning, though she did not make an explicit strategic competence evaluation. Nevertheless, she did diagnose a specific conceptual understanding (the relationship between multiplication and division) demonstrated by the student's work. Given the relationship between strategic competence and mathematical knowledge for teaching described by Copur-Gencturk and Doleck (2021), this statement could be interpreted as an implicit evaluation of the strategy as valid. However, Charlotte did not conduct a full breakdown of the learning goal into subconcepts and overgeneralized the student's conceptual understanding. At the conclusion of the evaluation, Charlotte stated, "overall, I think they understand the concept of division," but she did not elaborate on what other specific understandings of division the work demonstrated. Charlotte did consider the efficiency of the strategy in the solution in terms of ease for the student by stating, "with the 4.5 plus 4.5 equaling 9, it's good that they made it a whole number so they're not messing around with the decimals." Additionally, Charlotte commented that the student's final answer was correct, indicating an evaluation of accuracy (no errors). Finally, Charlotte discussed ways in which the work could have been made clearer, but claims, "I think they showed the work they needed for this problem. I understood it, and it seems like they understood it as a student." Charlotte's response exemplifies a generally productive application of the process in Figure 1.

However, not all PTs I interviewed exhibited this level of alignment with the process. The transcript below shows Jane's (pseudonym) response to the solution in Figure 2.

Jane: We started out with an equation of L times 4.5 equals 36, with 36 being how much of the rope we have, and the 4.5 being how much we want to cut it into, and L being our pieces. So, we know that 4.5 plus 4.5 would be 9, so we would add up 9 until we get relatively [close] or to the exact answer of 36, which they did. And we also know that 9 was the sum of 4.5 plus 4.5, so, because we had to use four 9's in order to get the 36, our answer would then be 8 pieces.

Interviewer: So because we had four 9's and each of those 9's was two 4.5's, that's where they got the 8 from?

Jane: Yes.

Interviewer: So, how would you evaluate the reasoning shown in this solution?

Jane: I think this is a really good method to use. It's quick and it seems pretty simple.

Interviewer: Okay. Any other thoughts or comments about this solution?

Jane: No.

Jane's reconstruction of the student's work had some similarities to Charlotte's in terms of attending to the mathematical details in the solution. However, unlike Charlotte, Jane did not apply metacognition to recognize that she filled in missing steps in the student's work and inferred that the student thought the same way. Also, although Jane described the meaning of each quantity in the original problem, she did not describe any conceptual understandings that the student demonstrated through their solution. Most of Jane's reconstruction is recitation of the steps the student did rather than interpreting the student's thinking, aligning with Level 1 in Fernández et al.'s (2013) framework: "interpretations of students' answers mainly rely on the description of the operations carried out and not on the meanings" (p. 453).

When I asked Jane to evaluate the student's work, she made evaluations of strategic competence ("I think this is a really good method to use") and efficiency/sophistication ("It's quick"). However, she did not provide any rationales for these evaluations, even after prompting. Specifically, she did not draw on any details from the work to justify her evaluations. In line with Bartell et al. (2013) and Spitzer et al. (2011), PTs may need more support to use the process in Figure 1 effectively and consistently (e.g., basing their evaluations on mathematical details).

Implications

Teacher Education

In the context of elementary teacher preparation, opportunities for PTs to evaluate written work (e.g., Fernández et al., 2013) and/or participate in simulated student interviews (e.g., Shaughnessy & Boerst, 2018a) can serve as initial exposures for PTs to consider how their future instruction can center student thinking. Such activities could be implemented in early content courses without a field experience component. The process in Figure 1 can also help MTEs understand how their PTs are engaging in productive (or unproductive) evaluations of student work and thinking. MTEs can then consider teaching PTs about evaluation criteria that are important but have not yet surfaced. For example, MTEs can purposefully select student work that is designed to elicit certain criteria for evaluation (e.g., a solution that does not clearly model the situation, yet reaches a correct numerical answer) as a basis for a classroom discussion.

The primary contribution of the evaluation process in Figure 1 is the elaborated synthesis of the construct of professional noticing of children's mathematical thinking (Jacobs et al., 2010) with Philipp's (2018) framework of teachers' diagnostic processes and the productive evaluation criteria from review of prior research in Table 1. Although professional noticing was one of the theoretical bases for Philipp's (2018) process, the elaborated framework clarifies the relationships between these ideas in a way that is useful to MTEs and teachers. MTEs can share this process with their PTs to guide them towards examining student work in ways that productively center student thinking. Additionally, in-service teachers can compare their own evaluation processes to the one in Figure 1 to help them develop new ways of thinking about student work. The questions listed in the "Evaluate the solution" row of Figure 1 could also serve as prompts during a professional development centered on examining student work.

Future Research

Figure 1 can also be used to characterize how PTs and in-service teachers evaluate students' mathematical work. Such investigations could lead to further refinements of this process. For example, tasks that ask students to execute a specified procedure may not provide any evidence of a student's conceptual understanding (Spitzer et al., 2011), so skipping some of the questions in the "Evaluate the solution" row would be appropriate. Additional arrows that explicate more specific paths through the process based on the nature of the underlying task could then be added to Figure 1. Creating and analyzing flowchart diagrams of PTs' reconstructions and evaluations could better emphasize the relationship between this process and evaluation criteria.

Evaluating students' work in ways that honor and center their thinking is complex. I hypothesize that opportunities to reconstruct and evaluate students' written work alongside instruction on processes for doing so could support PTs developing responsive teaching practices. However, more research is needed to ascertain how effective such instruction would be. Additionally, to tailor such instruction for PTs, it would be important to know the extent to which PTs' individual attributes (e.g., mathematical knowledge or beliefs) influence their selection or application of evaluative processes. Although these questions are the domain of future empirical studies, this work establishes a sound theoretical base for such research.

References

As'ari, A. R., Kurniati, D., & Subanji, S. (2019). Teachers expectation of students' thinking processes in written works: A survey of teachers' readiness in making thinking visible. *Journal on Mathematics Education*, *10*(3), 409–424. https://doi.org/10.22342/jme.10.3.7978.409-424

Association of Mathematics Teacher Educators. (2017). *Standards for preparing teachers of mathematics*. Available online at https://amte.net/standards.

Ayalon, M., & Rubel, L. H. (2022). Selecting and sequencing for a whole-class discussion: Teachers' considerations. *The Journal of Mathematical Behavior*, *66*, 100958. https://doi.org/10.1016/j.jmathb.2022.100958

Bartell, T. G., Webel, C., Bowen, B., & Dyson, N. (2013). Prospective teacher learning: Recognizing evidence of conceptual understanding. *Journal of Mathematics Teacher Education*, *16*(1), 57–79. https://doi.org/10.1007/s10857-012-9205-4

Clements, D. H., & Sarama, J. (2021). Sustainable, scalable professional development in early mathematics: Strategies, evaluation, and tools. In Y. Li, R. E. Howe, W. J. Lewis, & J. J. Madden (Eds.), *Developing mathematical proficiency for elementary instruction* (pp. 221–238). Springer. https://doi.org/10.1007/978-3-030-68956-8_10

Cohen, E. G., & Benton, J. (1988). Making groupwork work. *American Educator*, *12*(3), 10–17.

Copur-Gencturk, Y., & Doleck, T. (2021). Strategic competence for multistep fraction word problems: An overlooked aspect of mathematical knowledge for teaching. *Educational Studies in Mathematics*, *107*(1), 49–70. https://doi.org/10.1007/s10649-021-10028-1

Corven, J. (2021). Prospective elementary teachers' evaluations of student solutions to division story problems. In D. Olanoff, K. Johnson, & S. Spitzer (Eds.), *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1134–1135). Philadelphia, PA.

Dyer, E. B., & Sherin, M. G. (2016). Instructional reasoning about interpretations of student thinking that supports responsive teaching in secondary mathematics. *ZDM – Mathematics Education*, *42*(1), 69–82. https://doi.org/10.1007/s11858-015-0740-1

Fernández, C., Llinares, S., & Valls, J. (2013). Primary school teacher's noticing of students' mathematical thinking in problem solving. *The Mathematics Enthusiast*, *10*(1–2), 19. https://doi.org/10.54870/1551-3440.1274

Heinrichs, H., & Kaiser, G. (2018). Diagnostic competence for dealing with students' errors: Fostering diagnostic competence in error situations. In T. Leuders, K. Philipp, & J. Leuders (Eds.), *Diagnostic competence of mathematics teachers* (pp. 79–94). Springer. https://doi.org/10.1007/978-3-319-66327-2_4

Henningsen, M., & Stein, M. K. (1997). Mathematical tasks and student cognition: Classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education*, *28*(5), 524–549. https://doi.org/10.2307/749690

Hiebert, J., Morris, A. K., & Spitzer, S. M. (2018). Diagnosing learning goals: An often-overlooked teaching competency. In T. Leuders, K. Philipp, & J. Leuders (Eds.), *Diagnostic competence of mathematics teachers* (pp. 193–206). Springer. https://doi.org/10.1007/978-3-319-66327-2_10

Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education*, *41*(2), 169–202. https://doi.org/10.5951/jresematheduc.41.2.0169

Jessup, N. A. (2018). *Understanding teachers' noticing of children's mathematical thinking in written work from different sources* (Publication No. 10838739) [Doctoral dissertation, University of North Carolina at Greensboro]. ProQuest Dissertations & Theses Global.

Lee, M. Y. (2021). Using a technology tool to help pre-service teachers notice students' reasoning and errors on a mathematics problem. *ZDM – Mathematics Education*, *53*(1), 135–149. https://doi.org/10.1007/s11858-020-01189-z
Louie, N., Adiredja, A. P., & Jessup, N. (2021). Teacher noticing from a sociopolitical perspective: The FAIR framework for anti-deficit noticing. *ZDM – Mathematics Education*, *53*(1), 95–107. https://doi.org/10.1007/s11858-021-01229-2
Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Lawrence Erlbaum. https://doi.org/10.4324/9781410602589
Merriam-Webster. (n.d.). Evaluate. In *Merriam-Webster.com dictionary*. Retrieved January 17, 2023, from https://www.merriam-webster.com/dictionary/evaluate.
Michael, M. D. (2005). *Arithmetic and algebra word problems: Preservice teachers' content knowledge, attitudes, and appreciation of students' strategies* (Publication No. 3191240) [Doctoral dissertation, West Virginia University]. ProQuest Dissertations & Theses Global.
Moore, S. D., Morrow-Leong, K., & Gojak, L. M. (2020). *Mathematize it! Going beyond key words to make sense of word problems: Grades 3-5*. Corwin.
Moreno, M., Sánchez-Matamoros, G., Callejo, M. L., Pérez-Tyteca, P., & Llinares, S. (2021). How prospective kindergarten teachers develop their noticing skills: The instrumentation of a learning trajectory. *ZDM – Mathematics Education*, *53*(1), 57–72. https://doi.org/10.1007/s11858-021-01234-5
Morris, A. K. (2007). Factors affecting pre-service teachers' evaluations of the validity of students' mathematical arguments in classroom contexts. *Cognition and Instruction*, *25*(4), 479–522. https://doi.org/10.1080/07370000701632405
Morris, A. K., Hiebert, J., & Spitzer, S. M. (2009). Mathematical knowledge for teaching in planning and evaluating instruction: What can preservice teachers learn? *Journal for Research in Mathematics Education*, *40*(5), 491–529. https://doi.org/10.5951/jresematheduc.40.5.0491
National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all*. Author.
National Research Council [NRC]. (2001). *Adding it up: Helping children learn mathematics*. The National Academies Press. https://doi.org/10.17226/9822
Özdemir, I. E. F., & Pape, S. J. (2012). Supporting students' strategic competence: A case of a sixth-grade mathematics classroom. *Mathematics Education Research Journal*, *24*(2), 153–168. https://doi.org/10.1007/s13394-012-0033-8
Philipp, K. (2018). Diagnostic competences of mathematics teachers with a view to processes and knowledge resources. In T. Leuders, K. Philipp, & J. Leuders (Eds.), *Diagnostic competence of mathematics teachers* (pp. 109–127). Springer. https://doi.org/10.1007/978-3-319-66327-2_6
Robertson, A. D., Atkins, L. J., Levin, D. M., & Richards, J. (2015). What is responsive teaching? In A. D. Robertson, R. Scherr, & D. Hammer (Eds.), *Responsive teaching in science and mathematics* (pp. 1–35). Routledge. https://doi.org/10.4324/9781315689302
Schack, E. O., Fisher, M. H., Thomas, J. N., Eisenhardt, S., Tassell, J., & Yoder, M. (2013). Prospective elementary school teachers' professional noticing of children's early numeracy. *Journal of Mathematics Teacher Education*, *16*(5), 379–397. https://doi.org/10.1007/s10857-013-9240-9
Shaughnessy, M., & Boerst, T. (2018a). Designing simulations to learn about pre-service teachers' capabilities with eliciting and interpreting student thinking. In G. J. Stylianides & K. Hino (Eds.), *Research advances in the mathematical education of pre-service elementary teachers: An international perspective* (pp. 125–140). Springer. https://doi.org/10.1007/978-3-319-68342-3_9
Shaughnessy, M., & Boerst, T. A. (2018b). Uncovering the skills that preservice teachers bring to teacher education: The practice of eliciting a student's thinking. *Journal of Teacher Education*, *69*(1), 40–55. https://doi.org/10.1177/0022487117702574
Son, J.-W. (2013). How preservice teachers interpret and respond to student errors: Ratio and proportion in similar rectangles. *Educational Studies in Mathematics*, *84*(1), 49–70. https://doi.org/10.1007/s10649-013-9475-5
Spitzer, S. M., Phelps, C. M., Beyers, J. E. R., Johnson, D. Y., & Sieminski, E. M. (2011). Developing prospective elementary teachers' abilities to identify evidence of student mathematical achievement. *Journal of Mathematics Teacher Education*, *14*(1), 67–87. https://doi.org/10.1007/s10857-010-9141-0
Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning*, *10*(4), 313–340. https://doi.org/10.1080/10986060802229675

Stockero, S. L., Leatham, K. R., Van Zoest, L. R., & Peterson, B. E. (2017). Noticing distinctions among and within instances of student mathematical thinking. In E. O. Schack, M. H. Fisher, & J. A. Wilhelm (Eds.), *Teacher noticing: Bridging and broadening perspectives, contexts, and frameworks* (pp. 467–480). Springer. https://doi.org/10.1007/978-3-319-46753-5_27

Van Dooren, W., Verschaffel, L., & Onghena, P. (2002). The impact of preservice teachers' content knowledge on their evaluation of students' strategies for solving arithmetic and algebra word problems. *Journal for Research in Mathematics Education*, *33*(5), 319–351. https://doi.org/10.2307/4149957

van Es, E. A., & Sherin, M. G. (2021). Expanding on prior conceptualizations of teacher noticing. *ZDM – Mathematics Education*, *53*(1), 17–27. https://doi.org/10.1007/s11858-020-01211-4

Van Zoest, L. R., Stockero, S. L., Leatham, K. R., Peterson, B. E., & Ruk, J. M. (2020). Articulating the student mathematics in student contributions. In A. I. Sacristán, J. C. Cortés-Zavala, & P. M. Ruiz-Arias (Eds.), *Mathematics education across cultures: Proceedings of the 42nd meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 2105–2109). Mexico. https://doi.org/10.51272/pmena.42.2020-354

STAFF PERCEPTIONS OF THEIR ROLE AT A SUMMER STEM PROGRAM FOR MIDDLE SCHOOL GIRLS

Lynda R. Wiest
University of Nevada, Reno
wiest@unr.edu

Jessica Koltz
Utah State University
jessica.koltz@usu.edu

This interview research was designed to understand the perceptions of instructors and volunteers who worked at a one-week residential math and technology program for middle school girls regarding their experiences working with the program. Instructors and volunteers described their potential professional and/or personal growth, their challenges with the program, and suggested program improvements. The data indicate that personal benefits accrued from involvement in the program, in particular, enhanced knowledge and skills. Instructors and volunteers also reported professional growth in skills and dispositions. Instructor challenges included working with an especially broad ability range and insufficient collaboration with other instructors, and volunteers described uncertainty about their expected role. Some staff also suggested updates to technology used in the program. This study contributes to research on out-of-school-time STEM programs.

Keywords: Informal Education; Middle School Education; Teacher Knowledge; Teacher Beliefs

The purpose of this qualitative study was to gain insight into the perceptions of staff and volunteers regarding potential personally and professionally beneficial experiences and challenges faced while working at a one-week, residential summer math and technology program for middle school girls.

Women continue to be underrepresented in mathematics-based fields (e.g., Makarova et al., 2019). Various performance-related and social, psychological, and emotional factors can influence women's pursuit of and perseverance in mathematics. Unfavorable gender stereotypes in STEM (science, technology, engineering, and mathematics) are a major culprit (Cheryan et al., 2015; Makarova et al., 2019; Zhao et al., 2018). Girls and women thus tend to hold lower self-concepts, expectations for themselves, and career intentions in relation to STEM compared with boys/men (Robnett, 2016; Robnett & Thoman, 2017; Song et al., 2017).

Although societal change is necessary to address gender-STEM stereotypes, classroom teaching can also influence students' relationship with STEM in general and along gendered lines. One promising approach for supporting girls/women in STEM is out-of-school-time (OST) learning. OST programs have been shown to increase participants' STEM knowledge and performance, engagement, and career aspirations (Allen et al., 2019; Author2 et al., 2021; Demetry & Sontegrath, 2020). OST programs tend to use contemporary methods that support desirable outcomes. Whereas traditional mathematics teaching tends to be teacher-oriented with passive students who are expected to acquire prescribed knowledge and procedures, often through memorization, contemporary methods emphasize active student involvement that employs critical and creative problem-solving and sense-making approaches that make use of written and oral communication, interaction with others, hands-on approaches, productive struggle, and real-world applications (Li & Schoenfeld, 2019; Noreen & Rana, 2019; Yuanita et al., 2018). Importantly, OST programs have also been found to improve girls' STEM identities

(Allen et al., 2019; Hughes et al., 2013) and their dispositions toward STEM (Author2 et al., 2021).

A girls math and technology program, which has mainly consisted of a one-week summer residential program from its inception in 1998 to its recent coed version, provides learning opportunities, resources, and participation in a community of support that better prepares participants to enhance the quality of their academic, vocational, and everyday lives and to contribute to advancement of the wider society. The program targets girls at the critical middle school juncture to increase interest and expand STEM options that can serve them well in their personal life and future occupation. While staff are critically important to the success of such a program (Dai et al., 2021), little research has been conducted on the experiences and role of staff in out-of-school-time programs (Beharry et al., 2021; Spears et al., 2021).

Methods

Research Question

What perceptions do staff members and volunteers working at a one-week, residential summer math and technology program for middle school girls report regarding potential personally and professionally beneficial experiences and challenges faced in their role?

Procedures

A qualitative approach was deemed most appropriate to gather this information to identify individual points of view by having participants express their feelings and experiences. The purpose of qualitative research is to support participant voice, adding deeper understanding to a phenomenon of interest (Patton, 2002). Phenomenology can thus serve as an appropriate lens with which to view this research. Different groups can have differing behaviors and ideas that are important to understand to mitigate possible narrow thinking about a phenomenon (McLeod, 2008). The staff we interviewed provided personal perceptions of the program in which they participated that they described in terms of their self-awareness of their own experiences in relation to the program.

Individual interviews were considered to be most appropriate for conducting this research because of the intent to acquire individual responses from participants. Azzara (2010) notes, "When the research objective is to understand individual decision processes or individual responses, individual interviews are optimal" (p. 1). Phone interviews that averaged eight minutes each were conducted with 23 staff members (18 instructors, 5 volunteers), which reflects a response rate of 72% from 32 recruited staff members who worked with the program for two years or more from 1998-2018.

Interview questions were developed by the lead researcher, who served as Director of the program for its entire 20-year existence. These questions related to the interviewees' perspectives regarding their experiences with the program. Interview questions were semi-structured and addressed personal reflections on perceptions of and experiences with the program. These phone interviews were audio recorded by the second author and transcribed for later analysis of themes related to the research questions.

Four questions were posed during the interview conversations; (1) What do you believe you gained professionally and/or personally by working with the [program name]? (2) What were your greatest challenges in working with the [program name]? (3) Can you suggest any improvements to the [program name] that would enhance your role with the program? and (4) Is there anything else that you would like to add?

Data Source and Analysis

The source of data was participant comments made during interviews and documented on interview transcripts. Data analysis began with reading through the transcripts multiple times to identify themes related to experiences with and perceptions of the program. They were continually adjusted until both researchers agreed on the final themes. Example themes were "personal reward," "helping students," "improving skills," "collaborating," "personal growth," "professional growth," and "improving knowledge/skills." Open-ended coding was applied to all participants' responses, where data was broken into parts, examined, and compared by similarities or differences (Strauss & Corbin, 1998). Upon conclusion of reviewing the transcripts, coding was crucial to understanding key themes of across individual responses. The next step was to use axial coding, connecting the responses and forming categories within the interview data to identify overall themes for interpretation.

Results & Implications

Initial review of the data indicate that personal benefits accrued from involvement in the program, in particular, enhanced knowledge and skills. Instructors and volunteers also reported professional growth in skills and dispositions. They noted that their deepened mathematics understanding, insight into working with girls, enhanced their confidence in their role as K-12 teachers. They also reported the important of connecting with other women staff focusing on mathematics education. For example, one volunteer stated, "I'd like to give the girls any leg up that they can get. It was just really personally rewarding to give your time and make an impact on at least one person. And then professionally, it is always just good to be in a community and helping others."

Instructor challenges centered on an especially broad ability range, participant engagement during summer hours, and insufficient collaboration with other instructors, and volunteers described uncertainty about their expected role. For example, one instructor noted that the broad ability level, born of the program drawing from a large geographical region, led to some impatient students blurting out answers, thus compromising her ability to "give everybody that equal opportunity to really process what was going on." In terms of collaboration across instructors, this was an issue because each instructor taught a specific content area, such as geometry or algebra, but instructors did not communicate or collaborate with each other because they were only present to teach their own session. A challenge to volunteers was that they were uncertain about the level of involvement and input they should provide. In terms of open-ended program improvements suggested, the main recommendations were to update some of the technologies used (e.g., *Terrapin Logo* and *Geometer's Sketchpad*) and to better define and monitor the role of volunteers.

The results of this study show that out-of-school-time programs can offer important personal and professional benefits to staff but that challenges they face must be solicited and addressed by program directors. This study can contribute useful insights to the rapidly growing out-of-school-time STEM education movement because little research has been done in this area, especially in relation to underrepresented groups, staff turnover is high in youth programs, and the potential influence on mathematics might exceed that of some other subject areas (Koch et al., 2012; Saw et al., 2019). As noted earlier, staff are critically important to the success of out-of-school-time programs (Dai et al., 2021), but little research has been conducted on the experiences and role of staff in these programs (Beharry et al., 2021; Spears et al., 2021). Therefore, this study can be an important impetus to conduct further and deeper research on the experiences of staff in OST programs because insights staff provide can be integral to creating a more effective OST program that benefits all participants, including staff, volunteers, and students.

References

Allen, P., Chang, R., Gorrall, B. K., Waggenspack, L., Fukuda, E., Little, T., D., & Noam, G. G. (2019). From quality to outcomes: A national student of afterschool STEM programming. International Journal of STEM Education, 6(37). https://doi.org/10.1186/s40594-019-0191-2

Author2 et al. (2021).

Azzara, C. (2010). Qualitatively speaking: The focus group vs. in-depth interview debate. Retrieved from https://www.quirks.com/articles/qualitatively-speaking-the-focus-group-vs-in-depth-interview-debate

Beharry, Z., Johnson, K., & Frost, L. (2021). Girls in engineering, math, and science (GEMS): A successful program for middle school girls and college women in Southwest Florida. In L. R. Wiest, H. G. Crawford-Ferre, & J. E. Sanchez (Eds.), Out-of-school-time STEM programs for females: Implications for education research and practice (Vol. II: Short-Term Programs; pp. 71-96). Charlotte, NC: Information Age Publishing.

Cheryan, S., Master, A., & Meltzoff, A. N. (2015). Cultural stereotypes as gatekeepers: Increasing girls' interest in computer science and engineering by diversifying stereotypes. Frontiers in Psychology, 6(49), 1-8. https://doi.org/10.3389/fpsyg.2015.00049

Dai, K. Z., Kuprasertkul, N., & Liu, D. (2021). FEMMES STEM program for girls. In L. R. Wiest, H. G. Crawford-Ferre, & J. E. Sanchez (Eds.), Out-of-school-time STEM programs for females: Implications for education research and practice (Vol. II: Short-Term Programs; pp. 7-29). Charlotte, NC: Information Age Publishing.

Demetry, C., & Sontegrath, S. (2020). Longitudinal evaluation of a university-based outreach program for middle school girls yields evidence of positive engineering recruitment outcomes. Journal of Women and Minorities in Science and Engineering, 26(1), 31-60. https://doi.org/10.1615/JWomenMinorScienEng.2020028747

Hughes, R. M., Nzekwe, B., & Molyneaux, K. J. (2013). The single sex debate for girls in science: A comparison between two informal science programs on middle school students' stem identity formation. Research in Science Education, 43, 1979-2007. https://doi.org/10.1007/s11165-012-9345-7

Koch, M., Gorges, T., & Penuel, W. R. (2012). Build IT: Scaling and sustaining an afterschool computer science program for girls. Afterschool Matters, 16, 58-66.

Li, Y., & Schoenfeld, A. H. (2019). Problematizing teaching and learning mathematics as "given" in STEM education. International Journal of STEM Education, 6(44). https://doi.org/10.1186/s40594-019-0197-9

Makarova, E., Aeschlimann, B., & Herzog, W. (2019). The gender gap in STEM fields: The impact of the gender stereotype of math and science on secondary aspirations. Frontiers in Education, 4(60), 1-11. https://doi.org/10.3389/feduc.2019.00060

McLeod, S. (2017). Maslow's hierarchy of needs. Simply Psychology. Retrieved from https://www.simplypsychology.org/maslow.html

Noreen, R., & Rana, A. M. K. (2019). Activity-based teaching versus traditional method of teaching in mathematics at elementary level. Bulletin of Education and Research, 41(2), 145-159.

Patton, M. Q. (2002). Qualitative research and evaluation methods (3rd ed.). Thousand Oaks, CA: SAGE.

Robnett, R. D. (2016). Gender bias in STEM fields: Variation in prevalence and links to STEM self-concept. Psychology of Women Quarterly, 40(1), 65-79.

Robnett, R. D., & Thoman, S. E. (2017). STEM success expectancies and achievement among women in STEM majors. Journal of Applied Developmental Psychology, 52, 91-100.

Saw, G. K., Swagerty, B., Brewington, S., Chang, C.-N., & Culbertson, R. (2019). Out-of-school time STEM program: Students' attitudes toward and career interests in mathematics and science. International Journal of Evaluation and Research in Education, 8(2), 356-362.

Song, J., Zuo, B., Wen, F., & Yan, L. (2017). Math-gender stereotypes and career intentions: An application of expectancy-value theory. British Journal of Guidance and Counselling, 45(3), 328-340. http://dx.doi.org/10.1080/03069885.2017.1283680

Spears, J. D., Baird, C. L., & Montelone, B. A. (2021). In L. R. Wiest, H. G. Crawford-Ferre, & J. E. Sanchez (Eds.), Out-of-school-time STEM programs for females: Implications for education research and practice (Vol. II: Short-Term Programs; pp. 157-176). Charlotte, NC: Information Age Publishing.

Strauss, A., & Corbin, J. (1998). Basics of Qualitative Research: Techniques and Procedures for Developing Grounded Theory. Thousand Oaks, CA: SAGE.

Yuanita, P., Zulnaidi, H., & Zakaria, E. (2018). The effectiveness of realistic mathematics education approach: The role of mathematical representation as mediator between mathematical belief and problem solving. PLoS ONE, 13(9). https://doi.org/10.1371/journal.pone.0204847
Zhao, F., Zhang, Y., Alterman, V., Zhang, B., & Yu, G. (2018). Can math-gender stereotypes be reduced? A theory-based intervention program with adolescent girls. Current Psychology, 37(3), 612-624.

STUDENT ACCESS TO MATHEMATICAL LEARNING OPPORTUNITIES IN CO-TAUGHT ELEMENTARY MATHEMATICS

Laurel Dias
Utah Valley University
Laurel.dias@uvu.edu

Access is a critical component to consider in math learning because individual students' access to cognitively demanding mathematical tasks is often limited (Hiebert & Grouw, 2007; Lynch et al., 2018; Yeh et al., 2020). The success of all learners in mathematics requires that teachers work together to provide instruction, support, and interventions that enable every student to become an expert learner and doer of mathematics (Hiebert & Grouw, 2007; Lambert, 2020, 2021). Co-teaching is an instructional approach that draws on the expertise of two teachers as they deliver joint instruction to meet the needs of every learner in the classroom. Five commonly used co-teaching models are Parallel Teaching, Station Teaching, Teaming, Alternative Teaching, and One Teach, One Assist (Friend, 2008; Solis et al., 2012). Each model has variations in teacher instructional delivery, the size of student groups, and the recommended frequency of use. This research focuses on the question: how do various co-teaching models afford students access to mathematical learning opportunities? I conceptualize student accessibility through the Universal Design for Learning Math (UDL Math) framework (Lambert, 2021). The six design elements of the UDL Math framework delineate criteria for how math instruction should be designed to develop students into mathematical learners (Lambert, 2021).

In this exploratory case study, I conducted ethnographic classroom observations of co-taught mathematics instruction in a second-grade classroom of twenty-two students. The teacher pair who co-taught the class were selected through purposeful sampling: the teachers exhibited asset-based views of students on pilot survey data and revealed instructional strategies aligned with current educational research (Patton, 2002). My data sources are video-recorded lesson observations with accompanying field notes, instructional artifacts, teacher interviews, and reflective surveys. For my analysis, I first determined the co-teaching model used in each lesson through structural coding, a method for categorizing a data corpus (Saldaña, 2015). Then, I coded each co-taught lesson for accessibility using a coding scheme developed from the UDL Math framework (Lambert, 2021) with 36% of lessons coded for interrater reliability and achieving a moderate level of agreement (Cohen’s Kappa = .6) between coders.

My analysis reveals how the five different co-teaching models revealed evidence of different elements of the UDL Math framework when implemented with a diverse group of elementary students. When the Teaming model was observed, lessons revealed strong evidence of student collaboration and contextual connections, and all Teaming lessons consistently included problem-solving opportunities accessible to a wide range of students. The Alternative model showed discrepancies between student groups in the same lessons, with enrichment groups receiving more accessible instruction than intervention groups. This discrepancy was most evidently through the number of student choices offered and opportunities for student collaboration. One strength evident in the One Teach, One Assist model was a supportive classroom environment. Findings related to helpful feedback and accessibility of problem-solving opportunities in this model were impacted by which teacher was assisting and which was leading instruction. The Parallel model was the only model observed that had no instances of students choosing their solution paths, or agency to choose which representations they used.

Finally, Station model learning consistently included multiple representations but students lacked opportunities to make connections between the representations at each station.

References

Friend, M. (2008). Co-teaching: A simple solution that isn't simple after all. *Journal of Curriculum and Instruction, 2*(2), 9-19.

Hiebert, J., & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. *Second handbook of research on mathematics teaching and learning, 1*(1), 371-404.

Lambert, R. (2020). Increasing Access to Universally Designed Mathematics Classrooms. *Policy Analysis for California Education, PACE*.

Lambert, R. (2021). The Magic Is in the Margins: UDL Math. *Mathematics Teacher: Learning and Teaching PK-12, 114*(9), 660-669.

Lynch, S. D., Hunt, J. H., & Lewis, K. E. (2018). Productive struggle for all: Differentiated instruction. *Mathematics Teaching in the Middle School, 23*(4), 194-201.

Patton, M. Q. (2002). *Qualitative research and evaluation methods* (3rd ed.). Sage.

Saldaña, J. (2015). *The coding manual for qualitative researchers*. Sage.

Solis, M., Vaughn, S., Swanson, E., & McCulley, L. (2012). Collaborative models of instruction: The empirical foundations of inclusion and co-teaching. *Psychology in the Schools, 49*(5), 498-510.

Yeh, C., Sugita, T., & Tan, P. (2020). Reimagining Inclusive Spaces for Mathematics Learning. *Mathematics Teacher: Learning and Teaching PK-12, 113*(9), 708-714.

SUCCESSFUL IMPELEMENTATION OF EXPLICIT ATTENTION TO CONCEPTS (EAC) IN MIDDLE SCHOOL CLASSROOMS

Emily Leckie
Boise State University
emilyleckie@boisestate.edu

Angela Crawford
Boise State University
angelacrawford1@boisestate.edu

This study addresses the need to better describe instructional strategies used by middle grades mathematics teachers. After coding 177 videos of grades 6-8 mathematics instruction for indications of effective instructional practices, we further analyzed 8 of the highly scored videos with specific attention to teachers' implementation of strategies associated with Explicit Attention to Concepts (EAC) (Champion et al., 2020). We found that these effective teachers of mathematics tended to enact EAC by using a preferred strategy more predominantly than others, though all teachers used at least two EAC strategies during a lesson. Additionally, most participants in our study used an Initiate, Response, Evaluate (IRE) format (Mehan, 1979) when enacting EAC. We illustrate examples of their instruction with vignettes.

Keywords: Mathematical Representations, Instructional Activities and Practices, Middle School Education, Classroom Discourse

The implementation of instructional practices that help improve student achievement in mathematics has proven to be nuanced and difficult (Hill et al., 2005; Stein, et al., 2008). In a synthesis of studies, Hiebert and Grouws (2007) identified Explicit Attention to Concepts (EAC) and Students Opportunities to Struggle (SOS) as instructional approaches which were effective in increasing student mathematics achievement. Stein et al. (2017) found that students whose teachers' classroom instruction exhibited greater use of EAC and SOS had higher student math achievement gains than their peers on assessments that measured conceptual understanding and on assessments that measured procedural skills efficiency. These findings have been confirmed since 2007 (Flores et al., 2015; Fyfe et al., 2014; Kazemi & Stipek, 2009; Ng & Lee, 2009; Paliwal & Baroody, 2020; Stein, et al., 2017; Wilson et al., 2019). EAC and SOS are very general clusters of instructional strategies, though, and we need nuanced analysis of how teachers in particular contexts implement more specific instructional strategies to better design learning environments focused on student learning.

Theoretical Framework

Hiebert and Grouws (2007) focused on the importance of the presence of both EAC and SOS in mathematics classrooms for students to learn both procedural and conceptual mathematics. Considerable attention has been given as of late to SOS (Esmonde & Langer-Osuna, 2013; Jackson et al., 2013; Warshauer, 2015) and less attention has been given to EAC. For this study, we will focus on EAC. Hiebert and Grouws (2007) described what they mean by EAC with this statement: "By attending to concepts we mean treating mathematical connections in an explicit and public way" (p. 383). Hiebert and Grouws (2007) cited multiple studies spanning 50 years that indicated procedural and conceptual understanding of procedures is learned and retained better by students who learned from teachers who focused on both mathematical concepts and made connections between the procedure and conceptual understanding behind the procedure. Examples of EAC include attending to mathematical concepts, making connections among multiple representations, making connections across solution strategies, and reminding students of the main concept of the lesson and how it fits into a sequence of lessons.

Given the challenge of implementing instructional practices that help improve student achievement in mathematics (Hill et al., 2005), it is important to capture the scope and subtleties of effective mathematics instruction in order to better design learning environments that take student learning into account. Champion et al. (2020) created an EAC and SOS practice guide (see Figure 1; Champion et al., 2020) to add clarity and make EAC more actionable for teachers (while the practice guide addresses both SOS and EAC, our analysis focuses on EAC). The framework describes three overarching features derived from Hiebert and Grouws (2007) definition. Specifically**,** EAC is present in instruction that features: (a) a focus on mathematical concepts, (b) concepts that are made explicit and public, and (c) connections being emphasized between concepts and representations of ideas.

To make the EAC construct more actionable, the framework provides practical strategies that teachers could use to actualize the characteristics of EAC in their classrooms, each with two examples. Again, these strategies are identified from Hiebert and Grouws (2007) description of EAC. The Strategies for EAC are: (1) Specifically connecting to more than one representation of an idea, (2) Noting ways that different solution strategies are similar or different, (3) Discussing the mathematical reasoning that underlies a procedure, and (4) Pointing to a main idea in a lesson and how it fits into a bigger picture. Examples of each strategy are labeled A and B under each strategy (see Figure 1; Champion et al., 2020).

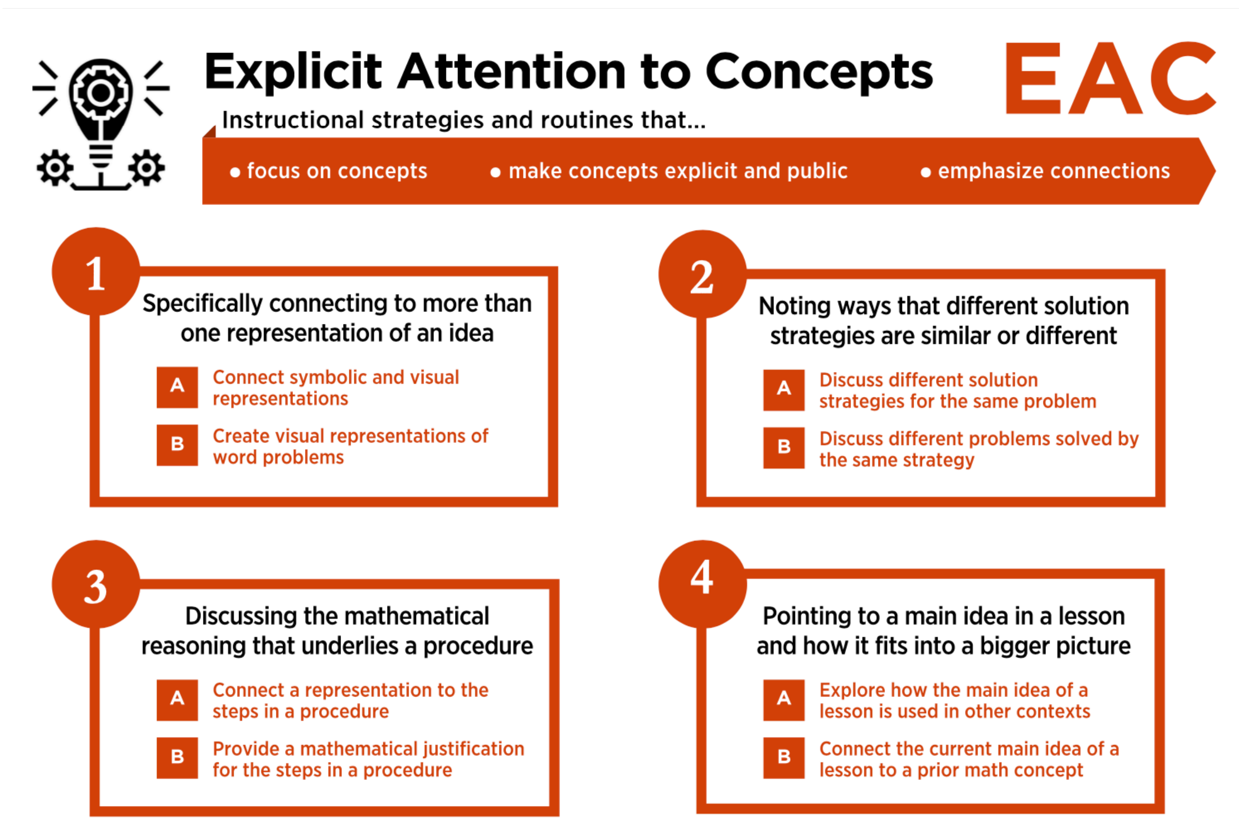


Figure 1: EAC/SOS Guide to Instructional Practices for Improving Math Achievement

The EAC Guide is a research-to-practice document, intended to provide actionable information for teachers. However, there is an ongoing need to better understand what teachers who do EAC well actually do, such as to what extent they use the strategies as described in the framework, and if there are things teachers do which are not included in the framework. Therefore, this study seeks to answer the following research questions: (1) What strategies do teachers use when effectively enacting EAC in whole class discussions following individual or group work time? (2) How do teachers enact those strategies?

Methods

We used essential qualitative analysis (Lahman, 2022) to describe how teachers successfully implement EAC in middle school classrooms.

Participants and Data Sources

This study is set within a larger three-year study of instructional sequences of EAC and SOS involving 100 middle grades mathematics teachers. The eight teacher participants in this study are grade 6-8 mathematics teachers from seven schools within four school districts in Idaho. Of the four school districts, two are at 27% low-income families, one is at 28% low-income families, and one is at 48% low-income families. Of the seven schools, one is rural and the other six are non-rural. One school is a K-12 charter school, one is a K-6 school, three are middle schools (6-8), and two are junior high schools (7-9).

The 8 videos of classroom instruction selected for this study are a purposeful sample of exemplary implementation of EAC. Teachers participating in the research project were asked to submit three videos of their instruction. A total of 177 videos were of high enough visual and auditory quality to be analyzed for levels of implementation of EAC. The videos were scored by ROOT project researchers. Videos were parsed into five-minute long segments. Segments were evaluated with a rubric scale comprising three levels: implementing, partially implementing, and not implementing (Crawford et al., 2021). The scale was applied to each of the features of EAC, which resulted in an overall EAC score for that segment. Since we do not expect EAC to be prominent throughout a lesson, we then calculated a mean for each video's five highest scoring EAC segments (representing 25 minutes of instruction). We selected the first 8 distinct teachers when ordering the means from highest to lowest.

Data Analysis

At the first level of analysis, the first author watched all eight videos and selected all segments where the teacher brought the whole class back together following student work time on a math task because we anticipated seeing EAC most prevalent during these sections of the lesson (Stein et al., 2008; Stein et al., 2017). Then these selected segments were parsed into talk turns which were captured and timestamped. Then each talk turn was coded for who was talking: teacher, focus student (sharing their own work/reasoning), student in class (answering a question posed by the teacher), student table conversation, and whole class response).

Three project researchers participated in the second round of coding. We used the four strategies in the EAC Framework as a priori codes (Saldaña, 2021) to coded talk turns where EAC was present. The first author developed a codebook with the four strategies, the code name, examples, and coding decision rules. The a priori codes were: *Strategy Zero:* Clarifying the mathematical meaning within one representation, *Strategy 1:* Specifically connecting to more than one representation of an idea, *Strategy 2:* Noting ways that different solution strategies are similar or different, *Strategy 3:* Discussing the mathematical reasoning that underlies a procedure, and *Strategy 4:* Pointing to a main idea in a lesson and how it fits into a bigger picture. To calibrate coding, the three researchers watched and coded 1 video together using the codebook. During the coding of the first video, a new strategy emerged and we added it to the codebook. One important decision rule we added was: Student responses are only coded when they are sharing their own ideas, not just giving the answer the teachers is looking for.

Then each of the three researchers watched and coded a second video separately. Inter-rater agreement was determined by identifying the proportion of talk turns assigned to each strategy. We agreed that Strategy 1 was the predominant strategy. Rater 1 applied this strategy in 87% of coded talk turns, rater 2 applied this strategy in 85% of coded talk turns, and rater 3 applied this strategy in 78% of coded talk turns. We discussed and reconciled disagreements until 100% agreement was reached. The code book was refined after this calibration by adding new decision rules.

After this calibration meeting, the remaining six videos were randomly assigned to each of the researchers. When a researcher felt uncertain about coding any talk turns, the team watched the segments and discussed until consensus was reached. This occurred three times.

The results were synthesized across the eight participant videos. We merged the observations together into a single data set. This allowed for cross-tabulation of strategy use across teachers, allowing us to answer the question of what strategies were used; as well as percentage distributions of each teachers' relative use of the five EAC strategies (four a priori and one emergent), to answer the question of how they were used. To further answer the question of how teachers implement the strategies, we looked for patterns of how teacher and student talk turns were coded.

Findings

This study analyzed videos with high levels of the features of EAC. To answer the question of what strategies exemplary teachers use when enacting EAC in whole class discussions following individual or group work time, we present strategies used by the eight participants. To answer the question of how teachers enact those strategies we present the predominance of one strategy used by teachers more than others strategies within a lesson and vignettes as examples of how EAC was effectively implemented.

Strategies exemplary teachers use when enacting EAC

Emergent Strategy. We found that teachers utilized five strategies when enacting EAC. Our initial coding scheme and the framework only included four strategies. However, when the team met to code the first video, we identified an emergent strategy. In analyzing strategy use, we found segments in which teachers were focusing on concepts and representations but were not fully aligned to the strategies as described in the framework and codebook. We recognized these segments as examples of an emergent strategy. We named and defined this strategy as *Strategy Zero:* Clarifying the mathematical meaning within one representation. The teacher was clarifying key mathematical concepts and making the concepts explicit and public, but doing so within one representation, instead of making connections between two representations, as expected based on the EAC framework. We chose to call this Strategy Zero instead of strategy five, because we inferred Strategy Zero had a foundational aspect of clarifying within one representation before making mathematical meaning across more than one representation or solution strategy.

We added Strategy Zero to the codebook. As we coded the seven additional videos, we saw Strategy Zero present in other participant videos as well. In total, six of the eight participant videos included Strategy Zero. In two of these 6 videos, it was not only present, but it was the prominent strategy used during the lesson. The following vignette comes from the video where we first discovered Strategy Zero.

Vignette 1: Example of Strategy Zero. In this portion of the lesson, the teacher is using the example of stacking disposable Styrofoam cups with wide rims to help students learn about non-proportional relationships. She has physical cups in the classroom and a picture of stacked cups projected on the board.

> Penelope: When we think about doubling the height of the cup, what do we lose when we drop those cups into the stack? We lose the base of the cup, right? That part that we would hold on to and we are only left with the lip. So, some people still tried to double 15 [teacher gestures stacking cups with her hands] and they got 30, and then they did another stack of 15 and they got 45. They were getting closer to 50. But all they were doing was taking their stack of 15 and trying to put it like this [teacher picks up actual cups and demonstrates stacking the cups one on top of another without nesting them inside one another], right? Because there's 15, and then 15 again, and then another 15. But when we stack the cups, I spent a lot of time at the back row, because we started to realize that we needed to remove something. So, first we looked at taking our whole 50 cm [teacher gestures, indicating the height of a large stack with her hands] and then dividing it. And we divided it into these pieces that were 1.41 cm [teacher points to something on the board then indicates 1.41 cm with her thumb and index finger]. Divide, divide, divide.

Strategy Use Across Participants. We identifying the proportion of talk turns assigned by researcher to each strategy to answer what strategies teachers used. We found most whole class discussions focused on one predominant strategy during their video lesson. The most used strategy across all eight participants was Strategy 1: Specifically connecting to more than one representation of an idea. Four of the eight participants used Strategy 1 as their predominant strategy. Interestingly, the second most predominant strategy was our emergent strategy, Strategy Zero: Clarifying the mathematical meaning within one representation. Two participants used Strategy Zero as their predominant strategy. Strategy 2: Noting ways that different solution strategies are similar or different, was the predominant strategy for one participant and Strategy 3: Discussing the mathematical reasoning that underlies a procedure, was the predominant strategy for one participant. It is noteworthy that Strategy 4: Pointing to a main idea in a lesson and how it fits into a bigger picture, was the only strategy that was not a predominant strategy in any of the eight participants' video lessons.

How exemplary teachers enact EAC strategies

Strategy Use Within a Lesson. Since we found that most whole class discussions focused on one predominant strategy within the lesson video, we calculated the percent of talk turns that were coded for each strategy. Figure 2 shows strategy usage as a percentage of talk turns by teacher. Strategies are presented in numerical order from left to right and teachers are presented in alphabetical order of their pseudo names. Predominate strategies ranged from 46% to 97% of the total strategies present with seven of the eight participants using the predominant strategy between 70% to 97% of the time. Six out of the eight participants used their predominant strategy much more than any other strategy. Specifically, six of the participants had 63% or more difference between their predominant strategy and their secondary strategy.

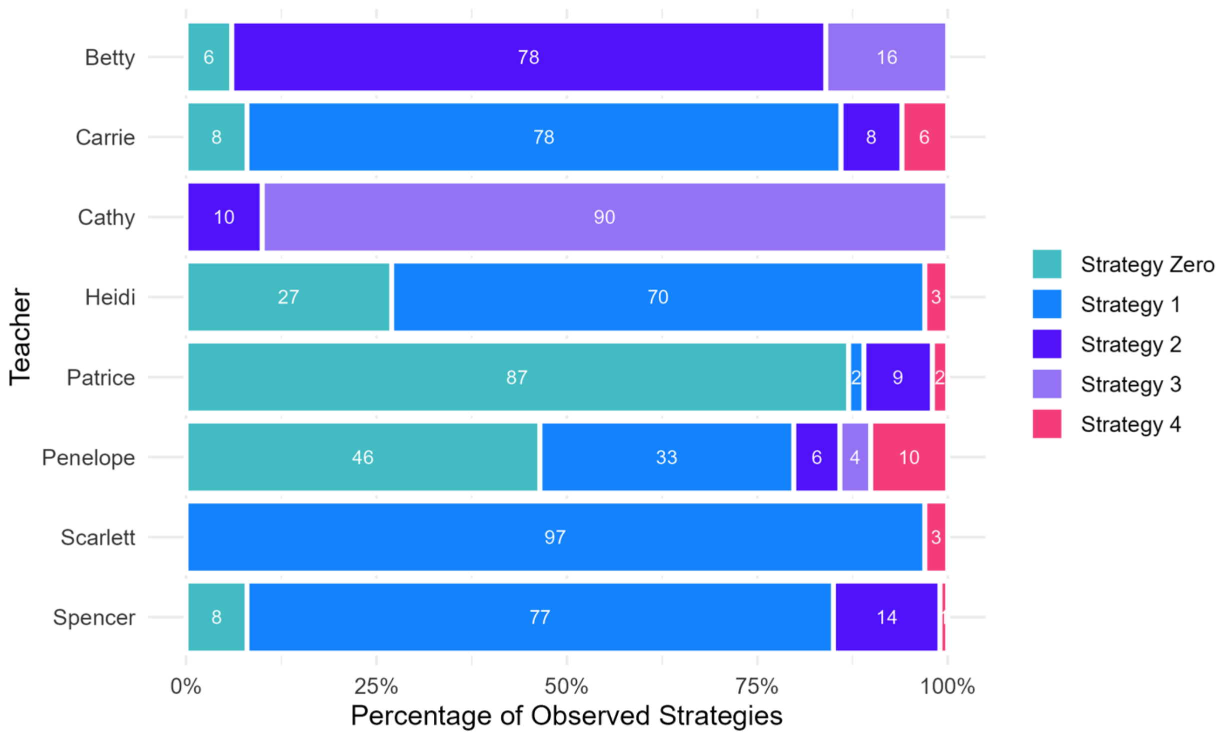


Figure 2: Percentage of Each Strategy by Teacher

We compared the number of strategies used by each participant. All participants used more than one strategy. One teacher used all five strategies, three teachers used four strategies, two teachers used three strategies, and two teachers used two strategies.

It is noteworthy that the pattern of use for Strategy 4 was distinct from that of other strategies. None of the eight videos included Strategy 4 as predominant strategy. Though, strategy 4 was identified in the majority of videos (six of eight), in every case this code was applied in no more than 10% of the talk turns. On average, in all eight videos, Strategy 4 was used 3% of the time.

Lesson Structure with EAC Present. When looking for patterns, we found a pattern in teacher – student interactions. We noticed that most videos had numerous uncoded student talk turns. We revisited the videos and identified these interactions as the Initiate-Response-Evaluation (IRE) structure where the teacher initiates a question with a right or wrong answer, a student responds, and the teacher evaluates their response (Mehan, 1979; Cazden, 1988).

Vignette 2: IRE Example of EAC. This vignette is from the video where we noticed the IRE structure showcases one example of an IRE exchange between the teacher and students. In this vignette, the teacher is asking students for the area of regions within a shape projected on the board. The teacher seems to be focused on getting answers using the area model to show the justification for the algorithm of multiplying decimals. All student names are pseudo names.

Cathy: Todd, what’s the width of region C?
Student: Zero point four.
Cathy: What’s the length?
Student: One.
Cathy: There you go.

We analyzed the coding data from the other 7 videos and found that seven of the eight participants used an IRE format while enacting EAC in their classroom video lessons. Spencer was the only participant who did not utilize an IRE structure in their video lesson. As we looked for patterns, we noted that though the teacher's pattern of strategy use was similar, the talk turn structure was different. Specifically, the types of questions the teacher asked, the expectation of an explanation in student responses, and duration of student responses. Vignette 2 illustrates the pattern of student questioning we found in Spencer's video. All student names are pseudo names.

Vignette 3: Non-IRE Example of EAC. After launching a task with their students, Spencer has given them some time to work in groups on vertical whiteboards. While working only one student has a marker and only the other two students in the group can speak. In this vignette, students are back at their desks and Spencer is pulling them together for a whole class conversation about the work they have done so far. Spencer is trying to get students to use a double number line to show their proportional reasoning.

Spencer: So, this is what I'm seeing. I saw a number line that looked like this [teacher is writing on the board]. Give me a thumbs up when you've thought about it...K, my first question is, how would I label this? How would I label this and why? Ready? Thank you.

[Students begin talking with their groups].

Spencer: Camille, go for it. What were you two talking about?

Camille: Umm, well we were talking about it would be like zero to ten miles.

Spencer: So, it would be important to put down miles here. How come?

Camille: Because it is how many, like it's ten miles that he ran. So, it's important to write it.

Spencer: Thank you. Bradly?

Bradly: You could maybe do like, one, two, three, four, five, six, seven, eight, and nine.

Spencer: So, there's some miles that we can put in here, that you would agree with. Are you okay with labeling it miles?

Bradly: Yeah.

Spencer: Mary, how come?

Mary: So, I'm…the miles is like one part, and then the…so, the miles is like the 10 miles and then…cause you labeled it 10. And then the next number line is probably gonna be…units is the four hours.

This vignette is structured differently than an IRE format not only because student talk turns are longer in duration. More importantly, the teacher asks students to justify their answer and to justify another student's answer, but only after giving all students a chance to discuss ideas and justifications with their peers.

Discussion

Current research gives insight on what supports student learning (e.g., focus on conceptual understanding), however, we know less about the intricate details of how effectively implemented mathematics instruction is done. In order to better design learning environments that take student learning into account, we describe both what strategies teachers use when implementing EAC and how they implement these strategies.

Strategies exemplary teachers use when enacting EAC

All four strategies described in the EAC framework were seen in these videos. In addition, a fifth strategy was present. We identified and defined Strategy Zero as: Clarifying the mathematical meaning within one representation. We identified this as an EAC strategy because of the focus on concepts, making these concepts explicit and public, and the emphasis on connections. The teacher was connecting an abstract mathematical concept to a single visual representation of the concept. This is different from Strategy 1 because Strategy 1 is about connections between more than one representation of a mathematical concept. It is important to note that teachers were taking the time to allow students to make sense of mathematical concepts within one representation. We believe teachers were doing this either because they recognized they needed to take the time to make sense of a mathematical idea in and of itself or to address student misconceptions as they came up. We find this noteworthy because this is implicit, not explicit, in Hiebert and Grouws (2007). The synthesis explicitly focuses on connecting multiple representation. Thus, it was not included in the Champion et al. (2020) framework.

The idea of this kind of conceptual work for students is consistent with the theory of concreteness fading which suggests meaning needs to be developed within one representation before moving onto others (Fyfe & Nathan, 2019). We intentionally names Strategy Zero because we believe Strategy Zero may be a foundational strategy in that students would struggle to make connections between representations (Strategy 1) or between strategies (Strategy 2) without first understanding the mathematical meaning within one representation (Strategy Zero). This recognition made us wonder if there was a hierarchy among the strategies in the EAC framework corresponding to a feature theory of concreteness fading which hypothesizes that transitions to symbolic procedural mathematics happen gradually. We are wondering if there is a progression from concrete to the abstract. When teachers feel the need to utilize Strategy Zero, should it always precede Strategy 1? Likewise, should Strategy 1 always precede Strategy 2 and so on? In order to add additional detail and clarity about these strategies for teachers in their classrooms, this possible hierarchy is worth exploring further.

How exemplary teachers enact EAC strategies

We found that teachers choose one predominant strategy to implement within a lesson, always with other strategies present, but, often with the other strategies much less present. This suggests that teachers find it beneficial to spend time within one strategy, to do it well, and not plan multiple strategies within a short period of time.

We found that teachers who used Strategy 4, only did so briefly. This strategy was used by six of the eight participants, but was never the predominant strategy. This suggests that teachers find Strategy 4 important enough to utilize, but that little time needs to be spent on Strategy 4 in order for Strategy 4 to be effective. Hiebert & Grouws (2007) point to the importance of reminding students of the main idea of the lesson and how it fits into the bigger picture and we found most of our teachers enacted this for at least a portion of their lesson.

We were surprised to find that seven of the eight participants used an IRE format while implementing EAC. Because these videos were selected as exemplar of EAC, we expected to see discussions that aligned more with the 5 Practices (Smith & Stein, 2011) which involve more student led discussions. However, this confirms previous claims that IRE is the default structure for classroom discourse (Cazden, 1988). We wonder if this is due to teacher training in IRE for making ideas explicit and public for students, and if this is the strategy teachers are familiar with and therefore use. The non-IRE strategy is of great interest to us. In this example of EAC without an IRE format, students may have had more voice and authority in the classroom (Cazden, 1988). It is also noteworthy that the teacher was able to implement EAC within the same timeframe as other teachers who were using an IRE format. Additional examples of EAC without an IRE format is worth exploring further.

References

Champion, J., Crawford, A. R., & Carney, M. (2020). Articulating effective middle grades instructional practices in a teacher-researcher alliance. In A. I. Sacristán, J. C. Cortés-Zavala, & P. M. Ruiz-Arias (Eds.), Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020-292

Crawford, A. R., Champion, J., & Carney, M. (2021, April). Operationalizing Constructs within Validity Arguments for K-12 Teacher Observation Measures. Paper presentation in symposium, Validity Arguments for K-12 Teacher Observational Measures: Issues and Future Directions, at the annual meeting of the American Education Research Association, virtual meeting.

Cazden, C. (1988). Classroom discourse: The language of teaching and learning. Heinemann.

Esmonde, I. & Langer-Osuna, J. M. (2013). Power in numbers: Student participation in mathematical discussions in heterogeneous spaces. Journal for Research in Mathematics Education (44), 1, 288-315. doi: 10.5951/jresematheduc.44.1.0288

Flores, R., Koontz, E., Inan, F. A., & Alagic, M. (2015). Multiple representation instruction first versus traditional algorithmic instruction first: Impact in middle school mathematics classrooms. Educational Studies in Mathematics, 89(2), 267-281.

Fyfe, E. R., & Nathan, M. J. (2019). Making “concreteness fading” more concrete as a theory of instruction for promoting transfer. Educational Review, 71(4), 403-422.

Fyfe, E. R., McNeil, N. M., Son, J. Y., & Goldstone, R. L. (2014). Concreteness fading in mathematics and science instruction: A systematic review. Educational psychology review, 26, 9-25.

Hiebert, J., & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students’ learning. Second handbook of research on mathematics teaching and learning, 1(1), 371-404.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers’ mathematical knowledge for teaching on student achievement. American educational research journal, 42(2), 371-406.

Jackson, K., Garrison, A., Wilson, J., Gibbons, L., & Shahan, A. (2013). Exploring relationships between setting up complex tasks and opportunities in concluding whole-class discussions in middle-grades mathematics instruction. Journal for Research in Mathematics Education, 44(4), 646-682.

Kazemi, E., & Stipek, D. (2009). Promoting Conceptual Thinking in Four Upper-Elementary Mathematics Classrooms. Journal of Education, 189(1–2), 123–137. https://doi.org/10.1177/0022057409189001-209

Lahman, M. (2022). Writing and Representing Qualitative Research. Sage.

Mehan, H. (1979). “What time is it, Denise?”: Asking known information questions in classroom discourse. Theory into Practice, 18(4), 285-294.

Ng, S. F., & Lee, K. (2009). The model method: Singapore children's tool for representing and solving algebraic word problems. Journal for Research in Mathematics Education, 40(3), 282-313.

Paliwal, V., & Baroody, A., (2020). Fostering the learning of subtraction concepts and the subtraction-as-addition reasoning strategy. Early Childhood Research Quarterly, 51, 403-415. https://doi.org/10.1016/j.ecresq.2019.05.008

Saldaña, J. (2021). The coding manual for qualitative researchers. SAGE.

Smith, M. S., & Stein, M. K. (2011). 5 Practices for orchestrating productive mathematical discussions. The National Council of Teachers of Mathematics, Inc.

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. Mathematical Thinking and Learning, 10(4), 313-340.

Stein, M. K., Correnti, R., Moore, D., Russell, J. L., & Kelly, K. (2017). Using theory and measurement to sharpen conceptualizations of mathematics teaching in the common core era. AERA Open, 3(1), 2332858416680566.

Warshauer, H. K. (2015). Productive struggle in middle school mathematics classrooms. Journal of Mathematics Teacher Education, 18(4), 375-400.

Wilson, J., Nazemi, M., Jackson, K., & Wilhelm, A. G. (2019). Investigating Teaching in Conceptually Oriented Mathematics Classrooms Characterized by African American Student Success. Journal for Research in Mathematics Education, 50(4), 362-400. https://doi.org/10.5951/jresematheduc.50.4.0362

TEACHER PRACTICES FOR CULTURALLY RESPONSIVE MATH MODELING IN GRADES K-2

Erin Turner
University of Arizona
eturner@arizona.edu

Mary Alice Carlson
Montana State University
mary.carlson5@montana.edu

Julia Aguirre
University of Washington
jaguirre@u.washington.edu

Jennifer Suh
George Mason University
jsuh4@gmu.edu

Jonathan Brown
University of Arizona
jebrown@arizona.edu

Mary Greene
University of Montana
marygreene1@montana.edu

Elzena McVicar
University of Washington
elzenam@u.washington.edu

There is growing recognition that mathematical modeling can be a lever for equity in elementary mathematics classrooms. This study focuses on the impact of a professional development program focused on culturally responsive mathematical modeling on 8 kindergarten through 2nd grade teachers' practices in modeling lessons. We use a project developed observation tool to evaluate two video recorded modeling lessons from each teacher (16 total). Findings focus on patterns in the strengths and challenges in primary grade teachers' practices for teaching modeling, including how teachers' practices align with culturally responsive teaching. We discuss implications of our findings for the design and refinement of professional development.

Keywords: Instructional Practice; Culturally Relevant Pedagogy; Modeling; Elementary School

Introduction

Mathematical modeling (MM) is an iterative process involving problem posing, testing, validation, and revision of mathematical models to inform decisions (Lesh & Zawojewski, 2007; Pollak, 2012). There is growing recognition that mathematical modeling can be a lever for equity in elementary mathematics classrooms. Modeling encourages diverse student contributions and gives teachers opportunities to "recognize and reward a broader range of mathematical abilities than those traditionally emphasized" (Lesh & Doerr, 2003, p. 23). When modeling tasks are grounded in culturally responsive contexts, teachers are empowered to build on the knowledge and cultural resources that students bring to the classroom and students are empowered to draw on their identities and experiences to inform mathematical work and take action (Aguirre et al., 2019; Suh et al., 2018; Turner et al., 2017). By building multiple connections to self, family, community, other subjects, and the world, mathematical modeling tasks humanize mathematics teaching and learning (Anhalt et al., 2018; Gutierrez, 2018; Suh et al., 2018).

While professional development initiatives have begun to support teachers' learning of culturally responsive mathematics modeling (Turner et al., 2022a), research on how teachers learn to enact practices for teaching modeling is limited, particularly in primary grades (kindergarten to second). This is because modeling includes practices that are not typical in primary grade mathematics classrooms like making assumptions, and testing and revising models (Niss, Blum, & Galbraith, 2007). Therefore, the primary aim of this study is to understand the potential impact of a professional development program focused on culturally responsive mathematical modeling on primary grade teachers' practices in modeling lessons. The following research questions guide our study: To what extent does a professional development program support teachers to enact practices for culturally responsive mathematical modeling lessons?

- What strengths and challenges do we notice in teachers' practices?
- How do teachers' practices for modeling lessons support culturally responsive mathematics teaching?

Culturally Responsive Mathematical Modeling Instruction

Our framework for culturally responsive mathematical modeling draws on mathematics education research that centers equity, namely work on culturally responsive mathematics teaching (Bartell et al., 2017; Leonard et al., 2010). Zavala and Aguirre (in press) highlight three essential strands of culturally responsive mathematics teaching, including: a) Connections to Knowledge and Identities –building on students' experiences, mathematical understandings, and cultural/community-based funds of knowledge to support their mathematics learning; b) Rigor and Support - maintaining high cognitive demand while simultaneously providing access points for learning including affirming multilingualism; and c) Power and Participation - teachers enhance equitable participation by disrupting status, distributing intellectual authority, and supporting student ownership of ideas. We see mathematical modeling instruction as a way to advance culturally responsive teaching because the relevant and cognitively demanding nature of modeling tasks cultivates space for diverse ideas and opportunities for connections and action (Anhalt et al., 2018; Cirillo et al., 2016; Turner et al., 2021, 2022a; Zavala & Aguirre, in press).

Teaching Practices for Culturally Responsive Mathematical Modeling in Primary Grades

There is growing consensus that teachers can support primary grades students to engage in mathematical modeling (English, 2012; Fulton, 2021; Wickstrom & Aytes, 2018). Researchers specifically highlight the importance of modeling tasks grounded in culturally relevant contexts to help students draw on their lived experiences (Albarracín, 2021; Albarracín & Gorgorió, 2020; Dindyal, 2010; Wickstom & Aytes, 2018), and the value of collaboration and dialogue with peers (Bonnotto, 2009; Mousoulides & English; 2008). Teacher practices which support primary grade students' engagement in mathematical modeling, include posing a series of smaller questions to help students make sense of a broader task (Albarracín, 2021), and introducing constraints one at a time so that students could adjust and refine their models without starting over (Osana & Foster, 2021). Other productive teacher moves include pausing small group work to share student strategies, and offering physical tools, graphic organizers, and sentence starters to scaffold students' modeling building work (Carlson, 2016 et al., 2016; Fulton, 2021; Wickstron & Aytes, 2018; Author). Teachers of young students also play an important role in helping students make assumptions about unknown quantities (Leavy & Hourigan, 2021; Stankiewicz-Van Der Zanden, Brown & Leavy, 2021). Yet most of these studies report on intensive efforts with one or two individual teachers and not on how teachers learn practices for modeling via participation in professional development. Our study aims to address this gap.

Research on the Impact of Professional Development on Teacher Practice

Project developed measures that relate to the specific focus of the professional development program are often used evaluate teacher practice, such as teachers' use of student thinking in instruction (Jacobs et al., 2007), or practices for facilitating classroom discourse (Cavanna, 2014). While studies sometimes find connections between ideas learned in professional development and teachers' subsequent classroom practice (Chen et al., 2020; Jacobs et al., 2007), others note impacts on teacher beliefs and perspectives, but not practices (Shirrell, Hopkins & Spillane, 2019). Despite these differences, there is consensus that understanding which practices teachers take up in their classrooms, and why, is essential, as it can inform revisions to professional learning programs (Caswell, Esmonde & Takeuchi, 2011; Franke et al, 2001).

Observation Tools for Measuring Teacher Practice in Mathematical Modeling Lessons

Observational tools to measure mathematics teaching practice tend to focus broadly on standards-based, or problem-solving oriented instruction (Walkowiak et al, 2014). A tool with a specific focus on mathematical modeling is important because scores would be interpreted as capturing aspects relevant and specific to teaching mathematical modeling. Developing an observation tool that attends, in substantive ways, to mathematical modeling and culturally responsive teaching has been a goal of our current project. We previously described (Turner et al., 2022b) our multi-step process for the initial development of this tool, including synthesizing key outcomes from relevant literature, generating initial validity evidence from an expert panel review, and testing the tool in diverse K-5 classrooms (Bostic et al, 2019). In this study, we use the tool to understand the strengths and challenges in primary grade teachers' practices for teaching modeling, and to explore how their practices align with culturally responsive teaching.

Methods

Professional Development Context

This study is part of a broader research and professional development program focused on culturally responsive mathematical modeling in elementary grades. Teachers participated in a year-long, professional development program that included both monthly in person sessions and asynchronous activities to deepen learning. In person sessions introduced frameworks for culturally responsive mathematics teaching (Zavala & Aguirre, in press), and included time to explore modeling tasks and routines, collaboratively plan activities, and reflect on classroom enactments. Asynchronous materials included readings, videos of modeling activities in K-5 classrooms, and collaborative reflection prompts. Teachers also had access to digital materials (modeling tasks, student work samples) to support classroom enactments.

Participants

This study focused on 8 primary grade teachers (kindergarten through grade 2) who participated in our broader professional development program at one of three research sites. Two of the teachers taught kindergarten, five taught first grade, and one taught 2nd grade. 6 of the teachers worked in schools that served racially and linguistically diverse students from underserved communities. Classrooms included migrant and refugee students from diverse countries of origin, and significant numbers of multilingual students. 2 teachers taught in predominantly white schools with a small but growing population of multilingual students.

Data Sources

Data sources included two video-taped modeling lessons from each teacher's classroom (16 lessons total). One lesson included a "snack sharing" modeling task that focused on making a plan to share snack items with classmates across one or more days. This lesson was recorded in the fall or winter of the school year. A second lesson focused on a "making" modeling task in which students generated a plan for making a set of items (e.g., picture frames, bird feeders,) for a school or community purpose. This lesson was recorded later in the school year. Lessons ranged in length from 54 to 187 minutes (average of 88), and often occurred over two days.

Classroom Observation Tool for Modeling Lessons

Each lesson was scored using the project developed classroom observation tool for culturally responsive mathematical modeling lessons (Turner et al., 2022b). The tool includes eight dimensions that focus on teaching practices for specific phases of the modeling process. Each dimension describes four levels of teacher practice (not present (0), emerging (1), proficient (2) and advanced (3). These levels are distinguished by the extent to which teachers use culturally responsive practices in that phase of the modeling cycle, such as maintaining high cognitive demand while simultaneously providing supports (Rigor and Support), soliciting diverse student ideas and allowing student ideas to drive decisions (Knowledge and Identities; Power and Participation). A ninth dimension focuses on culturally responsive practices that apply across the modeling process (i.e., connections to students' experiences and cultural/community contexts).

Table 1: EQ-STEMM Classroom Observation Tool for Elementary Modeling Lessons

Dimension	Focus of Dimension	Variation in Levels of Practice
1: Making Sense of the Context / Situation	Teachers offer supports to help students make sense of the context, solicit students' ideas or questions about the task context, and focus students on key considerations related to context.	•Presence of supports •Intensity of teacher solicitation •Focus on key considerations
2. Posing Problems	Teachers build on student ideas to pose the modeling problem. Teachers support students to ask and analyze mathematical questions.	•Student ownership of problem posing •Support for asking math questions
3. Identifying Important Quantities	Teachers support students to identify key quantities and to decide on a specific value for one or more quantities. Teacher asks students to explain the relevance of key quantities.	•Student ownership of quantities •Allowing variation in quantities •Support for explaining relevance
4. Making Assumptions	The teacher supports students to make /state assumptions, and to justify the relevance and reasonableness of assumptions.	•Student ownership of assumptions •Support for explaining relevance
5. Constructing and Operating on Models	The teacher facilitates student work as students create and operate on models, soliciting student ideas and supporting students to justify work.	•Student ownership of models •Support for justification
6. Analyzing or Interpreting Models	Teacher provides structures to support analyzing models or solutions and supports student participation so student ideas influence discussion.	•Presence of analysis supports •Student ownership of analysis •Support for justification
7. Revising Models	The teacher supports students in revising, ensuring that student contributions play a central role in model revisions, and supporting students to justify.	•Student ownership of revision •Support for justification
8. Reporting Out	The teacher supports students/groups to report and explain their work. The teacher provides students with options for reporting out their results.	•Student ownership of report out •Support for student choice
9: Connections to Students' Out-of-Class Experiences and Cultural and Community Contexts	Teachers support students to make connections to out-of-class experiences and/or cultural and community contexts. Connections inform modeling work - influencing decisions made or actions taken in any phase of modeling process.	•Connections present through the modeling cycle •Student ownership of connections •Connections inform modeling

Each lesson was scored by at least two members of the research team using a scoring method adapted from prior projects (Foote et al., 2020; Walkowiak et al., 2014). We watched lesson videos in eight-to-ten-minute segments, scripting teacher and student talk and actions to produce detailed lesson logs. After each segment, we noted evidence related to each dimension (including time stamps and examples) on a coding sheet. This process was repeated until the end of the lesson video. We reviewed all evidence against the dimension descriptors to assign a final score by dimension. Groups of two to four researchers reviewed scores for each lesson, and differences were resolved via discussion. Analysis focused on patterns of strengths and challenges by dimension. Patterns were examined within lessons (i.e., strengths and challenges in the snack sharing modeling lessons) and across lessons. Finally, we reviewed the lesson notes and coding sheet evidence to explore the specific teacher practices connected to each pattern.

Findings

Overview of Teacher Practice Scores

Table 2 includes average teacher practice scores for selected dimensions for each set of modeling lessons. For the purposes of this report, we focus on three dimensions that reflect key patterns related to our research questions, including consistent areas of strength (dimension 1), persistent areas of challenge (dimension 4) and dimensions where teacher practice seemed to shift between the first and second set of lessons (dimension 3).

Table 2: Teacher Practice Scores on Selected Dimensions

Dimension	Average Score on Snack Sharing Modeling Lessons	Average Score on Making Modeling Lessons
1. Making Sense of the Context	2.75	2.63
3. Identifying Important Quantities	1.63	2.38
4. Making Assumptions	0.88	0.75

Areas of Strength: Supporting Students to Make Sense of Real-World Contexts

We found that teachers enacted multiple strategies to help students make sense of the real-world context of the modeling tasks, including sharing images and realia related to the context, and inviting students to share observations, wonderings, and relevant experiences (e.g., “Have you ever had this snack? How do we usually share snacks at school?”). This strength was evident across all eight teachers, and across both sets of lessons observed. As an example, Ms. T, a first-grade teacher, launched the snack sharing modeling lesson by holding up two containers of small Madeline cakes and inviting students to share what they noticed. Students began by describing the physical attributes of the cakes (e.g., “they look like sweet bread”), which the teacher revoiced and recorded on a class chart. Students then posed questions about how the cakes were made, how large they were, whether the class would be able to eat them, and whether there were the same number of cakes in both boxes. Students passed around the containers as they observed and wondered, until most children in the class had the opportunity to share. After about ten minutes, the teacher noticed one student looking around the room and counting her classmates. The teacher paused the discussion and asked the class to reflect on her thinking.

> Ms. T: I want to go back to Deena’s thinking. Because she was looking at my box and then she was like… counting (gestures to show how she was counting the children) … then she looked at the box. What was she thinking?
>
> Student 1: Maybe she was wondering if there was less or more cakes than the children.
>
> Ms. T: That’s a really good math question, is that what you were wondering?
>
> (Deena confirmed that she was thinking about whether they would have enough for everyone, and the teacher then drew on this idea to pose the modeling problem.)

Ms. T: We have two boxes. I was wondering this question. You wanted to know if there was enough for all of us. What if I said: how many days will snacks last for our class? If we have this for snack every day, could we find out how many days it's going to last us?

This excerpt reflects patterns in the culturally responsive practices that teachers used to support students as they made sense of the modeling context. Teachers provided multiple supports for sense making, and repeated opportunities for students to share their ideas, both with partners and in whole group discussion. Teachers used targeted prompts to focus students' attention on key features of the context (i.e., in this case, Deena's idea that comparing the number of children to the number of cakes might be useful). Teachers also ensured students had ownership over ideas, recording their names alongside their comments, and explicitly connecting the modeling problem to their ideas. As a result, by the time teachers posed the modeling problem, students had generated a broad range of ideas about the context.

Areas of Shift: Identifying Important Quantities

The dimension where teacher practices shifted most from the first to the second set of lessons was Dimension 3, Identifying Important Quantities. In the snack sharing modeling lessons, half of the teachers evidenced emergent levels of practice (4 teachers), with fewer teachers scoring proficient (3 teachers) or advanced (1 teacher). Common patterns in teacher practices included providing time for students to discuss what they already knew that could help them with the modeling problem, and what they might need to find out or decide. For example, another first-grade teacher, Ms. B, posed a modeling problem related to how many boxes of hot chocolate they would need for all the first grade classes at a winter celebration. After posing the task, she asked students: "What might we need to know if we were going to do this project. Stand up and talk to your talking partner. We have some things we know and some things we need to decide." Following the partner talk, Ms. B solicited and recorded students' ideas. Students quickly identified one quantity that would be relevant– the number of students in each first-grade class. However, when students started to generate a range of ideas about this quantity ("I am estimating 16 or 19 kids in our class", and "We could count how many chairs there are or count the lockers") Ms. B responded by narrowing the conversation and directing students towards a specific value. She stated, "Could we decide that there are 16 kids in our class and the all the other classes too? I think that's fair. Let's write it down - '16 kids in every first-grade class.'"

This practice of encouraging students to list relevant quantities, but then funneling the class towards a single set of values so that all students worked with the same numbers as they built and operated on their models was common in this first set of lessons. These practices closed off space for variation and therein, limited students' opportunities to explain and justify their decisions, which reduced the cognitive demand. In other words, teachers' practices reflected tensions related to components of culturally responsive teaching – teachers provided supports, but these supports reduced the cognitive demand and minimized students' ownership over ideas.

In contrast, in the second set of modeling lessons – the making tasks – all teachers scored at the proficient (5 teachers) or advanced levels (3 teachers). A shift we noted is that teachers were more adept at efficiently focusing students' observations on relevant quantities, and importantly, on how quantities might vary. Teachers used strategically selected images or prompts that drew students' attention to variation. For example, Ms. J., a kindergarten teacher, taught a lesson about planning a set of materials for making a valentine craft (heart creatures) with their table groups. She launched the task by sharing images of heart creatures with different features. After students shared initial observations, Ms. J asked followed up with a more specific prompt to focus students on variation in quantities relevant to the problem (What is the same? What is different?)

Ms. J: (shows an image of different heart creatures) What is the same and what is different?
Student 1: (pointing to specific creatures) Big, little, little
Ms. J: So you mean the size is different, good.
Student 2: all of them are different
Ms. J: Ok, and how are they different?
Student 2: Eyes, and faces, and arms, and legs
Student 3: They all have heart heads.
Student 4: This one has 4 hearts on the arms and legs.
Student 3: They don't have the same eyes.
Ms. J: The same number of eyes? So they're different.

Students continued to identify consistencies and variations in the heart creatures, which Ms. J recorded on a class chart. Before she sent students off to work, she revisited the quantities they identified and the decisions they would have to make as they planned for the materials. This increased emphasis on variation and more consistent invitations for students to make decisions about quantities were notable patterns in the second set of lessons. These practices reflected components of culturally responsive mathematics teaching as teachers maintained high cognitive demand by holding space for variation and supporting students to explain and justify ideas.

Areas of Challenge: Making Assumptions

We found that supporting students to make assumptions was one of several dimensions that reflected challenges in teachers' emerging practices for mathematical modeling. In a few instances, teachers skipped conversations about assumptions, and instead directed students towards established quantities (e.g., there are this many students in our class) without consideration of how assumptions about the context could impact decisions. More often, teachers prompted students to consider potential assumptions, but students' role was limited to answering questions while the teacher-maintained ownership over stating and justifying the assumption. For example, in the snack sharing lessons, a second-grade teacher, Ms. F, asked students whether everyone in the class eats snack each day (focusing students' attention on potential assumptions).

Ms. F: But wait, do all of you eat the snack each day? How are we going to figure out how much we need to give out? It's [only] 18 if everybody eats the snack.
Student 1: maybe Ms. T [will want snack] too.
Ms. F: Ms. T is not here during snack time.
Student 2: How many people are missing [absent]?
Ms. F: There are 5 people missing.
Ms. F: So not everyone is going to eat snack. Do some of you bring snack from home? Raise your hand if you bring snack from home. (several students raise hands.)
Student 3: Only sometimes [do I bring snack from home]
Student 4: Since I really like those things that she brought I would eat that snack and save mine for tomorrow.
Student 5: I'd eat snack from home.
Ms. F: So we had 18 minus 1 person who won't eat snack, so we are going to say 17 people. And we know that is not true, some days people will be missing, but we are just going to say 17 people on most days.

While students responded to Ms. F's question with multiple ideas that could have supported assumptions (e.g., whether other teachers should be included, whether absent students should be counted, whether students who brought snack from home would always prefer the home snack), Ms. F directed the assumption making process and stated an assumption for students. We found a similar pattern in other lessons; students considered how different features of the context could impact the problem - they did not avoid this complexity – but teachers responded by making assumptions for students that narrowed the complexity, sometimes significantly. In other words, teachers' initial practices did not reflect key components of culturally responsive teaching. Students did not have ownership of ideas, as the teacher did not distribute intellectual authority. The cognitively demanding work of explaining assumptions was directed by the teacher.

Discussion and Conclusion

Enacting culturally responsive mathematical modeling in the elementary grades is ambitious teaching that has potential to humanize the teaching and learning of mathematics with young students. Practices for rehumanizing mathematics (Gutiérrez, 2018), such as helping students to make sense of and connect to the contexts of mathematical problem are common teaching practice in primary grade classrooms. We suspect that participating teachers already had many strategies for supporting this kind of sense making, which supported the strengths observed in dimension 1. Additionally, the professional development program provided multiple examples of visuals, tools and discussion prompts that teachers could use to introduce modeling contexts and foster sense making. While teachers' practices in this dimension reflected multiple components of culturally responsive mathematics teaching, there were also areas for growth. In particular, teachers' efforts to help students make sense of contexts did not include invitations for students to share related experiences from outside of school. We suspect this reflected the tasks themselves (which were grounded in school or classroom scenarios) but also points to an area of refinement for our professional development program.

Teaching culturally responsive mathematical modeling requires a new set of pedagogical skills. We found that teachers allowed more variation in quantities as they became comfortable with the openness and student decision making that characterizes modeling tasks. We suspect that the shifts in teacher practice in the important quantities dimension may have been related to two factors. First, in typical primary grade curriculum tasks, quantities are explicitly stated for students. In modeling tasks, teachers have a key role in supporting students to identify relevant quantities, and to make sense of potential variation (Anhalt, 2014). Given that all our participants were new to teaching modeling, these were new practices that teachers needed time to develop (English et al., 2005). Second, across the year we introduced various routines for supporting students to identify key quantities during professional development sessions. We also began to prompt teachers to reflect on the different decisions that student had to make in each modeling lesson. We recognized that this would be a challenging space for teachers and designed targeted tools and prompts to support these practices, which may have supported the shifts observed.

Finally, we suspect that teaching practices around assumptions were challenging for multiple reasons. The need to make assumptions to inform decisions is a unique feature of modeling (Galbriath, 2013; Suh et al., 2021), and as such, supporting students to make assumptions was a new teaching practice for all the teachers in our study. The fact that teachers entered this practice even in their first modeling lessons is promising. Teachers' initial attempts - which involved drawing students' attention to features of the context that might guide their decisions - suggest an emerging practice that could be further developed in professional development sessions. In other words, teachers likely need to play an active role in supporting students to state and explain assumptions, given that grappling with ambiguous or undefined information can be challenging for young children (English et al., 2005). Strategic prompts to focus children on the need for assumptions seem appropriate (i.e., What about people who bring snack from home, should we consider that?). However, professional development sessions could support teachers to respond to the diverse ideas that children generate in ways that distribute intellectual authority and maintain student ownership over ideas. Supporting teachers' culturally responsive practices for making assumptions is a key area of refinement for our professional development program.

Acknowledgments

This research was supported by grants NSF DRL 2008997, 2010202, 2010269, and 2010178.

References

Aguirre, J.M., Anhalt, C., Cortez, R., Turner, E.E., & Simi-Muller, K., (2019). Engaging teachers in the powerful combination of mathematical modeling and social justice. *Mathematics Teacher Educator, 7*(2). DOI: 10.5951/mathteaceduc.7.2.0007

Aguirre, J.M., Turner, E.E., Foote, M.Q., Roth McDuffie, A. (2019) *Leveraging student strengths with mathematical modeling in cultural and community contexts: Grades 3-5*. Invited Session. Annual Conference of the National Council of Teachers of Mathematics. San Diego, CA April 2019.

Albarracín, L. (2021). Large number estimation as a vehicle to promote mathematical modeling. *Early Childhood Education Journal, 49*(4), 681-691. DOI:10.1007/s10643-020-01104-x

Albarracín, L., & Gorgorio, N. (2020). Mathematical modeling projects oriented towards social impact as generators of learning opportunities: A case study. *Education Sciences, 8*(11), 1-20. DOI:10.3390/math8112034

Anhalt, C (2014) Scaffolding in mathematical modeling for ELLs. In M. Civil & E. Turner (Eds.) *The Common Core Mathematics Standards for English Language Learners: Grades K-8* (pp. 111-126). TESOL/NCTM.

Anhalt, C. O., Staats, S., Cortez, R., & Civil, M. (2018). Mathematical Modeling and Culturally Relevant Pedagogy. In Y.J Dori, Z.R. Mevarech, and D.R. Baker (Eds). *Cognition, Metacognition, and Culture in STEM Education* (pp. 307-330). Springer Publications.

Bartell, T., Wager, A., Edwards, A., Battey, D., Foote, M., & Spencer, J. (2017). Toward a framework for research linking equitable teaching with the standards for mathematical practice. *Journal for Research in Mathematics Education, 48*(1), 7-21.

Bonner, E. P., & Adams, T. L. (2012). Culturally responsive teaching in the context of mathematics: A grounded theory case study. *Journal of Mathematics Teacher Education, 15*, 25-38.

Bonotto, C. (2009). Working towards teaching realistic mathematical modeling and problem posing in Italian classrooms. In L. Verschaffel, B. Greer, W. Van Dooren, & S. Mukhopadhyay (Eds.), *Words and worlds: Modelling verbal descriptions of situations* (pp. 297–313). Rotterdam: Sense Publishers.

Bostic, J., Matney, G. & Sondergeld, T. (2019). A validation process for observation protocols: Using the Revised SMPs Look-for Protocol as a lens on teachers' promotion of the standards*. Investigations in Mathematical Learning, 11*(1), 69-82.

Cai, J., Cirillo, M., Pelesko, J. A., Borromeo Ferri, R., Borba, M., Geiger, V. Stillman, G., English, L. D., Wake, G., Kaiser, G., & Kwon, O. N. (2014). Mathematical modeling in school education: Mathematical, cognitive, curricular, instructional and teacher education perspectives. In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan (Eds.), *Proceedings of the joint meeting of PME 38 and PME-NA* 36 (pp. 145–172). Vancouver, Canada: PME.

Carlson, M.A., Wickstrom, M.H., Burroughs, B. & Fulton, E. (2016). A Case for Mathematical Modeling in the Elementary School Classroom. In C. Hirsch & A. Roth McDuffie (Eds.), *Annual perspectives in mathematics education (APME) 2016: Mathematical modeling and modeling mathematics* (pp. 121-129). 121–29. Reston, Va.: National Council of Teachers of Mathematics.

Caswell, B., Esmonde, I., & Takeuchi, M. (2011). Towards culturally relevant and responsive teaching of mathematics. In C. Rolheiser, M. Evans, & M. Gambhir (Eds.), *Inquiry into practice: Reaching every student through inclusive curriculum practices* (pp. 64-71). Toronto, CA: Ontario Institute for Studies in Education.

Cavanna, J.M. (2014). Shaping mathematics classroom discourse: Relating professional development and classroom practice. In C. Nichol, P. Liljedahl, S. Oesterle, & D. Allan (Eds.), *Proceedings of the joint meeting of PME 38 and PME-NA 36*, Volume 2 (pp. 257-264). Vancouver, Canada: PME

Chen, G., Chan, C. K., Chan, K. K., Clarke, S. N., & Resnick, L. B. (2020). Efficacy of video-based teacher professional development for increasing classroom discourse and student learning. *Journal of the Learning Sciences, 29*(4-5), 642-680.

Cirillo, M., Pelesko, J., Felton-Koestler, M., & Rubel, L. (2016). Perspectives on modeling in school mathematics. In C. R. Hirsch & A. R. Roth McDuffie (Eds.), *Mathematical Modeling and Modeling Mathematics* (pp. 3–16). National Council of Teachers of Mathematics.

Cirillo, M., Bartell, T. G., & Wager, A. (2016). Teaching mathematics for social justice through mathematical modeling. In C. Hirsch & A. Roth McDuffie (Eds.), *Annual perspectives in mathematics education: Mathematical modeling and modeling with mathematics* (pp. 87-96). National Council of Teachers of Mathematics.

English, L.D., Fox, J.L., & Watters, J.J. (2005). Problem posing and solving with mathematical modeling. *Teaching Children Mathematics, 12*(3), 156-163.

English, L. D. (2012). Data modelling with first-grade students. *Educational Studies in Mathematics, 81*, 15-30. DOI:10.1007/S10649-011-9377-3

Foote, M.Q., Aguirre, J.M., Turner, E.E., & Roth McDuffie, A. (2020). M2C3-Scan: Mathematical Modeling with Cultural and Community Contexts Observation protocol. Unpublished tool.

Franke, M. L., Carpenter, T. P., Levi, L., & Fennema, E. (2001). Capturing teachers' generative change: A follow-up study of professional development in mathematics. *American Educational Research Journal, 38*(3), 653-689.

Fulton, Elizabeth (2021). "Teachers' Use of Students' Mathematical Ideas in Mathematical Modeling." In J. Suh, M. Wickstrom, L. English (Eds). *Exploring Mathematical Modeling with Young Learners* (pp. 169-194). The Netherlands: Springer. DOI:10.1007/978-3-030-63900-6_11.

Galbraith, P. & Stillman, G. & Brown, J. (2013). Turning Ideas into Modeling Problems. Modeling Students' Mathematical Modeling Competencies: ICTMA 13.10.1007/978-1-4419-0561-1_11.

Gutiérrez, R. (2018). Introduction: The need to rehumanize mathematics. In I. Goffney & R. Gutiérrez (Eds.), *Rehumanizing mathematics for Black, Indigenous, and Latinx* Students (pp. 1–10). National Council of Teachers of Mathematics.

Jacobs, V. R., Franke, M. L., Carpenter, T. P., Levi, L., & Battey, D. (2007). Professional development focused on children's algebraic reasoning in elementary school. *Journal for Research in Mathematics Education, 38*(3), 258-288.

Leavy, A., & Hourigan, M. (2021). Data modelling and informal inferential reasoning: Instances of early mathematical modelling. Exploring Mathematical Modeling with Young Learners. In J. Suh, M. Wickstrom, L. English (Eds). *Exploring Mathematical Modeling with Young Learners* (pp. 67-93). The Netherlands: Springer. DOI:10.1007/978-3-030-63900-6_11.

Leonard, J., Brooks, W., Barnes-Johnson, J., & Berry III, R. Q. (2010). The nuances and complexities of teaching mathematics for cultural relevance and social justice. *Journal of Teacher Education, 61*(3), 261-270.

Lesh, R. A., & Doerr, H. M. (2003). *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching*. New York: Routledge.

Lesh, R., & Zawojewski, J. (2007). Problem solving and modeling. In F. K. Lester & National Council of Teachers of Mathematics (Eds.), *Second handbook of research on mathematics teaching and learning:Aa project of the National Council of Teachers of Mathematics* (pp. 763–804). Charlotte, NC: Information Age.

Mousoulides, N., & English, L. (2008). Modeling with data in Cypriot and Australian primary classrooms. In *Proceedings of the Joint Meeting of the International Group and the North American Chapter of Psychology in Education (PME 32)* (pp. 423-430). Centre for Research and Advanced Studies of the National Polytechnic Institute, Morelia, Mexico.

Niss, M., Blum, W., & Galbraith, P. L. (2007) Introduction. In W. Blum, P. L. Galbraith, H. W. Henn, & M. Niss (Eds.) *Modelling and Applications in Mathematics Education: The 14th ICMI Study* (pp. 3-32). New York: Springer.

Osana, H. P., & Foster, K. (2021). The Genesis of Modeling in Kindergarten. *Exploring Mathematical Modeling with Young Learners*. In J. Suh, M. Wickstrom, L. English (Eds). *Exploring Mathematical Modeling with Young Learners* (pp. 311-336). The Netherlands: Springer. DOI:10.1007/978-3-030-63900-6_11.

Pollak, H. (2012). Introduction. In H. Gould, D. R. Murray, & Sanfratello (Eds.), *Mathematical modeling handbook* (pp. viii–xi). Bedford, MA: Consortium for Mathematics and Its Applications.

Praetorius, A. K., & Charalambous, C. Y. (2018). Classroom observation frameworks for studying instructional quality: looking back and looking forward. *ZDM-Mathematics Education*, *50*, 535-553.

Shirrell, M., Hopkins, M., & Spillane, J. P. (2019). Educational infrastructure, professional learning, and changes in teachers' instructional practices and beliefs. *Professional development in education*, *45*(4), 599-613.

Stankiewicz-Van Der Zanden, R., Brown, S., & Levy, R. (2021). A Window into Mathematical Modeling in Kindergarten. In J. Suh, M. Wickstrom, L. English (Eds). *Exploring Mathematical Modeling with Young Learners* (pp. 267-310). The Netherlands: Springer. DOI:10.1007/978-3-030-63900-6_11

Suh, J.M., Matson, K., Birkhead, S., Green, S., Rossbach, M., Seshaiyer, P. & Jamieson, T.S. (2021). The Importance of Problem Formulation and Elementary Teachers as Designers of the Early Modeling Experiences for Elementary Students. In J.M Suh, M. Wickstrom & L. English (Eds.), *Exploring the Nature of Mathematical Modeling in the Early Grades* (pp. 113-145). Netherlands: Springer.

Suh, J., Britton, L., Burke, K., Matson, K., Ferguson, L., Jamieson, S., & Seshaiyer, P. (2018). Every penny counts: Promoting community engagement to engage students in mathematical modeling. In Goffney & Gutiérrez (Eds.) *Rehumanizing mathematics for Black, Indigenous, and Latinx students* (pp. 63-76). Reston, VA: NCTM.

Turner, E. & Bustillos, L. (2017). ¿Qué observamos aquí? ¿Qué preguntas tienen? Problem posing in Ms. Bustillos's Second-Grade Bilingual Classroom. In S. Celedón-Pattichis, D. White and M. Civil (Eds.) *Access and equity: Promoting high quality mathematics in grades Pre K-2* (pp. 45-62.). Reston, VA: National Council of Teachers of Mathematics.

Turner, E., Roth McDuffie, A., Aguirre, J., Foote, M. Q., Chapelle, C., Bennett, A., Granillo, M. & Ponnuru, N. (2021). Upcycling Plastic Bags to Make Jump Ropes: Elementary students leverage experiences and funds of knowledge as they engage in a relevant, community-oriented mathematical modeling task (pp. 235-266). In J. Suh, M. Wickstrom, L. English (Eds). *Exploring Mathematical Modeling with Young Learners*. The Netherlands: Springer. DOI:10.1007/978-3-030-63900-6_11.

Turner, E. E., Bennett, A. B., Granillo, M., Ponnuru, N., Roth Mcduffie, A., Foote, M. Q., ... & McVicar, E. (2022a). Authenticity of elementary teacher designed and implemented mathematical modeling tasks. *Mathematical Thinking and Learning*, 1-24.

Turner, E. E., Carlson, M. A., Fulton, E. A., Suh, J., Aguirre, J. (2022b). A Classroom Observation Tool for Equity-Oriented Teaching of Mathematical Modeling in the Elementary Grades. *Proceedings of the forty-fourth annual meeting of the North American Chapter for the Psychology of Mathematics Education* (pp. 1871-1876). Murfreesboro, TN: Middle Tennessee State University.

Walkowiak, T., Berry, R., Meyer, J., Rimm-Kaufman, S., & Ottmar, E. (2014). Introducing an observational measure of standards-based mathematics teaching practices: Evidence of validity and score reliability. *Educational Studies in Mathematics, 85*, 109–128.

Wickstrom, M. H., & Aytes, T. (2018). Elementary modeling: Connecting counting and sharing. *Teaching Children Mathematics, 24*(5), 300-307.

Zavala, M.R. & Aguirre, J.M. (in press) *Cultivating mathematical hearts: Culturally responsive math teaching in elementary classrooms*. Thousand Oaks, CA: Corwin.

TEACHERS' ORIENTATIONS TOWARD AND INTERACTIONS WITH STUDENTS' WRITING IN HIGH SCHOOL MATHEMATICS

Ethan P. Smith
Washington State University Tri-Cities
ethan.p.smith@wsu.edu

This report details a case study focused on different manifestations of teachers' orientations towards students' writing in high school mathematics. I make use of teacher interviews to unpack teachers' described orientations towards writing in mathematics, as well as their interpretations of interactions observed during recorded observations. These observations also illustrate the types of writing embedded in tasks enacted by these teachers and the nature of their interactions with students' writing during such tasks. Findings indicate participants' distinct understandings of "writing in math," show their flexibility in employing both teacher- and student-oriented interactions with student writing and suggest their desire to pursue more student-oriented interactions. Avenues for future research related to such findings are discussed.

Keywords: Communication, Instructional Activities and Practices, Teacher Beliefs, Teacher Noticing

Fostering students' effective communication of mathematical ideas is a key goal of educational reform movements and professional organizations in the United States (Common Core State Standards Initiative, 2010; National Council for Teachers of Mathematics, 2000). However, while supporting students' spoken communication has been a frequent focus of research in mathematics education research (Moschkovich, 2007; Munter et al., 2015; Webb et al., 2014), written communication has not received the same attention (Morgan, 1998; Pugalee, 2004). This relatively limited attention on writing is concerning because mathematical ideas are communicated not only through spoken language, but within and across the multiple sign systems of natural language (the written word), mathematical symbolism, and visual imagery – and these systems are often represented through written texts (O'Halloran, 2000, 2008). In other words, writing in mathematics is *multisemiotic*, or expressed within and across these multiple sign systems. Additionally, the language of mathematics itself evolved in part "as a response to the functions which were fulfilled symbolically and visually" (O'Halloran, 2008, p. 14). Therefore, mathematical meaning is not only being communicated multisemiotically through writing, but these sign systems also help inform the structure of spoken mathematical discourse. Student writing in the context of mathematics education thus warrants careful examination.

The overlapping nature of these multiple sign systems in mathematics also warrants the viewing of student writing expansively in the context of mathematics education. In this vein, I build from Draper and Siebert's (2010) framing of disciplinary literacy to broadly define writing in mathematics as the construction of any written text encompassing natural language, mathematical symbolism, or visual imagery that is used to understand or communicate mathematical ideas. This definition in turn conceives of a *text* as any representational object which is intended by its creator to communicate a meaning (Draper & Siebert, 2010; Schnotz et al., 2010).

My aim in the present study is to better understand teachers' orientations toward and interactions with students' writing in high school mathematics. In particular, I investigate instructional moments that arise at the intersection of students' written work and (spoken) discourse between students and teachers around such work, and address the ways that teachers conceive of students' writing in mathematics, attend to such writing, and interpret these interactions with such writing. I center this study on three questions: (1) Among a group of secondary mathematics teachers whose instruction is dialogically focused, what are their described orientations toward writing in school mathematics? (2) How do these teachers interact with their students' writing instructionally in relation to the writing expectations of the task? (3) How do these teachers describe their interpretation of students' writing and their interactions with students' writing?

Methods

This study employed an embedded multiple case study approach (Yin, 2018). Each of the teachers represented a case, while a subunit of teachers' observed interactions with students' writing was embedded within each of these three cases. Given that orientations arise at the intersection of beliefs, perceptions, and practices (Remillard & Bryans, 2004), this work made use of data from participant interviews, classroom observations, and collected samples of student written artifacts to form a nuanced reading of teachers' perceptions about student writing in mathematics and their observed interactions with such writing.

Participants and Setting

Three high school mathematics teachers in a Mid-Atlantic state participated in this study, with pseudonyms Mrs. Taylor, Mrs. Hudson, and Mrs. Barnett. All three teachers' instruction had previously been coded as dialogically focused during their participation in the SMiLES (Secondary Mathematics in-the-moment Longitudinal Engagement Study) project, which had used classroom observation data to identify potentially productive mathematics teaching practices (see Jansen et al., 2021). "Dialogically focused" instruction in this case entailed the use of (1) high-level, open-ended tasks, (2) opportunities for sharing multiple representations or strategies (e.g., graphs, tables, etc.), and (3) student discourse (Henningsen & Stein, 1997; Munter et al., 2015). Given the present study's focus on the relationships between student writing and spoken discourse, dialogically focused instruction was conjectured to be an opportune setting under which to potentially capture teachers interacting with students' writing.

Measures and Analysis

The data for this study came from two teacher interviews (pre- and post-observation) for each participant, recorded classroom observations (four for each participant during the Spring 2022 school year), and collected samples of student writing (captured during classroom observations). The pre-observation teacher interview drew on items adapted from Cantrell and colleagues (2008) meant to capture participants' described orientations towards writing in mathematics (the subject of the first research question). In particular, these questions addressed the extent to which participants descriptions about writing in mathematics are multisemiotic (i.e., consider different written sign systems) and whether participants oriented towards a writing-to-learn perspective (i.e., that writing is not merely a way to summarize what has already been learned, but that learning can occur through the practice of writing in mathematics; Connolly, 1989).

The second interview built off of this pre-observation interview, first allowing participants to member-check (Candela, 2019; Creswell & Clark, 2017) profiles created to summarize their described orientations about writing from the previous interview. Teachers were also asked to provide their noticings about samples of student writing captured during observations (see Goldsmith & Seago, 2011; Jacobs et al., 2010) and to engage in video viewing sessions (Erickson, 2007) of multiple interactions that they had with student writing as captured from their recorded classroom observations (described further below). Selection of these instructional video clips followed criteria similar to Sherin et al. (2009), namely that the chosen clips showed evidence of student thinking that was easily understandable and focuses on substantive mathematical ideas. In support of this, I particularly drew on clips that aligned with the captured samples of student writing and which addressed different types of interactions with student writing (described below). Participant responses to these interview components were used to address the third research question.

For the second research question, four recorded observations for each teacher were coded along dimensions of (1) the type of writing embedded in the instructional tasks enacted during the lesson and (2) the type of teacher interaction with students' writing. The categories of writing task was based on the work of Casa and colleagues (2016), and categorized the writing as either exploratory (to make personal sense of a situation or problem), procedural/symbolic (to calculate a numeric or algebraic solution by using procedures with mathematical symbolism), informative (to describe something), exclusive explanatory (to explain something, students executing "already-known" ideas), or inclusive explanatory (to explain something, students as active participants in forming ideas). The differentiation among types of explanatory writing tasks as either "exclusive" or "inclusive" was derived from the curriculum-focused work of Rotman (1988) and Herbel-Eisenmann (2007), where differences were noted in whether mathematical task prompts framed the reader (e.g., the student completing the task) as merely executing a defined set of statements to arrive at an already-known conclusion (exclusive, e.g., "Prove that $x = 7$ is a valid solution to the given equation.") or as an active participant in the living activity of "doing" mathematics (inclusive, "How might we find a solution to the following equation?"). The distinction between these explanatory types of writing were pursued under the conjecture that the framing of the prompt could foster different types of teacher interactions with the writing that students might produce during their engagement with such tasks.

The types of teacher interactions with students' writing, shown in Table 1, were based on literature concerning teachers' responses to student thinking in mathematics, as well as past research on instructional approaches to questioning. Firstly, an interaction could entail a teacher-driven response to a students' writing. This could involve a purely *evaluative* response (e.g., "Good job, that is correct" or "No, the solution is 12"), or a response that evaluates the student writing but also *describes* how the teacher sees (or do not see) their writing as communicating mathematical meaning (e.g., "I appreciate that you wrote out each step as you applied different operations to solve this equation, as that helps me understand how you arrived at your solution."). This latter type of interaction could be seen as a type of judicious telling (Smith, 1996) that uses students' own written or spoken ideas in order to focus those students onto specific mathematical ideas relevant to the learning goals (see Smith et al., 2023).

However, teachers could also more actively involve students in these interactions through a process that van Es and Sherin (2021) call *shaping*, which "involves teachers constructing interactions, in the midst of noticing, to gain access to additional information that further supports their noticing" (p. 23). This is aligned with what Boston et al. (2017) describe as *assessing* and *advancing* questions, with the former entailing the teacher assessing students' understanding of the mathematics behind their written work (e.g., "Where did this '*4' come from in your equation here?") and the latter entailing a noticing or wonderment meant to extend students' thinking (and writing) towards the learning goal (e.g., "Why did you draw the slope of this function as steeper than this other function? What about the function equation informs you how steep the slope might be when you graph it?"). With advancing questions – which I expand in this study to include suggestive statements that serve a similar point as questions (e.g., "See what happens to the function rule when you change the slope.") – the teacher is meant to then leave the student(s) to continue their thinking independently.

Table 1: Types of Teacher Interactions with Students' Writing in Mathematics

Type of interaction	Description
Evaluative Response	The teacher addresses a student's written text *without* then explaining how they see the writing as communicating mathematical meaning.
Descriptive Response	The teacher addresses a student's written text and/or states how they see the writing as communicating mathematical meaning.
Assessing Question(s) + Response	The teacher inquires the student about the mathematical meaning of their written text and then states how they see the writing as communicating mathematical meaning.
Response + Advancing Question(s)	The teacher states how they see the students' written text as communicating mathematical meaning and then ends the interaction with a question or suggested next step to advance or revise the students' writing.
Assessing Question(s) + Response + Advancing Question(s)	The teacher inquires the student about the mathematical meaning of their written text, states how they see the writing as communicating mathematical meaning, and then ends the interaction with a question or suggested next step to advance or revise the students' writing.

Such question-oriented interactions represent a more dialogic and student-focused approach to interacting with students' writing, as they allow students to retain at least some of their own voice in relation to the written work that they have produced, and to more accurately expand upon the mathematical meaning behind the written text that they had produced. I also anticipated and encountered mixed combinations of these sorts of questions with more teacher-focused evaluative and descriptive responses. For instance, a teacher may give descriptive feedback and follow this with an advancing question, or might ask an assessing question before steering the interaction towards an evaluative response. As such, Table 1 represents combinations of how assessing and advancing questions arose in the data.

Results

Teacher interviews indicated that the participants had distinct and nuanced described orientations towards writing in mathematics, with a clear emphasis on natural language writing (e.g., written explanations or justifications) but openness to multisemiotic perspectives towards writing and writing-to-learn. Classroom observation data indicated a variety of interactions with writing across a range of writing tasks, albeit with the most common interactions being teacher-drive (i.e., evaluative or descriptive responses) with procedural/symbolic tasks. Teachers' descriptions of selected interactions, however, did indicate that the participants had a desire for more student-centered interactions and exploratory or explanatory writing tasks.

Described Orientations Towards Writing in Mathematics

The participants' described orientations towards writing in mathematics are summarized in Table 2. This table shows the type of sign system(s) emphasized by participants in their interviews (i.e., "Semiotic Orientation") and the ways in which teachers affirmed a writing-to-learn (Connolly, 1989; Morgan, 1998) perspective, with writing framed as an instructional tool for supporting the learning of conceptual ideas (i.e., "Learning Orientation). All three of the participants described – or were receptive to – writing as arising through multiple sign systems in the doing and learning of mathematics (a multisemiotic perspective), and all participants were receptive to a writing-to-learn perspective. However, participants at times noted exceptions or reservations towards such receptive attitudes.

Table 2: Teachers' Described Orientations Towards Writing in Mathematics

Name	Semiotic Orientation	Learning Orientation
Mrs. Taylor	Natural language (written explanations) Receptive to a multisemiotic perspective*	Receptive to writing-to-learn*
Mrs. Hudson	Natural language (academic vocabulary) Receptive to a multisemiotic perspective*	Receptive to writing-to-learn
Mrs. Barnett	Writing as multisemiotic (words, visuals, symbols)	Receptive to writing-to-learn*

* (with exceptions or reservations)

Mrs. Taylor largely centered her descriptions about writing in mathematics in terms of writing to justify or explain mathematical processes and solutions through natural language (i.e., the written word). She saw such writing as complementary to (but distinct from) more symbolically focused, procedural tasks. While her descriptions did not emphasize connections across different sign systems of writing, Mrs. Taylor did state that more symbolic, procedural writing done by students and more explanatory, natural language writing were "both valuable" in mathematics. She also noted that, although students can learn mathematical ideas through the construction of written explanations, this is "the part that goes when you're on a time crunch." Thus, while she was sympathetic to a writing-to-learn perspective, she did not describe writing (via written explanations or justifications) as crucial to the learning process.

Mrs. Hudson's descriptions of writing centered largely on the idea of academic vocabulary (e.g., "monomial"). She noted how, because the language that we use to describe mathematical concepts or representations is rooted in the nature of such ideas, that by "breaking down English words and translating" them (e.g., discussing the "mono" in "monomial"), students can recognize "how they [the words] affect the math world." Thus, while her descriptions focused on natural language writing, Mrs. Hudson did see such writing as helping to develop students' understanding of other sign systems. She specifically described how, by frequently and publicly making use of students' written descriptions or explanations of relevant mathematical ideas, she saw writing as a way for students to gain comfort in both writing and speaking about mathematics with their peers, focus away from "if I'm correct or not," and value what they can learn from their peers' language. As such, she felt that writing "should go throughout" the learning cycle, including "warm up" activities, "rough draft" math, checks for understanding, and "exit ticket" routines, indicating a receptive stance towards writing-to-learn.

Mrs. Barnett held the most explicitly multisemiotic description of writing in mathematics, stating that "math is a form of communication in and of itself," where "I have this idea that I want to communicate to people, this pattern that I've noticed…and I just need to be able to demonstrate it for someone else in however way I see fit with words, visuals, [or] symbols." As mathematics and writing were both seen as forms of communication, Mrs. Barnett held that they were closely "intertwined" with one another. Because of this, she noted that "you could justify it [writing] anywhere in the learning process," although she also stated that her immediate thought was that writing best occurs at the end of a learning sequence "to summarize" what was learned. Such responses indicated that Mrs. Barnett was receptive towards writing-to-learn but held some reservations with that perspective.

Observed Teacher Interactions with Students' Writing in Mathematics

For each participant, four 90-minute lessons were recorded across the Spring 2022 semester and analyzed to identify types of writing tasks used by the teachers and types of interactions that they had with students' writing. These results are shown below in Table 3.

Table 3: Types of Writing Tasks and Teacher Interactions with Student Writing

	Procedural/ Symbolic	Exploratory	Informative	Explanatory (exclusive)	Explanatory (inclusive)	**TOTAL**
Evaluative Response	39 (6/23/10)	0 (0/0/0)	12 (3/4/5)	6 (5/0/1)	19 (7/12/0)	**76 (21/39/16)**
Descriptive Response	42 (3/12/27)	2 (0/0/2)	3 (1/2/0)	10 (8/1/1)	13 (8/5/0)	**71 (20/21/30)**
Assessing Question(s) + Response	11 (2/5/4)	3 (0/0/3)	1 (1/0/0)	2 (2/0/0)	6 (4/2/0)	**22 (9/6/7)**
Response + Advancing Question(s)	3 (0/2/1)	3 (1/0/2)	0 (0/0/0)	3 (2/1/0)	4 (3/1/0)	**13 (6/4/3)**
Assessing + Response + Advancing Questions	2 (0/1/1)	0 (0/0/0)	1 (1/0/0)	2 (2/0/0)	2 (2/0/0)	**7 (5/1/1)**
TOTAL	**97 (11/43/43)**	**8 (1/0/7)**	**17 (6/6/5)**	**23 (19/2/2)**	**44 (24/20/0)**	**189 (61/71/57)**

Note. Results show totals followed by individual teacher counts (Taylor/Hudson/Barnett) below.

These results show a variety of interactions that occurred across a range of writing tasks. For Mrs. Hudson and Mrs. Barnett, most of these interactions occurred during procedural/symbolic tasks (43 interactions each), while for Mrs. Taylor, a plurality of interactions occurred during explanatory (inclusive) tasks (24 interactions), followed by explanatory (exclusive) tasks (19 interactions). The results also show that evaluative and descriptive interactions (i.e., teacher-oriented interactions) composed the majority of interactions observed in every participant's lessons (41 out of 61 interactions for Mrs. Taylor, 60 out of 71 interactions for Mrs. Hudson, and 46 out of 57 interactions for Mrs. Barnett). Given the briefness of many such interactions (e.g., “That solution is correct” could be coded as an evaluative interaction), it is unsurprising that these interactions would be more frequent than those that required more extensive open-ended discourse (i.e., assessing and/or advancing questions) between the teacher and student.

Interestingly, the results also show that each participant did engage in student-oriented interactions (i.e., those that used assessing and/or advancing questions), and that such interactions occurred across the range of writing expectations embedded within tasks. Every teacher, for instance, was observed using such questions multiple times with procedural/ symbolic tasks. The results also show how, even as students were working on tasks that required explanatory writing, teachers continued to give quick evaluative or descriptive feedback to students in addition engaging them at times with assessing and/or advancing questions.

Teachers’ Described Interpretations of Students’ Writing in Mathematics

In their post-observation interviews, teachers were asked to interpret samples of students’ writing collected during the observations and captured examples of their interactions with students’ writing from those same lessons. These interpretations inform the third research question, and demonstrate important connections between participants’ described orientations and observed interactions with student writing.

Mrs. Taylor’s reactions to samples of student writing are in line with her focus on writing as a form of explanation and justification. She was asked to interpret samples of posters that student groups created to “prove” the classification of a quadrilateral (e.g., parallelogram) based on four given coordinates that represented the location of the quadrilateral’s vertices. She mentioned how, although the different student groups “got full credit because they did a fabulous job showing me their work and everything like that,” she wished that she had seen more students “showing me how you came up with those answers.” This again showed the way that she differentiated between more symbolic writing and written explanations.

Mrs. Hudson’s interpretations of students’ writing emphasized her previously described focus on building student comfort with sharing their mathematical thinking through both writing and speaking. For instance, she was asked to reflect on students’ writing from an activity where they were solving for unknown exponents (e.g., $3^b * 3^7 = 3^{11}$) and prompted to write out their solutions and reasoning through an app that Mrs. Hudson then used to display and comment on student responses whole class. She described how, even for students’ explanations that were imprecise or used incorrect grammar, she was “ok with that” because students were still “able to verbalize their thoughts or write out their thoughts and reasoning in a way that makes sense to them, and then we transition into some vocab that applies to what they’re saying.” This quote also alludes to Mrs. Hudson’s previously stated emphasis on academic vocabulary.

Mrs. Barnett’s interpretation of student writing samples appeared to reflect her described orientation towards writing as a form of communication. She reflected on a warm-up activity she had enacted where students were asked to describe what they noticed about a graph of the logarithmic function $f(x) = \log_b(x)$, the exponential function $g(x) = b^x$, and the line $y = x$. This was done to demonstrate the inverse relationship between logarithmic and exponential functions, and these responses were then displayed and discussed whole group. She recounted how she used to “get very just fixated on converting from logarithmic to exponential form,” and saw this activity as a way for students to grapple with seeing patterns across different sign systems. This emphasis on translating within and across sign systems appeared in line with Mrs. Barnett’s previous multisemiotic descriptions.

Mrs. Barnett was then asked to reflect on a recording of an interaction she had with students around this task. Her response was revealing in how she thought about her work with students around their writing and aligned with similar comments made by the other participants. The interaction in question was a descriptive response, but Mrs. Barnett stated how "I wish I had asked the students to make sense of each other's responses instead of reading through them myself." This indicated that she was aware of the teacher-oriented nature of the interaction, and expressed a desire for more student ownership and voice during such interactions.

Discussion and Conclusions

This study indicates the multiple dimensions through which teachers orient themselves towards writing in mathematics. Because teacher orientations arise at an intersection of beliefs, perceptions, and practices (Remillard & Bryans, 2004), this investigation is particularly helpful in shedding light on how teachers' descriptions of their orientation towards writing in mathematics compare to their observed interactions with students' writing and their interpretations of such interactions. Because this study employed a case study approach (Yin, 2018), the results are not generalizable. However, the categories devised for analyzing the observation data and the overarching study results offer a framework for future investigations. Importantly, cross-case analysis indicates some salient themes regarding how these teachers orient themselves towards writing in mathematics.

In terms of described orientations, each participants had a distinct semiotic orientation towards writing, from a focus on written explanations and justifications (Mrs. Taylor), to a focus on academic vocabulary (Mrs. Hudson), to a focus on communicating across sign systems (Mrs. Barnett). Although only Mrs. Barnett initially described writing in mathematics across multiple sign systems, all participants were receptive to a multisemiotic (O'Halloran, 2000, 2008) perspective on writing. Each participant was also receptive to a writing-to-learn (Connolly, 1989; Morgan, 1998) perspective, although Mrs. Taylor indicated that this was not essential for teaching mathematics and Mrs. Barnett noted that she generally thinks of writing as a way to summarize what has already been learned.

Although the teachers were receptive to these multisemiotic, writing-to-learn perspectives of writing in mathematics, their observed instruction focused primarily on procedural/symbolic tasks, and most of their interactions were evaluative or descriptive in nature. However, across all types of writing tasks teachers were still observed using assessing and/or advancing questions. This suggests that such teachers play a critical role as a facilitator in determining how students' writing is interpreted and interacted with during instruction. In other words, these teachers are able to enact even closed-ended, procedural tasks in ways that, at times, allow for student-oriented discourse around their written work to arise.

Participants' recognition of teacher-oriented interactions as being "missed opportunities" for student discourse also showed how these teachers could identify different types of interactions with students writing and signaled their preference for more student-oriented interactions. Given that participants' instruction had previously been found to be dialogically oriented (Munter et al., 2015), this could indicate the value that such teachers place upon student discourse. As such, supporting such teachers in building bridges between their orientations towards discourse and orientations towards writing in mathematics may be a compelling way to foster more consistent student-oriented interactions with students' written work.

Ultimately, participants in this study held distinct orientations towards "writing in math." However, their instruction and reflections suggested a strong desire to foster student ownership in the doing of and discourse around mathematics through such writing. By highlighting the connections between written and spoken communication in mathematics, as well as the multiple sign systems through which we communicate such ideas, such teachers might be better able to foster students' effective communication in the learning and doing of mathematics.

References

Boston, M. D., Dillon, F., Smith, M. S., & Miller, S. (2017). Taking Action: Implementing Effective Mathematics Teaching Practices in Grades 9–12. National Council of Teachers of Mathematics.

Candela, A. G. (2019). Exploring the function of member checking. *The Qualitative Report*, *24*(3), 619-628.

Cantrell, S. C., Burns, L. D., & Callaway, P. (2008). Middle-and high-school content area teachers' perceptions about literacy teaching and learning. *Literacy Research and Instruction*, *48*(1), 76-94.

Casa, T. M., Firmender, J. M., Cahill, J., Cardetti, F., Choppin, J. M., Cohen, J., … Zawodniak, R. (2016). Types of and purposes for elementary mathematical writing: Task force recommendations. https://mathwriting.education.uconn.edu. Retrieved Oct 22 2021.

Common Core State Standards Initiative. (2010). Common Core State Standards for mathematics. Retrieved from http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

Connolly, P. (1989). Writing and the ecology of learning. In P. Connolly & T. Vilardi (Eds.), *Writing to learn mathematics and science* (pp. 1-14). Teachers College Press.

Creswell, J. W., & Clark, V. L. P. (2017). Designing and conducting mixed methods research. Sage publications.

Draper, R. J., & Siebert, D. (2010). Rethinking texts, literacies, and literacy across the curriculum. In R.J. Draper, P. Broomhead, A.P. Jensen, J.D. Nokes, & D. Siebert (Eds.) *(Re)imagining content-area literacy instruction.* (pp. 20-39). Teachers College Press.

Erickson, F. (2007). Ways of seeing video: Toward a phenomenology of viewing minimally edited footage. *Video Research in the Learning Sciences*, 145-155.

Goldsmith, L. T., & Seago, N. (2011). Using classroom artifacts to focus teachers' noticing: Affordances and opportunities. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 169-187). Routledge.

Henningsen, M., & Stein, M. K. (1997). Mathematical tasks and student cognition: Classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education*, *28*(5), 524-549. https://doi.org/10.5951/jresematheduc.28.5.0524

Herbel-Eisenmann, B. A. (2007). From intended curriculum to written curriculum: Examining the voice of a mathematics textbook. *Journal for Research in Mathematics Education*, *38*(4), 344-369.

Jacobs, V. R., Lamb, L. L., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education*, *41*(2), 169-202.

Jansen, A., Smith, E., Middleton, J., & Cullicott, C. (2021). Opportunities for Mathematics Engagement in Secondary Teachers' Practice: Validating an Observation Tool. In D. Olanoff, K. Johnson, & S. Spitzer (Eds.), *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Philadelphia, PA.

Morgan, C. (1998). Writing mathematically: The discourse of investigation. Routledge.

Moschkovich, J. (2007). Examining mathematical discourse practices. *For the Learning of Mathematics*, 27(1), 24-30.

Munter, C., Stein, M., & Smith, M. (2015). Dialogic and direct instruction: Two distinct models of mathematics instruction and the debate(s) surrounding them. *Teachers College Record*, *117*(11).

National Council for Teachers of Mathematics. (2000). Principles and standards for school mathematics. Reston, VA

O'Halloran, K. L. (2000). Classroom discourse in mathematics: A multisemiotic analysis. *Linguistics and Education*, *10*(3), 359-388.

O'Halloran, K. L. (2008). Mathematical Discourse: Language, Symbolism and Visual Images. A&C Black.

Pugalee, D. K. (2004). A comparison of verbal and written descriptions of students' problem solving processes. *Educational Studies in Mathematics*, *55*(1), 27-47.

Remillard, J. T., & Bryans, M. B. (2004). Teachers' orientations toward mathematics curriculum materials: Implications for teacher learning. *Journal for Research in Mathematics Education*, *35*(5), 352-388.

Rotman, B. (1988). Towards a semiotics of mathematics. *Semiotica*, 72(1/2), 1–35.

Schnotz, W., Baadte, C., & Müller, A. (2010). Creative thinking and problem solving with depictive and descriptive representations. In *Use of representations in reasoning and problem solving* (pp. 21-45). Routledge.

Sherin, M. G., Linsenmeier, K. A., & van Es, E. A. (2009). Selecting video clips to promote mathematics teachers' discussion of student thinking. *Journal of Teacher Education*, *60*(3), 213-230.
Smith, E. P., Kelly, J., Sappington, S., Warren, K., & Jansen, A. (2023). Using judicious telling to strengthen literacy in mathematics. *Mathematics Teacher: Learning and Teaching PK-12*, *116*(2), 115-123.
Smith, J. P. (1996). Efficacy and teaching mathematics by telling: A challenge for reform. *Journal for Research in Mathematics Education*, *27*(4), 387-402.
van Es, E. A., & Sherin, M. G. (2021). Expanding on prior conceptualizations of teacher noticing. *ZDM–Mathematics Education*, *53*(1), 17-27.
Webb, N. M., Franke, M. L., Ing, M., Wong, J., Fernandez, C. H., Shin, N., & Turrou, A. C. (2014). Engaging with others' mathematical ideas: Interrelationships among student participation, teachers' instructional practices, and learning. *International Journal of Educational Research*, *63*, 79-93.
Yin, R. K. (2018). *Case study research and applications: Design and methods*. SAGE Books.

THE CONTEXT OF TIKTOK AND STUDENT LEARNING OF MATHEMATICS

Laura B. Kent
University of Arkansas
Lkent@uark.edu

This poster session presents the results of a sixth-grade students' use of story contexts to make sense of properties of operations. Results indicate that students found success solving problems based in real-world settings and that also connected to their personal experiences. One of the contexts included a story about the use of social media platforms. Recently, specific platforms, such as TikTok, have been banned by various organizations. This session addresses the use of potentially controversial contexts linked to banned social platforms in mathematics classrooms. Implications for mathematics teachers and teacher educators in terms of curricular choices and authentic practices are recommended for future study as issues of censorship permeate the current cultural and political environment.

The conceptual basis for this study included aspects of Realistic Mathematics Education (or RME) and dimensions of mathematically powerful classrooms. Van den Heuvel-Panhuizen, & Drijvers (2020) discussed several principles related to RME, including the *reality* principle which involves introducing mathematics problem situations that are meaningful to students.

Schoenfeld (2014) described the importance of cognitive demand, access to content, agency, authority & identity that are evident in powerful mathematics classroom settings. Giving students choices of how to approach and solve problems situated within contexts that are real-life and personally relevant to them provided opportunities to actively engage in these dimensions.

The questions addressed in this study were as follows:

- What strategies did students use to solve the TikTok problem?
- How did the social media context influence follow-up discussions between Mrs. G and her students?

This participants in this study included a sixth-grade mathematics teacher with 20 years of experience and her 38 students across three sections and three days of consecutive instruction. Prior to the enactment of the lesson, the teacher, Mrs. G, surveyed her students to determine what they were interested in and thinking about. The responses were, "talking to friends, TikTok, gaming, sports, shoes, fashion (which they call drip)". A percent problem was created using a TikTok context. The purpose of the problem was to assess whether students would use the associative property to facilitate solution to the problem (Barnett & Ding, 2019; Ding, et al., 2021).

The problem posed was, *"Bree spends 75 minutes per day watching TikTok videos. William spends 4% of the amount of time as Bree. How many minutes per day does William spend watching TikTok videos?"* The problem was posed in an open choice format to allow strategy choices and diverse approaches. The results showed that 63% of the students used a valid strategy which included 26% of the students applying the associative property of multiplication in their solution. The TikTok story context was central to students' access to and sense-making of the number relationships in the problem and reflections on the reasonableness of their answer.

There has been concern that the banning of TikTok could lead to further censoring of freedoms in teaching and learning (Koleson, 2020; Azman, et al., 2021). Given that TikTok is such a popular platform for so many it is worth considering whether banning this platform will also influence curricular choices of teachers. Additional research is needed to study the impact of potential censorship on classroom instruction and mathematics learning environments.

References

Azman, A. N., Rezal, N. S. A., Zulkeifli, N. Y., Mat, N. A. S., Saari, I. S., & Ab Hamid, A. S. (2021). Acceptance of TikTok on the youth towards education development. Borneo International Journal eISSN 2636-9826, 4(3), 19-25.

Barnett, E., & Ding, M. (2019). Teaching of the associative property: A natural classroom investigation. Investigations in Mathematics Learning, 11(2), 148-166.

Ding, M., Li, X., Hassler, R., & Barnett, E. (2021). Understanding the properties of operations: a cross-cultural analysis. International Journal of Mathematical Education in Science and Technology, 52(1), 39-64.

Koleson, Jenna, "TikTok is on the Clock, Will Democracy Stop?" (2020). SLU Law Journal Online. 45. https://scholarship.law.slu.edu/lawjournalonline/45

Schoenfeld, A. H. (2014). What makes for powerful classrooms, and how can we support teachers in creating them? A story of research and practice, productively intertwined. Educational researcher, 43(8), 404-412.

Van den Heuvel-Panhuizen, M., & Drijvers, P. (2020). Realistic mathematics education. Encyclopedia of mathematics education, 713-717.

THE DEVELOPMENT OF THE INSTRUCTIONAL VISION OF EARLY CAREER SECONDARY MATHEMATICS TEACHERS

Amanda Huffman Hayes
Purdue University
huffma33@purdue.edu

Using narrative inquiry, I share the beginning stories of the instructional visions of secondary mathematics teachers throughout their undergraduate studies of pedagogy and content-specific methods and evidenced in undergraduate coursework and student teaching experience. Guided by Munter's (2014) Visions of High-Quality Mathematics Instruction Role of the Teacher rubric, I answered the following research question: How do the instructional visions of secondary mathematics teachers develop throughout their undergraduate coursework and student teaching? Understanding the instructional visions of mathematics teachers as evidenced in undergraduate coursework and student teaching reveals the impact of discussions, activities, and readings in their education studies and field experiences.

Keywords: Instructional vision; Instructional activities and practices; Preservice teacher education

The work of teachers is like that of orchestra conductors using educational imagination and rules of thumb to continue conversation throughout the class, unlike that of technicians with a set of scientific prescriptions (Eisner, 1983). While conducting their orchestra, teachers make decisions about their teaching and student learning while continually working toward making music. To improve their practice, teachers make several conscious decisions (Leatham, 2006) while working toward *good* or *quality* teaching (Arbaugh et al., 2021; Beswick, 2007; Ernest, 1991; Hammerness, 2001). But what does this encompass? Ernest (1991) initially presented three variations to the definition of *quality* mathematics teaching: (a) students achieve success (e.g., achieving success on standardized assessments), (b) the teacher uses a wide array of teaching strategies, and (c) the teacher is influenced personal epistemology and mathematics-related beliefs system and its realization and transfer into practice. However, only in the third conceptualization does the teacher have much control. In the first two conceptions, there are several conditions outside the control of the teacher (e.g., changing standardized assessments, strategic compliance on practices used in the classroom). Thus, many mathematical researchers (e.g., Ball, 1990; Pajares, 1992; van Putten et al., 2014) focused on identifying teachers' beliefs and the impact of teachers' beliefs on mathematics instruction and students' learning.

Over time, as more researchers continued to study personal (i.e., teacher) impacts on teaching, some researchers (e.g., Arbaugh et al., 2021; Hammerness, 2001; Munter & Wilhelm, 2021) shifted to discussing teachers' vision, or more specifically instructional vision. This shift is a result of the lack of a clear definition of beliefs (Philipp, 2007), since some researchers (e.g., Pajares, 2014; van Putten et al., 2014) have used the term belief without defining it. For example, Parjares (1992) called defining beliefs "at best a game of player's choice" (p. 309). Munter (2014) addressed his use of the term "vision" instead of beliefs as "beliefs suggest a relatively static set of decontextualized ontological commitments" (p. 587), where vision is intended to communicate "a more dynamic view of the future" (p. 587). So while instructional vision and beliefs could be interpreted similarly, the focus of my research will be on the instructional visions of secondary mathematics teachers, including its dynamic view of the future, particularly when observing early career teachers striving to put their instructional vision into practice.

Vision has been consistently defined in the literature on mathematics education. Hammerness (2001) defined *teacher vision* as "a set of images of ideal classroom practice for which teachers' strive" (p. 143). Building on Hammerness's (2001) definition, Jansen et al. (2020) defined a *teacher's vision* as "an idealized image of classroom practice" (p. 184) encompassing "their aspirations or hopes of what could occur in their classrooms, schools, communities, and even society" (p. 184). In time, researchers started using more specific phrases, particularly *instructional vision*. Arbaugh et al. (2021) defined *instructional vision* as "a set of internal images that portray the work of teaching mathematics" (p. 449). Similarly, Munter and Wilhelm (2021) defined *instructional vision* "as the discourse that teachers or others currently employ to characterize the kind of 'ideal classroom practice' to which they aspire but have not yet necessarily mastered" (p. 343). These discourses include "both descriptions of what high-quality instruction looks and sounds like as well as rationales in terms of what it accomplishes" (p. 343). The instructional vision of teachers, particularly early career teachers, is significant as it depicts what teachers want to see and hear in their classroom.

Teachers' instructional visions matter (Feiman-Nemser, 2001; Viholainen et al., 2014; Woods & Wilhelm, 2020). Looking at the first stage of Feiman-Nemser's (2001) *Central Tasks of Learning to Teach*, learning to teach starts as preservice teachers critically contemplate their beliefs and how they are situated or related to their own vision of good teaching. Sfard (2007), in relation to students' mathematical understanding, found that a change in our discussion around how students develop mathematical understanding needs to happen before students can experience a change in something. The same can be thought of for good teaching; first, teachers discuss the changes they want to see, then they make them happen. Thus, adopting a reflective practice in teaching can lead to changes, often for the better.

Jansen et al. (2020) investigated the impact of teacher education programs on the instructional visions of mathematics teachers. Through online questionnaires, their data consisted of participants self-reported instructional vision for teaching elementary mathematics, the vision promoted by their teacher education program, the influence or lack of influence from their education program on their instructional vision, and if they thought their visions changed over time. Their findings raised awareness around how teacher education programs can impact the instructional visions of their graduates as they found an alignment related to developing students' conceptual understanding and engaging students in a productive struggle in their teachers' instructional visions and pre-service program experiences. However, they found these novice instructional visions can change with more classroom experience and professional development.

Arbaugh et al. (2021) and Munter and Wilhelm (2021) extended the body of research on the instructional vision of teachers. Arbaugh et al. (2021) examined, in a semester-long course, changes in the instructional visions of secondary mathematics preservice teachers of teaching mathematics. Although they found a positive change (e.g., moving from understanding teaching as telling to teaching as facilitating), they wondered about the permanence of such changes. This identified the need for longitudinal studies "to examine whether such changes in vision persist into teachers' early careers, including whether... [they] continue to change, stagnate, or revert" (p. 459). Munter and Wilhelm (2021), through the lens of "how teachers develop more sophisticated instructional visions" (p. 344), investigated instructional vision of middle school mathematics teachers. They found several factors contributing to the development of a more sophisticated instructional vision, including: (a) mathematical knowledge for teaching, (b) instructional practice, (c) instructional views of colleagues, and (d) frequency of interaction with colleagues. Jansen et al. (2020), Arbaugh et al. (2021), and Munter and Wilhelm (2021) encouraged looking to teachers' environments (e.g., school and community) when making sense of their instructional vision and any changes that might or might not be occurring.

My Positionality

As a former high school mathematics teacher, Sfard's (2007) notion of first changing how we discuss mathematical learning of students before changes take place resonated with me as I reflected on my own ideas before acting on them. I recall thinking about my future classroom as an undergraduate. Many professors challenged me in various courses to think about what we wanted our classroom to be like, what students would be doing, and what we, the teacher, would be doing. Woods and Wilhelm (2020) similarly noted the importance of early career teachers' instructional visions and how their initial instructional vision can impact the pedagogical resources they use. They also claimed that while teachers' instructional visions are often evolving, their early career instructional visions are often the most critical as early career teachers begin to implement various instructional practices. This resonated with me in reflecting on my own experience in the classroom. Just as I have had my own unique experiences related to the evolution of my instructional vision throughout nine years in a secondary mathematics classroom, so have other teachers. Thus, in my longitudinal research study, I examine the instructional visions of secondary mathematics teachers throughout their early career. Here, I share the initial phases of study (i.e., undergraduate coursework and student teaching) guided by my research question, *How do the instructional visions of secondary mathematics teachers develop throughout their undergraduate coursework and student teaching?*

Theoretical Framework

In analyzing secondary mathematics teachers' instructional visions, I used Munter's (2014) Vision of High-quality Mathematics Instruction (VHQMI) Role of the Teacher rubric. Munter's VHQMI rubrics focus on the development of teachers' visions of high-quality mathematics instruction. The development of teachers' visions is one key that differentiates Munter's definition and rubrics from others. Munter's VHQMI rubrics were developed from other theoretical perspectives, including Carpenter and Lehrer's (1999) notion of *learning with understanding*, Cobb and Bauersfeld's (1995) *emergent perspective* on learning, and Rogoff et al.'s (1996) *community of learners*. Through conducting research in their own classrooms, Carpenter and Lehrer (1999) found five forms of classroom activity through which mathematical understanding can emerge: "(a) constructing relationships between current and more sophisticated ways of thinking; (b) building rich, integrated knowledge structures; (c) reflecting; (d) articulating what one knows; and (e) making mathematical knowledge one's own by coming to perceive it as evolving and authoring one's own learning" (Munter, 2014, p. 589). Carpenter and Lehrer's (1999) five forms of activity align with Cobb and Bauersfeld's (1995) emergent perspective, as the five forms of activity are social practices through which individual knowledge construction, as well as mathematical learning, occur. Engaging in these social practices, we have the teacher and the students, or the community of learners as defined by Rogoff et al. (1996), existing somewhere on a continuum between teacher-led and student-led.

These combined perspectives contributed to Munter's (2014) VHQMI, which he divided into three dimensions: role of the teacher, classroom discourse, and mathematical tasks. In this study, I focus on Munter's (2014) role of the teacher: (a) *motivator*, (b) *deliverer of knowledge*, (c) *monitor*, (d) *facilitator of knowledge*, and (e) *more knowledgeable other* (see Table 1).

Table 1: Abbreviated VHQMI Role of the Teacher (Woods & Wilhelm, 2020, p. 117)

Role of the Teacher	Description
Motivator	Describes the teacher as being able to captivate and hold students' attention.
Deliverer of knowledge	Describes the teacher as the primary source of knowledge. Here, the teacher focuses on mathematical correctness and his/her thoroughness of explanations.
Monitor	Describes the teacher as the primary source of knowledge. However, the teacher emphasizes the importance of providing time for students to work together to make sense of what he/she has demonstrated.
Facilitator of knowledge	The teacher is focused on "reform instruction" but may not have a strong understanding of the pedagogical strategies that underlie those forms.
More knowledgeable other	The teacher describes his/her role as proactively supporting students' learning through participation. Further, the teacher emphasizes the importance of designing a learning environment that supports problematizing mathematical ideas, giving students voice and mathematical authority, and holding students accountable for their learning.

Munter (2014) used other existing research on effective mathematics instruction (i.e., Hiebert et al., 1997; Jackson et al., 2013; Lampert, 1990; Lobato, Clarke, & Ellis, 2005; Madsen-Nason & Lappan, 1987; Staples, 2007; Stigler & Hiebert, 1999) and data collected as part of the Middle School Mathematics and the Institutional Setting of Teaching (MIST; Cobb & Smith 2008) to identify three ways in which to characterize the role of the teacher: "(a) conception of typical activity structure (i.e., the teacher's general mode of instruction and role in classroom activities), (b) attribution of mathematical authority (i.e., students' roles with respect to the mathematics being learned), and (c) influence on classroom discourse" (p. 600). These three characterizations were then factored into the descriptions of the roles of the teacher.

First, Munter (2014) described a teacher as a *motivator*, or one who captivates and maintains the students' attention. For teachers who are *deliverers of knowledge*, instruction primarily encompasses teacher-to-student discourse with the teacher providing direct instruction with little discussion. The validity of mathematical tasks comes from the textbook or the teacher (Simon, 1994) when teachers are the *deliverers of knowledge*. Teachers who play the role of *monitors* supervise student-to-student discussion. If students stray toward inaccurate reasoning, the teacher resets them (Munter, 2014). According to Stigler and Hiebert (1999), the activity structure when teachers are *monitors* encompasses a two-phase *acquisition and application* lesson. The teacher leads a discussion or demonstration, and then the students work in groups with what was just demonstrated, while the teacher circulates around the classroom.

When looking at teachers who are the *facilitator of knowledge* and *more knowledgeable other*, the role of the teacher shifts as instruction becomes more student-led (Munter, 2014). Lobato, Clarke, and Ellis (2005) describe the classroom discourse with a *facilitator of knowledge* as including student-to-student talk with the teacher facilitating and hesitant to guide students so as not to interfere with the discovery process. Students are discovering and teaching themselves. Unlike when the teacher is a *monitor*, if students begin to stray toward inaccurate knowledge, the teacher attempts to guide students back on track through questioning (Munter, 2014). When teachers *facilitate knowledge*, Madsen-Nason and Lappan's (1987) *launch-explore-summarize* (LES) structure of a lesson unfolds with the teacher posing a task(s), the students working, typically in groups, to complete the task(s), and the students revealing and explaining their solutions at the end of the lesson. The teacher may also clarify the concept if necessary.

Teachers as *more knowledgeable others* go beyond facilitating (Munter, 2014), while continuing to model Madsen-Nason and Lappan's (1987) *launch-explore-summarize* lesson structure. Students' ideas are elicited to create a shared context through discussions (Staples, 2007). The teacher supports the learning of the students. Students work toward a shared goal through problematizing content (Hiebert et al., 1997), and instead of validity being provided by the textbook or teacher, it comes from the classroom community (Simon, 1994) as the students and teacher share the authority in learning (Lampert, 1990).

Methods

This project is currently in Phase III of a longitudinal case study (Merriam & Tisdell, 2016) in which I am investigating the instructional visions of six secondary mathematics teachers from their undergraduate studies into their early careers. The findings shared here will include data from both Phase I, undergraduate coursework, and Phase II, student teaching as data in Phase III, interviews during their first year of teach, is still ongoing in regard to data collection and analysis. To gain an understanding of each early career secondary mathematics teacher, I will compile an in-depth description and analysis in their respective contexts will be conducted as part of the case study (Merriam & Tisdell, 2016; Yin, 2014).

Participants, Context, and Data Sources

The six participants, identified using pseudonyms, completed their undergraduate studies in secondary mathematics education at a large midwestern university. The participants included three self-identifying males and three self-identifying females. One of the female participants is African American, while the other five participants are Caucasian. Two participants, one male and one female, completed their student teaching in a midwestern rural school, while the others completed their student teaching in midwestern urban schools. In addition to student teaching, each participant spent at least 30 hours in other middle school and high school mathematics classrooms and also individually taught a section of an entry-level college algebra course at their large midwestern university during the semester prior to student teaching.

The secondary mathematics education program in which the participants were enrolled was focused on collaborative, student-centered learning designed to develop students' conceptual understanding, mathematical reasoning, and problem solving in addition to their procedural fluency. Students wrote several lesson plans following the LES model of Madsen-Nason and Lappan (1987) and were asked to consider knowledge about their students that influences their planning, a universal design for learning with multiple learning supports through representation, action, expression, and engagement (Hunt, 2011), as well as an assessment plan (both summative and formative), differentiation, and accommodations. All of these elements of lesson planning, including how to successfully lead and launch, and what makes a good mathematical task, were discussed extensively in their two secondary mathematics methods courses. Students practiced peer teaching before teaching in their field experiences in conjunction with the methods courses.

Three sources of written data were collected during Phase I: (1) philosophy of education originally drafted in an introductory education course (Artifact I), (2) mathematics classroom vision written a year before student teaching in the first of two secondary mathematics methods courses (Artifact II), and (3) revised philosophy of education written the semester prior to student teaching in the second of two secondary mathematics methods courses (Artifact III).

The data collected in Phase II included two semi-structured interviews (Merriam & Tisdell, 2016) during student teaching, one about half-way through their experience (Interview IA) and another one at the conclusion of their student teaching (Interview IB). Additional data sources for the longitudinal study will include two semi-structured conversational interviews during each of their first two years of teaching mathematics (Phase III and Phase IV). Additionally, another source of data takes the form of research memos I continue to write throughout the data collection and analysis my study (Bogdan & Biklen, 2011).

Data Analysis: Interpreting Instructional Visions

As I began data analysis, I read, re-read, and took notes on key elements in the artifacts to make meaning of what was represented both implicitly and explicitly in the data (Bogdan & Biklen, 2011; Flick, 2014). In reading and re-reading the artifacts, I looked for themes around Munter's (2014) descriptors of the role of teacher, including student talk, launch, explore, summary, groups, and conversations related to the description of the discourse in the classroom, the structure of the activity, and who had the mathematical authority in the classroom. Triangulation amongst each of the artifacts and research memos was used to check the validity and reliability (Merriam & Tisdell, 2016). These key words and descriptors combined with my initial notes allowed me to determine how each of the instructional visions of the participants aligned with Munter's (2014) VHQMI Role of the Teacher rubric.

After collecting all the interviews during student teaching and transcribing them, I began the same process in looking for the same key descriptors. I completed this process for each participant before moving on to the interview transcripts of the next participant. Throughout all of the analysis I looked for reoccurring regularities or shifts in the key descriptors (Merriam & Tisdell, 2016) to uncover the alignment with Munter's (2014) VHQMI Role of the Teacher rubric.

Findings

Here, I share the development of the instructional vision of Allison and Corey as they represent the extremes in my sample. Allison's instructional vision was challenged in her students teaching, while Corey's instructional vision was refined. The obstacles Allison faced while further developing her instructional vision were unlike others, as was the freedom Corey experienced in trying new ideas while cultivating his own instructional vision. I focus on the key descriptors found in their undergraduate coursework (Artifact I, II and III) and the transcripts of the semi-structured interviews during their student teaching experience (Interview IA and IB) and how they correspond to Munter's (2014) role of the teacher rubric.

Allison

Allison, a Caucasian female, described her classroom as having students in clusters so that students could "communicate with each other" (Artifact I). In her philosophy (Artifact II), she described how "teachers can best help students learn by making them feel supported, rather than being dominated by an authority figure. Teachers should guide students toward success rather than forcing end results upon them." Similarly, she noted: "if the students have some input on how the class should run, they will be more likely to follow the rules and engage in the class because they will feel autonomous." These key descriptors focused on the active role students play in the mathematics classroom and Allison's role as the *more knowledgeable other*. Another key descriptor found in her philosophy of education was how she planned to "teach math using LES (launch-explore-summarize) and not focusing on lecturing for students to just understand the concepts surface-level and procedurally." This indicates her role as *a facilitator of knowledge* or a *more knowledgeable other* depending on the specific LES lesson.

Allison, who attended a large urban midwestern high school, completed her 16 weeks of student teaching at a small rural midwestern high school. She described in her first interview her cooperating teacher as almost completely opposite in regard to instructional vision, as she was a *motivator* or *deliverer of knowledge*. She found challenges with adjusting to this environment where many students do not graduate from high school and her cooperating teacher's style.

About a third of the way into her student teaching, Allison received new groups of students when the third trimester started in early March. Allison started teaching full-time and was hopeful that the student-centered strategies she had previously attempted would be more successful with new groups of students. Allison wanted to try the LES structure and wanted her students to work collaboratively. But while Allison's instructional vision still appeared to align with the characteristics of *facilitator of knowledge* and *more knowledgeable other*, her classroom became more of a place in which she was the *monitor* or *deliverer of knowledge*, the place at which her students were most comfortable.

Allison encountered additional challenges in all three of her classes (Algebra I Repeat, Algebra II, and Finite) where students did not have the requisite prior knowledge to learn the new content. For example, in Interview IB, Allison talked about her three-day plan for teaching piecewise functions in Algebra II. On the first day, the plan "exploded" as the students did not have a conceptual understanding of how to find the domain and range of functions on a graph. As a result, Allison shifted to direct instruction to help her students understand domain and range. Allison noted she wanted to "improve what I am doing,..., and to not have to teach to the test. I hate that. I like want the students to understand things more than just the problem that is on the test" (Interview IA). So, while Allison's student teaching turned into primarily direct instruction based on her circumstances and the role she exhibited was a *motivator* and a *deliverer of knowledge*, she was not happy and was eager to try something different in her own classroom. Following Allison into her first year of teaching will allow me to see how her instructional vision continues to develop.

Corey

"Accessibility and group dynamics will be very important to the physical layout," Corey, a Caucasian male, noted in his classroom vision (Artifact I). Corey's "goal is to lower as many physical barriers as possible in the way of students' conversations and ideations." These descriptors demonstrate Corey's eagerness for student participation and student-centered learning as the *facilitator of knowledge* or the *more knowledgeable other*. Corey continued to discuss student engagement by noting the importance of questions and how "students in my class should know that their ideas and questions are valuable." Corey envisioned his students discovering math by questioning and seeking their own answers. Corey went on to state, "rarely do I want to just tell students what the correct answer or correct method is; I'd like to encourage them to look inward to see if they can't follow their own intuition to a solution."

Corey's revised philosophy of education (Artifact II) echoed much of his previous classroom vision. In his philosophy of education, he added elements related to student learning, including productive struggle and "that (productive) struggle is necessary to conceptual understanding." He compared solving real world problems to that of productive struggle in a mathematics classroom: "very few problems in the real world come with pre-defined solutions, and most of them require some critical thinking, trial and error, and logical reasoning skills to solve." He concluded, "I still believe that my classroom can be a valuable asset to students' development as critical thinkers and persistent learners." These descriptions in his philosophy (i.e., questioning, student thinking and discovery, productive struggle, trial and error) situate Corey's instructional vision as that of a *facilitator of knowledge* or a *more knowledgeable other*.

Corey went on to student teach at a small primarily Caucasian rural midwestern junior/senior high school and worked with three cooperating teachers while teaching two sections of seventh grade math, a section of co-taught Algebra I with freshman, a section of Algebra I with eighth graders, and two sections of trigonometry with eleventh and twelfth graders. In working with three cooperating teachers and teaching three different courses, Corey reported learning a lot as he continued to refine his own instructional vision while trying to put some of his ideas into practice. In Interview IA, Corey mentioned was how he was finally starting to build student-teacher relationships with his students, and more so 7th graders. He found that his lessons were improving because he was able to relate to his students more. When talking about his lessons, Corey often talked about the launch, explore, and summary, providing evidence for his use of the LES model and situating himself as the *facilitator of knowledge* or the *more knowledgeable other*. In Interview IB, Corey described how his students would offer ideas and suggestions, but noted that he was leading them or acting as the *facilitator of knowledge*. Corey's variety of classes and cooperating teachers facilitated his exploration of activities and teaching styles, allowing him to take the role of *facilitator of knowledge* or even a *more knowledgeable other*.

Discussion and Implications

Perhaps not surprisingly, as Jansen et al. (2020) and Arbaugh et al. (2021) found similar results, the instructional visions of the secondary mathematics teachers were similar to and aligned with the ideals emphasized in their secondary mathematics education program, particularly relative to their undergraduate coursework (i.e., classroom vision and philosophy of education). The key words and phrases in their writing and in the interview transcripts aligned closely with the ideals of Munter's (2014) VHQIM role of teacher as *facilitator of knowledge* and the *more knowledgeable other.* Their undergraduate coursework emphasized the LES model, which was often mentioned in the interviews and in their descriptions of ideal lessons. In addition, collaborative learning through group work and class discussions with student talk was emphasized in their writing and interviews. Knowing the presence of these similarities between the instructional visions of secondary mathematics teachers and the approach of the secondary mathematics education program the teachers attended shows the impact of one's program on individual teachers. This stresses the importance for us as mathematics teacher educators to share with our future mathematics teachers how mathematics can be taught even though it may be different from how one learned mathematics or even observes mathematics being taught in the field. Readings, reflections, activities, and discussions in the undergraduate coursework impact the instructional visions of secondary mathematics teachers. The instructional vision of the cooperating teachers and classroom structures while student teaching also shaped their instructional visions. For Allison, she was challenged by this and experienced teaching in ways she herself was not happy with, but worked in the environment in which she was in. Corey, on the other hand, with multiple cooperating teachers, was able to try more while continuing to build on his instructional vision that he developed in his undergraduate coursework. As Arbaugh and colleagues (2021) questioned the permanence of the instructional visions of early-career mathematics teachers, there is still much to be explored as my study continues to follow these teachers into their early career in their own classrooms. One area of future exploration could be to include the instructional vision of the novice teachers' mentor teachers in order to see the possible impact of their cooperating teacher more clearly. Through continuing my study, I hope to see how these early career teachers begin to conduct their own orchestras (Eisner, 1983).

References

Arbaugh, F., Graysay, D., Freeburn, B., & Konuk, N. (2021). Investigating secondary mathematics preservice teachers' instructional vision: Learning to teach through pedagogies of practice. *Journal of Teacher Education, 72*(4), 448-462. https://doi.org/10.1177/0022487120965938

Ball, D. L. (1990). Breaking with experience in learning to teach mathematics: The role of a preservice methods course. *For the Learning of Mathematics, 10*(2), 10-16.

Beswick, K. (2007). Teachers' beliefs that matter in secondary mathematics classrooms. *Educational Studies in Mathematics, 65*, 95-120. https://doi.org/10.1007/s10649-006-9035-3

Bogdan, R. C., & Biklen, S. K. (2011). *Qualitative research for education: An introduction to theories and methods* (5th ed.). Pearson.

Carpenter, T. P., & Lehrer, R. (1999). Teaching and learning mathematics with understanding. In E. Fennema & T. A. Romberg (Eds.), *Mathematics classrooms that promote understanding* (pp.19-32). Erlbaum.

Cobb, P., & Bauersfeld, H. (1995). Introduction: The coordination of psychological and sociological perspectives in mathematics education. In P.Cobb & H. Bauersfeld (Eds.), *Emergence of mathematical meaning: Interaction in classroom cultures* (pp. 1-16). Erlbaum.

Cobb, P., & Smith, T. (2008). District development as a means of improving mathematics teaching and learning at scale. In K. Krainer & T. Wood (Eds.), *International handbook of mathematics teacher education: Vol 3. Participants in mathematics teacher education: Individuals, teams, communities, and networks* (pp. 231-254). Sense Publishers.

Eisner, E. W. (1983). The art and craft of teaching. *Educational Leadership, 40*(4), 4-13.

Ernest, P. (1991). Mathematics teacher education and quality. *Assessment and Evaluation in Higher Education, 16*(1), 56-65. https://doi.org/10.1080/0260293910160107

Feiman-Nemser, S. (2001). From preparation to practice: Designing a continuum to strengthen and sustain teaching. *Teachers College Record, 103*(6), 1013-1055. https://doi.org/10.1111/0161-4681.00141

Flick, U. (2014). Mapping the field. In U. Flick (Ed.), *The sage handbook of qualitative data analysis* (pp. 3-18). Sage.

Gougen, J.A., & Linde, C. (1993). Techniques for requirements elicitation. *Proceedings of the IEEE International Symposium on Requirements Engineering* (pp. 152-164). San Diego, CA. https://doi.org/10.1109/ISRE.1993.324843

Hammerness, K. (2001). Teachers' visions: The role of personal ideals in school reform. *Journal of Educational Change, 2*(2), 143-163. https://doi.org/10.1023/A:1017961615264

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K., Wearne, D., Murray, H., ... , Human, P. (1997). *Making sense: Teaching and learning mathematics with understanding*. Heinemann.

Hunt, J. H., & Andreasen, J. B. (2011). Making the most of universal design for learning. *Mathematics Teaching in the Middle School, 17*(3), 166-172. https://doi.org/10.5951/mathteacmiddscho.17.3.0166

Jansen, A., Gallivan, H. R., & Miller, E. (2020). Early-career teachers' instructional visions for mathematics teaching: impact of elementary teacher education. *Journal of Mathematics Teacher Education, 23,* 183-207. https://doi.org/10.1007/s10857-018-9419-1

Leatham, K. R. (2006). Viewing mathematics teachers' beliefs as sensible systems. *Journal of Mathematics Teacher Education, 9,* 91-102. https://doi.org/10.1007/s10857-006-9006-8

Lobato, J., Clarke, D., & Ellis, A. B. (2005). Initiating and eliciting in teaching: A reformulation of telling. *Journal for Research in Mathematics Education, 36*(2), 101-136.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal, 27*(1), 29-63. https://doi.org/10.3102/00028312027001029

Madsen-Nason, A., & Lapan, G. (1987). The middle grades mathematics project: Coaching as a strategy for changing teacher practice. In J.C. Bergeron, N. Herscovics, & C. Kieran (Eds.), *Proceedings of the 11th International Conference on the Psychology of Mathematics Education* (Vol. 2, pp. 100-106). http://files.eric.ed.gove/fulltext/ED383532.pdf#page=587

Merriam, S. B., & Tisdell, E. J. (2016). *Qualitative research: A guide to design and implementation.* Jossey-Bass.

Munter, C. (2014). Developing visions of high-quality mathematics instruction. *Journal for Research in Mathematics Education, 45*(5), 584-635. https://doi.org/10.5951/jresematheduc.45.5.0584

Munter, C., & Wilhelm, A. G. (2021). Mathematics teachers' knowledge, networks, practice, and change in instructional visions. *Journal of Teacher Education, 72*(3), 342-354. https://doi.org/10.1177/0022487120949836

Philipp, R. A. (2007). Mathematics teachers' beliefs and affect. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol. 1, pp. 257-315). Information Age Publishing.

Pajares, M. F. (1992). Teachers' beliefs and education research: Cleaning up a messy construct. *Review of Educational Research, 62*(3), 307-332.

Rogoff, B., Matusov, E., & White, C. (1996). Models of teaching and learning: Participation in community of learners. In D. R. Olson & N. Torrance (Eds.), *The handbook of education and human development* (pp. 388-414). Blackwell Publishers.

Sfard, A. (2007). When the rules of discourse change, but nobody tells you: Making sense of mathematics learning from a commognitive standpoint. *Journal of the Learning Sciences, 16*(4), 565-613. https://doi.org/10.1080/10508400701525253

Sherin, M. G. (2001). Developoing a professional vision of classroom events. In T. Wood, B. Scott Nelson, & J. Wafield (Eds.), *Beyond classical pedagogy: Teaching elementary school mathematics* (pp. 75-93). Erlbaum.

Simon, M. A. (1994). Learning mathematics and learning to teach: Learning cycles in mathematics teacher education. *Educational Studies in Mathematics, 26*(1), 114-155. https://doi.org/10.2307/749205

Staples, M. (2007). Supporting whole-class collaborative inquiry in a secondary mathematics classroom. *Cognition and Instruction, 25*(2-3), 161-217. https://doi.org/10.1080/07370000701301125

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom.* Free Press.

van Putten, S., Stols, G., & Howie, S. (2014). Do prospective mathematics teachers teach who they say they are?. *Journal of Mathematics Teacher Education, 17*, 369-392. https://doi.org/10.1007/s10857-013-9265-0

Viholainen, A., Asikainen M., & Hirvonen. (2014). Mathematics student teachers' epistemological beliefs about the nature of mathematics and the goals of mathematics teaching and learning in the beginning of their studies. *Eurasia Journal of Mathematics, Science, & Technology Education, 10*(2), 159-171. https://doi.org/10.12973/eurasia.2014/1028a

Woods, D. M., & Wilhelm, A. G. (2020). Learning to launch complex tasks: How instructional visions influence the exploration of the practice. *Mathematics Teacher Educator, 8*(3), 105-119. https://doi.org/10.5951/MTE.2020.0010

Yin, R. K. (2014). *Case study research: Design and methods* (5th ed.). Sage.

THE TEACHER'S ROLE IN FOSTERING COLLECTIVE CREATIVITY IN ELEMENTARY CLASSROOM SETTINGS

Ayman Aljarrah
Acadia University
ayman.aljarrah@acadiau.ca

Jo Towers
University of Calgary
towers@ucalgary.ca

In a research study designed to investigate the emergence of collective creativity in elementary classroom settings, and in which teachers' decision-making practices were analyzed alongside both the teachers' observed teaching practices in their classrooms and their students' problem solving actions, the first author developed four metaphors for collective mathematical creativity and linked the entailments of these metaphors to teachers' actions. In this paper, we discuss in detail these entailments and teacher actions, present a framework for collective creativity, and reflect on the implications for practice and further research.

Keywords: Elementary school education, Instructional activities and practices, Teacher beliefs

This paper is based on data collected by the first author in a research study designed to investigate the emergence of collective creativity in elementary classroom settings. In the study, teachers' decision-making practices were analyzed alongside both the teachers' observed teaching practices in their classrooms and their students' problem solving actions. Drawing on metaphors for collective mathematical creativity developed by the first author (Aljarrah, 2020; Aljarrah & Towers, 2019), and connecting teachers' responses to questions about their practice with both their observed teaching practices in their classrooms and their students' problem solving actions, in this paper we explore the entailments of the metaphors and their relationship to teachers' actions and present a framework for analyzing and understanding collective creativity in teaching and learning environments. We rely on data from interviews with the participating teachers, as well as examples of their students' mathematical activity. Elsewhere, we have presented detailed descriptions of the student actions and interactions that we claim showed evidence of the students' collective mathematical creativity (e.g., Aljarrah, 2020; Aljarrah & Towers, 2019, 2021, 2022). Here, we focus on the features of learning environments that foster such creativity.

Literature Review

Creativity—Individual and Collective

Researchers in the field of creativity have struggled to develop and sustain applicable pedagogical innovations that respond to the problem of fostering creativity within different contexts. In his discussion of the question, "How does a person learn to be creative?", Huebner (1967) saw that "the very question itself demands a definition of the word creative" (p. 134). While some see creativity as confined to special people, to particular arts-based activities, or to undisciplined play, Craft (2000) described it as "a state of mind in which all our intelligences are working together" (p. 38). The National Advisory Committee on Creative and Cultural Education (NACCCE, 1999) defined creativity as "imaginative activity fashioned so as to produce outcomes that are both original and of value" (p. 29). Another approach to creativity, which can be a good starting point for teachers who aim to teach for creativity, is Baer's (1997) conception of creativity as a continuum; that is, it is "not something that a person either has in abundance or lacks entirely" (p. 2). Very close to Baer's conception of creativity is Kaufman and

Beghetto's four-c model of creativity (Beghetto & Kaufman, 2007; Kaufman & Beghetto, 2009). In their model, Kaufman and Beghetto described four types/levels of creativity: interpretative or mini-c creativity, every-day or little-c creativity, expert or pro-c creativity, and legendary or big-c creativity. They refer to everyday creativity as the creative actions of the non-expert. According to Beghetto and Kaufman (2011), little-c and mini-c creativity are more appropriate in the classroom context. For example, elementary students' ideas for a science experiment can be considered creative in the context of the elementary school. Beghetto and Kaufman (2011) argued that "a students' [sic] novel and personally meaningful insight or interpretations (which occur with great frequency while learning) are important sources of larger-c creative potential" (p. 98).

Regarding collective creativity, Sawyer (2003) asserted the improvised and collective nature of group creativity. For Sawyer, group creativity is: (1) unpredictable, in that each moment emerges from the preceding flow of the performance, (2) collective, in that members of the group influence each other from moment to moment, and (3) emergent, in that the group demonstrates properties greater than the sum of its individuals. Based on the above ideas, Aljarrah (2018) defined collective creative acts as particular kinds of "(co)actions and interactions of a group of curious learners while they are working collaboratively on an engaging problematic situation. Such acts, which may include (1) summing forces, (2) expanding possibilities, (3) divergent thinking, and (4) assembling things in new ways, trigger the new and the crucial to emerge and evolve" (p. 136).

The Teacher's Role in the Development of Students' Creativity

According to Silver (1997), the contemporary view of creativity "is closely related to deep, flexible knowledge in content domains; is often associated with long periods of work and reflection rather than rapid, exceptional insight; and is susceptible to instructional and experiential influences" (p. 75). This view led Silver to advocate for creativity-enriched instruction for all students. To achieve this goal, Silver (1997) suggested the use of problem posing and problem-solving activities to stimulate inquiry-based mathematics instruction, in which "the responsibility for problem formulation and solution is shared between teacher and students" (p. 77).

Levenson (2013) explored teachers' perceptions of the cognitive demands associated with tasks that have the potential to promote mathematical creativity. Teachers identified several cognitive demands: the ability of the task to connect separate mathematical topics, the possibility to promote "non-algorithmic and nonstandard thinking skills" (p. 285), and the employment of "mathematical thinking freely and flexibly with the help of the mathematical skills acquired during [one's] life" (p. 286). According to Levenson (2013), equally important to the choosing of the task is how the task will be implemented.

Lev-Zamir and Leikin (2013) investigated the relationship between teachers' declarative conceptions of creativity and their conceptions-in-action. Lev-Zamir and Leikin noted that although teachers' theoretical background in creativity is important, it is not enough; equally important is how they implement their understandings of creativity in the classroom.

Davies et al. (2014) conducted a systematic literature review to explore the teacher's role in promoting creativity, and ways in which teachers can be best supported to develop the skills and confidence to facilitate creative learning environments. Their findings revealed that teachers play an important role in creating learning environments that promote their students' creativity through "building positive relationships, modelling creative behaviour, longer-term curriculum planning, striking a balance between freedom and structure, allowing flexible use of space, understanding learners' needs and learning styles, creating opportunities for peer collaboration and assessment, and effective use of resources" (Davies et al., 2014, p. 39). It was also noted that to fulfill such requirements, teachers need to have positive attitudes towards creativity and the self-confidence to teach in particular ways. This is a daunting list of expected actions and competencies for teachers. In what follows we present, explore, and examine entailments of various metaphors for collective mathematical creativity that suggests ways forward for teachers wishing to develop these skills and foster collective mathematical creativity in the classroom.

Methods

The data described below were collected during a design-based research study exploring collective creativity in elementary mathematics learning environments (Aljarrah, 2018). Two mathematics teachers and 25 of their sixth-grade students in a Canadian school setting participated in the study. The core research questions for the study were:*1) What can be learned from the process of developing and refining an emergent definition of collective creativity for the elementary mathematics classroom? 2) Does collective creativity emerge in elementary mathematics classroom settings? 3) How can we foster collective creativity in elementary mathematics classroom settings, and what is the teacher's role in this endeavor? and 4) What use might the construct of collective creativity be to teachers of mathematics?*

As the study aimed to explore collective creativity, the design included having the researcher (the first author) gather and design mathematics problems that it was hoped would allow for student collaboration, engagement, and group work and bring these problems forward for enactment by the teachers in their classrooms and in researcher-led, small-group, task-based interviews with students outside scheduled class times. Additional materials such as the teachers' and researcher's planning documents, copies of students' work, documented observations of classroom activity, and video-recordings of interviews with the participating teachers and task-based interviews with students were also collected. The processes of analysis followed Pirie's (1996) advice to "sit, look, think, look again" (p. 556) supported by Powell et al.'s (2003) analytical model for studying the development of mathematical thinking, which consists of seven interacting, non-linear phases: (1) viewing the video data, (2) describing the video data, (3) identifying critical events, (4) transcribing, (5) coding, (6) constructing a storyline, and (7) composing a narrative (p. 413). Following Flanagan (1954), an event was considered to be critical if it was helpful in triggering and/or explaining the emergence of collective creativity in elementary mathematics learning environments. An initial literature review in the field of creativity led the first author to characterize the existing literature within the four metaphorical domains named earlier (see also Table 1). Analysis of critical events within the data collected during the study, together with conversations between the authors of this paper and reference to Towers and Proulx's (2013) framework for teaching actions and Davis' (2018) work on entailments of grounding metaphors led to the remaining elements of the entailments chart (see Table 1).

Findings

As we noted in the literature review section, Aljarrah's (2018) description of collective creativity is based on four metaphors (summing forces, expanding possibilities, divergent thinking, and assembling things in new ways), which were used to describe students' creative acts during problem-solving sessions inside and outside their classrooms. The summing forces metaphor was used to encompass the ways in which learners coordinate their efforts to enable productive steering (Aljarrah, 2020) towards a mathematical understanding “that is not simply located in the actions of any one individual but in the collective engagement with the task posed” (Martin et al., 2006, p. 157). Expanding possibilities might be understood as broadening the learners’ horizon by gaining new insights based on previous insights. It is a kind of stretching of the space of the possible as a result of the evolving and the growth of the learners’ basic insights. Divergent thinking requires students to consider many potential pathways, look in many directions, journey outside a known content universe, go beyond the problem’s clearly given conditions and information, and think outside-the-box (Aljarrah & Towers, 2019). And, finally, the assembling (things in new ways) metaphor implies looking for associations and making connections. It is a vision of creativity based on an assumption that many educative things are within the reach of learners in their learning environment.

Since our focus as educators is students’ learning, a noteworthy question arose during the study: What do learning and teaching look like with(in) each metaphor of creativity? More precisely: If creative acts are summing forces, then what is learning and what is teaching? If creative acts are expanding possibilities, then what is learning and what is teaching? If creative acts are divergent thinking, then what is learning and what is teaching? If creative acts are assembling things in new ways, then what is learning and what is teaching?

In the following chart (Table 1), we present some entailments of the four metaphors of creativity generated during the research study. For each metaphor for collective mathematical creativity, the entailments include a visualization of the metaphor, the grounding metaphor on which the metaphor for creativity is built, and descriptors or metaphors that suggest the kind of learning and teaching anticipated for each way of thinking about mathematical creativity. Together, these entailments constitute the Collective Creativity Framework (CCF).

Table 1: The Collective Creativity Framework

Metaphors of Creativity	**Visualizing**	**Grounding Metaphor**	**Learning**	**Teaching**
Summing Forces		Sum of Forces	Productive Steering	Nudging
Expanding Possibilities		Inflating	Growth	Extending
Divergent Thinking		Multidirectionality	Widening Perception	Re-orienting Attention

Assembling Things in New Ways		(De)construction	Assembling	Triggering Disintegration

Our exploration of the complexities of these entailments is still ongoing, so they are offered here mostly as a provocation and speculation about the kinds of learning and teaching actions that observers might see when one of the four metaphors for mathematical creativity is shaping the classroom activity. Nevertheless, we did see examples of these entailments in action in the data and we present some of those below. These examples are drawn from data featuring a group of three sixth-grade students (assigned the pseudonyms Mark, Kyle, and Zaid) working togther on the following problem (which was designed based on examples from Empson and Levi, 2011):

Three children, Alex, Zac, and John, shared a chocolate bar. Explain in as many ways as you can how those children may divide the chocolate bar into three pieces such that Alex will get twice what John got, and John's part is no more than one-fourth of the original bar and no less than one-tenth of it.

Due to space limitations in this paper, we focus on addressing the entailments of only the first three metaphors of creativity, namely; summing forces, expanding possibilities, and divergent thinking.

Entailments of the Summing Forces Metaphor

The students began by familiarizing themselves with the problem. Mark was curious about John's part and commented, "But it is, [pause, 2 seconds] John barely gets any." Following this comment, Zaid asked, "Wait, but how much does Zac get?" Zaid's question initiated collective exploration and Mark suggested, "Just give him [i.e., Zac] a third," while Kyle responded to Zaid's wondering and Mark's suggestion by stating, "It is just whatever is left." Mark and Zaid agreed with him and stated, "Yeah, I guess so," and, "Okay, so, um" respectively.

Next, Mark suggested drawing the chocolate bar and on a shared piece of paper. He drew a rectangle and split it into quarters. Kyle then started to initiate a possible path to work on the task. "It does not say what Zac gets [pause 2 seconds] because we can do, like, John and Zac get twenty-five percent [each], and then, um, or sorry, no." Mark supported Kyle's suggestion by restating the given conditions of the problem: "Okay, Alex has to get twice what John got, and John's part cannot be more than one-fourth." Based on Mark's statement, Kyle refined his suggestion and stated with confidence, "Yes, like if John and Zac will get one-fourth [each], and then Alex gets, um, two-fourth—". Mark interrupted Kyle and stated, "Oh, yeah, it would work, yeah, because if John—". His statement seemed to not only represent a collective agreement, but also to express a collective aha moment when all the group felt as though the problem made sense. This, for example, can be seen through Mark's words, "Oh, yeah, it would work, yeah because if John" while he was reflecting on and reasoning an emerging pathway. Here, it is possible to hear their conversation as if just one person is speaking. This was evident in Kyle's words, when he interrupted Mark's statement and completed it by stating that "[because if John] gets twenty-five percent, Alex gets fifty percent, then there is twenty-five percent left from the bar, we just give that to Zac." It was evident that Zaid was also experiencing this collective aha moment, and he summarized the group's initial plan to proceed by stating, "I guess we just have to work with, um, Alex and John because Zac does not matter."

Turning now to examine the teaching that occasioned this learning, we note that an important aspect is the setting up of a learning environment within which it is possible to prompt, promote, and sustain effective interactions between all the agencies in the learning environment including the program of study, the people in the classroom, and the materials and tools. The choice of problem was a significant teaching act that nudged these students into a place where they were challenged, but at an appropriate mathematical level. According to one of the participant teachers, "the problem is inviting to all students at their own levels. Each brings their particular backgrounds and experiences to share their perspectives on the problem."

Group size was also considered by the teachers in this study to be an important teaching decision. Both participant teachers indicated that they prefer to divide the class into small groups (2-3 students in each group). One of the teachers noted, "I prefer the small groups. The bigger the group the less that gets done. Much more time is wasted with larger groupings." According to her, small groups allow students to share their ideas freely, and to ask questions comfortably: "Students have to be able to explain their understanding to their classmates so that they get the strategy being suggested. Kids are better at asking questions in small group settings and saying, 'I do not get it.'" The teacher also added that, "Often the students work with who they are comfortable with, which is often someone at their own level of comprehension. Elsewhere, Martin et al. (2006) have also recognized that "although [students] bring identifiable personal contributions to the collective action, it is through the coacting of these that collective understanding emerges and grows" (p. 157). In our example, students' productive steering toward "better" mathematical ideas and thoughts—that appear to be helpful for the group in developing a solution to the problem—"is not determined by all the individuals having the same understanding" (Martin et al., 2006, p. 157). Instead, and as seen in the above extract, it was a result of the summing of their diverse and different understandings and thoughts.

Another critical strategy to nudge the students towards effective collaboration and productive steering from the very beginning of their problem-solving activity was restricting the group to one shared document upon which to pool their ideas. Also noted as a strategy to promote collective mathematical understanding (Martin et al., 2006), here we acknowledge this strategy as a useful way to promote the kind of learning we identify as productive steering through its capacity to nudge students to collaborate. With one shared document, students can simultaneously see the developing problem-solving strategy and, if each has a writing tool, can equally contribute to the gathering of data for the problem or the steering of the solution.

Entailments of the Expanding Possibilities Metaphor

The entailment for learning for the Expanding Possibilities metaphor is Growth. In the following part of the excerpt, there are many occasions where students engage actively and collectively in processes that help them in growing their initial knowledge, thoughts, and strategies. As the problem solving progressed, Kyle summarized their different basic options for splitting the chocolate bar: "Okay, so, so we have our ninths, and we have our eighths, now sevenths, sixths, and fifths, yeah, these are our options." Zaid stressed the idea that "it goes on forever." Mark agreed with him and at that point, the task of the students expanded from finding as many ways as they could to divide the chocolate bar into three pieces to proving that the process of dividing the chocolate bar would go on forever.

The group collectively then developed the idea of "realistic" parts of the chocolate bar (the fractional parts they were considering—tenths, ninths, eighths, sevenths, sixths, and fifths) and "technical" parts (that could be divided forever). They then used quarters as a generator for an infinite series of options. On their working sheet they wrote, "last section always divided by four once you reach the last realistic idea." For example, one of their options was to give John two-ninths, Alex four ninths, and Zac three-ninths. They then expanded this "realistic" option. Kyle suggested, "As long as John always gets half of what Alex gets, Zac does not really matter." For them, since Zac does not really matter you can take one of his parts and divide it into four equal-sized sections, give one to John, two to Alex, and then divide the last section into four again; give one to John, two to Alex, and then keep zooming in to the last section and do the same thing an infinite number of times. According to them, this idea can be applied to all realistic options (i.e., the tenths, ninths, eighths, sevenths, sixths, and fifths).

Here we want to focus on the teacher's role as a participant in the learning endeavour who is responsible to make "judgments about when and how to intervene" (Martin & Towers, 2011, p. 275). Teachers' actions within the collective have the potential to either prompt or hinder the emergence of creative acts in mathematics learning environments. For example, while Mark, Kyle, and Zaid were discussing the possibility that the process of dividing the chocolate bar between the three children could go on forever, the first author suggested that they try to arrange the different possibilities in a table. After the first author's suggestion, Mark changed his mind and stated that, "*I do not think it can go forever. We cannot go more than one-fourth and we cannot go less than one-tenth*." On reflection, it is obvious that the intervention from the first author, while intended to extend the learning, actually shut down the growth of the students' thinking, blocked the emerging, collective structure, and directed the group to a misleading path. Realizing this, the first author decided to step back and watch for another opportunity to intervene for the purpose of furthering the evolving structure of mathematical thinking and understanding. As Martin and Towers (2011) note, "the teacher must continually listen to, and re-connect with, the improvisational actions of students, possessing a sophisticated capacity to step back until the collective action calls him or her forth" (p. 275).

Entailments of the Divergent Thinking Metaphor

In this section we show how the group's divergent thinking evolved while they were trying to find a *mathematical* way to show that the process of dividing the chocolate bar can go on forever. Kyle reflected on and expanded his initial suggestion: "Okay, so, so we have our ninths, and we have our eighths, now sevenths, sixths, and fifths, yeah, these are our options," and noted, "there is a way in between them." His statement that "there is a way in between [any two realistic possibilities]" indicates a moment where the group went beyond the stated conditions of the problem and started to think outside of its content universe. At this point, the first author intervened subtly by reorienting the group's attention to a tool that might help them think mathematically about their idea—the number line. Mark then initiated a new conversation about using the number line and decimals to explain why, *"mathematically,"* the process of dividing the chocolate bar could *"go on forever"*:

Mark: You could, you could also put it in decimals.

Zaid: But, I mean if you did decimal fractions then you can do anywhere from here [while pointing to a point on the number line that represents 0.1] to all the way to [while pointing to a point on the number line that represents 0.25]—

Mark: Look, if you did zero point two five to zero point one— [while he was drawing a line segment to represent a part of the number line with 0.1 and 0.25 as its endpoints].

Zaid: And, twenty-five hundredths, and umm.

Mark: Yeah.
Zaid: If you used the decimal fractions there will be a loooooot.
Mark: If you did this that is endless. You could do—
Zaid: Oh, yeah endless, because you could just keep adding like … point one, point one, point one, point one, point one, point one, one, one, one, one, one, one, one, one, one… [writing 0.1000000000000001 on their shared piece of paper]…
Mark: Exactly, for decimals it is endless.
Zaid: Yeah, okay.
Kyle: Yeah.
Mark: If you take two numbers, there is an endless list of numbers between them, even if it is one and two.
Kyle: Okay, um, we got it.

The intervention by the first author (offering a new tool with which to think) exemplifies the kind of re-orienting of attention that leads to a widening of the learner's perception about the problem and consequently to divergent thinking, the third of the metaphors for collective mathematical creativity. The students were familiar with the number line, of course, but it wasn't in play until proposed by the first author at a moment when the students had an idea about the space "between" the numbers they were considering but seemed not to have a concrete way to widen their perception of what this space between might contain. Reorienting their attention to something they knew about already but weren't capitalizing on was the teaching action that propelled them forward to think in divergent ways about the problem.

Discussion and Implications

The entailments of the metaphors for collective mathematical creativity that we have presented in the Collective Creativity Framework are significant ways to understand the underpinnings of effective creative learning in mathematics classroom settings, within which the teacher's role is understood as facilitating and scaffolding to provide "the student with room for the creative action required for learning" (Martin & Towers, 2011, p. 253). Davies et al. (2014), though, note a dearth of research that details how teachers can be supported in enhancing their practice to support such student creativity. Our work offers a framework for teachers to reflect on the opportunities they are offering to students to express their creativity through one of Aljarrah's (2018) four metaphors, and from there to glimpse, through the entailments chart that underpins the CCF, the teaching strategies that might enhance that creative activity in the classroom.

Davies et al. (2014) did note the importance of professional development that provides opportunities for reflection on practice and peer dialogue, and in addition reported that "external partnerships, especially with creative professionals, were seen to be beneficial as [they] led to co-creation of knowledge and exploration of conceptualisations of creativity" (p. 39). Our work shows how teachers can be supported in deepening opportunities for students to engage in collective mathematical creativity through the establishment of a partnership between classroom teachers and researchers with a shared goal of enhancing student experience and fostering mathematical creativity.

The design-based research model adopted for this project also provides opportunities for teachers to be introduced to new theoretical concepts (such as collective mathematical creativity), new forms of math problems (such as ones specifically designed to afford collective action), and alternative pedagogical structures, while at the same time providing researchers with authentic opportunities to interact with learners and teachers and to develop practice-informed theory.

References

Aljarrah, A. (2018). *Exploring collective creativity in elementary mathematics classroom settings*. Doctoral Dissertation. University of Calgary, Calgary, Alberta.

Aljarrah, A. (2020). Describing collective creative acts in a mathematical problem-solving environment. *The Journal of Mathematical Behavior, 60*, 1-14.

Aljarrah, A., & Towers, J. (2019, August). Discerning two creative acts: Expanding possibilities and divergent thinking. In M. Nolte (Ed.), *Proceedings of the 11th International Conference on Mathematical Creativity and Giftedness (MCG 11),* (pp. 103-108.) University of Hamburg, Hamburg, Germany.

Aljarrah, A., & Towers, J. (2021). Productive struggle leading to collective mathematical creativity. In D. Olanoff, K. Johnson, & S. M. Spitzer (Eds.), *Proceedings of the 43rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (PME-NA 43)* (pp. 1172-1182). PME-NA**.**

Aljarrah, A., & Towers, J. (2022). The emergence of collective mathematical creativity through students' productive struggle. *Canadian Journal of Science, Mathematics and Technology Education*. https://doi.org/10.1007/s42330-023-00259-0

Baer, J. (1997). *Creative teachers, creative students*. Allyn & Bacon.

Beghetto, R. A., & Kaufman, J. C. (2007). Toward a broader conception of creativity: A case for mini-c creativity. *Psychology of Aesthetics, Creativity, and the Arts, 1*(2), 73-79.

Beghetto, R. A., & Kaufman, J. C. (2011). Teaching for creativity with disciplined improvisation. In R. Keith Sawyer (Ed.), *Structure and improvisation in creative teaching* (pp. 94-109). Cambridge University Press.

Craft, A. (2000). *Creativity across the primary curriculum*. Routledge.

Davies, D., Jindal-Snape, D., Digby, R., Howe, A., Collier, C., & Hay, P. (2014). The roles and development needs of teachers to promote creativity: A systematic review of literature. *Teaching and Teacher Education, 41*, 34-41.

Davis, B. (2018). On the many metaphors of learning…and their associated educational frames. *Journal of Curriculum Studies, 50*(2), 182-203. 10.1080/00220272.2017.1330423

Empson, S. B., & Levi, L. (2011). Extending children's mathematics: Fractions and decimals. Heinemann.

Flanagan, J. C. (1954). The critical incident technique. *Psychological Bulletin, 51*(4), 327-358.

Huebner, D. E. (1967). Curriculum as concern for man's temporality. In V. Hills (1999). *The Lure of the Transcendent* (pp. 131-142). Lawrence Erlbaum Associates.

Kaufman, J. C., & Beghetto, R. A. (2009). Beyond big and little: The four C model of creativity. *Review of General Psychology, 13*(1), 1-12.

Levenson, E. (2013). Tasks that may occasion mathematical creativity: Teachers' choices. *Journal of Mathematics Teacher Education, 16*(4), 269-291.

Lev-Zamir, H., & Leikin, R. (2013). Saying versus doing: Teachers' conceptions of creativity in elementary mathematics teaching. *ZDM - The International Journal of Mathematics Education, 45*(2), 295-208.

Martin, L. C., Towers, J., & Pirie, S. E. B. (2006). Collective mathematical understanding as improvisation. *Mathematical Thinking and Learning, 8*(2), 149-183.

Martin, L. C., & Towers, J. (2011). Improvisational understanding in the mathematics classroom. In R. Keith Sawyer (Ed.), *Structure and improvisation in creative teaching* (pp. 252-278). Cambridge University Press.

National Advisory Committee on Creative and Cultural Education (NACCCE). (1999). *All our futures: Creativity, culture, and education*. Department of Education and Employment.

Pirie, S. E. B. (1996). What are the data? An exploration of the use of video-recording as a data gathering tool in the mathematics classroom. In E. Jakubowski, D. Watkins, & H. Biske (Eds.), *Proceedings of the sixteenth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Vol. II* (pp. 553–559). Eric Clearinghouse for Science, Mathematics, and Environmental Education.

Powell, A., Francisco, J., & Maher, C. (2003). An analytical model for studying the development of learners' mathematical ideas and reasoning using videotape data. *The Journal of Mathematical Behavior, 22*(4), 405-435.

Sawyer, R. K. (2003). *Group creativity: Music, theatre, collaboration*. Lawrence Erlbaum Associates.

Silver, E. (1997). Fostering creativity through instruction rich in mathematical problem solving and problem posing. *ZDM - The International Journal of Mathematics Education, 29*(3), 75-80.

Towers, J., & Proulx, J. (2013). An enactivist perspective on teaching mathematics—Reconceptualizing and expanding teaching actions. *Mathematics Teacher Education and Development, 15(1), 5-28*.

THE VARIETY AND COMPLEXITY OF TEACHERS' DESCRIPTIONS REGARDING ELEMENTARY MATH BLOCK STRUCTURE

Eric Partridge
University of Missouri
egpbk6@missouri.edu

Keywords: instructional activities and practices; elementary school education

In this paper, I refer to math block structure (or, in short, structure) as the patterns of action and interaction for teachers and students during instruction minutes dedicated to math. The theoretical framework for conceptualizing math block structure comes from Stodolsky's (1988) work on instructional formats, defined as "a global and familiar set of categories for well-known instructional arrangements" (Stodolsky, 1988, p. 13). Examples of instructional formats include recitation, group work, discussion, lecture, and seatwork (Stodolsky, 1981, 1988). This paper explores two research questions: What structures do elementary teachers describe within their math block? How, if at all, do teachers implement similar structures in different ways?

The data is from interviews with five elementary teachers who completed a master's program together, taught in the same district, and had five years of experience. This paper focuses on one interview question: *Describe the math block in your classroom on a typical day or week. How long is it, and what happens across those minutes*? Analysis consisted of qualitative coding of the interview transcripts. Table 1 summarizes instructional formats across classrooms, with an x indicating regularly use of that format and parentheses indicating occasional use.

Table 1: Instructional Formats Used in Teachers' Math Blocks

Name	Whole Group	Small Group	Independent work	Work with Classmates
Kylie	x	x	x	games
Alex	x	x	x	games
Jessie	x		(x)	main task
Jack	x		(x)	main task
Chrissy		x	x	games

Even when teachers utilized the same instructional format, they implemented the format differently. For whole-group time, Alex, Jessie, and Jack all began with a warm up, but followed that differently: Alex with a quick introduction to content; Jessie with a longer lesson; and Jack with a launch of a task (which paired with a later discussion after students solved it). Kylie either had the class complete an activity together or students worked on a task that was then discussed. At the teacher rotation during small-group time, Kylie supported individual students while others worked independently while Alex and Chrissy worked with the entire group on core instruction.

The variety in instructional format selection and implementation seen in this study indicates wide-ranging implications. For students, structure impacts the type of learning opportunities they will experience (e.g., no whole class discussion in Chrissy's class). For teachers, structure impacts the resources needed for planning as well as the opportunity to provide differentiation (e.g., Jack needs rich tasks, Jessie has limited opportunity for differentiation). For coaches and professional developers, shifting a teacher's practice would likely require the teacher to either assimilate new teaching practices into their current structure or to change their structure.

References

Stodolsky, S. (1981). *Subject matter constraints on the ecology of classroom instruction*. Paper presented at the Annual Meeting of the American Educational Research Association. Los Angeles.

Stodolsky, S. S. (1988). *The subject matters: Classroom activity in math and social studies*. University of Chicago Press.

TO USE OR NOT TO USE? TEACHERS' CONSIDERATIONS IN A CROWDED CURRICULAR LANDSCAPE

Doris Fulwider, M.S.Ed.
Purdue University
dfulwide@purdue.edu

Amy M. Olson, Ph.D.
Duquesne University
olsona@duq.edu

Keywords: Teacher Beliefs; Curriculum; Elementary School Education

Perspective(s) or Theoretical Framework

The number of online curricular resources has proliferated and there is evidence that teachers are making use of these resources to plan their lessons (Pepin, et al., 2013; Sawyer, et al., 2020). Silver (2020) characterizes teacher curriculum supplementation (TCS) as a widespread phenomenon spawned initially by the adoption of Common Core standards and accelerated by the COVID-19 pandemic. In his systematic literature review on the topic, he acknowledges TCS as underrepresented in scholarly literature due to the complexities surrounding ambiguous definitions of supplementation, the vast number and types of supplemental materials available, and the countless reasons teachers decide to source these alternative materials. Although complex, he argues that the widespread occurrence of TCS is an important one for researchers to study because the practice has consequential effects on both teachers and students.

Method and Analysis

Participants in this study included three Midwestern elementary teachers who participated in two individual interviews focused on their mathematics curricular decision making. Open coding (Saldaña, 2021) was used to code teachers' responses to the questions "*Why do you use this resource?"* and *"How do you use this resource?"*, allowing the researchers to synthesize TCS themes for individual teachers and across teachers for individual curricular resources.

Findings

The three teachers used between eight and thirteen curricular resources, including those they created. The following themes emerged from the data, providing a starting point for discussion and further systematized inquiry:

- Tension exists between mandates (expectations) and reality (classrooms).
- Use of the adopted curriculum varies (and the term "mandated' is fuzzy).
- Teachers make curricular decisions to support student engagement.
- Teachers make curricular decisions to support a wide range of learning levels.
- Teachers value other teachers' opinions and teacher-created materials.

References

Pepin, B., Gueudet, G., & Trouche, L. (2013). Resourcing teachers' work and their interactions: A collective perspective on resources, their use and transformation. *ZDM Mathematics Education*, 45, 929-943. https://www.doi.10.1007/s11858-013-0534-2

Sawyer, A. G., Dick, L. K., & Sutherland, P. (2020). Online mathematics teacherpreneurs developers on Teachers Pay teachers: Who are they and why are they popular? *Education Sciences, 10*(9), 1-21. https://www.doi.10.3390/educsci10090248

Saldaña, J. (2021). *Qualitative coding manual for researchers, 4th ed.* Sage.

Silver, D. (2020). A theoretical framework for studying teachers' curriculum supplementation. *Review of Educational Research, 92*(3), 455-489. https://www.doi.10.3102/00346543211063930

UNPACKING TEACHING INTERVENTIONS: FACILITATING CLASSROOM INTERACTIONS WHEN ENGAGING IN FERMI PROBLEMS

Eduan Martínez-Soto	Stephen Lewis	Azita Manouchehri
The Ohio State University	Columbus School for Girls	The Ohio State University
martinez-soto.1@osu.edu	slewis@columbusschoolforgirls.org	manouchehri.1@osu.edu

Keywords: Classroom Discourse, High School Education, Modeling, Problem Solving.

In recent years, the use of Fermi problems in mathematics education has gained popularity due to their perceived capacity for engaging students in reasoning and logical thinking (Barahmeh, Hamad & Barahmeh, 2017). The supporters of the inclusion of this type of task in K-12 curriculum capitalize on their value in building learners' socio-emotional learning competencies (Brave & Miller, 2022), problem solving (Taggart et al., 2007), and the development of social and cultural consciousness (Sriraman & Knott, 2009). Arlebäck (2009) positioned Fermi problems as a viable venue for bridging students' work on fermi tasks and mathematical modeling process, acknowledging that research is needed to better understand how students' work around Fermi tasks might be organized so modeling competencies are fostered in the course of their implementation. In this work, relying on a teaching experiment methodology (Steffe & Thompson, 2000) and using interactional analytical techniques (Schütte, Friesen & Jung, 2019), we examined over two instructional sessions, teacher-student interactions in a pre-calculus course as students worked on a Fermi problem. We sought to identify the type of interventions that the teacher used during both small and large group discussions in an effort to gauge the students' work towards production of a general model. The task asked: *How many steps would it take to burn off an entire bag of potato chips?*

Results

Dirks and Edge (1983, cited in Ärlebäck 2009) argued that successful work on Fermi tasks requires the individuals to possess "sufficient understanding of the problem to decide what data might be useful in solving it, insight to conceive of useful simplifying assumptions, an ability to estimate relevant physical quantities, and some specific scientific knowledge" (p. 602). In reporting the results of using Fermi problems in secondary mathematics classrooms in Sweden, Ärlebäck highlighted the critical role that students' personal knowledge played in not only the assumptions they made once encountering Fermi problems, but also in how they validated their answers. Ärlebäck (2009) cautioned that group composition and participants' preferences can strongly influence the variables they consider and assumptions they make, thus influencing the accuracy of models produced. In the sessions we examined, teacher interventions became central in focusing learners' work along some of the cognitive and social dimensions identified by Ärlebäck (2009). While in some cases the interventions were motivated by the teacher's perceived need to equalize the social capital of the participants in small groups, a majority of his comments aimed to provoke reflection, encouraging students to make connections among various answers obtained, addressing their implicit assumptions by introducing new constructs (e.g., age, gender, metabolism, height and weight of individuals), and confronting some false scientific assumptions they made (e.g., sweating means more calories are burnt). As such the implicit assumptions that the learners constructed became a source for group deliberation. These categories along with descriptions of students and teachers will be explained and illustrated with examples during the presentation.

References

Ärlebäck, J. B. (2009). Mathematical modelling in upper secondary mathematics education in Sweden: A curricula and design study. Linköping University, Linköping, Sweden Retrieved from http://liu.diva-portal.org/smash/get/diva2:302720/FULLTEXT02.

Barahmeh, H. M., Hamad, A. M. B., & Barahmeh, N. M. (2017). The Effect of Fermi Questions in the Development of Science Processes Skills in Physics among Jordanian Ninth Graders. *Journal of Education and Practice, 8(3)*, 186-194.

Brave, K. L., & Miller, J. (2022). Using Fermi Questions to Foster Community. *Mathematics Teacher: Learning and Teaching PK-12, 115(11)*, 801-807.

Dirks, M. K., & Edge, D. R., M. (1983). Problem solving: Enrico Fermi and the bull moose. *Vector, 40(3)*, 25-28.

Schütte, M., Friesen, R. A., & Jung, J. (2019). Interactional analysis: A method for analysing mathematical learning processes in interactions. *Compendium for early career researchers in mathematics education*, 101-129.

Sriraman, B., & Knott, L. (2009). The mathematics of estimation: Possibilities for interdisciplinary pedagogy and social consciousness. *Interchange*, 40, 205-223.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. *Handbook of research design in mathematics and science education*, 267-306.

Taggart, G. L., Adams, P. E., Eltze, E., Heinrichs, J., Hohman, J., & Hickman, K. (2007). Fermi questions. *Mathematics Teaching in the Middle School*, 13(3), 164-167.

USING BIG IDEAS OF MEASUREMENT AS A FRAMEWORK TO EXPLORE PROFESSIONAL NOTICING OF STUDENTS' THINKING

Mai Bui
Texas State University
mtb104@txstate.edu

This paper presents Big Ideas of Measurement as a framework of students' thinking about measurement. Drawing from research-based evidence, the framework is a collection of key concepts that students must develop for a robust understanding of measurement and, as such, are key aspects of students' thinking teachers should learn to notice. In a case study with four mathematics teacher educators, this framework was utilized to design an instrument to measure their professional noticing and to analyze the results. Findings provided snapshots of professional noticing of participants with varied expertise in content knowledge and student thinking. Additionally, the choice of artifacts appears to have influenced participants' noticing.

Keywords: Teacher Noticing, Measurement, Teacher Educators

Noticing students' mathematical thinking is fundamental to reform-based instruction which emphasizes adaptive and responsive teaching (Jacobs et al., 2022). While current literature has explored preservice and practicing teachers' professional noticing of student thinking in various content areas, such as arithmetic (Fisher et al., 2018; Jacobs et al., 2010; Schack et al., 2013), fractions (Jacobs et al., 2022), algebraic thinking (Walkoe, 2015), and multiple representations (Dreher & Kuntze, 2015), research on this topic is scarce in the domain of measurement (Caylan Ergene & Isiksal Bostan 2022), a crucial topic in the mathematics curriculum, particularly at the elementary school level. The purpose of this study is to investigate ways to assess the construct of professional noticing of students' thinking about measurement and to capture snapshots of varying levels of expertise in noticing.

According to NCTM (2000), measurement has widespread practical applications in everyday life, in other domains of mathematics, and in areas outside of mathematics. Therefore, it is essential for students to develop a robust understanding of this content domain. Nevertheless, research indicates that many students hold fragile and shallow understanding of measuring various attributes, such as length, angle, and area (Smith & Barrett, 2017). For example, students may know how to use a ruler to measure the length of an object, but they may struggle when the ruler is not aligned at the conventional zero-point (Kamii & Clark, 1997). To help students overcome the challenges of learning about measurement, teachers must be able to notice and use students' thinking to make instructional decisions. However, given the complexity of classroom instruction, teachers cannot pay attention to everything. Hence, they must be selective in their noticing. This raises the first question:

Q1: What are key aspects of students' thinking about measurement should teachers notice? This question was answered by conducting a literature review on students' thinking about measurement. Results are presented as a research-based framework entitled "Big Ideas of Measurement". This framework was then used to develop an instrument to explore teachers' professional noticing of students' thinking about measurement to answer the second question:

Q2: How do teacher educators with various expertise in measurement content and students' thinking notice key aspects of students' thinking about measurement when engaging with different instructional scenarios?A case study with four mathematics educators was conducted to investigate this research question.

Theoretical Framework

The foundational construct in this study is professional noticing of students' thinking about measurement. This section unpacks its essence by elucidating the two key terms "professional noticing" and "students' thinking about measurement".

Professional Noticing of Students' Mathematical Thinking

In this paper, I adapted Jacobs et al.'s (2010) concept of professional noticing of students' mathematical thinking, which comprises three interrelated skills: *attending to students' strategies details*, *interpreting students' understanding*, and *deciding how to respond on the basis of students' understanding*. This construct focuses on teachers' in-the-moment noticing when students explain their mathematical thinking verbally or in writing during classroom instruction rather than before or after instruction. For the purpose of this study, I focused on the first two component skills, attending and interpreting, because they serve as the foundation for the third.

Attending to students' strategies details pertains to the extent to which teachers identify mathematically significant details in students' strategies while *interpreting* students' understanding involves using these details to reason about their understanding in a way that is consistent with research on students' mathematical thinking.

Students' Thinking about Measurement

The term "student thinking about measurement" can encompass a wide range of students' conceptions related to various measurement quantities, and the development of their thinking regarding these quantities. This can include, for example, students' conceptions of angles, their struggles with distinguishing between area and perimeter, and the learning trajectory they follow in order to understand length. However, this study focuses on key concepts *across attributes* that researchers identified as foundational for students to develop a deep and robust understanding of measurement. This approach aligns with Smith and Barrett's (2017) call for prioritizing research on common concepts across measurement quantities.

Researchers have identified multiple lists of key measurement concepts that students should understand and named their lists using titles such as theories of measures (Lehrer, 2003), conceptual principles (Smith & Barrett, 2017), and essential understandings (Goldenberg & Clements, 2014). However, these lists are neither exhaustive nor unique; they contain overlapping and distinct ideas. To create a structured framework on students' thinking about measurement, I synthesized 16 key concepts from the literature and grouped them into five clusters. This research-based framework, called the *Big Ideas of Measurement*, is summarized in Table 1.[1] It is important to note that these big ideas are neither isolated nor "acquired in an all-or-none manner"; they are interrelated, forming a "web of connections" (Lehrer, 2003, p.182) that assist students in building a robust understanding of measurement across different attributes.

Methods

Based on the framework of *Big Ideas of Measurement*, a noticing instrument with three items, each featuring a different instructional scenario, was developed. A case study utilizing this instrument to measure professional noticing of four mathematics teacher educators was conducted in January 2023. The goal was to explore their professional noticing of students' thinking about measurement to answer Q2.

[1] I borrow the phrase *Big Ideas of Measurement* from Empson et al. (2006) who used it in course materials and interviews.

Table 1: Big Ideas of Measurement[2]

Clusters	Big Ideas	Description
The nature of measurement	*Assigning a number to an attribute*	Measurement can specify "how much" by assigning a number to attributes such as length, area, volume, angle, etc.
	Error & Precision	All measurement is inherently approximate.
Principles of Measurement	*Conservation*	Moving an object, or a shape does not change the measures of some attributes, for example, length, area, volume, angle, etc.
	Transitivity	Measures are transitive (e.g., If A>B and B>C, then A>C).
	Additivity	Units can be composed and decomposed (10 units is a composition of 6 units and 4 units or 10 groups of 1 unit). Most measures are additive (temperature is not).
Important Early Developmental Concepts/ Conceptions of Unit	*Identifying the attribute*	Does the child understand what they are measuring?
	Identical units	Is the same-sized unit used to create the measure?
	Exhaustive measure	All of the object has been measured without gaps between units or overlapping units
	Unit iteration	This includes making copies of units and arranging them, accumulating those units to obtain a measure, and eventually being able to reuse/copy a single unit
	Partitioning unit	Units can be partitioned into fractional amounts smaller than one unit
Standardization & Systems of Units	*Standard units*	Conventions/standard units facilitate communication
	Proportionality	The same attribute of an object can be measured with different units, and the different measurements produced are inversely proportional to the size of the unit.
	Composite units	Units of units—smaller units can be coordinated into a single larger unit. For example, measuring by groups of 100 centimeters is the same as a meter, seeing row and column structure in an array.
	Unit choice & Precision	The choice of unit in relation to the object determines precision.
The Use of Measurement Instruments	*Meaning of Instruments*	Instruments replace the need for use of multiple copies of units and move toward efficiency. Moreover, many big ideas of measurement are embedded in the design of instruments.
	Zero-point/ Origin	Each instrument has conventional zero point(s). Moreover, any point can serve as the origin or zero point on the instrument.

Note: These big ideas of measurement were drawn from and supported by existing literature on students' thinking about measurement including: Beckmann (2018); Bishop (2022); Dietiker et al. (2011); Goldenberg & Clements (2014); Jaslow & Vik (2006); Kamii (2006); Lehrer (2003); Lehrer et al. (2003); Lehrer et al. (2014); Lehrer et al. (1998); NCTM (2000); NGA Center & CCSSM (2010); Nitabach & Lehrer (1996); Piaget et al. (2013); Smith & Barrett (2017); Stephen & Clements (2003).

Participants

Purposeful sampling was used to select participants for this study. The intention was to capture a variety of noticing expertise. Four participants, Brielle, Luis, Hazel, and Mason (pseudonyms) are mathematics teacher educators who possess varied experiences in both the subject matter of measurement and students' thinking. They are enrolled in a doctoral program in mathematics education. Brielle has experience teaching developmental mathematics courses to

[2] My organization of the framework in this table was originated from Bishop (2022)'s MATH2312 course materials.

undergraduate students but has limited experience with regards to measurement content and students' mathematical thinking. Luis has taught a content course focusing on geometry and measurement for preservice teachers. He has expertise in measurement content but has little experience with students' thinking about measurement. Hazel has taught a content course on number systems and operations for preservice teachers and focused on students' mathematical thinking in her course. However, similar to Brielle, she also has limited exposure to measurement content. Lastly, Mason has experience with both measurement content and students' thinking about measurement, though he has not taught any courses for preservice teachers. He has, however, conducted clinical interviews with elementary students and analyzed their understanding of the big ideas of measurement for a course he was taking.

Noticing Instrument

This instrument was designed to measure professional noticing of students' thinking about measurement. It includes three items that prompt participants to engage with three instructional artifacts and respond to writing prompts to elicit their noticing. The artifacts were thoughtfully selected to represent a range of common scenarios in teaching practice and to cover multiple big ideas of measurement across quantities. Table 2 provides a summary of the key features of these three items.

Table 2: Descriptions of Three Items of the Noticing Instrument

	Item 1	**Item 2**	**Item 3**
Instructional Scenarios	Classroom Interactions	One-on-one Conversation	Students' Written Work
Artifacts	Video "Who's taller?: Measurement during circle time" (4 minutes)	Video "Helena measures Speedy with paper clips and a ruler" (5 minutes)	Three students' written work regarding finding area of a rectangle
Quantities	Length (Pre-measurement)	Length (Measurement)	Area
Description	During whole class instruction, the class engages in an activity to directly compare the heights of students and classify them into three sizes: Small, Medium, and Big	Helena, a first grader, measures Speedy, a string snake, with non-standard units (big and small paper clips), and standard units (inches, using a paper ruler)	Three students' written work shows how they draw tiles to cover a 6x5 rectangle to find its area.
Focused Big Ideas of Measurement	Identifying the attribute Conservation Transitivity	Identifying the attribute Identical units Exhaustive measure Unit iteration Partitioning unit Zero-point/ Origin Meaning of Instrument	Identical units Exhaustive measure Unit iteration Composite units

The writing prompts used in Item 2 (as an example) are:

Question 1: Please describe in detail what you think Helena did in response to each problem.

Question 2: Explain what you learned about Helena's understandings.

These prompts align with the two components, attending and interpreting, in the conceptualization of professional noticing in this study. The prompts for Item 1 and Item 3 are similar with a minor modification for Question 1 in Item 1. Specifically, this question was broader, "*Please describe in detail what you noticed when watching the video. Try to ask yourself "What else do you notice?" until you have nothing else to share*." This modification was made to investigate the potential impact of prompts on participants' noticing.

Data Analysis

To analyze participants' written responses for each item, I adapted the coding scheme developed by Jacobs et al. (2010, 2022). First, I created a rubric based on the framework of Big Ideas of Measurement. Particularly, I examined the artifact, identified big ideas of measurement which were present in students' strategies. I then identified significant details in students' strategies for each big idea and interpreted their understanding of the big idea (see examples in Row 2 of Table 3). Next, I looked at participants' responses for each item and assigned big idea codes whenever they appeared. If participants attended to and provided evidence for other significant mathematical details in students' strategies that were not included in the rubric, I assigned the code "Other." Finally, I gathered all details related to each big idea from participants' responses (see examples in Row 3 of Table 3) and compared them to the rubric, assigning a score of 0, 1, or 2 for their attending and interpreting skills based on the level of evidence in the participants' responses (see examples in Row 4 of Table 3).

Table 3: Example of Coding Process for Mason's Responses to Item 2

Big ideas of measurement	*Attending to details related to the big ideas in Helena's strategies*	*Interpreting Helena's understanding of the big ideas*
Use of identical units (Rubric)	Although there were small and big paper clips on the table, Helena consistently used the same-sized paper clips (5 big ones) until there was not enough space that she put a small one. However, she clearly named each unit in her measure (5 big and 1 small paperclips). She didn't say 6 paper clips long.	Helena mostly understood the big idea "use of identical unit". We can see that her understanding of identical units, exhaustive measure, and partitioned unit plays a role here: Helena understood that she needed to use the same-size (big) paperclips, but in the end, there was not enough space to fit a big paper clip. To maintain exhaustive measures, she used a small one.
Use of identical units (Mason's Response)	For the first problem, Helena started by taking big paper clips […]. After placing the last big paper clip, she realized a big paper clip would not fit in the small space left. She decided to try a small paper clip instead and found lining this next to big paper clips.	She also did not demonstrate the use of identical units and chose to instead, use a small paper clip to cover the last bit of the snake's length.
Use of identical units (Mason's scores & Reasons)	**Score: 1** Reason: Mason attended to some details related to this big idea in Helena's responses, but also missed some important details such as Helena clearly distinguished big and small paperclips as two different units.	**Score: 1** Reason: Although Mason was correct that Helena did not use all same-sized unit, she demonstrated some understanding of this big idea when consistently using big paper clips at the beginning.

For the attending component, a score of 0 indicated that participants did not attend to any details, or just very few details relevant to the big ideas in students' strategies. A score of 1 indicated that participants attended to some details in students' strategies but missed some important details. A score of 2 indicated that participants attended to almost all details related to the big ideas in students' strategies. Similarly, participants' interpreting component for each big idea was scored 0, 1, or 2 depending on how their interpretation aligned with research-based knowledge of students' measurement thinking. A score of 0 indicated that participants did not interpret students' understanding of the big idea, or their interpretation was misaligned with research findings. A score of 1 indicated that participants made inferences about students' understanding of the big idea, and their interpretation reflected students' thinking to some extent, but there were some minor misalignments with research, or the interpretation did not go in-depth. A score of 2 indicated participants' interpretation of students' understanding is thorough and consistent with research. After scoring participants' attending and interpreting components for each of the identified big ideas, I calculated their overall attending and interpreting scores for each item by taking the average.

For Item 1, participants' noticing was broader because the first question did not focus specifically on students' thinking. To further analyze their noticing, I used van Es and Sherin's (2008) coding framework. Specifically, I segmented participants' responses to Question 1 in Item 1 into smaller chunks based on shifts in the main ideas and assigned codes for four dimensions: *Actor* (codes: Teacher, Students, Other), *Topic* (codes: Mathematical thinking, Pedagogy, Climate, Management, Other), *Stance* (codes: Describe, Interpret, Evaluate), and *Specificity* (codes: General, Specific). The frequency of each code was then counted, and their percentages were calculated.

Findings

Table 4 summarizes participants' scores for attending (A) and interpreting (I) for three items in the instrument, along with their overall attending and interpreting scores. The last row shows the average scores for each item across all participants. It is important to note that these scores are not intended to classify participants as good or bad teachers, nor do they indicate what participants noticed to be good or bad. Rather, they measure the alignment of participants' noticing with the conceptualization of professional noticing in this study and help investigate varied expertise in noticing. In the following sections, I discuss what was learned about participants' professional noticing of students' thinking about measurement and how the choice of instructional artifacts influenced their noticing performance.

	Item 1	Item 2	Item 3	Participants' Overall Mean Score
Brielle	A: 1.2 I: 0.6	A: 1.25 I: 0.75	A: 0.75 I: 0.17	**A: 1.07** **I: 0.51**
Luis	A: 1 I: 0.6	A: 1.14 I: 0.86	A: 0.67 I: 0	**A: 0.94** **I: 0.49**
Hazel	A: 1.2 I: 0.6	A: 1.43 I: 1	A: 1.42 I: 0.83	**A: 1.35** **I: 0.81**
Mason	A: 1.4 I: 0.6	A: 1.43 I: 1.43	A: 1.58 I: 1.17	**A: 1.47** **I: 1.07**
Items' Overall Mean Score	**A: 1.2** **I: 0.6**	**A: 1.31** **I: 1.01**	**A: 1.11** **I: 0.54**	

Table 4: Summary of Participants' Noticing Scores

Snapshots of Varied Professional Noticing Expertise

Although the results of this case study with four participants cannot be generalized, there is a consistent pattern in Table 4: Mason has the highest performance in both attending and interpreting skills across three items, followed by Hazel, Brielle, and Luis. To gain more insight into their noticing, we can consider their responses to Question 2 in Item 3, where they interpreted Cassie's understanding from her work sample (See Figure 1).

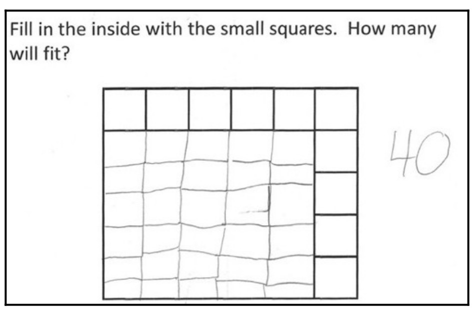


Figure 1: Cassie's Work Sample (Source: Empson et al., 2020)

Mason: Both Cassie and Eric demonstrate an exhaustive understanding of measurement, leaving no space inside the shape empty. Cassie demonstrates a partial understanding of identical units, as some drawn squares are the same, while others are not. […] Cassie demonstrates some understanding [of row and column structure] as her columns are lined up correctly.

Hazel: Cassie understands that the entire rectangle needs to be filled with squares. However, she does not understand completely that all the squares need to be the same size [..]

Brielle: Cassie uses **|** [*the vertical lines of squares in the first row*] as a reference point from the 1st row. Last row is much smaller than the rest but still counts each as a whole – Consistency.

Luis: Eric has a good understanding of partitioning…Cassie does as well, but something happened to where extra rows were made.

Mason's response showed that he paid attention to and interpreted Cassie' understanding of three big ideas of measurement: exhaustive measure, identical unit, and composite unit. While Hazel did not use the exact terminology from the framework like Mason, she did notice that Cassie filled the *"entire rectangles"* and did not use *"same size"* units, which showed evidence of Hazel's attention to and interpretation of Cassie's understanding of exhaustive measure and identical unit. However, she did not notice the big idea of composite unit like Mason did. Brielle attended to identical unit when she mentioned *"last row is much smaller than the rest"* while Luis paid attention to composite unit when he noticed the extra row in Cassie's work. Brielle and Luis received credit for attending to these big ideas, but their interpretations did not align with research on key concepts in students' thinking about measurement. Mason and Hazel's experience with students' thinking seemed to support their professional noticing. Especially for Mason, his knowledge of the framework on Big Ideas of Measurement seemed to give him a structure of what key aspects of students' thinking he should attend to and how to interpret their understanding. Moreover, Table 4 shows a pattern that all participants' attending scores are higher than their interpreting scores, which is consistent with existing literature in the field: participants cannot interpret something that they fail to notice.

Influences of Instructional Artifacts on Participants' Noticing

Notably, all participants had higher noticing scores for Item 2 compared to Items 1 and 3. It's worth considering that Item 2 utilized a video of a one-on-one conversation with a student, while Item 1 used a video of a whole-class interaction, and Item 3 used work samples from three different students. The video format and the focus on one student's thinking about measurement likely contributed to the participants' improved noticing performance in Item 2.

Comparing the performance of each participant in Items 1 and 3 in Table 4 reveals an interesting pattern: while Brielle and Luis had higher noticing scores for Item 1, Hazel and Mason performed better in Item 3. By looking at Table 5, which summarizes participants' more general noticing of Item 1 across four dimensions in addition to student thinking, we can get some interesting insight into this phenomenon. We can observe from this table that participants' focus for the two dimensions *Actor* and *Topic* differs. Brielle and Luis predominantly centered their attention on the students and mathematical thinking, while Hazel and Mason focused on the teacher and pedagogy. For instance, Mason initiated his response to Question 1 in Item 1 by noting, "*I noticed that the teacher was very good at using students' voice to center the discussion. Frequently, she would revoice what students said and asked."* The broadness of the prompt and the complexity of whole-class interactions led Hazel and Mason to attend to other actors and topics. This explains why their noticing scores for Item 1 were lower than their scores for Item 3.

Furthermore, the specificity of the prompts had an impact on the participants' inquiry stance towards their noticing. When the prompts were specific, such as in Items 2 and 3, participants took the corresponding inquiry stance of Describe for Question 1 and Interpret for Question 2. In contrast, when the prompt was broader, as in Question 1 in Item 1, participants' inquiry stance was more varied. For instance, in the previous example, Mason's noticing of the teacher's pedagogy took an evaluative stance, as he complimented the teacher's instruction.

Table 5: Participants' Noticing of Item 1 Across Four Dimensions

Dimensions	**Actor**			**Topic**					**Stance**			**Specificity**	
Codes	Tea	Stu	Oth	Mat	Ped	Cli	Man	Oth	Des	Int	Eva	Gen	Spe
Brielle	14	**100**	0	**100**	14	0	0	0	**71**	29	0	**71**	29
Luis	43	**86**	0	**71**	0	14	0	14	29	**57**	0	29	**71**
Hazel	**80**	40	0	20	**100**	0	0	0	40	**60**	0	20	**80**
Mason	**100**	14	0	43	**71**	14	14	14	29	29	**57**	14	**86**

Legend: Tea: *Teacher*; Stu: *Student*; Oth: *Other*; Mat: *Mathematical Thinking*; Ped: *Pedagogy*; Cli: *Climate*; Man: *Management*; Des: *Description*; Int: *Interpret*; Eva: *Evaluation*; Gen: *General*; Spe: *Specific*.
Note: Numbers in this table represent the percentage of segments getting a code out of the total segments

This analysis provided evidence that the instructional artifacts and the corresponding questions had an impact on participants' noticing. Specifically, the form of the artifacts, such as dynamic videos versus static student work samples; their nature, such as whole class interactions versus one-on-one conversations; and the level of specificity of the prompts used, all seemed to influence how participants noticed different aspects of the instruction. This finding highlights the importance of carefully selecting instructional artifacts and prompts in research on teacher noticing.

Brief Conclusions and Considerations

This paper makes three main contributions. First, it presents a synthesis of the *Big Ideas of Measurement* framework as a tool for understanding students' thinking about measurement across different quantities. Second, it introduces a noticing instrument and demonstrates how the framework can be used to analyze professional noticing of students' thinking about measurement. Third, a case study with four mathematics educators is presented to showcase the diverse expertise of professional noticing and to illustrate how the choice of instructional artifacts used in the instrument can influence noticing performance.

This case study suggests a potential relationship between participants' professional noticing and their prior experience with measurement content and students' thinking (both specific to measurement and a broader exposure and appreciation for student thinking in other content areas). Specifically, Mason's exposure to the framework appeared to support his attention and interpretation of students' thinking about measurement. However, further research with a larger sample size is needed to investigate this relationship more thoroughly. Additionally, future research could explore in a more systematic fashion how instructional artifacts influence noticing performance, and how the third component of professional noticing, deciding how to respond on the basis of student thinking, relates to attending and interpreting components in the context of measurement. These findings will be valuable for teacher educators and professional development facilitators to design effective courses and programs that support pre-service and in-service teachers in improving their professional noticing of students' thinking about measurement.

Acknowledgments

I am grateful to Dr. Jessica Bishop for her wholehearted support throughout this project, from inspiring me to start the idea for this research when observing her teaching, mentoring me with the instrument and research design, to encouraging me to write up this paper. I also thank Dr. Duncan Waite for his feedback to improve the quality of my writing.

References

Beckmann, S. (2018). Mathematics for elementary teachers with activities, 5th edition. Pearson.

Bishop, J.B. (2022). Handout "Big ideas of measurement" [Course curriculum for MATH2312, Texas State University]

Caylan Ergene, B., & Isiksal Bostan, M. (2022). Supporting Pre-service Mathematics Teachers' Professional Noticing of Students' Reasoning About Length. European Journal of Science and Mathematics Education, 10(1), 50–70. https://doi.org/10.30935/scimath/11384

Dietiker, L. C., Gonulates, F., & Smith, J. P. (2011). Understanding linear measure. Teaching children mathematics, 18(4), 252–259.

Dreher, A., & Kuntze, S. (2015). Teachers' professional knowledge and noticing: The case of multiple representations in the mathematics classroom. Educational Studies in Mathematics, 88(1), 89–114.

Empson, S. B., Turner, E. E., & Junk, D. (2006). Formative Assessment Instruments for Children's Mathematical Thinking in English and Spanish [Set of seven problem-solving interviews covering 1) addition and subtraction; 2) base-ten concepts and place value; 3) multiplication and division; 4) fractions; 5) multidigit operations; 6) measurement; and 7) algebra].

Empson, S. B. (2020). Course materials. Austin, TX: The University of Texas.

Fisher, M. H., Thomas, J., Schack, E. O., Jong, C., & Tassell, J. (2018). Noticing numeracy now! Examining changes in preservice teachers' noticing, knowledge, and attitudes. Mathematics Education Research Journal, 30, 209–232. https://doi.org/10.1007/s13394-017-0228-0

Goldenberg, E. P., & Clements, D. H. (2014). Developing essential understanding of geometry and measurement in grades PreK-Grade 2. National Council of Teachers of Mathematics.

Jacobs, V. R. & Spangler, D. A. (2017). Research on core practices in K-12 mathematics teaching. In J. Cai (Ed.), Compendium for Research in Mathematics Education, (p. 766–792). Reston, VA: National Council of Teachers of Mathematics.

Jacobs, V. R., Empson, S. B., Jessup, N. A., Dunning, A., Pynes, D. A., Krause, G., & Franke, T. M. (2022). Profiles of teachers' expertise in professional noticing of children's mathematical thinking. Journal of Mathematics Teacher Education, 1–30.

Jacobs, V. R., Lamb, L. L., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. Journal for Research in Mathematics Education, 41(2), 169–202.

Jaslow, L., & Vik, T. (2006). Using children's understandings of linear measurement to inform instruction. Teachers Engaged in Research: Inquiry into Mathematics Classrooms, 109–134.

Kamii, C. (2006). Measurement of length: How can we teach it better?. Teaching children mathematics, 13(3), 154–158.

Kamii, C., & Clark, F. B. (1997). Measurement of length: The need for a better approach to teaching. School Science and Mathematics, 97(3), 116–121.

Lehrer, R. (2003). "Developing understanding of measurement." In Jeremy Kilpatrick, W. Gary Martin, Deborah Schifter (Eds), A research companion to principles and standards for school mathematics (pp. 179–192). National Council of Teachers of Mathematics.

Lehrer, R., Jacobson, C., Thoyre, G., Kemeny, V., Strom, D., Horvath, J., Gance, S., & Koehler, M. (1998). Developing understanding of space and geometry in the primary grades. In R. Lehrer & D. Chazan (Eds.), Designing learning environments for developing understanding of geometry and space (pp. 169–200). Erlbaum.

Lehrer, R., Jaslow, L., & Curtis, C. (2003). Developing an understanding of measurement in the elementary grades. Learning and teaching measurement, 1, 100–121.

Lehrer, R., Slovin, H., Dougherty, B. J., & Zbiek, R. M. (2014). Developing essential understanding of geometry and measurement for teaching mathematics in grades 3-5. NCTM, National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics (NCTM). (2000). Principles and standards for school mathematics. Reston, VA: NCTM.

National Governors Association Center for Best Practices (NGA Center) and the Council of Chief State School Officers (CCSSO). (2010). Common Core State Standards for Mathematics. Washington, DC: NGA Center and CCSSO. http://www.corestandards.org.

Nitabach, E., & Lehrer, R. (1996). Research into practice: Developing spatial sense through area measurement. Teaching Children Mathematics, 2(8), 473–476.

Piaget, J., Inhelder, B., & Szeminska, A. (2013). Child's conception of geometry. Routledge.

Roth McDuffie, A., Foote, M. Q., Bolson, C., Turner, E. E., Aguirre, J. M., Bartell, T. G., ... & Land, T. (2014). Using video analysis to support prospective K-8 teachers' noticing of students' multiple mathematical knowledge bases. Journal of Mathematics Teacher Education, 17(3), 245–270.

Schack, E. O., Fisher, M. H., Thomas, J. N., Eisenhardt, S., Tassell, J., & Yoder, M. (2013). Prospective elementary school teachers' professional noticing of children's early numeracy. Journal of Mathematics Teacher Education, 16(5), 379–397. https://doi.org/10.1007/s10857-013-9240-9

Sherin, M. G., Jacobs, V. R., & Philipp, R. A. (2011). Situating the study of teacher noticing. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), Mathematics teacher noticing: Seeing through teachers' eyes (pp. 3–14). Routledge.

Smith, J. P., & Barrett, J. E. (2017). The learning and teaching of measurement: Coordinating quantity and number. In J. Cai (Ed.), Compendium for Research in Mathematics Education. (pp. 355–385): Reston, VA: National Council of Teachers of Mathematics.

Star, J. R., & Strickland, S. K. (2008). Learning to observe: Using video to improve preservice mathematics teachers' ability to notice. Journal of mathematics teacher education, 11(2), 107–125.

Stephen, M., & Clements, D. H. (2003). Linear and Area Measurement in Prekindergarten to Grade 2. Learning and Teaching Measurement, (August), 100–121.

van Es, E. A., & Sherin, M. G. (2008). Mathematics teachers' "learning to notice" in the context of a video club. Teaching and Teacher Education, 24(2), 244–276.

Walkoe, J. (2015). Exploring teacher noticing of student algebraic thinking in a video club. Journal of Mathematics Teacher Education, 18, 523–550.

USING SUNLIGHT TO MULTIPLY POSITIVE NUMBERS

Eric A Pandiscio
University of Maine
ericp@maine.edu

Justin K. Dimmel
University of Maine
justin.dimmel@maine.edu

Keywords: Geometry, Measurement, Design Experiments, Integrated STEM/STEAM.

This poster reports on the design, creation, and installation of a unique manipulative that engages learners using the natural environment. The SunRule is an interactive sculpture that harnesses the rays of the sun to perform multiplication. The sculpture tilts and rotates. Beams of sunlight shine through linear apertures and are scaled in proportions that enact multiplication. We also invented a hand-held version that is suitable for classroom use.

How can a physical, manipulable tool realize a continuous, scaling representation of multiplication? Most physical models are discrete, and most continuous models are not physical. Many represent multiplication as a version of repeated addition, as area or array, or occasionally as a combination. While all these representations are legitimate, and useful, they are incomplete. The SunRule demonstrates that multiplication can also be interpreted as a "stretching" operation, which lays the groundwork for ideas of ratio and proportion. We posit that the exploratory nature of the device, and its connection to an authentic real-world context, is engaging, and perhaps even inspiring, for all learners.

The project is a collaboration between two mathematics educators and two sculptors. Creating a sculpture of artistic, aesthetic beauty while preserving mathematical integrity was challenging. For the "installation" version (in a municipal park, accessible to all) we sacrificed continuity of the multiplicand for elegance, simplicity, and the minimum of moving parts.

We build on the work of McLoughlin and Droukjkova (2013) who developed a diagrammatic definition that models multiplication as continuous, directed scaling. We also extend the Genoa Group (Boero, Garuti, & Mariotti, 1996; Douck, 1999) who examined heights of objects and lengths of shadows to investigate a variety of research questions.

We have collected preliminary data from three pairs of preservice teachers in a scripted interview while using the SunRule. Our questions include: "How do students make sense of multiplication as a continuous operation with the assistance of a physical manipulative?", "Does working with a continuous model help students develop fraction number sense," and "Does embedding mathematics in a real-world environment, in which a physical phenomenon is used to do mathematics, affect students' engagement with mathematical ideas?"

Kinetic sculpture

_ x _ = ?

_ x _ = ?

Figure 1: Two versions of the SunRule multiplying numbers

References

Boero, P., Garuti, R., & Mariotti, M.A. (1996). Some dynamic mental processes underlying production and proving conjectures. In L. Puig & A. Gutierrez (Eds.), *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education* (pp. 121-128), Valencia, Spain.

Douek, N. (1999). Argumentation and conceptualization in context: a case study on sunshadows in primary school. *Educational Studies in Mathematics, 39*(1), 89-110.

McLoughlin, P. F., & Droujkova, M. (2013). A geometric approach to defining multiplication. arXiv preprint, 1301.6602.

USING VIRTUAL REALITY TO EXPLORE MAGNITUDE OF NUMBERS

David Kirkland
University of Nevada, Reno
davidkirkland@nevada.unr.edu

Teruni Lamberg
University of Nevada, Reno
terunil@unr.edu

Keywords: PMENA – Instructional Activities and Practices

Understanding powers of ten requires numerical understanding of magnitudes. This process involves developing number sense (Laski and Siegler, 2007; Siegler & Opfer, 2003; Ellet, 2005). Students need to engage in multiplicative reasoning to make sense of powers of 10.

Number lines have been typically used for students to represent the magnitude of 10. Even adult pre-service teachers struggle to represent the magnitude of 10 using additive reasoning and tend to use additive reasoning as opposed to multiplicative reasoning (Brass & Harkness, 2016). The challenge is that it is hard to visualize very large or small numbers. Virtual reality is an immersive technology that can immerse students in a three-dimensional sensory experience, and they can manipulate the environment. Researchers have found that this technology enhances student motivation to learn (Osypova et al, 2020).

This study investigated 11 middle students' experiences on how virtual reality impacted their understanding of the powers of 10 magnitude. Students participated in a weeklong summer camp on nanotechnology and math. Powers of 10 was one of the key concepts to understand how materials performed in the nano scale. Activities related to the powers of 10 first included videos of what happens to materials in the nano scale, students ordered number cards based on the magnitude of powers of 10 and watched a video on the powers of 10. After these activities, students spent one and a half hours in a virtual lab looking at inside of the hand where they experienced an immersive experience of going inside the hand to being able to investigate protons and electrons. They were able to navigate the virtual environment at their own pace and click on things based on their curiosity. They got to view DNA strands, how the blood vessels worked inside the hand and keep going deeper while displays of magnitude appeared on the screen. Students individually explored while some of the students watched on the screen what the student was doing, and the teacher scaffolded the conversations to discuss what they were seeing scientifically and mathematically.

After the session, the students were asked to reflect on their experiences of how the Virtual Reality emersion experience enhance or did not enhance their understanding of the magnitude of 10. The data collected included audio recordings and fieldnotes. Strauss and Corbin, (1998) Constant Comparative method was used to identify themes that emerged from the student reflections. Students found the experience engaging and felt that they were able to visualize the powers of 10 in a way that solidified their understanding of magnitude and connected it to the context of the hand and zooming in. For example, a student said: "I see the blood streams, I have zoomed in 100,000 times and see the blood stream." The students not only thought about magnitude but connected it into the context of how the body works. A student explained she had a better understanding of the nano scale and learned how different cells and how the body works.

In conclusion, the virtual reality context enhanced students' ability to visualize the powers of 10 multiplicatively. Connecting to a real-world context, enhanced students' ability to to understand the magnitude of numbers and develop a better number sense.

References

Brass, A., & Harkness, S. S. (2017). Pre-service teachers' conceptions of the magnitude of large numbers. Investigations in Mathematics Learning, 9(2), 53-68.

Laski, E. V., & Siegler, R. S. (2007). Is 27 a big number? Correlational and causal connections among numerical categorization, number line estimation, and numerical magnitude comparison. Child Development, 78(6), 1723–1743. doi:10.1111/cdev.2007.78.issue-6

Ellet, K. (2005). Making a million meaningful. Mathematics Teaching in the Middle School, 10(8), 416–423.

Siegler, R. S., & Opfer, J. (2003). The development of numerical estimation: Evidence for multiple representations of numerical quantity. Psychological Science, 14(3), 237–250. doi:10.1111/1467-9280.02438

WHEN WHOLE-CLASS DISCOURSE PREDICTS POOR LEARNING OUTCOMES: AN EXAMINATION OF 47 SECONDARY ALGEBRA CLASSES

Samuel Otten
University of Missouri
ottensa@missouri.edu

Zandra de Araujo
University of Florida
zdearaujo@coe.ufl.edu

Ze Wang
Ashburn, VA
zewang@gmail.com

Ruby L. Ellis
North Carolina State University
rellis7@ncsu.edu

We observed lessons and collected student pre- and post-test data from 47 algebra classes in a variety of school districts. To our surprise, we found that time spent in whole-class discourse was negatively correlated with students' learning gains, both on a procedural knowledge measure and a conceptual measure. In looking more closely at some of the quality indicators of the whole-class discourse, such as the presence of mathematical justifications and the integration of multiple representations, analysis still revealed no positive relationship with student learning gains. We discuss whether these results may be related to the lack of ambitious instruction in the data set, the tendency of students to disengage when whole-class discourse is occurring, or the possibility that the whole-class discourse is happening too early in the lesson.

Keywords: classroom discourse; algebra and algebraic thinking; instructional activities

As part of a multi-year project focused on algebra instruction, we led a team that observed 47 secondary algebra classes in a wide variety of school districts and gathered pre- and post-test data of students' procedural performance and conceptual understanding (de Araujo et al., 2018). Based on three lesson observations in each class, we compiled class profiles that included a variety of structural features and quality indicators for the algebra instruction. These features included the activity structures used, the behavioral engagement level of students, the conceptual development of the mathematical ideas, the integration of mathematical representations, the type of student public contributions, and more (see Otten et al., 2023, for additional details).

One particular feature of interest was the scope and nature of whole-class discourse (WCD). What is the relationship between WCD in Algebra 1 classes and students' performance on procedural and conceptual measures of algebra learning? Based on past research (e.g., Herbel-Eisenmann et al., 2013; Staples, 2007), we hypothesized that WCD would play an important role in the growth of students' conceptual understanding, if not also the procedural performance, as the class community drew connections and explicated meanings of the algebraic content. We further hypothesized that those classes with WCD rich in conceptual development, integrated mathematical representations, minimal mathematical errors, and connections to other lessons (Smith & Stein, 2011; Xu & Mesiti, 2021) would fare especially well on the conceptual measure.

In this brief report, we share results that reveal neither of our hypotheses were confirmed by the data from these 47 algebra classes. Correlational analysis shows that more time spent in WCD predicted lower gains on both the procedural and conceptual measures. Even when accounting for the quality indicators of hypothesized productive WCD, there was still no positive relationship to the learning gains. After a brief presentation of this analysis, we discuss possible explanations for the lack of expected results.

Theoretical Perspective

Although we administered individual assessments of students' procedural performance and conceptual understanding, our theoretical perspective on learning is not strictly cognitive. We view students' mathematical learning as a sociocultural process whereby they come to participate actively in a mathematical community (Vygotsky, 1978). Thus, the lesson observation protocol (Otten et al., 2023) attends to the nature of the classroom discourse (univocal versus dialogic), the interactions of the students with one another and with the teacher, the authority dynamics at play with regard to the mathematical content, and some of the social patterns exhibited by the teacher (initiating feedback to students versus responding reactively to student questions). Although we view learning as occurring through and being defined by students' participation in forms of mathematical discourse (Lemke, 1990), we nevertheless expect that this learning will be reflected on certain individualized types of mathematical activity such as paper-and-pencil content assessments. In other words, though we view learning through a sociocultural perspective and designed our overall study from that standpoint, we still recognize that students can demonstrate certain forms of knowledge and skill through individual written responses that also happen to be valued by teachers, school leaders, and community members. This brief report focuses on those pre- and post-test data, described below, although we note that the broader study also included a qualitative survey for students and some of our analysis has focused on the classroom interactions on their own merit (e.g., de Araujo et al., 2017; Otten et al., 2021).

Method

The study involved lesson observations, teacher survey data, and student survey data from 47 Algebra 1 (or Algebra 1 equivalent) classes in Midwestern U.S. school districts that ranged from rural to urban contexts and from small to large enrollments. The larger project focused on Algebra 1 classes using a flipped instructional model ($n = 22$) as compared to those using non-flipped instruction ($n = 25$). For this brief report, we do not focus on the flipped/non-flipped variable but instead pool the classes together to examine general associations between features of the lesson observations (in particular, the resulting class profiles that were produced based on the lesson observations) and the students' pre- and post-test scores.

The 47 algebra classes included 541 students for whom we had complete data on the pre- and post-tests. These students were predominantly in grade 9 but there was a notable subgroup in grade 8 and a few in grades 10 and 11. The pre-test was administered within the first few weeks of the school year and the post-test (identical to the pre-test) was administered within the last few weeks of the school year. They included two instruments embedded within the same assessment—a procedural knowledge instrument (PK) and a concept-of-variable inventory (CoV), both of which were adapted from Genareo and colleagues' (2021) work on algebra progress monitoring, though we expanded the timeframe for the assessment from 14 minutes to 30 minutes to allow for more variation in responses while still maintaining a time limit so as to avoid a ceiling effect (the CoV items are designed so that students with conceptual understanding can answer them rapidly whereas those relying on procedural knowledge are likely to take longer as they work through an inefficient solution strategy). The PK instrument entails multiple-choice items (e.g., "Simplify $3(m + 2) + 2(m – 1)$" with answer options $5m + 1$, $5m + 4$, $5m + 8$, $6m + 4$) and yields a score out of 16, whereas the CoV instrument entails a mixture of multiple-choice and open-response items (e.g., If $n + 878 = 915$, what does $n + 880$ equal?) and yields a score out of 39. Some CoV items allowed for partial credit.

The observation data consisted of live field notes generated across three lesson observations throughout the school year, typically with two project members observing each lesson. (In a few situations, a solo observer conducted the third observation after a consistent instructional profile had already emerged from the first two observations.) The observers then completed an observation protocol (Otten et al., 2023; Zhao et al., 2018) and reconciled any coding disagreements. The completed observations protocols for the three lessons for each class were then discussed with the full project team and compiled into an instructional profile involving the use of class time by activity structure, features of the mathematical development, student involvement, tasks, etc.

Due to space restrictions, we focus here only on certain aspects of the whole-class discourse (WCD) coding. In particular, we demarcated the segments of *class time* that were spent in a WCD activity structure, meaning everyone in the class was expected to attend to the same public discourse, often verbal discourse led by the teacher but also written text on the front board or students presenting to the class. We coded for students' *behavioral engagement* during WCD based on students' gaze and apparent attentiveness (mostly on task, mixed, or mostly off task). We also drew from past classroom observation research (e.g., Boston, 2012; Hill et al., 2012) to code on a 4-point scale several aspects of the *quality of WCD*, namely, the conceptual development and justification of the mathematical ideas, integration of mathematical representations, connections to past or future mathematics lessons, and the absence of unmitigated mathematical errors (Otten et al., 2023), which for this brief report were merged into a single variable.

To answer the research question about the relationship between WCD in Algebra 1 classes and students' PK and CoV performance we used the class profiles and the student pre- and post-test data to run two-level (i.e., within-level and between-level) growth models (Table 1).

Table 1: Models for the relationship between WCD and student learning

Within-Level	Between-Level
$y_{tij} = i_{ij} + s_{ij}{*}(\text{time})_{tij} + e_{tij}$	$\beta_{00j} = \gamma_{000} + \gamma_{001}{*}(\text{WCD})_j + u_{00j}$
$i_{ij} = \beta_{00j} + r_{0ij}$	$\beta_{10j} = \gamma_{100} + \gamma_{101}{*}(\text{WCD})_j + u_{10j}$
$s_{ij} = \beta_{10j} + r_{1ij}$	

y_{tij} is the outcome variable for student *i* in class *j* at time *t* (*t* = 0 is pre-test; *t* = 1 is post-test). i_{ij} and s_{ij} are the intercept and slope parameters for student *i* in class *j*. Since there are only two time points, to identify the model, we fixed the residual variances of the pre and post scores to be zero at both levels. Fixed parameter estimates are for the pre-test score (i.e., intercept growth parameter), the gain from pre- to post-test (i.e., slope growth parameter), as well as the effect of WCD on the pre-test score and on the gain.

Findings

Overall, there were improvements over time on both the PK and CoV measures. With regard to the time spent on WCD, however, there was a negative relationship with both the PK and CoV gains. For PK, the students' pre-test score estimate was about 6.67 (out of 16) and the gain was about 2.65 points. Classes with a higher percentage of time spent on WCD tended to have a lower pre-test score ($p = 0.041$) and tended to have a lower gain over time ($p = 0.045$) than classes with a lower percentage of time spent on WCD. More specifically, as the percentage of time spent on WCD increased by 1, the gain tended to decrease by about 1.99 points. The situation was similar for the CoV measure. Students' pre-test score estimate was about 16.52 (out of 39) and the gain was about 1.97 points, but classes with a higher percentage of time spent on WCD tended to have a lower pre-test score ($p = 0.024$) and tended to have a lower gain over time ($p = 0.013$). More specifically, as the percentage of time spent on WCD increased by 1, the gain tended to decrease by about 5.06 points. Note that this is a correlational finding, not causal—a fact underscored by the lower pre-test scores for classes that tended to use more WCD.

Moving from quantity to quality of WCD, we found there to be no statistically-significant relationship between quality of WCD and the average pre-test scores ($p = 0.182$ on PK; $p = 0.383$ on CoV) nor the improvement over time on either measure ($p = 0.246$ on PK; $p = 0.356$ on

CoV). In other words, even for the classes wherein WCD was characterized by aspects of discourse that are hypothetically productive (e.g., justifications, integrating representations, making connections, avoiding errors), we did not detect a positive relationship with the procedural or conceptual learning gains.

Discussion

We found, in examining 47 algebra classes in a wide variety of school settings, that the time spent on whole-class discourse (WCD) was negatively correlated with both procedural and conceptual learning gains and that the quality of WCD did not have a statistically-significant relationship with the gains. It is possible that the failure to detect positive relationships is due to flaws in study design, limitations of the particular measures used, or contextual factors beyond our control (e.g., the COVID-19 pandemic began during post-test data collection for some of the classes). But it is also possible these results are indicative of something meaningful that is worth considering. We close with a few speculations about why spending time in WCD, even when achieving some of the aspects of research-recommended discourse, might be negatively (or not positively) associated with learning gains. (We should note that we saw very few cognitively-demanding tasks in any of the 47 algebra classes, and we saw almost no instances of shared mathematical authority or dialogic discourse, so the WCD present here is not necessarily emblematic of the discourse-rich instruction proscribed by mathematics education researchers.)

One possibility is that it might be more about the benefits of other activity structures rather than drawbacks of WCD. Providing ample time for students to work independently and engage directly in the mathematical problems, for instance, might be especially worthwhile. Indeed, that is what we found in our data, so it could be that WCD's negative correlation comes from the fact that class time is a zero sum situation and time in WCD means less time in independent work. Nevertheless, when classes exhibit the "good" features of WCD, we hypothesized that it would be adding value beyond what the students accomplish in their independent work, at least with regard to conceptual learning, and yet we still did not detect a positive correlation in that case.

Another possible explanation is that student engagement is an important underlying factor. Perhaps students are less engaged during WCD and so time spent in other activity structures is more valuable. We acknowledge that it is impossible to know for sure if students are attentive during WCD, but according to our engagement coding, there was a positive relationship between WCD engagement and PK gains ($p = 0.010$), though it was not statistically significant for CoV gains ($p = 0.160$). It is plausible that, as WCD spans a longer duration, engagement might wane. One might also wonder if WCD's relation to learning depends on the number of students involved or the extent of students' contributions, but we did not find any statistically-significant associations between those student-involvement variables and the PK or CoV learning gains.

It might also be that it is not about the duration or the quality of the WCD but rather its placement in the lesson. Sinha and Kapur (2021) conducted a meta-analysis and found that it is important to place the WCD, teacher-led instruction *after* students have had an opportunity to problem solve and make initial attempts. Perhaps this placement variable will be more predictive than the quality indicators in this brief report, and perhaps long segments of WCD tend to occur at the start of lessons, which would explain the negative correlation. Further analysis is required.

In conclusion, we think discourse-oriented scholars should take seriously these findings because there are many reasons to promote WCD (e.g., inclusion of ideas, student empowerment, collective sensemaking and dialogue), but when working with school partners, it is important to be clear-eyed about the potential drawbacks with regard student outcomes they may care about.

Acknowledgments

This study was funded by the National Science Foundation (#1721025). Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of NSF. We thank Courtney Vahle, Jaepil Han, Wenmin Zhao, and Jessica Kamuru for their help with data collection.

References

Boston, M. (2012). Assessing instructional quality in mathematics. *The Elementary School Journal*, *113*(1), 76–104. https://doi.org/10.1086/666387

de Araujo, Z., Otten, S., & Birisci, S. (2017). Mathematics teachers' motivations for, conceptions of, and experiences with flipped instruction. *Teaching and Teacher Education*, *62*, 60–70. http://dx.doi.org/10.1016/j.tate.2016.11.006

de Araujo, Z., Tarr, J., Wang, Z., & Otten, S. (2018, August 15). *Examining relationships between flipped instruction and students' learning of mathematics*. National Science Foundation. https://www.nsf.gov/awardsearch/showAward?AWD_ID=1721025&HistoricalAwards=false

Genareo, V. R., Foegen, A., Dougherty, B. J., DeLeeuw, W. W., Olson, J., & Karaman Dundar, R. (2021). Technical adequacy of procedural and conceptual algebra screening measures in high school algebra. *Assessment for Effective Intervention*, *46*(2), 121–131.

Herbel-Eisenmann, B., Steele, M., & Cirillo, M. (2013). (Developing) teacher discourse moves: A framework for professional development. *Mathematics Teacher Educator*, *1*, 181–196.

Hill, H. C., Charalambous, C. Y., Blazar, D., McGinn, D., Kraft, M. A., Beisiegel, M., Humez, A., Litke, E., & Lynch, K. (2012). Validating arguments for observational instruments: Attending to multiple sources of variation. *Educational Assessment*, *17*(2–3), 88–106. https://doi.org/10.1080/10627197.2012.715019

Lemke, J. L. (1990). *Talking science: Language, learning, and values*. Greenwood Publishing.

Otten, S., Ellis, R. L., Wang, Z., & de Araujo, Z. (2021). Comparing motivations for flipped instruction to data on flipped implementations in algebra. In D. Olanoff, S. Spitzer, & K. Johnson (Eds.), *Proceedings of the 43rd meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1554–1558). Philadelphia, PA: PME-NA.

Otten, S., de Araujo, Z., Sherman, M., & Birisci, S. (2023). A framework for capturing structural variation in flipped mathematics instruction. *International Journal of Mathematical Education in Science and Technology*, *54*(5), 639–670. https://doi.org/10.1080/0020739X.2021.1958945

Sinha, T., & Kapur, M. (2021). When problem solving followed by instruction works: Evidence for productive failure. *Review of Educational Research*, *91*(5), 761–798.

Smith, M. S., & Stein, M. K. (2011). *5 practices for orchestrating productive mathematics discussions*. National Council of Teachers of Mathematics.

Staples, M. (2007). Supporting whole-class collaborative inquiry in a secondary mathematics classroom. Cognition and Instruction, 25, 161–217. http://dx.doi.org/10.1080/07370000701301125

Vygotsky, L. (1978). *Mind in society: The development of higher psychological processes*. Harvard University Press.

Xu, L., & Mesiti, C. (2022). Teacher orchestration of student responses to rich mathematics tasks in the US and Japanese classrooms. *ZDM Mathematics Education*, *54*, 273–286. https://doi.org/10.1007/s11858-021-01322-6

Zhao, W., Han, J., de Araujo, Z., & Otten, S. (2018, November). *Flipped observation protocol*. Poster presented at the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Greenville, SC.

POLICY, INSTRUCTIONAL LEADERSHIP, TEACHER EDUCATORS

A MATHEMATICS MENTORING PROGRAM: POST-PANDEMIC

Jenna Tague
Clovis Community College
jenna.tague@cloviscollege.edu

The purpose of this report is to share the supports created as part of a mathematics mentoring program in a mid-sized western university. The research is longitudinal, and the current paper will share results from the fourth (of six) years of a project. I use a mixed method approach to examining the efficacy of the mentoring program, and the results indicate that the program may have helped students through the pandemic and to shift back afterwards.

Keywords: Instructional Leadership; Undergraduate Education, Equity and Justice

Increasing retention in mathematics majors for people of color and women remains essential. One way to advocate for this issue is to develop more supportive undergraduate mathematics departments through mentorship programs (Mondisa & McComb, 2018). Research shows many STEM majors leave because they have a bad experience, and more troubling that people of color and women are more likely to leave than their white counterparts (e.g., Anderson & Kim 2006; Hill, Corbett, & Rose, 2010; Griffith, 2010). These results and experiences indicate that research is needed on how mathematics departments might be more inclusive, and that mentoring programs might be a good way to approach this issue (Mondisa & McComb, 2015).

The mentoring program in this proposal was set at a mid-sized western comprehensive university. The supports created for the program were designed to address challenges faced by low-income students, students of color, and women in our mathematics department. The purpose of this report is to share ongoing research results of what's working and what's not, and to describe alterations to the mentoring program that have been made over time, particularly in response to the pandemic, and the resulting effects.

Literature Review

There are two important facets of literature to examine around mathematics mentoring programs: (1) which supports are most helpful to students, and (2) what qualitative and quantitative measures are valid in examining the efficacy of support programs?

In studying facets of support programs for STEM majors, Lisberg and Woods (2018) found four areas that were essential: (1) peer and faculty mentorship, (2) familiarity with programs and faculty, (3) student mindsets, and (4) student learning techniques. In the first, peer and faculty mentorship, it is essential that students are paired with a peer or faculty that they see as a mentor. Other research indicates that having mentor(s) increases retention rates in STEM programs (Campbell & Campbell, 1997). The second category is about becoming familiar with the college structure and connecting students with the necessary resources on campus. For example, many first-generation college students could struggle with applying to financial aid. The third category, student mindset, is about educating students on growth mindsets (Dweck, 2008) as well as providing students with examples of how individuals could overcome academic difficulties (Walton & Cohen, 2011). The last category, student learning techniques, focuses on the idea that many students do not know *how* to study for STEM classes, and that given specific instruction in that area, students can improve their learning. These results mirror previous results in mathematics indicating the importance of study groups (Triesman, 1992). Lisberg and Woods

(2018) found that students in their support program were more likely to pass mathematics courses in their first year, and, if they did not pass them, were two times as likely to retake them.

In a metanalysis of non-academic supports that were most helpful to retaining STEM community college students, Karp (2011) found they could be sorted into four main categories: *social relationships*, *career options*, *college structure*, and *life issues* (italics added for emphasis and to denote each category). The *social relationship* category included those supports that helped students connect with one another and faculty members on campus, which echoes the mentorship category from Lisberg and Woods' (2018) study. The second category included educating students about *career options* both in terms of helping them choose appropriate majors, but also providing information on what careers were possible with which majors. The *college structure* category aligned with the familiarity with programs category from Lisberg and Woods (2018). Lastly, the *life issues* category was identified as the most difficult one to support students through. However, when students had resources to connect to on campus, they were more likely to persist and remain enrolled.

Having examined research on the supports that were helpful, I will move on to discuss the how these programs have been studied and compared. Karp (2011) remarked that the majority of the research from the metanalysis made use of Likert-style questions that asked how helpful the students thought particular supports were. The students were not asked why the supports were helpful. Similarly, Mondisa and McComb (2018) commented on how many support programs relied solely on quantitative comparisons like comparisons of students' GPA or attrition rates. Both papers noted that there is a lack of research on support programs as a whole because they often are created, but not studied, and so more research is needed on support programs. Lastly there was evidence that there is a need for more qualitative data to examine what is most helpful and why for supporting underrepresented students in STEM majors (Estrada et al., 2017).

In designing our program, we centered our supports around Karp's (2011) four categories: *social relationships*, *career options*, *college structure*, and *life issues*, while also including mindset and study skills from Lisberg and Woods (2018) categories. The research design includes quantitative methods, such as comparison of graduation rates, and qualitative methods to examine the social aspects of the program.

Context

The support program was created at a mid-sized western comprehensive university where the overall student population consisted of 67% first generation students, 61% Pell-eligible, and has overall student population demographics 3% African American, 14% Asian (mainly Southeast Asian: Hmong and Cambodian), 49% Hispanic, 6% non-resident students, 3% two or more races, 5% unknown, and 20% white.

During the preparation year, focus groups were held with current students to identify areas of support for our student populations (Tague, 2021). The focus groups were analyzed for overall themes and those themes informed how the supports were built around Karp's (2011) categories. The team consisted of three mathematicians, who I will refer to as mentors, and myself. The mentoring program began in Fall 2018 and has shifted to meet the students' needs over time.

In the *social relationships* category, we designed scholarships, weekly meetings, and required office hours or tutoring. It may strike the reader as odd to include scholarships in this category, however, we have found that without financial support, students work two or more jobs, which prevents them from spending time on campus and making connections. We work on educating students on *career options* through advising sessions, workshops, and guest speakers.

These became more prevalent in the last few years as the students reached their junior and senior years. The workshops in the later years have been focused on graduating, and research experiences to emphasize the *college structure*. Lastly, it is difficult to plan ahead for what *life issues* students might face, however, we hoped that through the community building, students would have the social network to persevere through these.

Theoretical Framework and Methods

I used a mixed methods approach to this research. For the quantitative methods, I have been tracking longitudinal data including graduation rates. For the qualitative methods, I used the framework of Social Community (Mondisa & McComb, 2018) to measure the impact of the program at an individual level in terms of *connectedness, resiliency,* and *communities of practice*. I measured how *connected* each student was to the program by asking, for example, do you have a mentor? I measured *resiliency* by asking the students, for example, if they faced any obstacles and if so, how they worked to overcome them. The framework defines *communities of practice* as “collections of like-minded individuals sharing similar experiences and social resources as they interact with and support each other (Eckert, 2006; Wenger, 2000)” (Mondisa & McComb, 2018, p. 98). For this question, I asked students if they felt like they belonged as part of the mathematics department, and why or why not. All interview questions were adapted from Mondisa and McComb’s (2018) framework.

The current report is from the fourth (out of six) years of the program. In this year, there were 14 scholars (5 seniors and 9 juniors). There were 12 students recruited in the control group (6 juniors and 6 seniors). All participants in the study were mathematics majors and both groups consisted of a majority of people of color. All participants, scholar or control, participated in semi-structured interviews, which, on average, lasted 27 minutes during October 2021-January 2022. The interviews were recorded, transcribed verbatim using software, and then altered for correctness. The resulting transcriptions were organized into a spreadsheet and then separated by theme: *connectedness*, *resiliency*, and *communities of practice* (Mondisa & McComb, 2018).

Results

I will present the quantitative results first, followed by the qualitative results by theme. Because of the longitudinal nature of the study, this was the first year that I could report graduation rates. They are shown in Table 1 below, where the starred year represents the first year the program was running. It shows that the support program might be helping a larger percentage of mathematics students graduate in four years.

Table 1. Rate of Mathematics Majors Graduating in Four Years

Year of Entry	Percent of Majors Graduating in Four Years
2013	11.1%
2014	18.2%
2015	27.6%
2016	27%
2017	21.6%
2018*	37.8%

Connectedness

In previous years, the scholars were significantly more likely to show connectedness by stating that they had a mentor that was a faculty member (Tague, 2021; 2022). However, in the current year 11 of 14 scholars versus 8 of 12 control students reported having a faculty member as a mentor (χ^2=0.1153; p=0.7341), which was not significant. It makes sense that in the later years of the majors, the students who have persisted would have a connection to a faculty member. There was a difference between how the control group talked about their faculty mentors versus how the scholars did. The control group mainly stated that they could go to the mentor if they had questions about their academic paths or about a course. For the scholars, many mentioned that they felt they could talk to their mentor "about anything". One even stated, "*I feel like he's my academic dad*." In summary, there was no difference in the connectedness as measured by if they had a mentor, but qualitatively the scholars described closer relationships with their mentors.

Resiliency

Both groups struggled with time management and switching back to in person from being online. Many of them described how they struggled the previous year with a lack of socializing and how they valued being back in person for their math courses. Much of the new time management issues had to do with readjusting to commuting to campus again. One theme that appeared in the control group was that of mental health linked to motivation. The majority of the control group mentioned that mental health centered around motivation was an obstacle for them in the current semester, whereas only two of the scholars mentioned these specific obstacles. These results indicate that both groups displayed resiliency, however the scholars seemed to have used their social networks to overcome motivation struggles.

Communities of Practice

In previous years, there was no difference in the rate of belonging between the control group and scholars. This year, out of 14 scholars, 12 stated they felt like they belonged versus 6 out of 12 of the control group. This indicated a statistically significant difference in communities of practice between the groups (χ^2=3.869; p=0.0491). There was also a difference in how the groups talked about belonging. The control group referenced that some of their belonging was based on academic achievement, whereas for the scholars, their belonging was based on their involvement and contribution to the department community. As a representative sample of the control group, one student articulated, "*Do I have the credentials to be in the math department? I'm just a student here.*" This contrasts with the representative sample from the scholars saying, "*I didn't in the beginning, but I feel like I do now…I feel like I contribute a part of the department*."

Conclusions

All of the students, scholar and control group, displayed connectedness and resiliency. The scholars' connectedness was deeper in their descriptions of what issues they could ask their mentors about. The scholars' resiliency in coming back after the pandemic was less plagued with motivation issues than the control group. Lastly, the scholars displayed significantly higher levels of communities of practice than the control group. These results are consistent with results from the prior three years in the project (Tague, 2021; 2022) where the scholars were more comfortable in the first year going to office hours, and in the second year were more likely to have a faculty member as a mentor. Results from each year of the study are promising, and provide evidence that with support, students might feel more belonging and persistence in mathematics departments.

Acknowledgments

The funding for this research was provided by grant 1742236.

References

Anderson, E., and Kim, D. (2006). *Increasing the Success of Minority Students in Science and Technology*. Washington, DC: American Council on Education.

Campbell, T. A., & Campbell, D. E. (1997). Faculty/student mentor program: effects on academic performance and retention. *Research in Higher Education, 38*(6), 727–742.

Dweck, C. S. (2008). Mindsets and math/science achievement. Carnegie Corp. of New York–Institute for Advanced Study Commission on Mathematics and Science Education, New York.

Eckert, P. (2006). Communities of practice. *Encyclopedia of language and linguistics, 2*(2006), 683–685.

Ehrich, L. C., Hansford, B., & Tennent, L. (2004). Formal mentoring programs in education and other professions: A review of the literature. *Educational Administration Quarterly, 40*(4), 518–540.

Estrada, M., Burnett, M., Campbell, A. G., Campbell, P. B., Denetclaw, W. F., Gutierrez, C. G., Hurtado, S., Gilbert, J. H. Matsui, J., McGee, R., Moses Okpodu, C., Robinson, T. J., Summers, M.F., Werner-Washburne, M., & Zavala, M. (2017). Improving underrepresented minority student persistence in STEM. *CBE-Life Sciences Education*, *15*(3).

Gershenfeld, S. (2014). A Review of Undergraduate Mentoring Programs. *Review of Educational Research, 84*(3), 365–391.

Griffith, A. (2010). Persistence of women and minorities in STEM field majors: Is it the school that matters? *Economics of Education Review, 29*(6): 911–922.

Higher Education Research Institute. (2010). Degrees of Success: Bachelor's Degree Completion Rates Among Initial STEM Majors. Los Angeles: Author.

Hill, C., Corbett, C., and Rose, A.S. (2010). *Why So Few? Women in Science, Technology, Engineering, and Mathematics*. Washington, DC: American Association of University Women.

Hrabowski, F. A., & Pearson, W., Jr (1993). Recruiting and retaining talented African–American males in college science and engineering. *Journal of College Science Teaching, 22*(4), 234–238.

Karp, M. M. (2011). *Toward a New Understanding of Non-Academic Student Support: Four Mechanisms Encouraging Positive Student Outcomes in the Community College*. CCRC Working Paper No. 28. Community College Research Center, New York, New York.

Lisberg, A., & Woods, B. (2018). Mentorship, mindset and learning strategies: An integrative approach to increasing underrepresented minority student retention in a STEM undergraduate program. *Journal of STEM Education*, *19*(3), 14-20.

Maton, K. I., Hrabowski, F. A., & Schmitt, C. L. (2000). African American college students excelling in the sciences: College and postcollege outcomes in the Meyerhoff Scholars Program. *Journal of Research in Science Teaching, 37*(7), 629–654.

Mondisa, J. L., & McComb, S. A. (2015). Social Community: A Mechanism to Explain the Success of STEM Minority Mentoring Programs. Mentoring & Tutoring: Partnership in Learning, 23(2), 149–163.

Mondisa, J. L. & McComb, S.A. (2018). The role of social community and individual differences in minority mentoring programs. *Mentoring & Tutoring: Partnership in Learning*, *26*:1, 91-113.

Seymour, E., and Hewitt, N.M. (1997). *Talking About Leaving: Why Undergraduates Leave the Sciences*. Boulder, CO: Westview Press.

Snyder, T.D., and Dillow, S.A. (2011). *Digest of Education Statistics, 2010* (NCES 2011-015). National Center for Education Statistics, Institute of Education Sciences, U.S. Department of Education. Washington, DC.

Tague, J. (2021). The Social Community of a Mathematics Support Program. Proceedings of the 2021 Psychology of Mathematics Education- North America Conference, Philadelphia, Pennsylvania, USA.

Tague, J. (2022). A Mathematics Support Program: Modifications and Responses to the Pandemic. Proceedings of the 2022 Psychology of Mathematics Education – North America Conference, Nashville, TN, USA.

Treisman, Uri (1992). Studying students studying calculus: A look at the lives of minority mathematics students in college. *The College Mathematics Journal, 23*(5), 362–372.

Walton, G. M., & Cohen, G. L. (2011). A brief social-belonging intervention improves academic and health outcomes of minority students. *Science, 331*(6023), 1447-1451. doi:10.1126/science.1198364.

Wenger, E. (2000). Communities of practice and social learning systems. *Organization, 7*(2), 225–246.

DILEMMAS AS A SITE FOR MATHEMATICS TEACHER EDUCATOR LEARNING

Michael Jarry-Shore
North Carolina State University
mjarryshore@ncsu.edu

Keywords: Teacher Educators, Preservice Teacher Education, Professional Development

For this self-study (Vanassche & Kelchtermans, 2015), I examine the following question: how might unpacking the dilemmas that mathematics teacher educators (MTEs) encounter support MTE learning? I root the study in a dilemma that arose as I was modeling a Number Talk (Humphreys & Parker, 2015). For this talk, I had asked a class of pre-service elementary teachers (PSTs) to solve 75 – 29 using mental math. The first student to share used a strategy I had expected: 75 – 25 = 50; 50 – 4 = 46. The second student shared a strategy I had also expected: 75 – 30 = 45; 45 + 1 = 46. Then a third student shared the following: 75 – 29 → 80 – 30 = 50; 50 – 5 – 1 = 44. And I was stuck. I had not anticipated this strategy and could not figure out, on-the-spot, why the strategy did not yield the correct answer of 46. I found myself faced with a dilemma, having to choose between two equally undesirable options (Lampert, 1985). While we could try to figure out the strategy as a class, that risked causing confusion given my lack of understanding and could potentially leave PSTs thinking that centering student strategies is a daunting task. Alternatively, we could bypass the strategy, but that risked conveying that only student thinking that is anticipated and easily comprehended is to be centered. Ultimately, I pursued the latter option. We later considered the lesson from the teacher's perspective. I shared that I was unsure if I had handled this moment well. I also saw a look of concern on PSTs' faces, as they seemed to recognize that similar episodes might arise in their own mathematics teaching.

After the lesson, I shared this dilemma with another MTE in my department. We discussed how a number line could have demonstrated the source of the error in this strategy. We then drew a number line showing the difference between 75 and 29. Next, we placed the numbers 80 and 30 on this number line, which showed that the difference in this new problem had increased by five at the upper end, yet by only one at the lower end. As this number line demonstrated, changing 75 – 29 to 80 – 30 had resulted in the PST finding a difference that was four units too large. In unpacking this dilemma with a colleague, I developed my knowledge of "the ways of representing and formulating the subject that make it comprehensible to others" (Shulman, 1986, p. 9). I also found myself feeling better able to respond to this particular dilemma – to pursue or bypass an unanticipated solution strategy – in the future. Finally, I recognized that unpacking such dilemmas with supportive colleagues could develop my practice as a MTE.

Scholars have described the dilemmas that classroom teachers face (Lampert, 1985) and my experience suggests that MTEs not only face their own dilemmas but may also learn a great deal by unpacking these with others. There are thousands of MTEs across the globe teaching courses like mine or facilitating professional development with practicing teachers, yet little research has explicitly examined their preparation and learning (Lloyd, 2020; Tzur, 2001). I propose that collectively reflecting on the dilemma's MTEs encounter, with the support of other MTEs, could support MTE learning. In this poster presentation, I will make this case by homing in on the example described above. I will also describe plans for future research, in which I propose developing criteria for identifying the dilemmas MTEs encounter. Specifically, I will describe plans to ask MTEs to bring their dilemmas to gatherings with other MTEs, then study the nature of these dilemmas and the MTE learning that results from unpacking these collectively.

References

Humphreys, C., & Parker, R. (2015). Making number talks matter: Developing mathematical practices and deepening understanding, grades 4–10. Stenhouse Publishers.

Lampert, M. (1985). How do teachers manage to teach? Perspectives on problems in practice. Harvard Educational Review, 55(2), 178–195. https://doi.org/10.17763/haer.55.2.56142234616x4352

Lloyd, G. M. (2020). Collaborations and contexts for participation and learning in mathematics teacher education. In G. M. Lloyd & O. Chapman (Eds.), International handbook of mathematics teacher education: Volume 3. Participants in mathematics teacher education (pp. 1–14). Brill.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. Educational Researcher, 15(2), 4–14. https://doi.org/10.3102/0013189X015002004

Tzur, R. (2001). Becoming a mathematics teacher-educator: Conceptualizing the terrain through self-reflective analysis. Journal of Mathematics Teacher Education, 4(4), 259–283. https://doi.org/10.1023/A:1013314009952

Vanassche, E., & Kelchtermans, G. (2015). The state of the art in Self-Study of Teacher Education Practices: A systematic literature review. Journal of Curriculum Studies, 47(4), 508–528. https://doi.org/10.1080/00220272.2014.995712

EXAMINING MATHEMATICS TEACHER EDUCATORS' RATIONALES WHEN IMPLEMENTING A COMMON FUNCTIONS TASK

Jonathan D. Watkins
Ball State University jdwatkins@bsu.edu

Alees Lee
Weber State University
aleeslee@weber.edu

Nicholas Kochmanski
University of North Carolina at Greensboro
nmkochmansk@uncg.edu

Terrie M. Galanti
University of North Florida
terrie.galanti@unf.edu

In this study, we examine the knowledge, purposes, and rationales of four mathematics teacher educators (MTEs) as they implemented a common mathematical task in content and methods courses for prospective secondary teachers. When implementing the common task, the MTEs made decisions to foreground the learning of mathematics or the learning of pedagogy that were situated in their knowledge of their respective university programs, their knowledge of prospective teachers and their needs, as well as the MTEs' beliefs and backgrounds.

Keywords: Teacher Educators; Instructional Activities and Practices

Preparing prospective teachers (PTs) to enact task-based, student-centered instruction in secondary classrooms is challenging work. It involves providing PTs with opportunities to develop both robust mathematical understandings and effective pedagogical strategies in their university programs (AMTE, 2017). One-way MTEs can provide mathematical and pedagogical learning opportunities is to integrate rich mathematics tasks into their courses (Thanheiser, 2015; Watson & Mason, 2007). Doing so can provide PTs with opportunities to experience effective teaching strategies first-hand as students and to engage with important mathematical ideas (e.g., Kochmanski et al., 2022). Yet, the coursework required for PTs can look vastly different depending on the structure and design of teacher licensure programs. Some programs, for example, require mostly traditional mathematics courses along with methods courses, while others require mathematics content courses specific to the teaching of secondary mathematics. MTEs are often asked to make principled decisions about whether (and how much) to foreground mathematical understandings or effective teaching practices in the courses they are designing for PTs. Though these decisions are likely influenced by the constraints of PT program structures and designs, there may be other factors that also contribute to MTEs' decisions. Currently, we are unaware of any studies that have looked closely at how individual MTEs foreground mathematics over pedagogy (or vice-versa) in their mathematics education courses.

Further, several scholars in mathematics education have called for increased attention to MTEs' knowledge, purposes, and rationales as they prepare PTs (Appova & Taylor, 2019; Jaworski & Huang, 2014). The primary goal of this study is to contribute to research on MTEs' knowledge, purposes, and practices by closely examining the rationales of four MTEs working in different institutional contexts. We focused on the foregrounding of the learning of mathematics or the learning of pedagogy as the MTEs planned and implemented a common mathematics task in their courses for PTs. The following questions guided our analysis:

1. To what extent did MTEs foreground the learning of mathematics compared to the learning of pedagogy in their enactment of a common functions task with prospective secondary teachers?
2. Why did the MTEs foreground the learning of mathematics and the learning of pedagogy in the ways that they did?

Theoretical Framework

Our analysis was informed by prior research examining MTEs' thinking and reasoning. In particular, we follow Appova and Taylor (2019) in focusing on several factors that can influence MTE decision-making, including: (a) MTEs' goals for teachers' learning; (b) their purposes for engaging teachings in particular tasks; and (c) their knowledge of content, pedagogy, and students' learning. Additionally, teacher education, as with classroom teaching, occurs in context, and as such we also attend to the features of the institutional context that may influence MTEs' decision-making, such as, for example, the nature of teacher education program designs, whether the teacher education program is situated in a school of education or in a mathematics department, and the educational history of the institution in which the courses take place.

Methods

This study examines the rationales of four MTEs across four different U.S. postsecondary institutions as they implemented a common mathematics task adapted from Beckman (2022). The Coffee Task (see Figure 1) focuses on the topic of function, which is an "essential understanding" topic in secondary mathematics (Zhang et al., 2020). The task is intended to elicit ideas of discrete or continuous functional relationships, differences between codomain and range, and approaches to graphing functions rooted in real-world situations or contexts.

> An online coffee company charges $12 for each bag of coffee plus $5 for shipping (no matter how many bags are ordered). Model this (situation/function) in as many ways as you can.

Figure 1. The Coffee Task

One MTE with an appointment in a mathematics department implemented the task in a K-12 math content course for teachers. Another MTE with an appointment in a mathematics department implemented the task in two separate courses: a K-12 math content course for teachers and a secondary methods course. The two MTEs with appointments in colleges of education implemented the task in methods courses for secondary mathematics teachers. In all, the four MTEs implemented the same task in five different courses.

We collected the following data as part of the study: video recordings of each MTE's implementation of the focal task, classroom artifacts from those sessions (e.g., students' work and written reflections), and MTE interviews regarding instruction. Together, these data enabled us to look for and understand the differences in the extent to which the MTEs foregrounded the learning of mathematics and the learning of pedagogy. The interviews were particularly helpful in building our understanding of *why* the MTEs chose to foreground mathematics or pedagogy. We conducted thematic analysis across the memos to surface similarities and differences in the MTEs' rationales for the ways in which they implemented the task.

Findings

To What Extent Did MTEs Foreground the Learning of Mathematics Compared to the Learning of Pedagogy in Their Enactment of a Common Functions Task?

There was variation in how much the MTEs chose to foreground the learning of teaching compared to the learning of mathematics in the implementation of the focal mathematics task. In the sections that follow, we discuss the balance of content and pedagogy in all five classes, organized by the types of courses in which the MTEs implemented the focal task.

Methods courses. Two of the three MTEs who implemented the Coffee Task in methods courses foregrounded the learning of pedagogy over the learning of mathematics. One used the task to elicit student thinking about elements of the task they would need to attend to effectively launch mathematics tasks. The second MTE modeled the enactment of the task in a reform-oriented launch-explore-summarize lesson and had PTs examine his implementation. In contrast, the third MTE foregrounded the learning of content in her methods course. She prompted students to identify the five "big ideas" of functions from NCTM's Developing Essential Understandings series within their task solutions. That said, she did engage with pedagogy by framing the task as a context for discussing how to facilitate meaningful mathematical discourse—one of the eight Principles to Actions effective teaching practices (NCTM, 2014).

Content for teaching courses. Both MTEs who taught content for teaching courses foregrounded the learning of mathematics. In one class, the MTE used the Coffee Task primarily as a formative assessment. to assess understanding of multiple function representations. The other MTE foregrounded the learning of content by asking PTs to not only model the situation but to also describe the relationship between the quantities; explain whether the relationship was a function; and describe the domain, codomain, and range of the relation. In both cases, the MTEs also discussed pedagogy, but it was ancillary the mathematics.

Why Did the MTEs Foreground the Learning of Mathematics and the Learning of Pedagogy in the Ways that They Did?

In interviews before and after the implementation of the Coffee Task, all four MTEs emphasized their intent to provide PTs with opportunities to learn about both mathematics and pedagogy. However, as noted above, the extent to which the MTEs foregrounded these different but overlapping domains of our discipline differed. We identified several factors that appeared to influence the extent to which the MTEs foregrounded the learning of content versus the learning of pedagogy. These factors included the MTEs' knowledge of their program structure and course sequencing; knowledge of PTs and course composition; and the MTEs' personal experiences, interests, and educational/professional background.

Knowledge of program structure and course sequencing. The MTEs' understandings of PTs' prior and expected future experiences in mathematics content and methods courses influenced the extent to which they focused on content and pedagogy in their courses. For example, one of the MTEs placed a heavy emphasis on mathematics in his content course because he knew that his PTs would be exploring pedagogy in three math methods courses throughout their program. A different MTE foregrounded mathematics in her methods course because her university program of study has one three-credit mathematics methods course and no content courses for mathematics teaching.

Additionally, the MTEs considered when the task was being implemented in the course sequence, so that it could be coherently integrated within the existing sequence. For example, one of the MTEs who implemented the task in his methods course did so at a time in the course when students were learning about launching tasks. This contributed significantly to his decision to use the task as a context for exploring how to launch a task effectively. A different MTE, on the other hand, implemented the task during the final weeks of a third (and final) methods course, so he used the task as an opportunity to "put the pieces of the puzzle together" by modeling and discussing the entire launch-explore-summarize process.

Knowledge of prospective teachers and course composition. All of the MTEs drew upon their knowledge of PT development, as well as the general composition of their courses, when deciding on how much to foreground mathematics or pedagogy. For example, one of the MTEs taught a methods class that contained PTs from several different programs, including middle grades, secondary, and alternative licensure pathways. In order to ensure the class was relevant to all PTs, the MTE made a concerted effort to foreground issues of pedagogy, as opposed to content. Another MTE taught a synchronous, online course. The MTE did not model instruction of the Coffee Task for her PTs, as she knew they would be primarily teaching in face-to-face settings and the ways in which she would enact the task in an online setting would differ from how it might look in person. As such, the MTE foregrounded the learning of content over pedagogy by focusing on highlighting the mathematical ideas teachers can surface with this task, as this was the element she saw as most relevant to the PTs' future teaching.

MTEs' personal experiences, interests, and backgrounds. Decisions to foreground mathematics or pedagogy were also influenced by MTEs' personal experiences, interests, and educational/professional background. While each of the instructors articulated a vision of effective teaching focused on supporting students' development of conceptual understanding, their differing backgrounds and experiences were reflected in their instructional choices. For example, one of the MTEs drew on her experiences as a professional engineer and a high school mathematics teacher; she emphasized mathematical modeling aspects of function in her teaching and our ability to understand quantifiable phenomena. Another MTE, who was an undergraduate English major, focused on the practices of launching a task and the more apparent mathematical ideas in the task rather than using the task to discuss deeper mathematical ideas inherent in the task.

Discussion

Secondary MTEs' instructional decisions are highly influenced by the perceived purposes and roles the courses hold within the larger university programs. While MTEs can learn about content and pedagogy during their training as MTEs, it is also vital to consider how we are preparing MTEs to uncover and navigate the perceived purposes of their courses. As MTEs reflect on their own mathematics experiences and their beliefs about effective practices in preparing PTs, they are better positioned to create and support high-quality mathematics teacher preparation programs.

References

Appova, A., & Taylor, C.E. (2019). Expert mathematics teacher educators' purposes and practices for providing prospective teachers with opportunities to develop pedagogical content knowledge in content courses. Journal of Mathematics Teacher Education, 22, 179-204. https://doi.org/10.1007/s10857-017-9385-z

Association of Mathematics Teacher Educators. (2017). Standards for preparing teachers of mathematics. Retrieved from https://amte.net/standards

Beckmann, S. (2022). Mathematics for elementary and middle school teachers with activities (6th ed.). Pearson.
Jaworski, B., & Huang, R. (2014). Teachers and didacticians: Key stakeholders in the processes of developing mathematics teaching. ZDM, 46, 173-188. https://doi.org/10.1007/s11858-014-0574-2
Kochmanski, N., Galanti, T. M., Lee, A. S, and Watkins, J. D. (2022, November 17 - 20). Modeling ambitious teaching in synchronous online mathematics education. In A. E. Lischka, E. B. Dyer, R. S. Jones, J. Lovett, J.,
J. Strayer, J., & S. Drown, S. (2022). Proceedings of the 44th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (pp. 2006 – 2014). Nashville, TN: Middle Tennessee State University.
National Council of Teachers of Mathematics. (2014). Principles to actions: Ensuring mathematical success for all. Author.
Thanheiser, E. (2015). Developing prospective teachers' conceptions with well-designed tasks: Explaining successes and analyzing conceptual difficulties. Journal of Mathematics Teacher Education, 18, 141-172. https://doi.org/10.1007/s10857-014-9272-9
Watson, A., & Mason, J. (2007). Taken-as-shared: A review of common assumptions about mathematical tasks in teacher education. Journal of Mathematics Teacher Education, 10, 205-215. https://doi.org/10.1007/s10857-007-9059-3
Zhang, P., Brown, A., Joseph, M., & He, J. (2020). What content preK-8 prospective teachers should know and why: Topic recommendations for content courses. The Mathematics Enthusiast, 17, 403-434. https://doi.org/10.54870/1551-3440.1493

EXPLORING THE VALIDITY OF A MEASURE OF INSTRUCTIONAL VISION

Cathy Holl-Cross
UNC, Charlotte
chollcro@uncc.edu

P. Holt Wilson
UNC, Greensboro
phwilson@uncg.edu

Olanrewaju Oriowo
UNC, Charlotte
ooriowo@uncc.edu

Christine Fisher
UNC Greensboro
cakierna@uncg.edu

Olu Adefope
East Carolina
adefopeo19@ecu.edu

Keywords: Instructional Vision, Measurement

Established and emerging research suggests the importance of a shared instructional vision for collective efforts to improve instruction (e.g., Cobb et al. 2013). Instructional vision refers to images of ideal practice that teachers aspire to (Hammerness, 2001). Munter (2014) used this idea to articulate a framework for vision of high-quality mathematics instruction (VHQMI) and developed an interview protocol and set of rubrics to track educators' descriptions of instruction and their alignment toward research-based descriptions of high quality math instruction along dimensions of the *role of the teacher*, *classroom discourse*, and *mathematics tasks*. The VHQMI rubrics have since proven useful in investigations related to teacher learning and improvements in instructional practice (Munter & Correnti, 2017; Munter & Wilhelm, 2021).

In the context of a statewide partnership to support improvements in mathematics instruction, our research aims to test the conjecture that developing a shared VHQMI is foundational to successful implementations of STEM education innovations at scale. To investigate this conjecture, our team extended Okeyo's (2019) VHQMI item development study to design a survey instrument based on Munter's (2014) rubrics that approximates educators' VHQMI. Our goal was to develop an instrument to characterize educators' VHQMI quickly and efficiently at scale to inform the team's design efforts. The online survey consists of twelve pairwise comparison questions for each dimension (total of 36 questions) where respondents indicate which of two choices more accurately describes their VHQMI.

This study investigates the validity of the VHQMI survey instrument as an approximation to Munter's (2014) VHQMI rubric scores and identify factors that may be significant predictors of VHQMI survey scores through examining two questions: (1) Is there a significant positive correlation between VHQMI survey and interview rubric scores? and (2) What factors are significant predictors of educators' vision score? To investigate these questions, we conducted VHQMI interviews and administered the survey to seventy K-12 mathematics teachers, coaches, and curriculum leaders from the partnership. Analyses of interview and survey scores indicated a moderate correlation (Spearman's rho=.3367, p=.004) and significant predictor variables of role group in a multiple regression model to predict VHQMI. ANCOVA test also showed significant differences between VHQMI scores among role groups but not among grade bands. Although these results provide modest evidence that survey scores may serve as rough approximations for respondents' VHQMI score. Additional investigations of paired interview and survey scores with larger variability are needed to strengthen a validity argument for the VHQMI survey.

Acknowledgements

This work was supported by the National Science Foundation under grant DRL - 2100895 awarded to East Carolina University, DRL-2100903 awarded UNC Greensboro, DRL - 2100947 awarded UNC Charlotte, and DRL - 2100833 awarded to Appalachian State University. Any opinions, findings, and conclusions or recommendations expressed herein are those of the principal investigators and do not necessarily reflect the views of the National Science Foundation

References

Baguley, T. (2012). Serious Stats: A guide to advanced statistics for the behavioral sciences. London: Springer.

Cobb., P., Jackson, K., Smith, T., Sorum, M., & Henrick, E. (2013). Design research with educational systems: Investigating and supporting improvements in the quality of mathematics teaching and learning at scale. National Society for the Study of Education, 112(2), pp. 320-349.

Hammerness, K. (2001). Teachers' visions: The role of personal ideals in school reform. Journal of Educational Change, 2(2), 143-163. https://doi.org/10.1023/A:1017961615264

Munter, C. (2014). Developing visions of high-quality mathematics instruction. Journal for Research in Mathematics Education, 45(5), 584–635. https://doi.org/10.5951/jresematheduc.45.5.0584

Munter, C., & Correnti, R. (2017). Examining relations between mathematics teachers' instructional vision and knowledge and change in practice. American Journal of Education, 123, 171–202. https://doi.org/10.1086/689928

Munter, C., & Wilhelm, A. G. (2021). Mathematics teachers' knowledge, networks, practice, and change in instructional visions. Journal of Teacher Education, 72(3), 342–354. https://doi.org/10.1177/0022487120949836

Okeyo, S.R. (2019). Development and validation of an instrument to assess mathematics educators' visions of mathematics instruction. [Doctoral dissertation, University of North Carolina at Greensboro].

INVESTIGATING A SURVEY MEASURING VISION OF HIGH-QUALITY MATHEMATICS INSTRUCTION

P. Holt Wilson
UNC Greensboro
phwilson@uncg.edu

Robert Henson
UNC Greensboro
rahenson@uncg.edu

Allison McCulloch
UNC Charlotte
allison.mcculloch@uncc.edu

Cathy Holl-Cross
UNC Charlotte
chollcro@uncc.edu

Research has shown that instructional vision is related to teacher learning, instructional quality and change, and an important consideration for large scale improvement efforts. To date, these studies have primarily relied on resource-intensive methods to measure instructional vision. In this paper, we detail the development of a survey instrument to measure vision of high quality mathematics instruction and explore its relation to established rubrics for assessing vision in interview settings.

Keywords: Instructional Vision, Measurement.

Over the last two decades, researchers investigating mathematics teacher learning and professional development have offered a number of cognitive, social, historical, organizational, and contextual explanations for variations in what and how teachers learn in different settings. One of these explanations that has proven useful is the idea of instructional vision. Defined as "a set of images of ideal classroom practice for which teachers strive" (Hammerness, 2006, p.143), a teacher's vision represents a form of practice to which they aspire, and researchers have documented its association with instructional quality (Munter & Correnti, 2017; Wilhelm, 2014), changes in classroom over time (Munter & Correnti, 2017), students' academic outcomes (Chance & Segura, 2009), and successful implementation of new policies (Gamoran et al., 2003). Some have used vision to understand teachers' participation in their professional communities, showing that vision influences the ways one filters competing messages about practice in different social contexts (Tichnor & Schwartz, 2017), shapes interactions within professional networks (Munter & Wilhelm, 2021), and facilitates collaborations among teachers and other instructional support professionals when it is shared (Peterson et al., 1996). Others working with districts to create structures, resources, and systems to support teacher learning have noted the importance of a shared vision in organizational improvement efforts (Cobb & Jackson, 2011).

The field's understanding of instructional vision and the role it plays in teacher learning and professional growth was established through exploratory studies using qualitative methods such as interviews (e.g., Hammerness, 2001; Munter, 2014) and document analysis (e.g., Tichnor & Schwartz, 2017). As the knowledge base matures, researchers interested in using this knowledge to design innovations for improvements at scale will need additional methods and instrumentation to characterize the visions of those in educational systems. Efficient methods that yield relevant and timely analyses of instructional vision would be useful for assessing needs, designing implementation supports, and monitoring progress over time. In this paper, we describe a survey instrument designed to measure instructional vision in mathematics developed in the context of a statewide partnership of researchers and state leaders to support the implementation of new state mathematics standards. We then report findings from an analysis of survey responses to assess the degree to which the instrument's measure of vision is consistent with the established VHQMI interview rubrics (Munter, 2014) on which it is based.

Theoretical Perspectives

Significant advances have been made in characterizing instruction that supports all students in learning mathematics. Described by some as high-quality mathematics instruction, instruction that meets these goals aims for teachers to be intentional in supporting students by problematizing ideas, supporting students in developing mathematical authority, and scaffolding classroom discussions in ways that formalize learning goals for students. Though this characterization summarizes the research base and may represent Hammerness's (2006) notion of "images of ideal classroom practice" for some, what mathematics teachers and leaders see when imagining ideal instruction varies considerably (Munter, 2014).

Building from Hammerness' (2006) conception of vision, Munter (2014) and colleagues (Munter & Wilhelm, 2021) describe instructional vision as a discourse used to characterize ideal teaching practice. As a discourse, what one believes to be high quality mathematics teaching is reproduced in, and shaped by, professional and other discourses in which teachers participate. From his characterization of a vision of high quality mathematics instruction based on a synthesis of mathematics education research, he developed and tested an interview protocol and accompanying rubrics to track educators' descriptions of instruction and their alignment with research-based descriptions of high-quality mathematics instruction. These rubrics articulate qualitatively distinct levels of sophistication of vision of high-quality mathematics instruction for three dimensions of classroom practice – the *role of the teacher*, *classroom discourse*, and *mathematics tasks* – and have been used in studies focused on prospective teachers (Walkowiak, et al., 2015), practicing teachers (Munter, 2015; Wilhelm, 2014); mathematics leaders (Jackson et al., 2015); and principals (Katterfeld, 2015; Munter & Wilhelm, 2021).

Since 2016, our partnership has used Munter's definition of instructional vision as a discourse and the VHQMI rubrics to guide our efforts to design resources for implementing new mathematics standards. This paper focuses on a survey we developed to provide the partnership with descriptions of the instructional visions held by educators across levels of the state education system to inform its designs and assess the degree to which they promoted a more sophisticated and shared vision.

VHQMI Survey Instrument Development

The VHQMI survey was developed in the context of a partnership of mathematics education researchers and leaders from the state education agency and school districts to support the implementation of new state mathematics standards. Formed in 2016, the partnership identified disparate instructional visions as a key impediment to standards implementation and set a goal of developing a shared vision of high-quality mathematics instruction across levels of the state educational system. To meet this goal, partners collaboratively designed an array of implementation resources and professional learning experiences for teachers, coaches, and school and district leaders that embodied high quality instruction and distributed them to all districts through the state agency. To inform these efforts and to monitor progress, the research team recognized that qualitative methods for measuring instructional vision at the scale of a state system would be impractical and fail to provide formative information in a timely manner.

The VHQMI survey is designed to measure educators' visions of high-quality mathematics instruction along the same dimensions of practice as the VHQMI rubrics for the role of the teacher, mathematics tasks, and classroom discourse (Munter, 2014). Similar to Munter's interview protocol, the survey is framed with a question about what respondents might observe when visiting a mathematics class that would suggest the instruction was of high quality. The survey items present pairs of observations one might make in a classroom, and respondents indicate which choice better describes higher quality mathematics instruction. The observations are short statements created by the research team and are based on levels of the VHQMI rubrics. Through a series of cognitive interviews, expert reviews, large scale pilot administrations, and item analyses from 2017 – 2019 (Okeyo, 2019), the partnership tested and refined approximately 60 of these observations aligned with the different levels for the rubrics.

From this pool, we selected a set of four observations, one corresponding to each level of a VHQMI rubric, to create six items that paired each observation with the others and asked respondents to indicate which observation they felt better suggested high quality instruction. To construct the instrument, we repeated this process twice for each rubric to create a total of 36 pairwise comparison items. We then used an online survey management service to organize the items into six sets of comparisons, two for each corresponding VHQMI rubric, and specified that the order in which these sets, items, and observations are presented is random. In addition to these items, the VHQMI survey also includes a set of demographic questions that collect respondents' roles in an educational system (e.g., classroom teacher, instructional coach, school administrator, district specialist), the grade band focus of their role (e.g., elementary grades, high school), and the school district in which they work.

Methods

The purpose of this study is to assess the extent to which the VHQMI survey measures are consistent with the VHQMI rubrics upon which it is based. To do so, we examined responses from a pilot administration to determine if patterns of responses reflected the ordered levels of the VHQMI rubrics. We recruited participants to assist us in piloting the survey over a three-week period using the partnership's social media accounts. Participants received a message with an anonymous link to the survey to complete during the three-week period. A total of 65 participants completed the survey including 59 prospective elementary grades and high school teachers, 2 math coaches, 3 district specialists, and 1 university-based teacher educator.

Survey responses were organized into three sets corresponding to the three VHQMI rubrics and imported into R Studio. To analyze each set of responses, we used dual scaling, a technique for quantifying relationships among categorical data (Nishisato, 1994). For paired comparison data, dual scaling produces optimal weights for both items and individual respondent choices. Item weights represent the order respondents used to express preferences across sets of items. Analyses that yield sufficiently distinct and ordered weights that reflect the theory used to create items indicate that respondents perceived and used that ordering to express their preferences.

Findings

Summarized in Table 1, item weights and squared correlation ratios from our analyses indicate that the VHQMI survey items measured instructional vision in ways that were either consistent or somewhat consistent with the VHQMI rubrics. For the mathematics tasks items, item weights were distinct from one another, suggesting that respondents perceived meaningful distinctions among the observations. Moreover, the weights strictly decrease as the rubric level increases and explain 43% of the total variability between these items. Together, these results suggest that the task portion of the survey measured instructional vision in a way that is consistent with the VHQMI tasks rubric.

Responses to the role of the teacher items were somewhat consistent with the corresponding VHQMI rubric. The weights for these items explained 50% of the variability in responses and reflected a similar pattern as the task items. Item weights strictly decrease as the rubric level increases with the exception of levels 3 and 4. That these weights were much closer in value than others suggest the respondents did not perceive a difference between the level 3 and 4 choices.

For classroom discourse items, weights accounted for 41% of the variability in responses and indicated that respondents perceived qualitative differences among items. However, the change in direction of weight values from decreasing to increasing between levels 3 and 4 suggests that respondents viewed level 3 choices as more extreme than those aligned with level 4. Thus for these items, responses were less consistent with the VHQMI rubric than the other sets of items.

Table 1: VHQMI Survey Item Weights and Squared Correlation Ratios

	Item Weights				
Rubric Levels	1	2	3	4	η^2
Mathematical Tasks	0.75	0.14	-0.37	-0.53	0.43
Role of the Teacher	0.86	-0.16	-0.35	-0.34	0.50
Classroom Discourse	0.72	0.18	-0.59	-0.32	0.41

Discussion

Our goal is to refine the survey into a valid measure that can complement the VHQMI interview protocol and rubrics and enable new directions for research on instructional vision. To that end, we shared mixed results from a study exploring how the survey measured instructional vision relative to the rubrics. To varying degrees, the results indicated that responses to the survey items were consistent with the VHQMI rubrics. For items related to mathematics tasks, responses reflected the corresponding rubric. Responses to items related to the role of the teacher were similar with the exception of levels 3 and 4, which were indistinguishable to respondents in this study. However, responses to the classroom discourse items suggested that respondents perceived level 4 choices to be less extreme than those aligned with level 3.

In interpreting these results, it is important to note the uniqueness of the sample. The overwhelming number of responses from prospective teachers and lack of responses from classroom teachers might explain the issues with the teacher and discourse items. For example, recognizing the qualitative distinctions related to providing instructional guidance or the nuances of classroom discussions described by levels 3 and 4 of the corresponding VHQMI rubrics likely requires significant experience leading a classroom of students or facilitating a class discussion. Though we suspect the inconsistencies revealed by our analysis are a product of our sample, we nonetheless are undertaking an additional round of cognitive interviews with a focus on the teacher and discourse items to better understand our results.

Acknowledgments

This work was supported by the National Science Foundation under grant DRL - 2100895 awarded to East Carolina University, DRL-2100903 awarded UNC Greensboro, DRL - 2100947 awarded UNC Charlotte, and DRL - 2100833 awarded to Appalachian State University. Any opinions, findings, and conclusions or recommendations expressed herein are those of the principal investigators and do not necessarily reflect the views of the National Science Foundation.

References

Chance, P. L., & Segura, S. N. (2009). A rural high school's collaborative approach to school improvement. *Journal of Research in Rural Education (Online)*, *24*(5), 1-12.

Cobb, P., & Jackson, K. (2011). Towards an empirically grounded theory of action for improving the Quality of mathematics teaching at scale. *Mathematics Teacher Education and Development, 13*(1), 6-33.

Gamoran, A., Anderson, C., Quiroz, P., Secada, W., Williams, T., & Ashmann, S. (2003). *Transforming teaching in math and science: How schools and districts can support change*. Teachers College Press.

Hammerness, K. (2006). *Seeing through teachers' eyes: Professional ideals and classroom practices*. Teachers College Press.

Honig, M. I., & Hatch, T. C. (2004). Crafting coherence: How schools strategically manage multiple, external demands. *Educational Researcher*, *33*(8), 16-30.

Jackson, K., Cobb, P., Wilson, J., Webster, M., Dunlap, C., & Appelgate, M. (2015). Investigating the development of mathematics leaders' capacity to support teachers' learning on a large scale. *ZDM*, *47*(1), 93-104.

Katterfeld, K. (2011). *Principal leadership for instruction: associations between principal vision, principal involvement in instruction, and teachers' perceptions of expectations for standards-based instructional practice* (Unpublished doctoral dissertation). Vanderbilt University, Nashville, TN.

Munter, C., & Correnti, R. (2017). Examining relations between mathematics teachers' instructional vision and knowledge and change in practice. *American Journal of Education*, *123*(2), 171-202.

Munter, C. (2014). Developing visions of high-quality mathematics instruction. *Journal for Research in Mathematics Education*, *45*(5), 584-635.

Munter, C. (2015). Envisioning the role of the mathematics teacher. *NCSM Journal of Mathematics Education Leadership, 16*(1), 29–40.

Munter, C. & Wilhelm, A.G. (2020). Mathematics teachers' knowledge, networks, practice, and change in instructional vision. *Journal of Teacher Education*, 1-13.

Nishisato, S. (2014). *Elements of dual scaling: An introduction to practical data analysis*. Psychology Press.

Peterson, P. L., McCartney, S. J., & Elmore, R. F. (1996). Learning from school restructuring. *American Educational Research Journal*, *33*(1), 119-153.

Ticknor, A. S., & Schwartz, C. S. (2017). "It just got real:" Navigating the affordances and constraints of school-based learning in a mathematics-specific induction program. *Action in Teacher Education*, *39*(2), 138-152.

Wilhelm, A. G. (2014). Mathematics teachers' enactment of cognitively demanding tasks: Investigating links to teachers' knowledge and conceptions. *Journal for Research in Mathematics Education*, *45*(5), 636-674.

RESEARCH EXPECTATIONS FOR MATHEMATICS EDUCATION FACULTY IN US INSTITUTIONS OF HIGHER EDUCATION

Blake E. Peterson
Brigham Young University
peterson@mathed.byu.edu

Steven R. Williams
Brigham Young University
williams@mathed.byu.edu

Keith R. Leatham
Brigham Young University
kleatham@mathed.byu.edu

This paper reports the results of a survey of 404 US mathematics education faculty regarding the research expectations for obtaining tenure. Survey questions asked about expected numbers of publications per year, how much different types of publications (e.g., journal articles, book chapters) and scholarly activities (e.g., giving presentations, obtaining funding) were valued. Statistical analyses were used to examine differences in these results across three demographic characteristics (institution type, research commitment, department). We found statistically significant differences related to each of these variables. Research expectations varied substantially across institution type. For example, the average expected number of yearly publications was 2.23, 1.63, and .99 papers at R1, R2, and Other institutions respectively. By contrast, research expectations seldom varied by department.

Keywords: research expectations; tenure; scholarly activity

In 1987 the Carnegie Foundation published a report of a multi-site study of the American academic profession (Clarke, 1987). It documented the three-fold division of academic work into research, teaching, and service that remains to this day as a framework for decisions about promotion, and compensation based on performance in these three areas (Barat & Harvey, 2015; Hardré et al., 2011; Needham, 1982; O'Meara, 2011; Schmidgall & Woods, 1994).

Of these three areas, research is often considered of most importance, being both highly valued and rewarded. Youn & Price (2009) pointed out that in the 1980s, research "became the dominant basis for academic rewards" (p. 205) and remains an important consideration in tenure and promotion decisions (Barat & Harvey, 2015; Kruger & Washburn, 1987; Fairweather, 1993; 2005; Hardré et al., 2011; Park & Gordon, 1996; Price & Cotton, 2006). Moreover, there is a documented positive effect of scholarship on salary (Barat & Harvey, 2015; Fairweather, 1993).

Despite the importance of scholarship in the lives and livelihoods of academics, it must compete with teaching and service obligations in the expectations for faculty positions. Those in academic positions need to know how to balance these obligations (Trower, 2010; Barat & Harvey, 2015). Unfortunately, studies have documented that many faculty feel expectations are not clearly communicated and that assessment is often subjective and/or politicized (Acker & Webber, 2016; Barat & Harvey, 2015; Hardré & Kollman, 2012: Lawrence et al., 2014; Park & Gordon, 1996; Price & Cotton, 2006; Schmidgall & Woods, 1994; Walker et al., 2010). Often, no official documents are available that specify expectations, or they may exist at administrative levels that can provide only very general guidance (Hardré & Cox, 2009).

These difficulties are complicated further because expectations for research vary substantially among disciplines and institutions (Acker & Webber, 2016; Brewer & Rickels, 2011; Clarke, 1987; Gardner & Veliz, 2014; Pellegrino et al., 2018; Price & Cotton, 2006; Schmidgall & Woods, 1994). Although some literature has explored expectations in specific disciplines (e.g., education, accounting, public administration, sociology), little is known about research expectations in comparatively small and relatively new fields like mathematics education. Information related to research expectations for mathematics education researchers from across the US would be of immense value to faculty in the ongoing processes of hiring, setting tenure expectations, mentoring new faculty, and refining department-level expectations documents, as well as its obvious value to faculty who are undergoing job searches, or tenure and promotion decisions. The purpose of this paper is to report the results of our investigation into the typical research expectations for mathematics education faculty in the United States.

Literature Review

Although research on expectations in mathematics education is sparse, a robust literature from other disciplines provides guidance on how research expectations vary across disciplines and what institution, or position-specific characteristics are likely to influence such expectations.

Demographic Characteristics Affecting Expectations for Scholarship

Perhaps the clearest results of research into expectations in higher education is that they vary across institutions. Indeed, the most used classification of institutions of higher education is the Carnegie classification, in which research productivity plays a prominent role. There is evidence to suggest that expectations can increase in institutions who are "striving" (Gardner & Velize , 2014) to emulate more prestigious institutions. Greene and colleagues (2008) found that some faculty reported spending 120% of their time on fulfilling their academic expectations. Significantly, these overloads occurred in R2 institutions, in which research might become more highly valued, but expectations for teaching and service may not be concomitantly adjusted.

A second major characteristic that may affect expectations is the type of department in which the faculty member is housed. In our field, faculty are typically housed either in a department within an education college, or in a department of mathematics, although of course other options are possible. A study by Shih et al. (2018) found that about half of the recent mathematics education PhDs in their sample were housed in mathematics departments, the other half being housed in schools or colleges of education with half of the education positions in R1 institutions, and three-quarters of the mathematics positions in neither R1 nor R2 institutions. Given results such as these and the hybrid nature of the discipline of mathematics education, we concluded it was important to explore potential differences in research expectations at the department level.

Finally, the fraction of time spent doing research is the third major characteristic we assumed would affect participant responses. Greene and colleagues (2008) noted that an R1 university's typical division is 40% research, 40% teaching, and 20% service. In an examination of expectation documents in academia, Hardré and Cox (2009) found significant variability in whether these percentages were specified, as well as variability in what the percentages were in cases where they were quantified. As another example, Davis et al. (2006) found that counseling program faculty reported spending 49% - 55% of their time teaching, 26% - 27% of their time on research and 18% - 21% of their time on service. Hardré et al. (2011) found that effort expended on research was positively correlated with productivity, whereas teaching load—likely time spent on teaching—had a significant negative correlation. We take from these results that, as with institution type and department type, the fraction of their time committed to research is an important consideration when seeking to understand research expectations.

Examples of Research Expectations from the Literature

A clear message from the literature is that scholarship is often measured by the quantity and sometimes quality of research publications. A number of studies have reported on expected numbers of published papers. Hardré and Cox (2009) reported that across disciplines, departments that were categorized as having high research expectations generally expected "averages from one to two articles published every year" (p. 409) for tenure and promotion. Davis et al. (2006) reported that in counseling programs, "the perceived scholarly productivity for tenure and promotion is the equivalent of about one scholarly publication per year" (p. 152). Greene and colleagues (2008) found that the typical expectations for an education faculty member at an R1 university equates to "two publications per year" (p. 432). It is difficult to tell, however, whether the variation was a product of the discipline or the level of institution because of the lack of clarity on the level of institutions involved in the studies.

Studies have also explored the roles that different types of scholarly activities play in evaluating faculty work. In general, the extent to which scholarly activities are valued differs across disciplines and institutions (Hardré & Cox 2009; Price & Cotton, 2006). For example, Barat and Harvey (2015) report that in business schools, a wider variety of scholarly work (books, book chapters, book reviews) has recently come to be seen as valuable. Acker and Webber (2016) found that the perceived relative values of publications in research journals, books, and eternal funding differed by discipline. In nursing Kruger and Washburn (1987) found a hierarchy in the perceived values of activities, ranging from the most valued (publications in refereed journals, funded grants, and sole-authored publications) to the least valued (published teaching materials, publications in non-refereed venues). Goodnight and colleagues (2003) found that expectations for marketing faculty varied according to the highest degree offered in the program (Bachelor, MBA, or PhD).

Summary

We conclude that institution type, department type, and percent of time assigned to research are likely to affect faculty members' perceptions of research expectations. In order to measure research expectations, we chose to look at expected number of papers, types of publications, and other scholarly activities that may be counted as research in our field.

Defining and Measuring Expectations

We asked two research questions: (a) What are the typical research expectations for mathematics education faculty in the United States? (b) How do research expectations differ across institutions, types of departments, and proportion of time devoted to research? By the term *research expectations* we mean what is expected by those making tenure and promotion decisions. In some cases, these expectations may be codified in written documents. However, the literature suggests that such documents do not always exist, and when they do exist, they may not provide much detail to guide faculty members. In this paper, we report instead on the perceptions of mathematics education faculty members regarding expectations at their institutions. A second paper is under preparation that discusses what can be learned from an examination of official documents.

Because expectations are often not spelled out for faculty, building an argument for tenure or promotion is a matter of showing that the candidate has successfully engaged in 1) a sufficient number of activities that are 2) sufficiently valued by the institution. The relationship between the *number* of activities and the *value* of the activities is illustrated by a single research journal article possibly outweighing two or three practitioner journal articles. Thus measuring expectations must take into account both what is seen as necessary, and what is seen as valuable. We hope our data will provide guidance to faculty members about how to spend their time, and to departments as they seek to clearly communicate research expectations.

Measuring what our participants saw as *necessary* was straightforward. In cases where some activities (such as publishing research papers) were seen as necessary, we were also able to obtain clear-cut numerical measures (such as the number research papers that should be published in a given time period). Measuring the activities our participants saw as valuable typically took the form of asking them to judge whether an activity would count *a lot*, *some*, or *little or none*. These Likert-type scales allowed us to gather information on the relative value of various activities in the minds of our participants.

Methods

Participants

The target population for our study was US mathematics education faculty with research expectations. We chose to focus on US contexts because we were familiar with the US higher education system in general, and there is evidence that substantial variation exists between countries when it comes to academic positions and tenure at institutions of higher education (European University Institute, 2022; Pietilä, 2019). To create our sample, we first identified authors who had published in a collection of 55 mathematics education journals (see Williams & Leatham, 2017 for a list of these journals) from 2015 to 2017 and who were associated with a 4-year institution of higher education in the United States at the time of publication. To augment this list with researchers whose focus may have led them to publish in other venues during that same timeframe, we identified authors of chapters in the *Research in Mathematics Education* edited book series by Springer, the National Council of Teachers of Mathematics (NCTM) edited book series titled *Annual Perspectives in Mathematics Education*, and NCTM's practitioner journals *Teaching Children Mathematics*, *Mathematics Teaching in the Middle School*, and *Mathematics Teacher*. Once authors were identified, we verified email addresses and, in some cases, determined that individuals were no longer in a university position and removed them from our list. In total, we identified 1593[8] US university faculty members who had published in mathematics education from 2015 to 2017 and for whom we could find a current email address.

Although we knew that some of these individuals were likely not in our target population because they were coauthors but not mathematics education researchers, we decided to err on the side of inclusion and allow individuals to decide whether they were indeed in our target population. Of the 1593 to whom the survey was sent, 821 responded[9]. Of these 821, 411 identified themselves as not being part of the target population. When these were eliminated, along with six incomplete survey responses, there were 404 respondents from our target population. Based on the fact that half of those who responded were not in the target population, we infer that at least half of those to whom we sent the survey were not in the target population. Thus, it seems reasonable to assume a response rate of roughly 50% of the target population.

Survey

Based on our review of the literature on research expectations, we devised a set of questions in three categories: 1) demographics; 2) the number of publications expected each year; and 3) the types of scholarly publications and activities that would meet research expectations. Category 1 questions were designed to classify participants' institution and department as well as the relative time commitment for teaching, research, and service. For category 2 we asked both

[8] Due to clerical errors, 119 participants who should have been on the list of those who received surveys were not on the list. Unfortunately, the error was identified long after the survey had closed.
[9] Respondents are those who responded to the survey either directly or indirectly through email.

about the number of expected publications and whether that number was explicitly stated (e.g., in an official document) or implicitly understood. The types of publications we included for consideration in category 3 were research journal articles, books or book chapters, textbooks, conference proceedings papers, and practitioner journal articles. We also asked about pursuing and obtaining external funding, giving presentations, and providing professional development. To measure the extent to which these scholarly publications and activities were valued, we asked respondents to indicate whether they counted toward tenure and how much they counted. A draft survey was piloted with a group of 24 colleagues and as a result, items were extensively edited for clarity (e.g. clarifying that *citizenship* was the same as *service*).

Analysis

Our three key dependent variables are represented in our data by the demographic variables *institution type* (i.e. *R1*, *R2*, and *other*—meaning neither *R1* nor *R2*), *department* (i.e. *education* and *math*), and *research commitment* (i.e. percent time devoted to research). We began by counting frequencies of responses to the various survey questions, and then disaggregated the results across the demographic variables.

To assess the influence of the variables *institution type, department,* and *research commitment*, we conducted multinomial logistic regressions using these three variables as predictors. For example, we looked at the effects of these variables on how publications in practitioner journals were valued. The response variable was categorical with three levels—*a lot*, *some*, or *little or none*—so in the model we compared both *a lot* and *little or none* to *some* in the odds ratios. Because *institution type* had three levels, we compared both *R1* and *other* to *R2*. In the cases for which the response variables were numeric, such as the number of published papers expected per year or *research commitment*, the model was appropriately modified.

In our analysis significant effects were determined for each of our three dependent variables as the others were held constant. Since the literature review made it clear that our three demographic variables are likely related, it might be expected that respondents in R1 institutions would have higher research commitments. In looking for significant effects of *institution type*, therefore, we looked particularly at effects holding *research commitment* and *department* constant, thus minimizing the effects of possible correlations among our dependent variables.

In order to better understand the size and direction of the significant effects, we also conducted a series of *post-hoc* analyses further probing relationships shown to be significant by the regression analyses. Again, using the example of effects on reports on expectations regarding practitioner journal articles, we looked at pairwise comparisons among reported levels of importance (e.g., *a lot*, *some*, and *little or none*) and noted changes in the odds ratios.

Results

The primary purpose of this study was to learn what the typical research expectations are for mathematics education faculty in the United States. As conjectured, expectations often vary by institution type, department, and research commitment. We describe the data set as a whole and then synthesize the results according to these demographic variables, focusing initially on institution type as this variable most often resulted in statistically significant differences.

Respondents are located in a wide range of institutions, evenly split between departments of education and departments of mathematics. The proportion of time they are expected to devote to research is typically 20-40%, but the range varies from 0-70%. On average faculty are expected to publish about 1.64 papers per year where research journal articles are the most valued publication type and count a lot toward meeting expectations. Most view book and book chapters, practitioner journal articles, and conference proceedings papers as counting some, and there is not clear consensus on the value of publishing textbooks. With respect to other scholarly activities beyond publication, typically there is an expectation to give presentations, although such presentations likely only count some (not a lot). For a majority, faculty are expected to pursue external funding, and this pursuit counts some or a lot even if the funding is not obtained. Over two-thirds of faculty report that obtaining external funding is not necessary but obtaining it counts a lot. Providing professional development is neither necessary nor much valued. With respect to authorship, first authorship is seen as a crucial element of the publication portfolio.

With respect to Carnegie classifications, mathematics education faculty at R1 institutions are more likely to be in departments of education than of mathematics. Typically, they are expected to devote 40% of their time to research endeavors. R1 faculty are expected, on average, to publish about 2.25 papers per year. That said, holding all else equal, R1 faculty are less likely than their R2 counterparts to have this expected number of papers articulated explicitly in a written expectations document. With respect to the types of publications that count, as with all respondents, research journal articles count a lot. For R1 faculty, however, all other types of publications that we asked about are less valued than at R2 and Other institutions. R1 faculty are more likely to be required to give presentations than R2 faculty. R1 faculty are more likely than their R2 and Other counterparts to be required to pursue and obtain external funding. Finally, R1 faculty were more likely than R2 faculty to report sole authorship as important.

Mathematics education faculty at R2 institutions are fairly evenly split between departments of education and of mathematics. About 65% of faculty at R2 institutions are expected to devote at least 40% of their time to research endeavors. R2 faculty are expected, on average, to publish about 1.5 papers per year. Furthermore, holding all else constant, R2 faculty are more likely than their R1 and Other counterparts to have this expected number of papers articulated explicitly in a written expectations document. Among types of publications, research journal articles count a lot, while all other types of publications that we asked about likely would count some. R2 faculty were less likely than R1 faculty but more likely than Other faculty to be required to pursue and obtain external funding. Finally, R2 faculty were less likely than R1 faculty but more likely than Other faculty to report sole authorship as important.

Mathematics education faculty at Other institutions are more likely to be in departments of mathematics than of education. Typically, they are expected to devote about 20% of their time to research; over 80% of respondents report 10-30% expected research time. Other faculty are expected, on average, to publish about 1 paper per year. That said, holding all else constant, Other faculty are less likely than their R2 counterparts to have this expected number of papers articulated explicitly in a written expectations document. Research journal articles count a lot. All other types of publications that we asked about likely would count some. Other faculty were more likely than R2 faculty to value giving presentations and providing professional development. Other faculty were more likely than their R1 and R2 counterparts to report that there was no expectation to pursue or obtain external funding. Finally, Other faculty were less likely than R2 faculty to report sole authorship as important.

With respect research commitment, for a 10% increase in the reported time spent on research there is a 0.184 increase in the reported number of expected papers. Moreover, respondents reporting higher research commitments (holding all else constant) were less likely to have this expected number of papers articulated explicitly in a written expectations document. Research journal articles count a lot independent of research commitment. Respondents with higher research commitments did report that books or book chapters and practitioner journal articles were less valued than those with lower research commitments. Faculty with higher research commitments are more likely to report being required to pursue and obtain external funding and are less likely to report that providing professional development is valued. Finally, faculty with higher research commitments are more likely to report both sole authorship and lead authorship as being valued.

With respect to department type 50% of respondents belonged to education departments, 43% belonged to mathematics departments, and the remaining 7% were in departments with a mixture of disciplines like education, arts, and sciences. Over 50% of our respondents who were in education departments were in R1 institutions while half of our respondents in mathematics departments were from Other institutions. Regarding research commitment, 64% of respondents from education departments spend 40% of their time or more doing research while 55% of respondents from mathematics departments spend less than 40% of their time doing research. In general our results showed few differences in the expectations of mathematics and education departments. That said, holding all else constant, respondents in mathematics departments reported that 0.309 fewer papers were expected than was reported by those in education departments. The only difference in department expectations regarding types of publication is conference proceedings papers—respondents from mathematics departments were more likely than respondents in education departments to report that conference proceedings papers were valued. Regarding the other types of scholarly activities (giving presentations, providing professional development, pursuing or obtaining funding), mathematics departments valued pursuing funding and obtaining funding more than those in education departments. On the other hand, mathematics departments are less likely to require providing professional development and require giving presentations. Finally, respondents in mathematics departments were less likely than those in education departments to report that first authorship was important.

The results confirmed a number of suspicions we had going into this study about research expectations in mathematics education. There were, however, some results that were surprising. With respect to workload distribution, we suspected that the most common workload distribution would be a 40-40-20 split and that there would be variability in that distribution at R2 institutions. We were surprised by the variability at R1 and Other institutions. With respect to the expected number of yearly publications, we anticipated that the expectation would be highest for R1 faculty and lowest for faculty at Other institutions but we were surprised that 11% of R1 faculty reported the expectation of 3 or more papers per year.

With respect to publication types, we anticipated that research journal articles would be the most valued regardless of institution type, department or research commitment. Although we suspected that textbooks and practitioner journal articles would be less valued at R1 institutions than at R2 institutions, we were surprised that books or book chapters or conference proceedings papers are also less valued. We were surprised that mathematics department faculty were more likely than education department faculty to report that conference proceedings papers counted.

With respect to other scholarly activities, we suspected that pursuing funding would be most likely to be expected at R1 institutions and least likely to be expected at Other institutions. What surprised us is that over 75% of respondents indicated that presentations are necessary and that R1 faculty are more likely to be expected to give presentations than R2 or Other faculty.

Finally, with respect to whether the expected number of publications was in a written document, we did not expect to see any differences depending on demographic characteristics. We were thus surprised that the number was more likely to be in a written document at R2 institutions than at R1 or Other institutions. In addition, those with higher research commitments were less likely to have the expected number of publications in a written document.

Conclusion

The results of this study show some of the ways that research expectations for mathematics education faculty in the US vary based on institution type, department type, and research commitment. Knowing both that and how research expectations vary provides valuable information for individual faculty, mathematics education programs, and to the broader field of mathematics education. Individual faculty could profit from seeking clarification from their own department regarding the categories of expectations discussed herein, asking questions like:

- Does our department have an expected number of publications?
- To what extent are book chapters or conference proceedings papers valued?
- Is pursing funding, even if it is not funded, valued?
- To what extent is sole authorship expected?

Mathematics education programs could locate themselves within these results and consider questions like the following:

- To what extent do we align with what is typical of our particular combination of institution type, department type, and portion of time devoted to research?
- If we vary substantially from what is typical, do we want to make adjustments or provide further justification for the variation?
- To what extent are our expectations, whether they align or not with what is typical, made explicit in written documents?
- Might there be areas where we should articulate expectations that have thus far been left unaddressed? For example, do we make clear whether practitioner journal articles count and how much they count relative to other types of publications?

As a field of mathematics education, these results tell us something of the types of scholarly activities that are being expected of mathematics education faculty in the US. These results could prompt us to consider questions like the following:

- Are there other scholarly activities that, as a field, we feel should be valued more or less than they are? What might we do as a community to influence an increase or decrease in emphasis on such activities?
- Are there sufficient quality publication venues and outlets for other scholarly activities to support the needs of the field?
- Might mathematics education faculty benefit from supporting documentation from the field that articulates the value-added for certain scholarly activities (such as professional development activities) that would benefit the field but are not typically valued much?

We hope the results of this study will motivate all mathematics education faculty and the departments where they work to have an open, transparent dialogue about details related to research expectations and to create or refine their expectations documents. Given the variation that exists across institutions, departments, and research commitments, it would seem beneficial for expectations documents to make explicit mention of expected number of publications, how much various types of publications count, how much other types of scholarly activities count and whether those activities are necessary. Such work would benefit individual faculty, mathematics education programs, and the entire field of mathematics education. We also hope that the results of this study begin a broader conversation in the field to consider not just what *is* expected, but what *could* or *should* be expected in order to continue to move the field of mathematics education forward.

References

Acker, S., & Webber, M. (2016). Discipline and publish: The tenure review process in Ontario universities. In L Shultz & M. Viczko, (Eds), Assembling and governing the higher education institution. (pp. 233-255). Palgrave Studies in Global Citizenship Education and Democracy. https://doi.org/10.1057/978-1-137-52261-0_13

Barat, S., & Harvey, H. (2015). A benefit-maximization solution to our faculty promotion and tenure process. Journal of Educational Issues, 1(2), 20-34. https://doi.org/10.5296/jei.v1i2.7747

Brewer, W. D., & Rickels, D. A. (2012). Demographics and faculty time allocation of music education professors in the United States. Journal of Music Teacher Education, 22(1), 63-74. https://doi.org/10.1177/1057083711424504

Clarke, B. R. (1987) The academic life. Small worlds, different worlds. A Carnegie Foundation special report. Princeton University Press.

Davis, T. E., Levitt, D. H., McGlothlin, J. M., & Hill, N. R. (2006). Perceived expectations related to promotion and tenure: A national survey of CACREP program liaisons. Counselor Education and Supervision, 46(2), 146-156. https://doi.org/10.1002/j.1556-6978.2006.tb00019.x

European University Institute. (2022). Careers by country. https://www.eui.eu/Programmes AndFellowships/AcademicCareersObservatory/AcademicCareersbyCountry

Fairweather, J. S. (1993). Faculty rewards reconsidered: The nature of tradeoffs. Change, 25(4), 44-47. https://www.jstor.org/stable/40165072

Gardner, S. K., & Veliz, D. (2014). Evincing the ratchet: A thematic analysis of the promotion and tenure guidelines at a striving university. The Review of Higher Education, 38(1), 105-132. https://doi.org/10.1353/rhe.2014.0045

Goodnight, J. E., Elam, E. R., & Emmert, P. (2003). Marketing department chairpersons' views on promotion requirements for marketing faculty. Marketing Education Review, 13(2), 43-54. https://doi.org/10.1080/10528008.2003.11488829

Greene, H. C., O'Connor, K. A., Good, A. J., Ledford, C. C., Peel, B. B., & Zhang, G. (2008). Building a support system toward tenure: Challenges and needs of tenure-track faculty in colleges of education. Mentoring & Tutoring: Partnership in Learning, 16(4), 429-447. https://doi.org/10.1080/13611260802433791

Hardré, P. L, & Kollman, S. L. (2012). Motivational implications of faculty performance standards. Educational Management Administration & Leadership, 40(6), 724-751. https://doi.org/10.1177/1741143212456913

Hardré, P. L., Beesley, A. D., Miller, R. L., & Pace, T. M. (2011). Faculty motivation to do research: Across disciplines in research-extensive universities. Journal of the Professoriate, 5(1), 35-69.

Hardré, P., & Cox, M. (2009). Evaluating faculty work: Expectations and standards of faculty performance in research universities. Research Papers in Education, 24(4), 383-419. https://doi.org/10.1080/02671520802348590

Kruger, S., & Washburn, J. (1987). Tenure and promotion: An update of university nursing faculty. Journal of Nursing Education, 26(5), 182-186. https://doi.org/10.3928/0148-4834-19870501-04

Lawrence, J. H., Celis, S., & Ott, M. (2014). Is the tenure process fair? What faculty think. The Journal of Higher Education, 85(2), 155-192. https://doi.org/10.1080/00221546.2014.11777323

Needham, D. (1982). Improving faculty evaluation and reward systems. The Journal of Economic Education, 13(1), 6-18.

O'Meara, K. (2011). Inside the panopticon: Studying academic reward systems. In J. Smart & M. Paulsen (Eds.), Higher education: Handbook of theory and research (pp. 161-220). Springer. https://doi.org/10.1007/978-94-007-0702-3_5

Park, S. H., & Gordon, M. E. (1996). Publication records and tenure decisions in the field of strategic management. Strategic Management Journal, 17(2), 109-128. https://doi.org/10.1002/(SICI)1097-0266(199602)17:2%3C109::AID-SMJ796%3E3.0.CO;2-P

Pellegrino, K., Conway, C. M., & Millican, J. S. (2018). Tenure and promotion experiences of music teacher educators: A mixed-methods study. Journal of Music Teacher Education, 27(2), 82-99. https://doi.org/10.1177/1057083717730085
Pietilä, M. (2019). Incentivizing academics: Experiences and expectations of the tenure track in Finland, Studies in Higher Education, 44(6), 932-945. https://doi.org/10.1080/03075079.2017.1405250
Price, J., & Cotten, S. R. (2006). Teaching, research, and service: Expectations of assistant professors. The American Sociologist, 37(1), 5-21. https://doi.org/10.1007/s12108-006-1011-y
Schmidgall, R. S., & Woods, R. H. (1994). CHRIE member perceptions of tenure requirements in hospitality education programs. Hospitality Research Journal, 18(1), 101-120. https://doi.org/10.1177/109634809401800108
Shih, J. C., Reys, R. E., Reys, B J. , & Engledowl, C. (2018). Examining the career paths of doctorates in mathematics education working in institutions of higher education. Investigations in Mathematics Learning, 12(1), 1-9. https://doi.org/10.1080/19477503.2018.1514954.
Trower, C. A. (2010). A new generation of faculty: Similar core values in a different world. AAC&U Peer Review, 12(3), 27-30.
Youn, T. I. K., & Price, T. M. (2009). Learning from the experience of others: The evolution of faculty tenure and promotion rules in comprehensive institutions. The Journal of Higher Education, 80(2), 204-237. https://doi.org/10.1080/00221546.2009.11772139
Walker, K. B., Fleischman, G. M., & Stephenson, T. (2010). The incidence of documented standards for research in departments of accounting at US institutions. Journal of Accounting Education, 28(2), 43-57. https://doi.org/10.1016/j.jaccedu.2011.02.003
Williams, S. R., & Leatham, K. R. (2017). Journal quality in mathematics education. Journal for Research in Mathematics Education, 48(4), 369–396. https://doi.org/10.5951/jresematheduc.48.4.0369

THE STORY OF VCAST-SIG: COLLABORATIVE APPROACHES THAT ENHANCE MTE PROFESSIONAL KNOWLEDGE AND TEACHER LEARNING

Nirmala Naresh	Marta Magiera	Kanita DuCloux
University of North Texas	Marquette University	Western Kentucky University
Nirmala.naresh@unt.edu	marta.magiera@marquette.edu	kanita.ducloux@wku.edu

Laurie Cavey	Theresa Hopkins	Lee Roberson
Boise State University	University of Tennessee	University of Colorado Boulder
lauriecavey@boisestate.edu	thopkins@utk.edu	lee.roberson@colorado.edu

Like K-12 mathematics teachers, for whom effective instruction requires a full suite of pedagogical and mathematical knowledge and skills, Mathematics Teacher Educators (MTEs) also need a professional knowledge base to facilitate the development of future educators capable of supporting K-12 students' mathematical learning. In this paper, we theorize about contributors to the development of a Community of Practice (CoP) who were engaged with developing and implementing a common set of instructional materials. We discuss how using the materials in various contexts and interacting with the user community provided a much-needed network of MTEs focused on enhancing their students' mathematical learning experiences as well as their own professional development in terms of teaching, research, and curriculum development.

Keywords: Teacher Educators, Professional Development, Preservice Teacher Education, Mathematical Knowledge for Teaching Teachers.

Introduction

University-based mathematics teacher educators (MTEs) play a critical role in enhancing the quality of mathematics teachers. Like K-12 mathematics teachers, for whom effective instruction requires a full suite of pedagogical and mathematical knowledge and skills, MTEs also need a complementary set of knowledge, practices, and skills to facilitate the development of future educators capable of supporting K-12 students' mathematical learning. MTEs' specialized knowledge, termed as mathematical knowledge for teaching teachers (MKTT), for example, includes an understanding of how teacher candidates (TCs) best learn mathematics and how they think about students' mathematical thinking. MTEs need to have an awareness of and be able to notice the beliefs about mathematics teaching and learning that teacher candidates hold (Beswick & Goos, 2018; Superfine & Li, 2014a). They also need to understand how pedagogical modeling in university mathematics and mathematics education classrooms can support the preparation of TCs for teaching K-12 mathematics.

Research indicates that MTEs learn through reflecting on their practice as well as from their own research with teachers (e.g., Jaworski, 2001; Masingila et al., 2018). By reviewing their teaching practices and reflecting on their beliefs, assumptions, and outcomes of student learning, MTEs have the potential to understand and improve their practice (Appova & Taylor, 2019; Chapman, 2008; Zaslavsky & Leikin, 2004). While past research has documented the nature of MTEs' knowledge and the benefits of MTE's professional development, how MTEs come to engage in activities that help them grow in their practice has received less research attention.

In this paper, we discuss the development of a Community of Practice (CoP) within a group of MTEs engaged with VCAST learning materials and how the community organically emerged within the group. We describe and explore factors that appear to have contributed to the development of a CoP (referred to here as the VCAST-SIG) through the collaborations of a group of nine MTEs from seven different institutions across the United States. The factors we describe emerged from broad themes identified across members' reflections on their individual practice and how those practices were impacted through participation in the VCAST-SIG. Findings highlight how the VCAST-SIG stimulated an agentive process for MTEs' thinking about their practice and continuous improvement focused on students' learning and engagement with mathematics.

Context of the Study

VCAST is an NSF-supported design-based curriculum project which drove the emergence of our CoP, and has, to date, been implemented in 16 universities across the United States. VCAST project was designed to support MTEs in their efforts to prepare secondary mathematics teachers by providing materials through which MTEs can engage secondary TCs with meaningful opportunities for learning from student thinking. A key focus of the project is to support TCs' development of both the knowledge and the dispositions needed to attend to and productively respond to the depth of students' mathematical thinking. This often entails mathematical flexibility and a willingness to learn from one's own students, especially when those students' ideas differ from the more conventional mathematical ideas that dominate traditional mathematics coursework and instruction.

VCAST curriculum materials are organized into modules, each of which focuses on the mathematical concepts embedded within a non-standard task and has three distinct parts, an asynchronous Online Component, a synchronous In-Class Component, and an asynchronous Exit Ticket (Cavey et al., 2020). Each module engages TCs in the following activities: a) solving a challenging mathematics task, b) watching video clips of secondary students solving the same task, c) engaging in repeated cycles of analysis, inference, prediction, and reflection about student thinking, and d) solving an adjusted version of the task. While TCs are expected to work individually during both the Online Component and Exit Ticket, they engage in collaborative analysis, inference, prediction, and reflection about student thinking during the In-Class Component. The curriculum project, with its emphasis on TCs' mathematical development, engaged MTEs in a variety of professional development opportunities, including summer training prior to module implementation and ongoing asynchronous training support during implementation.

Theoretical Background

Mathematical Knowledge for Teaching Teachers

MTEs require a specialized and multi-dimensional knowledge base, mathematics knowledge for teaching teachers, for effective work and preparation of K-12 mathematics teachers (e.g., Beswick & Goos, 2018; Superfine & Li, 2014a; Jackson et al., 2020). Many TCs enter their college mathematics coursework with a procedural focus and views on mathematics teaching which stem from their past experiences with procedure-oriented mathematics learning (Mason, 2010). To better prepare TCs to meaningfully support K-12 students' mathematical learning, MTEs need to know which mathematical ideas should be made explicit for TCs and how to raise awareness of how these ideas are represented, communicated, and co-constructed during instruction. MTEs' knowledge base should encompass the challenges or misconceptions TCs might have, the range of ways in which TCs think about students' mathematical work, what TCs value in the work of mathematics teaching, and the views on mathematics and mathematics teaching TCs might hold (Superfine & Li, 2014a).

Before MTEs can support TCs and help them make a shift toward more student-centered instructional practices, they need to understand their TCs' thinking about mathematics and mathematics teaching. MTEs also need to know how to engage TCs in mathematical explorations in ways that supports TCs' learning of mathematics and connects their experiences as mathematics students to their future work as mathematics teachers. To support TCs' mathematical reasoning at high cognitive levels, MTEs need to know the mathematics content TCs learn and to understand how TCs engage with this content (e.g., Beswick & Goos, 2018; Superfine & Li, 2014b).

In this paper, we describe how participation in the broader curriculum project and the subsequent VCAST-SIG provided a platform for MTEs developing the varied aspects of MKTT. The professional development opportunities not only allowed the members of the VCAST-SIG to expand their mathematical knowledge for teaching (Ball et al., 2001) but also further develop their mathematical knowledge for teaching teachers (MKTT). In this context, MKT is the content that MTEs aimed to help TCs learn, while MKTT refers to developing TCs' understanding of and connection among ideas and developing awareness of student thinking (Superfine & Li, 2014b).

Framework of Inquiry

VCAST-SIG was initiated organically by MTEs through informal relationships that were built as we participated in the larger curriculum project. To document and theorize how participants of the curriculum project became inducted and immersed in the VCAST-SIG, we adopt a sociocultural approach, particularly drawing from Lave and Wenger's (1991) situated learning theory. Within this domain, we use the concept of a CoP as a framework to describe how the VCAST-SIG fostered a robust community of scholars focused on advancing prospective teachers' mathematical learning. Learning within a CoP is stimulated by members' engagement in a social and co-constructed activity situated in a particular context (Lave & Wenger, 1991). To document and theorize how members of the CoP grew in their MKTT knowledge, we use the following components of the knowledge strategy framework (Wenger, 2000): a) Build a community, b) develop an inclusive social learning system to strengthen the community, and c) sustain momentum by applying knowledge, reassessing, reflecting, and renewing engagement.

Methodology

A group of nine MTEs that participated in the VCAST curriculum project came together to form the VCAST-SIG. After a year of involvement with the initial curriculum project, we were struck by what appeared to be a CoP that developed among both the creators and implementers of the VCAST curriculum modules. This led us to wonder, "What are the characteristics of our CoP? What factors contributed to the development and success of our CoP? How does participation in a CoP impact us professionally?" To address such questions, we selected a methodology which would involve members of VCAST-SIG acting as both investigators and participants. We chose a phenomenographic approach as it "describes the meaning for several individuals of their lived experiences or a concept or a phenomenon" (Creswell, 2007, p. 57). In this study, the phenomenon was the emergence of a CoP through our work with VCAST. In developing an understanding of our "common experiences" in the emergence of VCAST-SIG, we strive to determine the factors that led to our CoP and to share those practices with others. Members of the VCAST-SIG come from a variety of demographic and professional backgrounds that included teaching-intensive, research-intensive, and teaching-research-balanced positions. Four out of nine participants were a single MTE in their respective departments.

Data Collection and Analysis

Our data comes from semi-structured (Orgill, 2002) interviews conducted with each participant. The interviews were conducted and recorded using Zoom. The goal of the interviews was to better understand, through the experiences and narratives of the members, how the CoP developed, understand the impact of our interactions within the CoP, as well as how the CoP impacted our individual professional growth. The interview protocol was formulated drawing upon the concept of building professional learning communities (Kruse et al., 2009). The protocol consisted of nine questions that elicited information about members' demographics and academic background, collaboration within the CoP, and reflection on members experiences with the CoP.

Consistent with the phenomenographic research methods, we engaged in multiple stages of data analysis including familiarization, compilation, and classification (Han & Ellis, 2019). We first watched all nine interviews and read the interview transcripts. Next, each member compiled participants' responses to interview questions on a separate tab on a shared spreadsheet. We each consolidated the raw data by documenting our preliminary observations on the spreadsheet, highlighting quotes from the interview that we found relevant to the study questions. This process enabled us to delineate codes specific to themes deduced from the interview protocol (Kruse et al., 2009). As we engaged in this process, we noticed that the deduced themes did not accurately reflect the depth and complexity of participant responses and that the categories also fell short of addressing the key study questions. Hence, we adopted a more open and rigorous approach to further classify data and induced additional themes to the spreadsheet based on participants' initial categorization of the interview responses. These additional themes were developed using codes derived from the data. Below, we present examples of coding related to the induced and deduced themes.

Deduced Theme (DT): Knowledge for teaching teachers.
Code: MTE noticed, described and reflected on TCs engagement with modules
MTE Quote: TCs were very engaged with some materials, and they took it very seriously and they themselves were very reflective, and so they thought a lot about their own practice. One of my students from the first year did her senior capstone project by taking the hexagon task and interviewing students in a local high school.

MTE Quote: [I'm] using what we've done in VCAST with student analysis. I have put that into my practice a lot more than I ever did in the past. I do that now with my little seventh graders.
Code: Transfer of VCAST Pedagogy to another teaching context
Induced Theme (IT): Pedagogical shift.

Shifting our approach to classify data allowed for a more focused and richer analysis, resulting in the development and refinement of codes and themes. Data analysis revealed more than twenty-five codes related to induced themes and deduced themes.

Findings

Informal opportunities to share what we had learned with other MTEs in the VCAST curriculum project led to a series of joint professional endeavors to engage in a deep reflection of our learnings. Partnering with other instructors across the U.S. who shared the same vision, work ethic, and positive commitment to helping improve the secondary mathematics teacher preparation was powerful, especially since the range of universities and teacher candidates was so much broader than what is typically found in one campus setting. It was our commonalities that brought us together at first, and then the variation between us enriched that experience. This variability proved to be beneficial and offered a variety of opportunities for members' professional growth, which we highlight in the following paragraphs.

Pedagogical Shifts

The first theme, which highlighted how membership in the VCAST-SIG afforded participants with learning opportunities, stemmed from MTE observations of shifts in their own instructional practices, especially regarding course assessments. MTEs reported designing and implementing tasks that encouraged TCs to avoid a "tunnel vision" that can occur when TCs try to solve a problem the "right way." MTEs prompted TCs to analyze student work samples and make connections between TCs' own and the student's understanding. Furthermore, MTEs reported making conscious decisions to engage TCs in exploring theoretical underpinnings of mathematical concepts and discussing mathematics education research. These instructional shifts were intended to leverage the ideas TCs recognized in students' solutions to support TCs' understanding of students' mathematical development. The extension of VCAST pedagogy and assessment to other mathematics courses involving non-teacher education majors was another unforeseen but welcome outcome of VCAST curriculum implementation.

Knowledge for Teaching Teachers

The second theme outlines how MTEs enhanced their own knowledge of teaching teachers by becoming more aware of what and how their TCs engaged with mathematics and student thinking. Regarding TC learning as students, MTEs reported an increased awareness of multiple problem-solving approaches to tasks and their own ability to make connections among them. When TCs engaged with the VCAST asynchronous online tasks, they experienced productive struggle which deepened their mathematical content knowledge. MTEs also reported changes in the depth of TC conversations about different approaches, reflecting a better conceptual understanding of the key ideas of a task. Regarding TC development as potential teachers, MTEs saw a change from concentrating on student errors to speculating and pondering their thought processes. Though TC development was a key outcome of VCAST material implementation, noticing this growth, being able to use in-class conversations to amplify TC learning, and the VCAST material's emphasis on evidence-based analysis and humility have impacted MTEs and their own capacity to support TC development. Involvement with VCAST materials, allowing TCs to engage with student thinking, and engagement in discussions and reflection sessions, have helped MTEs bolster questioning strategies, become better listeners, better understand how TCs engage in productive mathematical struggle and develop additional directions for future teaching.

Mathematical Learning

This theme involves MTE observations about their own mathematical learning. Although MTEs reported a range of new understandings specific to each of the four modules, three broad categories related to that learning emerged. First, when MTEs felt they already possessed a high level of expertise with the mathematics of a task, implementation of the modules revealed unanticipated conceptual complexities for students. This served to either expose or reorient MTEs’ attention toward the foundational concepts embedded within each task. Second, analyzing the modules’ range of student evidence with candidates deepened MTEs’ capacity to describe, draw connections between, and compare different types of student reasoning. Lastly, for module tasks that were less familiar and more difficult, MTEs reported learning additional mathematics along with their candidates and being humbled by the challenges they themselves encountered.

Engagement in Scholarship

The final theme pertains to MTE's acknowledgment of heightened interest in research. While some MTEs' workloads or work contexts did not necessitate their participation in research, their involvement in VCAST-SIG allowed them to occasionally venture outside of their comfort zones and attempt scholarly activities. Their engagement in the research was sparked by the desire to improve their classroom instruction and the act of seeing how well the VCAST research complemented and improved their pedagogical goals. Members’ differing research backgrounds brought forward ideas for improving the implementation of the VCAST modules. By infusing aspects of culturally responsive pedagogy, student identity, and affective domain, MTEs sought to strengthen TCs’ reasoning and explanation skills. Furthermore, the diverse research interests increased VCAST-SIG's awareness of opportunities to disseminate their work to broader audiences. Having access to the different perspectives of MTE instructors, the ability to “peek inside” the classrooms of community peers, and to read literature related to other MTEs’ research agendas has broadened participants’ own perception about what it means to prepare teacher educators well.

Implications

Stepping back and reviewing how the VCAST-SIG became a CoP reveals factors that mirror the “knowledge strategy” framework for building a CoP (Wenger, 2000).

Build a Community

The VCAST-SIG members continued to build on the foundation provided by the curriculum project through a) reflective discussions sharing their module implementation experiences, b) refining and finding additional application for the VCAST modules (Cavey et al, 2021) and c) by gathering, analyzing, and documenting evidence (Chapman, 2008) that demonstrated a growth in their TCs' mathematical knowledge MKT, and teaching dispositions (Cavey et al., 2020). This sense of accomplishment strengthened MTEs’ connection to the VCAST-SIG. Consistent with research findings (Olanoff et al., 2021; Goos & Bennison, 2019), in our study, we noticed that effective CoPs are grounded in participants' common goals. *For those pursuing their own CoPs, we suggest finding a common need or goal that will bring together professionals who subscribe to the goal and then provide opportunities for contributions that advance these goals.*

Develop an Inclusive Social Learning System to Strengthen the Community

VCAST-SIG members hold different roles within their mathematics departments, so the group provided an opportunity for members, including lone MTEs in their respective departments to engage in cross-institutional peer interaction and collaboration. In addition to the variation in MTE roles, there were also significant differences in the student populations that the members taught. Due to the diversity of the student groups, the VCAST-SIG was able to assess the effectiveness of the VCAST modules for students with diverse learning goals and the potential for expanding the applications of the skills to non-TCs and math courses that are not specifically geared toward education. Members of the group who are primarily involved in teaching have expanded their areas of interest in research, either by broadening the scope of their existing research or by concentrating on scholarly work that addresses the intersection of the VCAST curriculum and its implementation and reflection on their pedagogical practices. Thus, we emphasize *the significance of collaborating beyond members' departmental responsibilities and experiences to provide perspective to discussions and uncover connections and applications that other group members might not have considered.*

Sustain Momentum

Beyond the curriculum project, the reflective structures and discussions continued to be an integral part of the VCAST-SIG. Community development came from the inclusion of multiple points of contact to connect through both synchronous modalities, such as Zoom meetings, and asynchronous modalities, such as Google docs and spreadsheets and resources posted on the VCAST-SIG's shared folder. The monthly Zoom meetings encouraged MTEs to reflect on their practice, established a sense of accountability for the group, and served as a springboard for developing new goals. Each member chose their own path to address these goals with support from other group members. Members added personal reflections, pedagogical activities, and manuscripts on MTE professional development to the shared online folder and it gradually transformed into a rich repository of knowledge. *Once a community has been established, it is vital to put in place structures for reflection, repository for ideas, and establish accountability to prime members for upcoming community discussions.*

Conclusion

The VCAST-SIG continues to expand on their work that first brought them together by focusing on both teaching and scholarship. This momentum was built on various achievements, interactions, and reflections that occurred throughout the knowledge strategy cycle. Participation in the project is different from other venues of professional collaboration due to its unique blend of opportunities to engage in the design-based cycle of curriculum development, scholarship, and a community of practice which shares the same goals while also highlighting a diverse range of paths toward those goals. It has broadened participants' perspectives on mathematics, teaching, and the preparation of teachers by layering in new focuses on culturally responsive pedagogy, the benefits of self-study and reflection, different instructional strategies, and tools suitable for remote use. There is also a renewed focus on the value of attending to an individual's mathematical understanding - whether it's the MTEs' own understanding, the VCAST material's featured secondary students' understanding, or the understanding of teacher candidates.

Engaging in this work has strengthened our community by clarifying for ourselves, (1) what factors drew us together (institutional isolation, opportunity to work on a common problem of interest, and the added isolation of the COVID-19 pandemic), (2) what factors contributed to our continued growth and development (relationship-building, contributions respected and valued, diversity of perspectives), and (3) what outcomes are associated with our participation (shifts in teaching and scholarship, observed shifts in TC learning, and shifts in our perspectives on the teaching and learning of mathematics). By sharing our work, we aim to advance our collective knowledge of ways to support MTEs' professional development, thereby enriching the mathematical and pedagogical learning experiences of our TC's.

References

Appova, A., & Taylor, C. (2019). Expert mathematics teacher educators' purposes and practices for providing prospective teachers with opportunities to develop pedagogical content knowledge in content courses. *Journal of Mathematics Teacher Education*, *22*(2), 179-204.

Cavey, L.O., Naresh, N., DuCloux, K.K. & Totorica, T. (2021). Eliciting dispositions for teaching in the context of a video-based intervention for secondary teacher candidates. The 43rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Philadelphia, PA, United States.

Cavey, L. O., Libberton, J., Totorica, T., Carney, M., & Lowenthal, P. R. (2020). VCAST learning modules: A functions & modeling course innovation. In J. Goodell & S. Koc (Eds.), Preparing STEM teachers: A replication model (pp. 259-275). Information Age Publishing.

Ball, D. L., Lubienski, S., & Mewborn, D. S. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), Handbook of Research on Teaching (pp. 433–456). Washington, DC: American Educational Research Association.

Beswick, K., & Goos, M. (2018). Mathematics teacher educator knowledge: What do we know and where to from here? *Journal of Mathematics Teacher Education*, *21* (5), 417-427.

Chapman, O. (2008). Mathematics teacher educators' learning from research on their instructional practices: A cognitive perspective. In B. Jaworski & T. Woods (Eds.). *The international handbook of mathematics teacher education: The mathematics teacher educator as a developing professional* (Vol 4, pp.115-134). Rotterdam: Sense Publishers.

Creswell, J. W. (2007). *Qualitative inquiry & research design: Choosing among five approaches*, (2nd Ed.). Thousand Oaks: Sage Publications.

Goos, M. & Bennison, A. (2019). A zone theory approach to analysing identity formation in mathematics education. *ZDM Mathematics Education 51*, 405–418. https://doi.org/10.1007/s11858-018-1011-8

Han, F., & Ellis, R. A. (2019). Using phenomenography to tackle key challenges in science education. Frontiers in Psychology, 10, https://doi.org/10.3389/fpsyg.2019.01414

Jackson, B., Hauk, S., Tsay, J.J., & Ramirez, A. (2020). Professional development for mathematics teacher educators: Need and design. *The Mathematics Enthusiast 17*(1 & 2), 537-583.

Jaworski, B. (2001). Developing mathematics teaching: Teachers, teacher educators, and researchers as co-learners. In F.-L. Lin & T. Cooney (Eds.), *Making sense of mathematics teacher education* (pp. 295–320). Dordrecht: Kluwer.

Kruse, S., Louis, K. S., & Bryk, A. (2009). Building professional community in schools. 13 Parameters: A Literacy Leadership Toolkit, *Research Resource Book*. (pp. 159-163). Pearson Education.

Lave, J., & Wenger, E. (1991). *Situated learning. Legitimate peripheral participation*. Cambridge: University of Cambridge Press.

Masingila, J. O., Olanoff, D., & Kimani, P. P. (2018). Mathematical knowledge for teaching teachers: Knowledge used and developed by mathematics teacher educators in learning to teach via problem solving. *Journal of Mathematics Teacher Education*, *21*(5), 429-450.

Mason, J. (2010). Attention and intention in learning about teaching through teaching. In R. Leikin & R. Zazkis (Eds.), *Learning through teaching mathematics: Development of teachers' knowledge and expertise in practice* (pp. 23-48). Dordrecht, Netherlands: Springer.

Olanoff, D., Masingila, J. O., & Kimani, P. M. (2021). Supporting mathematics teacher educators' growth and development through communities of practice. In M. Goos, K. Beswick (eds.), *The learning and development of Mathematics Teacher Educators* (pp. 147-166), Research in Mathematics Education, https:://doi.org/10.1007/978-3-030-62408-8_8

Orgill, M. K. (2002). *Phenomenography*. Retrieved January 2008 from http://www.minds.may.ie/~dez/phenom.html

Superfine, A.C. & Li, W. (2014a). Developing mathematical knowledge for teaching teachers: A model for the professional development of teacher educators. *Issues in Teacher Education*, *23*(1), 113-132.

Superfine, A.C. & Li, W. (2014b). Exploring the mathematical knowledge needed for teaching teachers. *Journal of Teacher Education*, *65*(4), 303-314.

Wenger, E. (2000) Communities of practice: The key to knowledge strategy. In *Knowledge and Communities* (pp. 3–20). https://doi.org/10.1016/B978-0-7506-7293-1.50004-4

Zaslavsky, O., & Leikin, R. (2004). Professional development of mathematics teacher educators: Growth through practice. *Journal of Mathematics Teacher Education*, *7*(1), 5-23.

TOOL, GATEKEEPER, POWER, VEHICLE: MATHEMATICS TEACHER EDUCATORS' VIEWS OF MATHEMATICS AND ITS PURPOSES

Amy Brass
Pennsylvania State University
aqb6161@psu.edu

What conceptions do mathematics teacher educators (MTEs) hold about mathematics and its purposes? The study reported here explored metaphors for mathematics used by MTEs and the purposes of mathematics these metaphors described. Without explicit prompting, MTEs used metaphors to describe mathematics in survey responses for a larger study. The data revealed four distinct metaphors used by three or more MTEs: mathematics is a tool, mathematics is a gatekeeper, mathematics is power, and mathematics is a vehicle. These metaphors and the different purposes of mathematics they describe reflect complex conceptions of mathematics held by MTEs and provide opportunities to consider how conceptions and purposes of mathematics may influence instructional practice.

Keywords: Teacher Educators; Affect, Emotion, Beliefs, and Attitudes

Mathematics teacher educators (MTEs) are called to enhance and deepen preservice teachers' (PSTs) mathematical and pedagogical knowledge [e.g., Association of Mathematics Teacher Educators (AMTE), 2017]. Yet, relatively little is known about MTEs' knowledge, practices, and beliefs (e.g., Appova & Taylor, 2019; Cross Francis et al., 2022), including how MTEs view mathematics. Given that mathematics teachers' conceptions of mathematics influence their instructional practices (Thompson, 1984), it could be reasoned that MTEs' conceptions of mathematics similarly influence their instructional practices with PSTs. The work I report here provides insights about MTEs' conceptions of mathematics and its purposes by focusing on metaphors used by MTEs to describe mathematics. In doing so, I sought to address this research question: What do MTEs' metaphors for mathematics reveal about their conceptions of mathematics and its purposes?

Theoretical and Conceptual Framings

A metaphor is a conceptual tool which allows us to compare something that is abstract (e.g., mathematics) to something more concrete (e.g., jigsaw puzzle). Lakoff and Núñez (2000) describe a conceptual metaphor as an "inferential structure of a concrete source domain [that] gives structure to an abstract target domain" (p. 42). This inferential structure can be represented through a unidirectional mapping (Lakoff & Núñez). In an example metaphor, *mathematics is a jigsaw puzzle,* the source domain is a jigsaw puzzle, and the target domain is mathematics (see Figure 1).

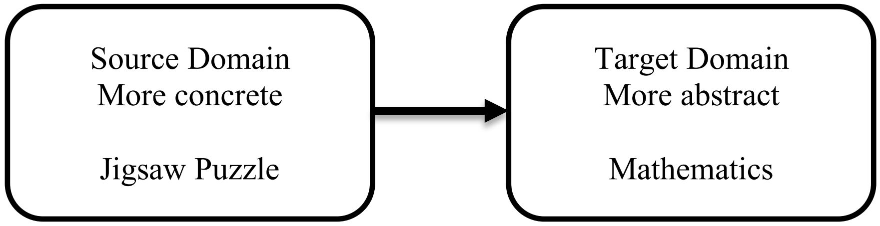


Figure 1: Unidirectional Mapping of a Metaphor

By interpreting one thing in terms of another, a metaphor draws attention to the features and qualities of the more abstract target domain (i.e., mathematics) (Lakoff & Johnson, 1980). For example, by interpreting mathematics in terms of a jigsaw puzzle, the text surrounding the metaphor might emphasize the features and qualities one sees in mathematics that are similar to completing a jigsaw puzzle, such as different approaches, frustration with the challenge, interconnectedness of pieces, etc. In this way, the metaphors we use help to reveal different conceptions, ways of thinking, and actions (Sfard, 1998).

Sfard (1998) called metaphors "the most primitive, most elusive, and yet amazingly informative objects of analysis" (p. 4). As such, metaphors used by MTEs in which mathematics was the target domain (Lakoff & Núñez, 2000) and the text surrounding these metaphors were the unit of analysis in this study.

Methods

This study is part of a larger, qualitative study focused on MTEs' conceptions of the relationships across mathematics, mathematics education, and citizenship. For this report, I focus specifically on the metaphors for mathematics that were used by MTEs during the larger study.

Participants and Data Collection

In 2022, AMTE members were emailed a link to an anonymous Qualtrics survey, which consisted of 15 questions. There were four multiple-answer, semi-closed-ended questions for demographic information and 11 open-ended response questions. The open-ended response questions asked participants about their views on citizenship in multiple contexts (e.g., classroom, local, national), views of mathematics and mathematics education for citizenship (e.g., using math as a citizen), and views of their professional practice (e.g., engaging PSTs in connecting mathematics and citizenship). None of the questions explicitly prompted participants to use metaphors in their responses. Out of the 90 MTEs who participated in the Qualtrics survey, 24 participants used a metaphor for mathematics in their survey responses.

Data Analysis

Qualitative data analysis of survey responses involved open coding and analytic memos (Saldaña, 2021). Through the open coding process, metaphors for mathematics were identified in participants' responses. The target domain for each metaphor was mathematics, and four source domains were used by three or more MTEs (Lakoff & Núñez, 2000). This led to four distinct metaphors: mathematics is a tool, mathematics is a gatekeeper, mathematics is power, and mathematics is a vehicle.

I created a document that grouped quotes containing metaphors from MTEs' survey responses by the source domain (e.g., tool, gatekeeper) (Lakoff & Núñez, 2000). This allowed me to assess the prevalence of different metaphors used by participants. I also analyzed the text surrounding each metaphor for how mathematics was described in each quote, revealing the purposes of mathematics illustrated through the metaphors. Six purposes of mathematics emerged through participants' metaphors: success and opportunity, social justice, information and data, making sense of the world, and general purpose. Each quote was cut-out, color-coded by metaphor, and physically arranged into categories by metaphor and purpose. This helped me determine how the metaphors were being used by participants and how various purposes of mathematics were represented by the same metaphor across MTEs' responses.

If a metaphor was used more than once by a participant with the same mathematical purpose, it was counted only once. There were two MTEs who used the same metaphor throughout their survey responses but for different mathematical purposes, and these metaphors were counted according to each purpose they reflected. There was also one MTE who used three different metaphors in their responses for different purposes of mathematics. This led to 28 metaphors that were analyzed. Table 1 displays the counts for the different metaphors and purposes.

Table 1: Metaphors and Purposes of Mathematics

Purposes of Mathematics	Metaphors of Mathematics			
	Mathematics is a Tool (n = 13)	Mathematics is a Gatekeeper (n = 7)	Mathematics is Power (n = 5)	Mathematics is a Vehicle (n = 3)
Success and Opportunity (n = 10)	1	5	3	1
Social justice (n = 5)	3	1	0	1
Information and Data (n = 5)	5	0	0	0
Making Sense of the World (n = 4)	2	0	2	0
Reflecting Democracy (n = 2)	1	0	0	1
General Purpose (n = 2)	1	1	0	0

Results

Survey responses revealed four distinct metaphors describing six different purposes of mathematics. The metaphors – mathematics is a tool, mathematics is a gatekeeper, mathematics is power, and mathematics is a vehicle – and the purposes they describe are discussed below.

Mathematics is a Tool

The most prevalent metaphor used by MTEs in their survey responses was *mathematics is a tool* (n = 13), and all of the different purposes for mathematics were described with this metaphor. The most frequently mentioned purpose of mathematics with this metaphor was to use information and/or data. Interestingly, no other metaphor described this purpose. For example, MTE 59 responded, "Mathematics is a form of information as well as a tool for processing information." Some MTEs, such as MTE 11, described mathematics as "a tool for making sense of the world around us." MTE 11 elaborated, "Math helps to understand/identify, interpret, and respond to many aspects of a society." Other MTEs emphasized social justice as a purpose for mathematics through this metaphor, such as MTE 31 who noted, "Mathematics is the tool that can be used to challenge the status quo and support social justice." There was also one participant who saw "mathematics as a tool for democratic citizenship" (MTE 3) and one participant who described mathematics as generally being a useful tool by noting that they needed to "convince students it [mathematics] is a useful tool" (MTE 55).

Mathematics is a Gatekeeper

MTEs' metaphors of *mathematics as gatekeeper* (n = 7) most frequently mentioned the role of mathematics and/or mathematics achievement in granting or limiting opportunities, including those related to higher education and careers. For example, MTE 43 noted how mathematics is "used by society as a marker of educational attainment and educational readiness [and] the ways it has served as a gatekeeper to higher education." Similarly, MTE 25 mentioned helping PSTs to see how mathematics "has been used as a gatekeeper to college and careers." Strikingly, only one MTE used the gatekeeper metaphor to talk about social justice by responding:

> Mathematics is often used as a weapon and/or a gatekeeper - keeping some people from achieving and maintaining the status quo. If mathematics is taught with a lens of social justice proving math is for all, then our society would benefit. (MTE 44)

There was also one MTE described the general purpose of learning mathematics through the mathematics is a gatekeeper metaphor by saying, "math is a gatekeeper for so many who want to be an integral part of society. Therefore, believing that all people can learn and do mathematics is essential" (MTE 38).

Mathematics is Power

When MTEs used the *mathematics is power* metaphor (n = 6), it was usually to describe the role of mathematics in being successful or gaining opportunities, although not all MTEs saw this as a positive trait. MTE 19 lamented, "mathematics is unfortunately power in addition to other things, like beauty. So, you have to open up access to mathematics learning and equip everyone with the power to be successful." Other MTEs viewed mathematics as power to make sense of the world. For example, MTE 61 stated, "Mathematics is power to make sense of issues and situations. It is important knowledge for citizens to be able to engage with their community fully."

Mathematics is a Vehicle

There were three MTEs who used the metaphor of *mathematics is a vehicle*, each reflecting a different purpose of mathematics. In discussing their classroom environment, MTE 71 noted, "Math is a vehicle to demonstrate what equitable and democratic organizations can look like." MTE 68 also described how they used mathematics as vehicle with PSTs by saying, "I like to use mathematics as a vehicle to discuss social problems/issues in our society." The third MTE described mathematics as a vehicle to success and stated, "Mathematics is a vehicle that provides citizens the opportunities in this world and the future world to do what you want in a career!" (MTE 27).

Discussion

The metaphors MTEs used for mathematics provide a nuanced view of MTEs' conceptions of mathematics and its purposes. Indeed, the metaphors and purposes of mathematics they described emphasize the complex, and sometimes conflicting, relationships MTEs have with mathematics. For example, some of the metaphors describe a positive connotation of mathematics that emphasize its use (e.g., mathematics is a tool). Yet, other metaphors used by MTEs reflect a negative connotation (e.g., mathematics is a gatekeeper) or ambiguity in interpretation (e.g., mathematics is power).

The prevalence of the metaphors reflecting the mathematical purpose of success and opportunity provides our field with an opportunity to reflect upon the complexities of this purpose. Results suggest that MTEs support success and opportunity as well as social justice as purposes of mathematics. Although power and success can conflict with social justice, our field wants to commit to both purposes. Yet, primarily describing mathematics as a tool, gatekeeper, power, and/or vehicle for success and opportunity suggests a mainstream view of equity that looks only at access and achievement (Gutiérrez, 2018). These results suggest the need for further inquiry into how these complex, and sometimes conflicting, conceptions and purposes of mathematics might impact MTEs' instructional practices and engagement with PSTs.

References

Appova, A., & Taylor, C. E. (2019). Expert mathematics teacher educators' purposes and practices for providing prospective teachers with opportunities to develop pedagogical content knowledge in content courses. *Journal of Mathematics Teacher Education, 22*, 179-204. https://doi.org/10.1007/s10857-017-9385-z

Association of Mathematics Teacher Educators. (2017). *Standards for preparing teachers of mathematics*. Association of Mathematics Teacher Educators.

Cross Francis, D., Kaur Bharaj, P., Liu, J., Phillips, A., Park Rogers, M., Zhong, Q., Cesljarev, C., & Lloyd, K. (2022). Questioning our credibility: An exploration of the professional identity development of mathematics teacher educators. *Mathematics*, *10*(1), Article 66. https://doi.org/10.3390/math10010066

Gutiérrez, R. (2018). Political conocimiento for teaching mathematics: Why teachers need it and how to develop it. In S. E. Kastberg, A. M. Tyminski, A. E. Lischka, & W. B. Sanchez (Eds.), *Building support for scholarly practices in mathematics methods* (pp. 11-37). Information Age Publishing, Inc.

Lakoff, G., & Johnson, M. (1980). *Metaphors we live by*. Chicago: University of Chicago Press.

Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being.* Basic Books.

Saldaña, J. (2021). *The coding manual for qualitative researchers* (4th ed.). SAGE.

Sfard, A. (1998). On two metaphors for learning and the dangers of choosing just one. *Educational Researcher*, *27*(2), 4–13. https://doi.org/10.3102/0013189X027002004

Thompson, A. G. (1984). The relationship of teachers' conceptions of mathematics and mathematics teaching to instructional practice. *Educational Studies in Mathematics*, *15*, 105–127. https://doi.org/10.1007/BF00305892

TRANSITIONING ACROSS LANGUAGES: HOW LANGUAGE AND ACCOUNTABILITY POLICIES SHAPED MATHEMATICS INSTRUCTION IN DUAL-LANGUAGE CLASSROOMS

TRANSICIÓN ENTRE IDIOMAS: COMO LAS NORMAS LINGÜÍSTICAS Y DE CONTABILIDAD INFORMARON LA ENSEÑANZA DE LAS MATEMÁTICAS EN AULAS BILINGÜES

Rosa Chavez
Texas Tech University
rosa.d.chavez@ttu.edu

Paola Montufar
New York University
Paola.montufar@nyu.edu

This study explores how dual-language teachers' decision-making was shaped by language policies, accountability policies, and standardized assessments. The study draws on interviews with third and fourth grade elementary teachers in California and Texas (N=17) to understand how they balanced accountability policies and assessments that worked in opposition to being able to provide mathematical learning opportunities within a dual language context. In particular, this study highlights how teachers contended with tensions when the language of instruction was different from the language of assessment. These findings raise the need for coherence across federal, state, and local policymaking for dual language programs to remain student-centered in the implementation of mathematics curriculum to meet the needs of all learners.

Keywords: Policy, Assessment, Elementary School Education

Introduction and Framework

Language education programs have been in place for decades to support linguistically diverse students (Lindholm-Leary, 2012; Valdés & Figueroa, 1994), yet "schools in the United States have not been truly interested in developing bilingualism; in fact, public schools often operate to promote immigrant students' shift to English" (Sánchez, García, & Solorza, 2018, p. 39). While No Child Left Behind policies upheld "monolingual, monoglossic, standardized middle-class norms onto multilingual communities" (Palmer et al., 2019, p. 125), a different program structure– dual language education (DLE), began to gain popularity for educating linguistically diverse students (Lindholm-Leary, 2005; Palmer, et al., 2019; Sánchez, García, & Solorza, 2018; Valdés, 1997). Additionally, research on mathematics classroom learning opportunities shows that student's experiences with mathematics affects not only what they learn, but also how they view themselves as learners and doers of mathematics. Opportunities to learn (OTL) arise from the norms and activities that shape how students engage with mathematics (Boaler, 2016; Jackson, 2009; Langer-Osuna, 2016). Certain classroom structures may provide (or constrain) OTL, leading students to develop vastly different mathematical identities (Gresalfi & Cobb, 2006; Greeno & Gresalfi, 2008). Students will thus experience different learning opportunities dependent in part on the ways that teachers make instructional decisions. These interactions have consequences for student mathematical learning.

This study examined how third and fourth grade dual-language and bilingual elementary teachers in California and Texas made sense of layered language policies, accountability policies, and standardized assessments to shape mathematics instruction. This study looked at this relationship by answering the following: How did third and fourth grade teachers in California and Texas make sense of educational policies to inform their mathematics instructional decisions?

Accountability and testing policies have been a landmark of U.S education for over two decades and have made high-stakes tests routine in schools. Teachers make-sense of these policies in various ways to make instructional decisions and provide OTL in classrooms, and ultimately students perceive the OTL made available in ways that shape the mathematical dispositions that they develop to see themselves as mathematics learners. For teachers in dual language programs, the language of instruction creates an additional complex layer for decision making.

There are many other factors that influence how teachers make sense of policy as a resource for planning, particularly for dual-language and bilingual teachers. Most notably, teachers' knowledge, beliefs and dispositions about subject content and their students has been shown to influence how they make sense of policies (Coburn, 2005; Horn, 2018; Spillane, 2002) and language ideologies further shape decision-making for teachers (Rosa & Flores, 2017). Ultimately, these instructional decisions impact the learning opportunities teachers provide linguistically diverse students in their classrooms.

Thus, it is difficult to separate what students are learning and the ways in which they are learning it (Boaler, 2016; Cobb & Bowers, 1999; Lave, 1988) without considering how teachers make decisions (Shavelson & Stern, 1981; Borko & Shavelson, 1990). Indeed, the research on mathematics classroom learning opportunities shows that students' experiences with mathematics affect not only what they learn (Boaler & Greeno, 2000), but also how they view themselves as learners and doers of mathematics (Boaler, 2016; Jackson, 2009; Langer-Osuna, 2016). This body of work frames classrooms as sites of social practice with patterns of interaction, assumptions, and norms (Cobb & Bauersfeld, 1995; Cobb, Gresalfi, & Hodge, 2009; Cobb & Yackel, 1996; Engestrom, 1999). Within DLE, certain classroom structures may provide more equitable OTL.

Methods

Data sources

Given that federally mandated assessments were not required during the 2019-2020 school year, this study focused on 3rd and 4th grade since this would be the first year that students in these grades took a federally mandated state assessment for accountability purposes. Teacher participants for this study (N=17) were predominantly recruited from large, urban school districts in California and Texas. Due to the onset of the COVID-19 pandemic, additional teachers were recruited through social media platforms using the hashtags of #texasteachers, #iteachthird, #duallanguage, and #bilingualeducator. Attention to other teacher characteristics like years of teaching experience, school type, and other self-reported demographics were also considered to ensure a more representative sample. Once several teachers agreed to participate, snowball sampling (Emerson, 2015) gave me the opportunity to recruit teachers from the same campuses or districts to be able to note patterns or differences in teacher decisions making for teachers working under similar educational policies. All interviews were conducted over zoom by the first author and lasted anywhere from 45 minutes to two hours, based on teacher responses and availability.

Data Analysis

Semi-structured interviews allowed for understanding how teachers' sense-making shaped their instruction (Glesne, 2005). During interviews, the first author took note of what teachers referenced as important for their mathematics instruction (ex. structure or classroom, planning resources mentioned) and hand-wrote quick memos reflecting on my notes and questions that were coming up (Emerson, Fretz, & Shaw, 1995). This process further informed the initial codes that were developed alongside a-priori codes from the literature (e.g., beliefs, planning- long-range vs. daily, change to practice). Four interviews were transcribed and uploaded to Nvivo for initial coding by both authors. Interviews were open-coded focused on six categories: 1) policies and mandated assessments, 2) assessment for learning, 3) distance learning, 4) teacher agency, 5) math instruction, and 6) other - anything else that came up in the interview that stood out but did not fall under the previous categories. We began with these six categories to "summarize interviews into manageable vignettes for review" (Saldaña, 2021, p. 60) while ensuring that the main tenets that addressed the phenomenon of interest were captured. Brief one-page narratives were then constructed related to these categories and the research question. Memos were then categorized by region (state, and geographical location), and by district to look at patterns across. This process allowed for emergent codes to surface (Corbin & Strauss, 1990), helping to refine and finalize a codebook. All interviews were then uploaded to Nvivo for a second round of coding.

Findings

Across the entire set of interviews, there were notable differences in how teachers talked about how they were making sense of policies and assessments to shape their instructional practices. In particular, there were clear differences in the knowledge that the teachers in this study had about federal, state, and district policies. This knowledge informed what teachers reported needing to focus on for their decisions for teachers in Texas, and a lack of knowledge provided opportunities for teachers in California to exercise more agency in their instruction. Even though almost all teachers reported feeling that policies focused on assessments had negative consequences such as narrowing their instructional focus and the effect on student morale, they still felt that assessments provided necessary data for holding teachers and students accountable.

The varying language policies that teachers navigated highlighted the complex practice of sense-making that DLE teachers further engaged in. Teachers felt constrained when the language of assessments differed from the language of instruction. One teacher, Brenda, spoke to having to shift away from teaching mathematics in Spanish due to the assessment.

> "The problem is it's given in English. And my students have been having Spanish this whole time...that's why we started transitioning from Spanish to English because of the state exam. I was like, I think we need to start transitioning them, but I really wanted to stick to Spanish. But because of the pressure from the district to make sure they do well."

Here, Brenda was responding to how the state assessments shaped her planning, and if she thought the knowledge students need to be successful long term is the same as the knowledge they need for the assessment. Throughout her interview, she referenced knowledge of studies that reported dual immersion kids are able to transfer information from one language to the other, she still felt pressure from administration at her campus (and from the district) to ensure that students did well on the assessments that were in English, pushing her to transition teaching in

English as the state assessment became more imminent. While Brenda felt strongly about wanting to stick to teaching in Spanish given the dual-language policies and practices, the disconnect between the language of the assessment and the language of instruction led to prioritizing the English language in her instruction, affecting the learning opportunities that were provided for students in the dual language context. This was one example of how teachers were having to contend with the tension of various policies, and how state assessments that are used for accountability ultimately overpowered other possibilities in her instructional decisions.

While Brenda didn't have a choice on the language of the assessment, there were other teachers who were able to provide the assessment to students in either language. Yet, even for those teachers that had the option, accountability and language policies still led to tensions due to pushes from administration to transition students to testing in English due to the accountability systems in place. Sammie stated:

> "The whole thing about bilingual education comes from a different game. The official discourse of the bilingual department is that we should tend to both languages…The department will say, you should not force a kid to take the exams [in English] if they're not ready. But then the administration comes and tells you, 'Well, why has this kid [not taken] the test in English yet?' And you have to justify and fight it [because] once you transition language, you can't go back."

Here, Sammie was sharing how conflicting policies created tensions for teachers at her campus. While the bilingual education department at the district level was emphasizing both languages due to their use of a 50/50 model, her campus administrators were pushing for testing students in English due to the accountability structures in place for Texas. She further shared that once students transitioned to testing in English, they were no longer able to test in Spanish. Thus, even for students in a dual-language program that provided the option to select the language of assessment, that option was no longer available once students transitioned to testing in English.

These two examples highlight how state assessments, and the language in which they were administered, shaped not only the instructional decisions that teachers felt they could make in the case of Brenda, but also the contradicting policies that DLE teachers have to contend with. This highlights the need for there to be more coherence in how policies are not only enforced but also in how they are developed. If we want to truly emphasize dual language as an asset when learning mathematics, policies should make space for students to not only engage in Spanish but to also test in Spanish. And ideally, students should have the option to use both languages.

Conclusion

The complexities of the multi-layered policies that teachers for culturally and linguistically diverse students contend with daily are overlooked when we fail to communicate across departments within districts. As several teachers in this study noted, there were conflicting messages that teachers received that shaped how they were thinking about mathematics curriculum, language practices, and assessments for accountability. For some teachers, there was a lack of coherence between the bilingual department and accountability department. For others there was a lack of coherence between the mathematics curriculum department and the accountability department. Thus, it raises the need for coherence across departments that maintains the integrity of language programs and implementations of mathematics curriculum that are student-centered while meeting the accountability policies. Additionally, accountability policies for English language proficiency as they are currently structured are inherently at odds with supporting students to become biliterate (Sánchez, García, & Solorza, 2018) and develop productive mathematical disposition. This study pushes the field to consider adapting policies *for* learning that center and support linguistically diverse students.

References

Boaler, J. (2016). Mathematical mindsets: Unleashing students' potential through creative math, inspiring messages and innovative teaching. John Wiley & Sons.

Boaler, J., & Greeno, G. (2000). Identity, Agency and Knowing in Mathematics Worlds. In J. Boaler (Ed.), Multiple Perspectives on Mathematics Teaching & Learning(pp. 181–210). Greenwood Publishing Group.

Borko, H., & Shavelson, R. J. (1990). Teacher decision making. In Dimensions of Thinking and Cognitive Instruction, 311-346.

Cobb, P., & Bauersfeld, H. (Eds.). (1995). Emergence of mathematical meaning: Interaction in classroom cultures. Hillsdale, NJ: Erlbaum.

Cobb, P., & Bowers, J. S. (1999). Cognitive and situated learning perspectives in theory and practice. Educational Researcher, 28(2), 4-15.

Cobb, P., Gresalfi, M.S., & Hodge, L. (2009). An interpretive scheme for analyzing the identities that students develop in mathematics classrooms. Journal for Research in Mathematics Education, 40, 40-68.

Cobb, P., & Yackel, E. (1996). Constructivist, Emergent, and Sociocultural Perspectives in the Context of Developmental Research. Educational Psychologist, 31(3/4), 175–190.

Coburn, C. E. (2005). Shaping teacher sensemaking: School leaders and the enactment of reading policy. Educational Policy, 19(3), 476–509. https://doi.org/10.1177/0895904805276143

Corbin, J. M., & Strauss, A. (1990). Grounded theory research: Procedures, canons, and evaluative criteria. Qualitative sociology, 13(1), 3-21.

Emerson, R. W. (2015). Convenience sampling, random sampling, and snowball sampling: How does sampling affect the validity of research?. Journal of Visual Impairment & Blindness, 109(2), 164-168.

Emerson, R., Fretz, R., & Shaw, L. (1995). Writing Ethnographic Fieldnotes. Chicago: University of Chicago Press. Engestrom, Y. (1999). Activity theory and individual and social transformation. In Y. Engestrom, R. Miettinen, & R. Punamaki (Eds.), Perspectives on activity theory(p. 19-38). New York: Cambridge University Press.

Glesne, C. 2005. Making Words Fly: Developing understanding through interviewing. In Becoming Qualitative Researchers: An Introduction. Pearson Publishing: Boston.

Greeno, J. G., & Gresalfi, M.S. (2008). Opportunities to learn in practice and identity. In P. A. Moss, D. C. Pullin, J.P. Gee, E. H. Haertel, & L. J. Young (Eds.), Assessment, Equity, and Opportunity to Learn. New York: Cambridge University Press.

Gresalfi, M. S. (2009). Taking up opportunities to learn: Constructing dispositions in mathematics classrooms. The Journal of the learning sciences, 18(3), 327-369.

Gresalfi, M. S., & Cobb, P. (2006). Cultivating Students' Discipline-Specific Dispositions as a Critical Goal for Pedagogy and Equity. Pedagogies: An International Journal, 1(1), 49–57.

Horn, I. S. (2007). Fast Kids, Slow Kids, Lazy Kids: Framing the Mismatch Problem in Mathematics Teachers' Conversations. Journal of the Learning Sciences, 16(1), 37–79.

Horn, I. S. (2018). Accountability as a Design for Teacher Learning: Sensemaking About Mathematics and Equity in the NCLB Era. Urban Education, 53(3), 382–408.

Jackson, K. (2009). The social construction of youth and mathematics: The case of a fifth-grade classroom. Mathematics Teaching, Learning, and Liberation in the Lives of Black Children, 175–199. http://doi.org/10.4324/9780203877708

Langer-Osuna, J. M. (2016). The social construction of authority among peers and its implications for collaborative mathematics problem solving. Mathematical Thinking and Learning, 18(2), 107-124.

Langer-Osuna, J. M. (2011). How Brianna Became Bossy and Kofi Came Out Smart: Understanding the Trajectories of Identity and Engagement for Two Group Leaders in a Project-Based Mathematics Classroom. Canadian Journal of Science, Mathematics and Technology Education, 11(3), 207–225.

Lave, J. (1988). Cognition in practice. Cambridge, England: Cambridge University Press.

Lindholm-Leary, K. J. (2005). The rich promise of two-way immersion. Educational Leadership, 62(4), 56-59. Lindholm-Leary, K. (2012). Success and challenges in dual language education. Theory Into Practice, 51(4), 256-262.

Moss, J., & Case, R. (1999). Developing children's understanding of the rational numbers: A new model and an

experimental curriculum. Journal for research in mathematics education, 122-147.
Palmer, D. K., Cervantes-Soon, C., Dorner, L., & Heiman, D. (2019). Bilingualism, biliteracy, biculturalism, and critical consciousness for all: Proposing a fourth fundamental goal for two-way dual language education. Theory into Practice, 58(2), 121-133.
Saldaña, J. (2021). The coding manual for qualitative researchers. Sage.
Sánchez, M. T., García, O., & Solorza, C. (2018). Reframing language allocation policy in dual language bilingual education. Bilingual Research Journal, 41(1), 37-51.
Shavelson, R. J., & Stern, P. (1981). Research on teachers' pedagogical thoughts, judgments, decisions, and behavior. Review of educational research, 51(4), 455-498.
Spillane, J. P. (2002). Local Theories of Teacher Change: The Pedagogy of District Policies and Programs. Teachers College Record, 104(3), 377–420. https://doi.org/10.1111/1467-9620.00167
Rist, R. C. (2000). Student social class and teacher expectations: The self-fulfilling prophecy in ghetto education with an introduction by the author. Harvard Educational Review, 70(3), 257-301.
Rosa, J., & Flores, N. (2017). Unsettling race and language: Toward a raciolinguistic perspective. Language in society, 46(5), 621-647.
Valdés, G. (1997). Dual-language immersion programs: A cautionary note concerning the education of language-minority students. Harvard Educational Review, 67(3), 391-430.
Valdés, G., & Figueroa, R. A. (1994). Bilingualism and testing: A special case of bias. Ablex Publishing.