Tychonov's Solution: An Overlooked Opportunity to Blend Pure Mathematics into Mechanical Engineering Education

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Abstract: Tychonov's 1935 solution of the heat equation, exhibiting nontrivial heat fluxes spontaneously appearing along an isolated conducting rod initially held at zero degrees, has intrigued some specialists for almost a century. No doubt those practicing heat engineers who took mathematics seriously were initially relieved to learn that the construction was valid only for infinitely long rods; the integrity of their published exchanger designs could be defended by citing this weakness, together with the known discrepancies between the heat equation and physical reality. Some mathematicians contrived additional hypotheses to disqualify the Tychonov solution. Recently a computer simulation was executed, revealing just how astonishingly unbridled the solution is. But there remain incongruities in this singular example that invite metaphysical speculation. We fuel the latter with a recap of the history from a lighthearted perspective, providing heat transfer engineering students with a rare insight into the practical value of the mathematicians' exacting obsession with generality.

Keywords: Heat transfer, Nonuniqueness, Singular boundary conditions, Engineering mathematics

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Introduction and Background

In this paper we shall attempt to summarize some of the mathematical investigations spawned by Tychonov's discovery (Tychonoff, 1935) of a counterintuitive solution of the heat equation, share some recent insights, and reexamine the significance of the studies from the perspective of the student of heat transfer engineering. By so doing we hope to dispel the skepticism that such students acquire regarding the practical value of the mathematicians' exacting obsession with rigor and generality.

Synopsis of Heat Flow Physics

The heat equation (customarily attributed to Fourier (1822) governs the evolution of temperature T as a function of time t and position (x, y, z) in a conductor with uniform diffusivity (set equal to 1 by choice of units). The flow can be one-dimensional $T(t,x)$ if the conductor is a laterally insulated rod $x_1 < x < x_2$ with uniform cross section. The typical initial-boundary value problem encountered by engineers is expressed

$$\frac{\partial T}{\partial t} - \frac{\partial^2 T}{\partial x^2} = 0\,,\ x_1 < x < x_2,\ t > 0\,; \tag{1}$$

$$\mathrm{T}(0,\mathrm{x}) = \mathrm{T}_0(\mathrm{x}) \text{ (initial condition) ;} \tag{2}$$

$$\alpha_1 T(t,x_1) + \beta_1 \frac{\partial T(t,x_1)}{\partial x} = A_1\,; \quad \alpha_2 T(t,x_2) + \beta_2 \frac{\partial T(t,x_2)}{\partial x} = A_2 \quad \text{(boundary conditions).} \tag{3}$$

A and B are arbitrary *known* functions of x and t. If $\beta = 0$ the "Dirichlet condition" is specifying the temperature at the end of the rod; if $\alpha = 0$ the "Neumann condition" is specifying the heat flux; and if neither is zero the "Robin condition" is modeling leaky insulation. Technical issues, such as the precise nature of the continuity at $t = 0$ and the end points, do not concern us here.

(Since $\mathrm{e}^{-\pi^2 t} \cos \pi \mathrm{x}, \mathrm{x}_1 = -\mathrm{x}_2 = -1/2$ is a solution, a unit rod possessing this diffusivity with its ends packed in ice and an initial half-sine-wave temperature profile would cool by a factor e^{-1} in $\pi^{-2} \sim 0.1$ time units.)

For rods of finite length (and reasonable initial values $T_0(x)$) the equations have one and only one solution. In fact the system is so well-behaved that it is often proffered as the inaugural example in textbooks studying partial differential equations. Indeed, this solution can be explicitly displayed as a Fourier series: for $x_1 = -L$, $x_2 = L$, and homogeneous Dirichlet boundary conditions ($\beta_1 = \beta_2 = A_1 = A_2 = 0$), we have (Nagel et al, 2018)

$$T(t,x) = \sum_{n=1}^{\infty} e^{-(\frac{n\pi}{2L})^2 t} \sin \frac{n\pi(x+L)}{2L} \int_{-L}^{L} T_0(\xi) \sin \frac{n\pi(\xi+L)}{2L} d\xi / L\,. \tag{4}$$

The convergence of the sum is at least as strong as that of the Fourier series representation of $T_0(x)$.

Uniqueness

For our purposes the crucial point of these deliberations is that the solution is *unique*. Now uniqueness of the solution has a special significance to engineers that may not occur to mathematicians. It implies that *they have got the physics right*; once they have measured the initial temperature T_0 and the two end temperatures or fluxes A_1 and A_2, the behavior is completely determined. There are no more "clean-up" or "fine-tuning" measurements that need to be made. Hadamard (1902) expressed three conditions for well-posedness of a system - existence, uniqueness, and continuous dependence on data. Of these, *uniqueness* is the most relevant to engineers. They need to predict the performance of a heat pipe precisely, without any extraneous possibilities lurking about.

A well-known property that is logically equivalent to uniqueness when all the governing equations are *linear* (as for (1-3)) is the following: the *only* solution of the associated *homogeneous* system

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}\,; T(0,x) = 0\,;$$

$$\alpha_1 T(t,x_1) + \beta_1 \frac{\partial T(t,x_1)}{\partial x} = \alpha_2 T(t,x_2) + \beta_2 \frac{\partial T(t,x_2)}{\partial x} = 0 \tag{5}$$

is $T(t,x) \equiv 0$. After all, if $T_1(t,x)$ and $T_2(t,x)$ were different solutions to (1-3), then $T_1(t,x)$ - $T_2(t,x)$ would be a non-identically-zero solution to (5). This restatement of uniqueness is usually easier to apply.

The Infinite Rod and Poisson's Formula

If we let the (half-)length L go to infinity, the solution expression (4) approaches the *Poisson formula* (Poisson, 1835),

$$T(t,x) = (4\pi t)^{-1/2} \int_{-\infty}^{\infty} T_0(\xi) e^{-(\xi - x)^2/4t} \, d\xi \quad . \tag{6}$$

(The derivation is quite similar to the familiar extrapolation of the Fourier series to the Fourier transform.) So formula (6) gives a solution to the infinite-rod problem

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} : \ -\infty < x < \infty, t > 0 ; \quad T(0,x) = T_0(x) \text{ (initial condition).} \tag{7}$$

But as we shall see, Tychonov constructed another, non-identically-zero, solution to the homogeneous version of (7),

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} : \ -\infty < x < \infty, \ t > 0 ; \quad T(0,x) = 0. \tag{8}$$

This would predict that a rod, initially at 0 degrees and isolated from any external heat sources, can spontaneously attain nonzero temperatures – if Fourier's heat equation is valid! Even worse, it casts a shadow on all engineering designs premised on the Fourier equation (homogeneous or nonhomogeneous) since, as we indicated, it implies that solutions to eq. (7) are not unique(!)

Lest heat-exchanger consultants panic at the thought of having to refund their commissions, they can take some solace in the fact that these anomalies only apply to infinite rods. No one will ever build an infinitely long heat exchanger (Fig. 1). (*We shall see that this statement is not as inane as it sounds*.) And finite rods are described by systems (1) having unique solutions.

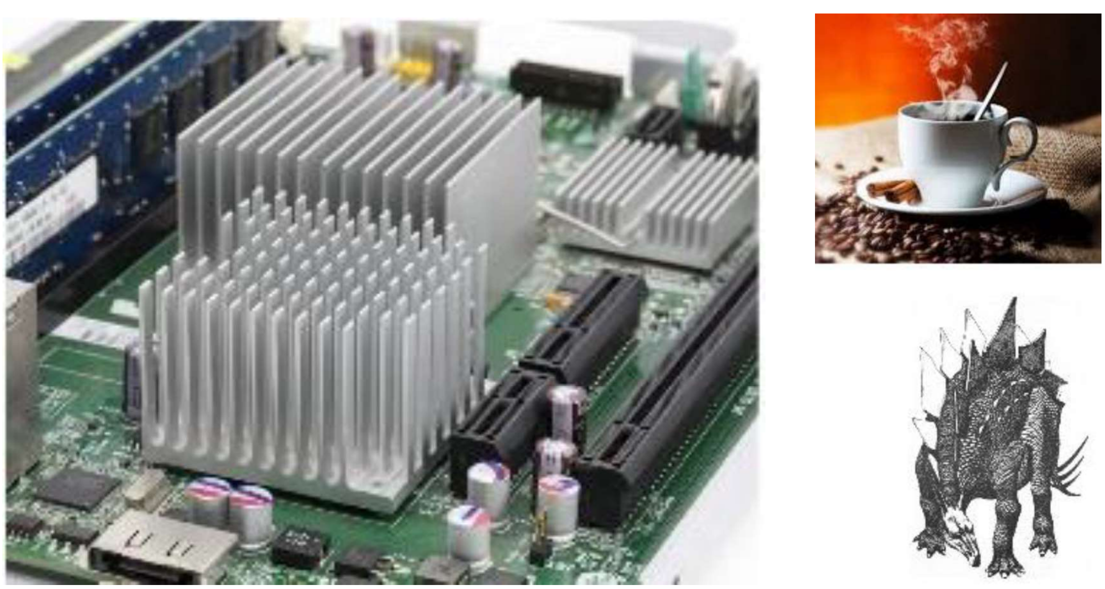


Figure 1. Heat Exchangers. (a) Electronic heat sink
(b) Metal spoon in a cup of coffee (c) Platelets on a stegosaurus.

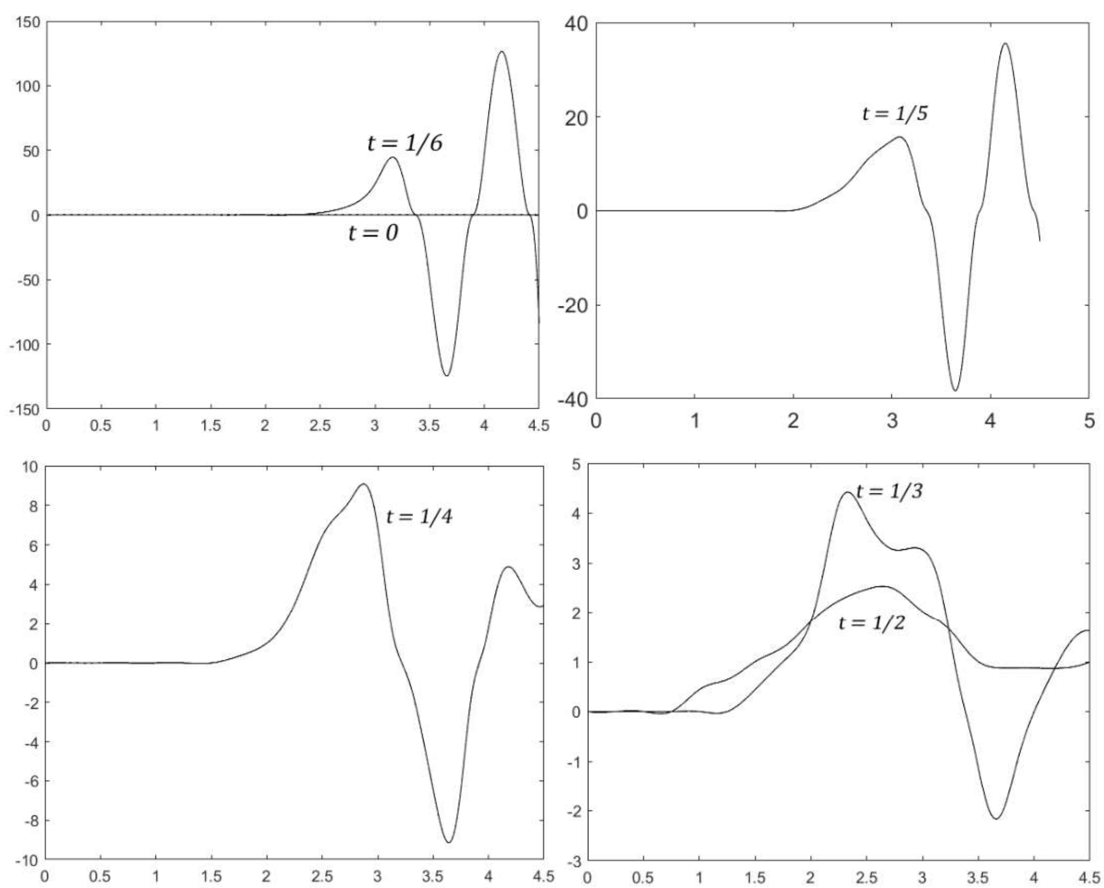
Admittedly, Poisson's formula (4) does find its way into respectable analyses, when it is invoked as a convenient approximation to these finite-length, unique, solutions. The logic is secure. But nothing in the statement of (7) dictates that its $T(t,x)$ has to be the limit of finite-rod solutions. (Forgive me, I can't resist: *Poisson's formula does not have to take the heat for Tychonov's epiphany*.)

Tychonov's Solution

Tychonov constructed his example as the sum of a series:

$$T_{Tychonov}(t,x) = \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k!)} x^{2k} \quad \text{where} \quad g(t) = \begin{cases} e^{-t^{-\alpha}}, & t \geq 0 \\ 0, & t < 0 \end{cases} \quad (\alpha > 1). \tag{9}$$

He proved that the series converged and that $T_{Tychonov}$ was smooth – infinitely often termwise differentiable for *all t* and *x*, in fact. But although $T_{Tychonov}(x,t)$ is identically zero initially, it immediately fluctuates - quite violently, in fact. Rodland (2017) has meticulously computed some snapshots of the profiles (and we have brutally compressed them for display in Fig. 2, where the abscissa is length and the ordinate is temperature); note the different temperature scales on the graphs.



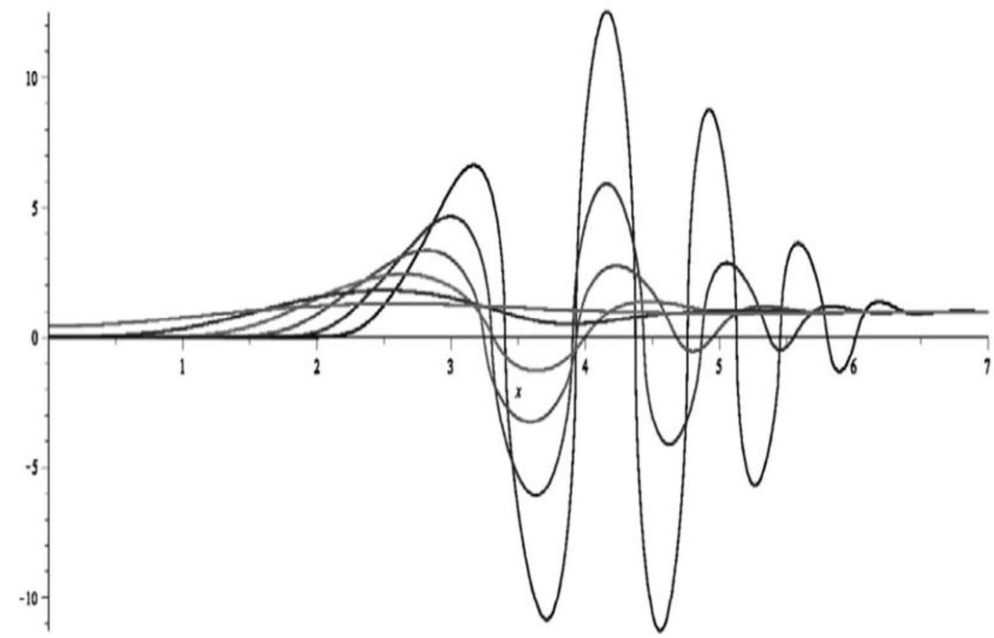


Figure 2. Sample snapshots of the Tychonov solution

Boundary Conditions at Infinity

The contradictory nature of the finite and infinite rod problems compel us to scrutinize the destiny of the boundary conditions as L approaches ∞. Physics is of no help. After all, what is the significance of the flux out of the end of an endless rod?

An intriguing observation emerges if we examine the evolution of the finite-rod solutions to the *Neumann* (insulated tip) and *Robin* (leaky tip) problems as the length increases. They both approach the same limit as the Dirichlet (fixed temperature) solution - i.e. the Poisson formula (6)! The *infinite-rod approximation* to the solutions of the finite-rod Dirichlet, Neumann, and Robin problems is immune to the choice of the boundary condition. But if we simply drop the boundary condition altogether as in (7), the Tychonov solution rears its ugly head and we forfeit uniqueness. The boundary condition at infinity is certainly an enigma.

Contrived Boundary Conditions

There are other mathematical conditions which, if imposed, would restore the uniqueness property (Doetsch, 1936; Täcklind, 1936). Tychonov himself proved, in his 1935 paper, that if we insisted that for some positive M and m

$$|T(t,x)| \leq M \exp(m|x|^2), \quad (10)$$

then the solutions to (7,8) would be unique. Chung and Kim (1994, 1999) have shown that if condition (10) is weakened to either

$$|T(t,x)| \le M \exp(\frac{m}{t} + m|x|^2) \quad (11)$$

or

$$|T(t,x)| \le M \exp\left(\left(\frac{m}{t}\right)^{\mu} + m|x|^2\right), \quad 0 < \mu < 1, \quad (12)$$

then, too, (7,8) would possess unique solutions.

Sketches of (10, 11) are displayed in Fig. 3 for $M = m = 1$ (so they can be amplified to any degree). Also note the log scales. The Chung-Kim restrictions (11, 12) effectively append a vertical asymptote to the time-independent Tychonov "umbrella" (10) as $t \downarrow 0$.

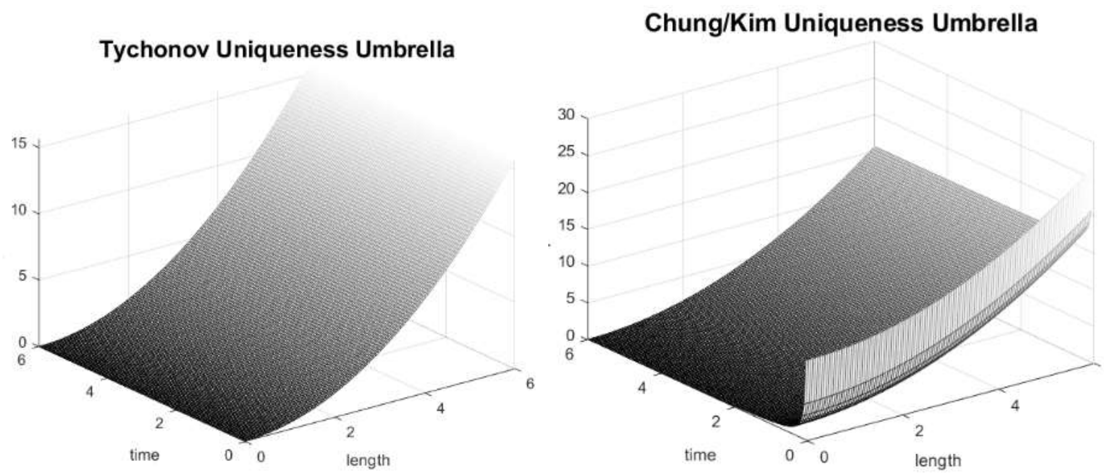


Figure 3. Tychonov and Chung/Kim Uniqueness Umbrellas

An alternate way of phrasing these infinite-rod uniqueness theorems is:

> If there *is* a non-identically-zero solution to (8), then for any positive M and m and any μ in (0,1) the magnitude of this solution will exceed $M \exp(\left(\frac{m}{t}\right)^{\mu} + m|x|^2)$ somewhere, at some time (x,t).

But none of these repairs (10-12) impress the practicing engineer, because they are academic; they can not be tested *a priori*. They would have to be checked all along the rod, for all times. No retrofitting can be applied to the rod to ensure their compliance.

A completely different constraint ensuring uniqueness was announced by Widder (1944).

> If we impose the additional constraint on (8) that its solutions must be nonnegative, then the only solution is identically zero. Alternatively, any non-identically-zero solution to (8) must be negative somewhere, sometime.

This may resonate with practitioners because of the physical interpretation of temperature as mean kinetic energy; absolute 0 degrees Kelvin is the lowest *possible* temperature, and no thermal governors have to be jury-rigged to ensure it is not undercut. Thus Widder proved that the solution to the infinite heat rod is unique among all *physically possible* solutions (i.e. those consistent with kinetic theory). However this is not totally satisfactory; after all, we did not have to restrict ourselves to "physically possible solutions" to establish uniqueness for finite length rods. And how did kinetic theory find its way into the heat equation?

Boundary Conditions for the Wave Equation

It is enlightening to compare this behavior with that of the wave equation system for, say, electromagnetic voltage *V*, whose one-dimensional homogeneous form reads

$$\frac{\partial^2 V}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 V}{\partial x^2}, \; V(0,x) = \frac{\partial}{\partial t}V(0,x) = 0 . \tag{13}$$

The general solution of the differential equation is the superposition of a waveform propagating to the left plus one propagating to the right, at speed *c* (*D'Alembert,* 1747):

$$V(t,x) = f(x+ct) + g(x-ct) . \tag{14}$$

This implies that the solution of the homogeneous system at every point *x* will remain zero until the nearest *non*zero initial disturbance reaches it, traveling at speed *c*; but there *is* no initial disturbance for (13), anywhere. So the homogeneous solution is identically zero; and as we have seen, that means the solution to the (nonhomogeneous) wave equation system is unique. (!)

Conclusions and Speculation

Why does the infinite homogeneous heat equation system have nonidentically zero solutions, but the homogeneous wave equation does not? Two fanciful observations have evolved to help us live with this dichotomy:

(*i*) Since voltage disturbances can propagate no faster than *c*, there is no electromagnetic disturbance *within range* of a point *x* to "rattle" it at any finite time. However, it can be argued (from Poisson's formula) that an isolated *thermal* disturbance produces a nonzero effect everywhere, instantaneously - if the Fourier heat equation is to be taken as gospel. (Of course this reveals that the heat equation is nonrelativistic.) (A popular quip notes that if both the heat and wave equations were accurate, then when we strike a match we would feel the heat before we see the light.) So we can imagine Tychonov's thermal storm sitting out there at infinity, waiting for the right moment *t*=0, and *instantly* rushing in.

(*ii*) As noted, no one will ever build an infinitely long heat rod. Big Bang theorists assure us that the number of fundamental particles in the universe is limited, so we'll run out of material before we get to infinity. Equation

(7) is only valid as an approximation for long, finite, rods. So cosmology has insulated us from the Tychonov storms at infinity. Now this is not as inane as it sounds; we radiate *electromagnetic* waves to infinity every time we turn on our car radios, and we don't need to build conductors to escort them. We have no cosmological savior to protect us from electromagnetic storms at infinity; the uniqueness theorem is our salvation. (Antenna specialists note: Sommerfeld's radiation condition, which banishes incoming waves located at finite distances, is not connected to the uniqueness theorem.)

In a slightly more serious vein: the electromagnetic wave equation is a rigorous mathematical consequence of the electrodynamic laws of Coulomb, Ampere, Gauss, Faraday, and Maxwell. But a detailed derivation of the heat equation is less "clean", invoking assumptions about statistical ensembles over atomic particles (Williams, 1985). It is conceivable that the germ of an absolute zero temperature is implicit in these deliberations - rendering Widder's condition as the most appropriate to censure the Tychonov solution.

At any rate, an exposition of the startling nature of Tychonov's discovery and its perceived significance - ranging from serious/academic to speculative/frivolous - should go far in resolving the most hardened engineering student's contempt for the mathematician's obsession with rigor.

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