Students' Learning Obstacles in Solving Early Algebra Problems: A Focus on Functional Thinking

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Abstract: This study describes students' learning obstacles in solving early algebra problems requiring functional thinking ability. To reach this aim, qualitative research was conducted in this study. Participants of this study were 39 ninth graders and a mathematics teacher at one of the lower secondary schools in Bandung, Indonesia. The data were collected through the written test about early algebra problems, interviews, and document study. The findings revealed that fewer students achieve the correspondence level in their functional thinking ability. Many of them are on covariation or recursive patterns level. The variety of students' functional thinking levels in solving the problem is influenced by their previous learning experiences with early algebra, mainly functions. By exploring students' learning experiences, this study shows that students have some learning obstacles, including ontogenic obstacles due to students' lack of prerequisite knowledge about the concept of variables, didactical obstacles due to the teacher's teaching implementation focusing solely on the operational rather than the structural conception of functions, and epistemological obstacles due to students' limited knowledge in the concept of variables and functions. Therefore, the identified learning obstacles can be one of the references when developing a lesson design about functions for enhancing students' functional thinking ability.

Keywords: Functional Thinking, Early Algebra, Functions, Learning Obstacle

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Introduction

Learning obstacle is a condition that makes the students' new knowledge acquisition during the learning process run slowly or restricted, allowing students to experience problems in learning. These problems can be seen in errors made by the students (Brousseau, 2006). Moreover, the notion of *learning obstacle* itself initiated by Bachelard & Piaget's discursion (in Brousseau, 2006) that mistakes conducted by students during learning are not solely sourced from themselves. However, mistakes can result from their prior knowledge, which seems right and proper to solve specific problems, but that knowledge becomes useless for solving another problem with the same characteristic. One example is Adu-Gyamfi et al.'s (2015) study, which reported that most students wrote *6s=p* instead of *6p=s* in the question "*the students' population is six times the professors*" because they thought variables as representing an object or abbreviation rather than quantities that vary. This kind of learning obstacle does not happen without *causal factors,* which Kansanen and Meri (1999) proposed can be observed based on the teacher-student-topic relationship.

According to its source, learning obstacles can be categorized into three types: ontogenic obstacle, didactical obstacle, and epistemological obstacle (Brousseau, 2006). The ontogenic obstacle occurs due to learning activity or the tasks given are incompatible with students' cognitive development. The task can be too difficult for the student to solve, or else it can be too easy to solve, causing them to lose the sense of being challenged. The didactical obstacle occurs due to the teacher's teaching sequences. Teachers might focus on the *know-how* aspect, such as ensuring that students can memorize formulas and problem-solving procedures, and pay less attention to the *know-why* aspect, which fosters students' reasoning. The epistemological obstacle can be identified when students' knowledge is limited to a particular context. All these obstacles can happen during the student's learning process, especially when they learn about algebra. This study focuses on early algebra since it is considered a well-known topic that causes many student problems (Kieran et al., 2016).

The Background of Study

Learning arithmetic and algebra separately can cause students to experience difficulties in developing their cognitive process from concrete to abstract (Carraher & Schliemann, 2007; Kieran et al., 2016). This issue makes the notion of early algebra learning within school mathematics noteworthy among researchers. Early algebra is not *the formal algebra* but more like a bridging topic for students, mostly in elementary school, to prepare them to learn the formal one. According to Kaput (2008), the main aspect of early algebra is generalization and symbolization. From Kaput's idea, Blanton et al. (2018) proposed the math content area for early algebra: generalized arithmetic; equivalences, expressions, equations, and inequalities; and functional thinking. This study will focus on functional thinking as it is the bridge for students to learn and understand functions.

Functional thinking (FT) is an individual cognitive process to generalize functional relations between varying quantities (variables) in mathematics (Lichti & Roth, 2018). Students' FT can be examined based on how they identify a relation between quantities, which is categorized into three levels (Confrey & Smith, 1994; Doorman et al., 2012; Smith, 2008). The lowest level is *recursive patterns*; students see the quantities' relation as the input-output process. Following this process, the students may identify that the value change in a variable influences the value change of the other variable. This level of thinking is included in *covariation*. At the end of the students' generalization process, they can determine the relationship between variables in general (applies to any value of the existing variables). This process involves the highest level of thinking, namely *correspondence*.

In mathematics learning, FT is essential as one of the core abilities students require to master. Since elementary school, FT has been involved in learning object configurations and number patterns (Kaput & Blanton, 2005; National Council of Teachers of Mathematics [NCTM], 2000). The learning continues to high school about linear and non-linear functions and calculus for higher education (NCTM, 2000).

Although the importance of FT is often discussed in the literature, many studies revealed students' challenges in solving problems requiring FT (Pinto & Cañadas, 2021; Ramírez et al., 2022). The frequent issue in studies is students' inability to recognize a pattern, whether in a sequence of numbers or object configurations (El Mouhayar, 2018; Wilkie & Clarke, 2016). Students who have better FT ability often use natural language to express the functional relation between variables due to a lack of understanding of the variable's notation (Ayala-Altamirano & Molina, 2020; Lucariello et al., 2014; Wilkie, 2016a). In FT, and therefore in functions, students ought to understand that a variable represents a *varying quantity* (Doorman et al., 2012; Kleiner, 1989). However, the students' restricted image of a variable is commonly influenced by their prior learning experiences, for instance, in linear equations, which hold an understanding that a variable is *the unknown* (representing a single quantity).

Following these issues, some authors try to analyze further how students work on problems that require FT, particularly through the early algebra lesson. Many of them emphasize their study on the exploration of students' FT in generalizing geometrical shapes and object configurations (Pinto & Cañadas, 2021; Ramírez et al., 2022; Wilkie, 2016a) and word problems (Ayala-Altamirano & Molina, 2020; Blanton et al., 2017; Pinto et al., 2022). Moreover, some studies focus on designing the early algebra lesson to foster FT in elementary school students (Stephens et al., 2017; Wilkie, 2016b). Although these studies significantly contributed to mathematics learning, few addressed the FT on secondary school students. İndeed, early algebra is intended to be introduced and has been implemented at the elementary school level in some developed countries (Pinto & Cañadas, 2021; Watanabe, 2011). Nevertheless, early algebra is not yet presented in Indonesian elementary school mathematics, resulting in students lacking preparation in learning formal algebra in high school. Another research gap is that existing studies focus their analysis on students' FT, with less attention to the sources of why they only reach a certain level of FT. We regard the sources as learning obstacles.

Considering the essential role of FT in mathematics learning, students' difficulties with FT, and the research gap, there is a need to conduct a study to investigate high school students' FT further. Therefore, this study aims to describe secondary school students' learning obstacles in solving early algebra problems whose solutions require FT ability. Two questions are addressed in this study: 1) How is the students' FT level in solving early algebra problems? And 2) How are the learning obstacles experienced by the students in solving the problems? Analyzing these aspects of students' FT can aid other researchers, curriculum developers, and teachers. For researchers, the recent study might provide a new perspective on different learning obstacles that students experienced in early algebra learning, focusing on FT. For curriculum developers, it gives suggestions on how early algebra should be presented for curriculum design. For teachers, it provides suggestions to teach their students about early algebra and perhaps more broadly.

Method

Research Design

The purpose of this study was to investigate students' learning obstacles in solving early algebra problems whose solution requires FT. Therefore, this study was conducted under the interpretive paradigm with a qualitative method. By tracing students' prior learning experiences in early algebra, particularly algebra as the study of functions, the study aimed to decipher all of the meanings of what causes them to encounter learning obstacles.

Participant and Data Collection

The participants in this study were 39 ninth-grade students and a mathematics teacher, and the concept of functions was the subject under investigation. The selection of this topic was driven by the absence of explicit early algebra lessons in the Indonesian curriculum. Typically, formal algebra is taught in junior high school. Nonetheless, we selected the concept of functions as a part of early algebra because it is related to the FT.

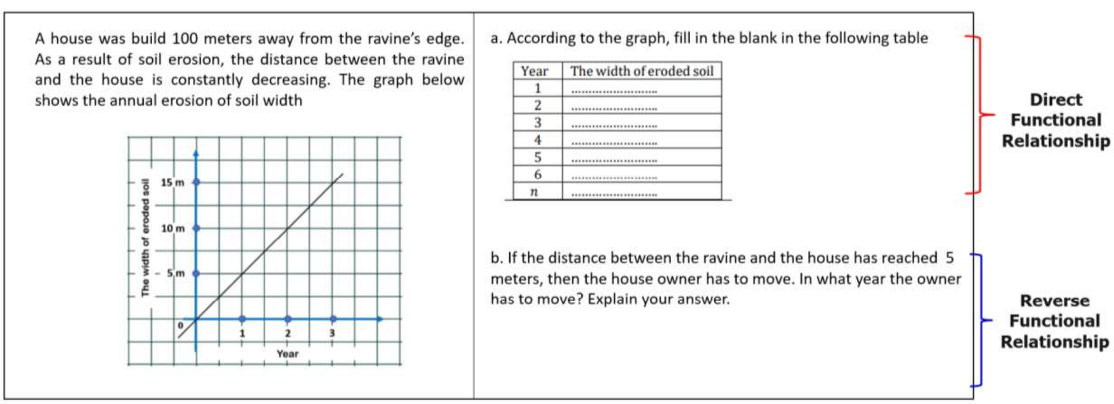


Figure 1. The Example of Problems in the Student's Written Test

Furthermore, written tests, interviews, and documentation studies were used to collect data for this study. The written examination consists of three problems in early algebra focusing on FT. Each question consists of two parts: (a) students are asked to generalize a pattern (direct functional relationship), and (b) students are asked to determine a specific value from a given output (reverse functional relationship). Specifically, each question was designed based on all levels of FT, so students answering the same question could reveal different levels of FT. Figure 1 depicts an example of questions given to the student. In addition, the interview included ten representative students and the math instructor. A documentation study was conducted to analyze the teaching and learning documentation, such as the class's mathematics textbook.

Data Analysis

The data analysis used qualitative data analysis techniques containing three stages: preparing, managing, and interpreting the data.

Preparing the data

At the stage of primary data analysis, all collected information was gathered for analysis. The data prepared for analysis in this study are the results of students' written tests, students' and the teacher's interview transcripts, and a documentation study on the series of mathematical textbook tasks.

Managing the data

We attempted to analyze the data from the written tests of the students. We categorized the students' written test responses based on FT levels in each type of problem (direct and reverse functional relationships) as the point in obtaining additional information about their learning obstacles. The interview results of the students were also classified based on their FT levels. It was meant to explore more about why the student chose a particular strategy to solve the problem. The results of the teacher's interview regarding how he designed and implemented the learning of the concept of function were also included, as the purpose of this study was also to determine the impact of students' prior learning experiences. Since the teacher stated that the majority of students' learning paths for the concept of function were based on the textbook, the analysis of tasks presented in the math textbook on the related topic was also considered.

Interpreting the data

The final stage of analysis is data interpretation. This stage was done by matching each data with a connection or an explanation of the other data. For instance, the teacher's interview supported the outcome of a student's interview regarding their previous learning activity in class, which explained the rationale for their chosen strategy in problem-solving. At this stage of the analysis, all interconnected data will be interpreted and categorized based on three types of learning obstacles experienced by students: ontogenic, didactical, and epistemological.

Results

The results of this study will be presented in accordance with the proposed research questions from the previous section. First, this section will discuss the students' FT level in solving early algebra problems. Following that discussion, the students' learning obstacles identified in solving the problems will be explained.

Students' Functional Thinking in Solving Early Algebra Problems

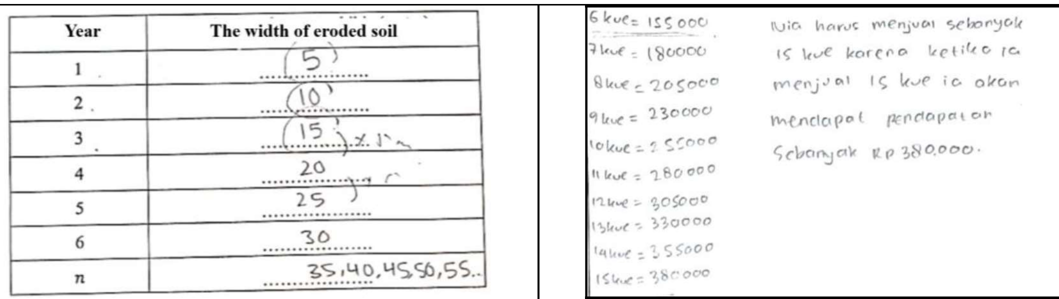
In investigating students' FT, they were asked to solve three problems which implicitly used three different types of functions, that is *y=ax* (question number 1), *y=ax+b* (question number 2), and *y=ax-b* (question number 3. Each question consists of two parts, namely direct and reverse functional relationships (FR). From 39 participating students, the result shows that most students are in the recursive level of FT. The description of students' written test and FT level categorizations are described in Table 1 as follows.

Table 1. Description of Students' FT in Solving Early Algebra Problems

Problem	Type of function	Type of functional relationship (FR)	Num. of students evidenced for FT (Total students: 39)			Num. of students who are not evidenced for FT
			Recursive Patterns	Covariation	Correspondence	
1.	*y=ax*	Direct FR	32	-	7	-
		Reverse FR	6	3	7	23
2.	*y=ax+b*	Direct FR	35	-	3	1
		Reverse FR	23	3	2	11
3.	*y=ax-b*	Direct FR	20	-	5	14
		Reverse FR	8	4	3	24

Table 1 shows that students' way of solving all three problems is categorized into the recursive patterns level in FT, followed by the correspondence and the covariation. In addition, we have observed that the number of students who solve direct FR problems is always greater than the number of students who solve reverse FR problems. These results indicate that students found it more difficult to solve reverse FR problems than direct ones. A further explanation of how students solve the direct FR (generalize the patterns and determine the formula) and solve the reverse FR problems are described below.

Students who demonstrate recursive pattern level in FT always use a similar method to solve problems, according to our investigation. In order to solve direct FR questions, they determined the value of each dependent variable by adding the same number to the previous term's value. Similarly, recursive patterns level students also count the number in consecutive terms when solving reverse FR problems. Figure 2 below illustrates examples of student responses at this level



(a) (b)

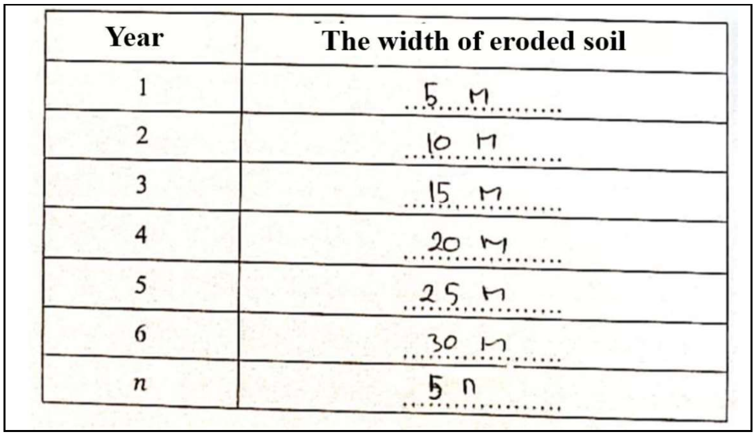
Figure 2. Examples of Students' Answers in Recursive Pattern Level: Direct FR (a) and Reverse FR (b)

In Figure 2a, students could find the width of eroded soil in the fourth, fifth, and sixth years by adding 5 meters to the previous year. As a result, students wrote 35, 40, 45, and so on to indicate the width in the nth year instead of 5n. In solving reverse FR (Figure 2b), similar to solving direct FR, the student listed one by one until he found that profit would reach 380,000 rupiahs if the seller sold 15 cakes (*kue* in Indonesian).

Furthermore, students that reach the covariation level of FT can be evidenced by their way of solving the reverse FR problems. In their strategy of finding the independent variable value from the given dependent variable value, they used trial-and-error as their solution. For instance, in Figure 3, to find how many days are needed for the boy to save 130,000 rupiahs money, the students chose to *guess* the amount of money collected on certain days. Although this strategy is also valid, their answer indicates that they cannot reverse the relationship and opt for working from the input to the output (direct FR). In Figure 3, the function type is $y=ax+b$, and the students cannot do the reverse calculation, $x=(y-b)/a$.

Figure 3. An Example of The Student's Answer in Covariation Level (Hari means day in Indonesian)

Students with the highest level of FT, namely the correspondence, managed to identify the relationship between independent and dependent variables, symbolize the relationship, and do the reverse calculation. In Figure 4 below, students with a correspondence level of FT can determine the width in the nth year as *5n* instead of stating the exact value in Figure 2a. They understand that the width is 5 times a year and not merely *add 5* to each width. Likewise, in the reverse problem (Figure 1), the students understand that if the width is 5 times a year, then the year is one-fifth the width. So, they used the formula instead of listing one by one to find the year when the width reached 95 meters.



Year	The width of eroded soil
1	5 m
2	10 m
3	15 m
4	20 m
5	25 m
6	30 m
n	5 n

Figure 4. An Example of The Student's Answer in Correspondence Level

Nevertheless, not all students who solve direct FR problems with correspondence can solve reverse FR problems at the same level. Despite being categorized with the correspondence level in solving direct FR, some only reach the covariation or recursive patterns level in solving reverse FR ones.

Students' Learning Obstacles in Solving Early Algebra Problems with A Focus on Functional Thinking

Based on the previous section, this study highlights two significant issues that students deal with when solving early algebra problems. Those three issues came from students who (1) applied the recursive pattern strategy to solve direct FR problems, thus, did not make it to the algebraic expression of the generalization, and (2) managed to work on direct FR problems but not with reverse FR ones. As a further investigation, this study explains the reasons for the three existing issues. This study considers the reasons as students' learning obstacles (LO).

As the primary stage of students' LO identification, the researcher (R) interviewed two representative students with the first issue. One student (S1) guessed the n symbol as the pattern, while the other one (S2) worked on the n symbol as the next term. The interview of both students is transcribed as follows.

Transcript 1: The interview with S1

R : When you solve question 1A, can you see the pattern in this problem?

S1 : Yes, it is 5 meters (*the width of eroded soil always increases by 5 meters per year*)

R : Right. Can you tell me how you find the width in the fourth until the sixth year?

S1 : I just added 5 to the previous width.

R : Do you understand the meaning of the nth year here?

S1 : I do not miss.

R : Okay, what do you mean 5 in the width for the *n*th year you wrote here?

S1 : I just thought *n* was the pattern, so I wrote 5.

Transcript 2: The interview with S2

R : Okay, when you solve question 1A, can you see the pattern?

S2 : Yes, the width always adds 5 meters per year.

R : Good. How can you find the width in the fourth until the sixth year?

S2 : Adding 5, so 20 is from 15 plus 5, 25 is 20 plus 5, 30 is 25 plus 5.

R : Do you understand the meaning of the nth year here?

S2 : It is the width of the following year.

R : So, you mean that n is the width of the seventh year? That is why you wrote 35?

S2 : Yes.

According to the interview transcribed above, there are two ways of students' understanding about the *n* letter. First, the student's answer in Transcript 1 indicates that neither S1 knows about the letter *n* referring to nor does S1 know that *n* is a variable. The student's lack of understanding of variables resulting S1 producing incomplete solutions. The fact that the problem requires the student to have a sufficient understanding of the variable, which S1 has not, according to Brousseau (2006), is categorized as having an *ontogenic obstacle*. Unlike S1, who does not know what the *n* stands for, S2 sees the *n* as a single value. Thus, S2 has a limited understanding of a variable representing a *specific unknown* but not with *things that vary*. According to Brousseau (2006), having a restricted image of a concept that is only useful for specific problems but not for others is categorized as having an *epistemological obstacle*.

Furthermore, the researcher managed to dig for more explanation from a representative student (S3) with the second issue (solved direct FR but did not solve reverse FR problems. An example of an interview transcript with S3 is described below.

Transcript 3: The interview with S3

R : Do you understand the problem in question 1B?

S3 : No, Miss.

R : Okay, look. The starting distance between the ravine and the house is 100 meters. In this question, the distance becomes 5 meters. How much is the width of eroded soil?

S3 : It is 95 meters, Miss.

R : How can you find the year when the width reaches 95 meters?

S3 : Just add 5 meters one by one.

According to the interview, S3 did not understand what to do with the reverse FR problem on his first attempt. However, with a bit of help from the researcher, S3 solved the problem. Moreover, notice that even S3 make it to solve the reverse FR problem; he treats the reverse FR like the direct one. Instead of using the shorter way, namely, 95 divided by 5, he chooses to add 5 in consecutive terms until he finds 95. According to Brousseau (2006), this type of student who has the inability to do the inverse operation of the variable relationships indicates that he has an *ontogenic obstacle*.

Besides the finding of two types of students' learning obstacles, namely ontogenic and epistemological, how students solve the problem might be influenced by their previous learning activity. The students' FT is highly related to how they construct and understand the concept of function. Therefore, this study seeks to investigate how students previously learned this topic by interviewing the teacher and analyzing the tasks given to the students during the learning. The researcher conducted an interview with the students to obtain information about the learning implementation of the function's concept. The interview is shown as follows.

Transcript 4: The interview with the teacher (T)

R : How is the learning process in the concept of function before?

T : At that time, we still implemented online learning due to Covid-19. Consequently, not all lessons about the function were learned by the students. For the concept of function, the students were only expected to know the function's definition, identify function and non-function, represent a function with a Venn diagram, an algebraic symbol $f(x)=y$, and find the image of a function from a given function's equation.

R : Okay. Also, if you introduced the students to $f(x)=y$ as the function notation, did you mention using other letters as the variables?

T : We did not have much time, so I focused on using $f(x)=y$ as the formal way to represent the function.

R : You said that the students learned about finding the image of a function. Did you also mention how to find the x when $f(x)$ is given?

T : No, only until the image of a function.

The interview result with the teacher gives more insight into the possible reasons why several students have ontogenic and epistemological obstacles. According to Transcript 4, three main points should be underlined regarding the prior learning implementation: the learning of function focuses on definition and representations, the students are only taught $f(x)=y$ as the only algebraic symbol to express a function, and the learning ends with finding the function's image.

These points obtained from the teacher's interview come up with different connections with the identified LO of the students. As the result of learning the function's symbol $f(x)=y$ as the only way to represent a function algebraically, the students could not understand n as a variable in direct FR problems (ontogenic) and had a restricted image of n as a specific unknown, not a variable (epistemological). Since the teacher only taught how to find the image of function (direct FR), it is understandable that students encountered greater difficulty when solving problems involving reverse FR (ontogenic). Students' limitation in understanding the function concept correctly because of the teaching material implemented by the teacher, according to Brousseau (2006), is categorized as a *didactical obstacle*. Nevertheless, this teaching and learning limitation happens due to the pandemic, where the teacher has no other option but to conduct the learning activity in a short time frame.

Furthermore, this study also analyzes the tasks used during the learning activity to help students construct knowledge about the function's concept. According to the teacher's interview, the student's mathematics textbook is the primary source for learning the concept. Therefore, the student's learning obstacle analysis proceeds to the series of tasks presented by the textbook in function.

Based on the analysis, the textbook begins the introduction of function by its definition in terms of pairing between sets: *a function from Set A to B is a unique relation that pairs each element in Set A to only one element in Set B.* Following this definition, the textbook presents tasks that help students learn different representations of a function. However, we identify that the textbook does not provide connections between the representations, that is, to understand that a function is an object whose values remain the same (across its domain) despite its representation changes. In this case, the textbook does not provide how the operational conception of function (as the input-output process) can lead to the structural conception (function as an object). The missing task to link these dual concepts of a function is depicted in Figure 5.

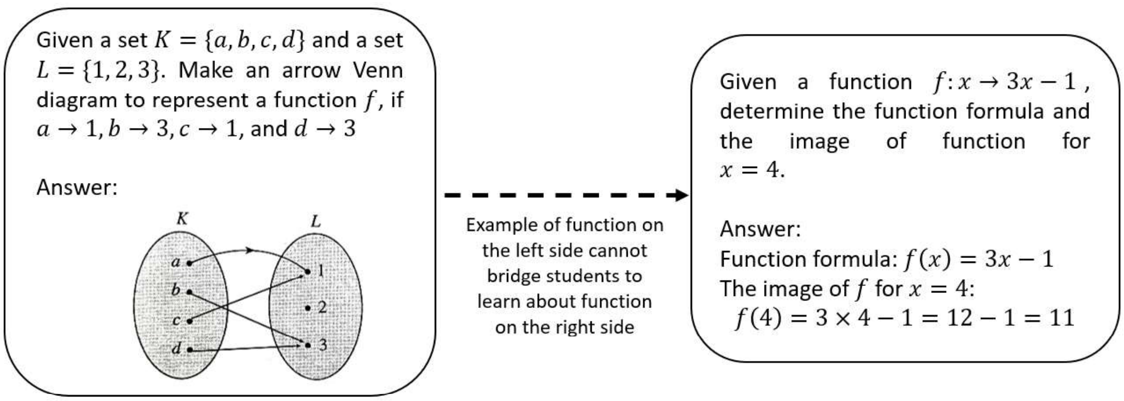


Figure 5. The Examples of Tasks in the Textbook

In Figure 5, the task on the left only presents a function as pairings between sets while the task on the right side presents a function in analytical expression (algebraic). The missing link between the left and the right tasks is the function rule, which can be used to determine the image of the function. The left task does not help students to grasp an understanding that a function expresses a generality. This understanding is needed for them to solve the right task. Nevertheless, the absence of tasks linking the left and the right side might result in the students having a limited understanding of the meaning of the algebraic representation of a function. This finding supports our previous investigation about students' epistemological, ontological, and didactical obstacles; that is, all the learning obstacles experienced by the students are sourced from the textbook they use during the learning. Students' limited understanding of the function's concept due to the tasks within the textbook, which does not consider their cognitive development progression, based on Brousseau (2006), is categorized into the *epistemological obstacle.*

To conclude, Figure 6 shows the linkage between learning obstacles experienced by the students in solving the early algebra problems with a focus on FT.

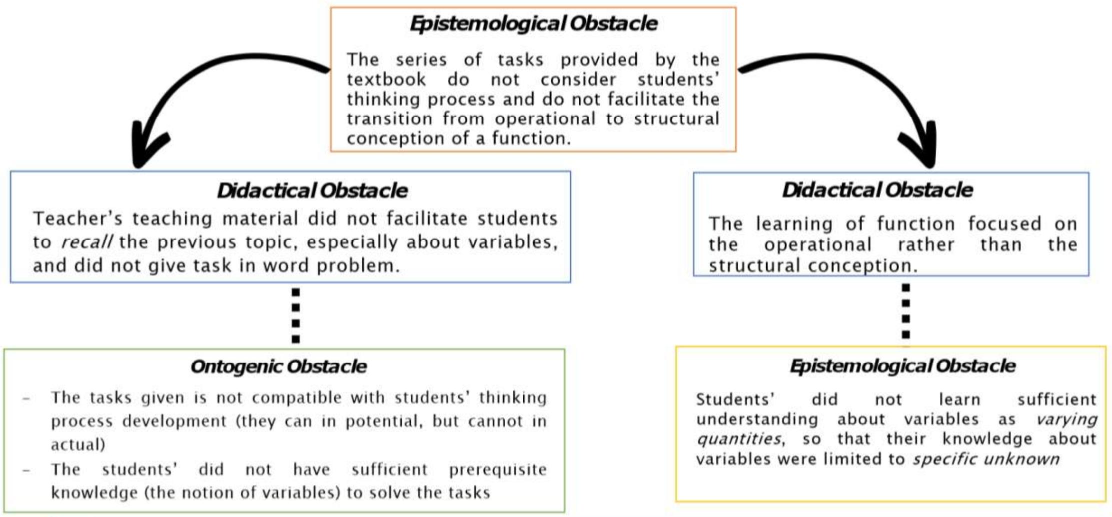


Figure 6. Students' Learning Obstacles in Solving Early Algebra Problems with a Focus on FT

Discussion

This paper refers to the FT's level proposed by Doorman et al. (2012), Smith (2008), and Wilkie (2014), namely *recursive patterns, covariation, and correspondence.* According to the finding, students who worked on the problems (direct or reverse FR) with the correspondence strategy managed to understand algebra, as a study of function, as a mathematical object (Doorman et al., 2012). Nonetheless, few of these students can symbolize the generalization due to their limitedness in identifying the patterns in word problems. This study's result complements prior research that generalizing patterns from word problems are more challenging than object configurations (Pinto & Cañadas, 2021; Ramírez et al., 2022; Wilkie & Clarke, 2016).

Unlike students who make it to the correspondence level, students with recursive pattern levels can only identify a limited variation within a sequence of numbers or objects. Doorman et al. (2012) and Lichti and Roth (2018) associate the recursive patterns level with the input-output assignment aspect, the student's ability to pair each element in the domain to only one element in the codomain; however, this pairing is local. Moreover, in identifying the pattern, previous studies explain the finding of this research that students with recursive patterns level focus on the values' change of one variable without paying attention that the values of both variables change simultaneously (Pinto & Cañadas, 2021; Wilkie, 2016a).

Furthermore, students with the correspondence level solved the reverse FR problems by performing Polya's (2004) working backward strategy. In working backwards, the student determines the primary goal and then starts working backwards to the initial condition. This problem-solving strategy suits the strategy to solve reverse FR problems by knowing the goal (the given value of the dependent variable) and beginning the reverse operation to find the value of the independent variable.

On the other hand, students who reached the recursive patterns and covariation level of FT did not work with the working backward strategy in solving reverse FR problems. Students with the recursive patterns level tend to solve the problem by repeating addition from y_1 to y_n (y_n is the given value of the dependent variable). When they find the y_n, they need to see which x's value (the independent variable) corresponds to the y_n. Another strategy was conducted by the covariation level students. They apply the trial-and-error strategy to find the independent variable's value: if x_1, then y_1; if x_2, then y_2. Indeed, this kind of strategy is often performed by students in solving problems related to pattern generalization (Malisani & Spagnolo, 2009; Radford, 2010).

Moreover, this study also reports that students find solving the reverse FR problems more challenging than the direct FR ones. According to the study of Callejo and Zapatera (2017), elementary school students found difficulties in solving reverse problems, even though they met no challenges in solving the direct ones. In fact, the study of Wilkie (2016a) that was conducted a year before revealed that even solving the reverse problems still exists as an issue for secondary school students. Therefore, this study complements the previous research that in solving reverse FR problems, not only do students complicate it, but also they tend to downgrade their FT level from solving the direct FR problems. For instance, some students show the corresponding level in solving direct FR but only reach the recursive patterns level in solving reverse ones.

This study also put efforts into investigating the possible students' learning obstacles. We refer to the three learning obstacles stated by Brousseau (2006), namely ontogenic, didactical, and epistemological. According to the finding, two main issues experienced by the students are categorized into having ontogenic obstacles. First, their insufficient understanding of variables affected their strategy in solving the problems given; they recursively determined the value of each term without being able to represent the generality with algebraic symbols. Second, their unfamiliarity towards working on a reverse operation to the functional relationship between variables. Based on Brousseau (2006), students with ontogenic obstacles encounter an imbalance between the given task and their existing knowledge or cognitive development.

In addition to students' ontogenic obstacles, students with the epistemological obstacle, who reach the recursive patterns or covariation level, have a restricted image of variables. Küchemann (1978) and Usiskin (1988) stated that judging from its usefulness, a variable exhibits different meanings: generalized number, specific unknown, and varying quantities. In a function, students should understand a variable representing varying quantities. However, the students have not yet reached this level of comprehension and still depend on a variable representing a specific unknown. Likewise, some students consider x the only letter to denote a variable and have difficulties when the problem requires them to use n as the variable. These findings support the prior study, which reported that the number of students who obtain knowledge of a variable as varying quantities is low, and the students' possibility of denoting a variable with x might be caused by their unfamiliarity with using another letter except x (Sajka, 2003; Trigueros & Ursini, 1999). According to Brousseau (2006), students with epistemological obstacles construct a limited understanding of knowledge, which may be affected by their previous learning experiences, so their obtained knowledge is only helpful for specific problems.

Relating to the students' ontogenic and epistemological obstacles, this study also reveals that during the learning process in the function's concept, the teacher only introduces a function through mathematical problems with less attention to the word problems. Meanwhile, according to Wilkie (2016b), helping students to learn about functions and nurturing their functional thinking can be done by teachers through giving contextual problems. Moreover, this study also reports that learning functions emphasize the operational rather than the structural conception of a function. However, understanding a function as an analytical expression $f(x)=y$ requires students to grasp both conceptions; operational as finding the image of function (y) from specific x within the domain, and structural as knowing the formula $f(x)=y$ is valid for every x in the domain (Sfard, 1991). Thus, students' obstacles caused by their previous learning experiences, which are less effective in helping them acquire the attained knowledge, is called didactical obstacle (Brousseau, 2006).

Finally, this study reveals the primary source of all students' learning obstacles discussed above. By analyzing the series of tasks presented in the textbook used during the learning activity, this study reveals that the order of tasks within the textbook does not present connectivity between tasks and students' thinking processes. The tasks do not bridge the students to construct knowledge about functions in which algebraic symbols can represent the rule. Thus, students' restricted image of a function caused by the series of tasks provided by the textbook, according to Brousseau (2006), is called an epistemological obstacle. The tasks within textbooks are essential in shaping students' knowledge since the knowledge they learn is highly dependent on the series of tasks given to them (Fitriati et al., 2020; Henningsen & Stein, 1997; Hiebert & Wearne, 1993).

Conclusion

This study provides evidence of lower secondary school students' FT in solving early algebra problems and the types of learning obstacles they experience when attempting to solve these problems. Comparing the covariation and recursive pattern levels, the findings revealed that only a small percentage of students have already attained the correspondence level. Due to the fact that the presented problem is a word problem, this study also revealed that students find it more difficult to generalize patterns in word problems than in object configurations or geometrical shapes. This study also determined how the type of functional relationships affected students' ability to solve problems. According to the test answers, direct functional relationship problems are easier to solve than reverse ones.

Furthermore, three types of students' learning obstacles were identified. The ontogenic obstacle experienced by the students as a result of the assigned tasks is incompatible with their existing knowledge, both in the notion of variables and the concept of functions. The didactical obstacle is found due to the teacher's teaching material and implementation of the concept of function, which did not focus on the role of variables in functions and only attained the operational rather than structural conception of a function. The epistemological obstacle is identified based on the series of tasks in the students' textbook as their primary sources of learning, which failed to take into account the students' thought process during the transition from an operational to a structural conception of a function. Existing studies have made significant contributions to the examination of students' FT in different types of problems; however, this study explains why students only attain a certain level of FT by analyzing their learning obstacles.

Recommendations

Based on these findings, we recommend that the development of students' FT be taken into account when designing early algebra learning activities for students. Similarly, considering the students' learning obstacles in that topic should be taken into consideration in the learning design so that the design can overcome those obstacles and meet the student's needs.

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