EXPERIENCING EQUIVALENCE WITH GRASPABLE MATH: RESULTS FROM A MIDDLE-SCHOOL STUDY

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Understanding equivalence is fundamental to STEM disciplines, but many students struggle with the concept. We present a novel method for students to explore ideas of equivalence where students dynamically transform expressions from initial states to goal states in the web-based app Graspable Math. The structure of the goal state activities implies that any initial and goal state pair represent the same quantity (or varying quantity). We propose that for students, the physical experience of moving algebraic objects and observing how the initial state transforms into the goal state helps generalize notation mechanics. In fall of 2019, we will test this supposition in a randomized control trial of 1500 students in which student performance in pre- and post-tests will be compared to an online problem set control.

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Student misunderstandings or misconceptions about the equivalence and the equals sign have been noted as inhibiting success in upper-level mathematics and other STEM disciplines (Kieran, 2007; Knuth, Stephens, McNeil, & Alibali, 2006; Stephens, Knuth, Blanton, Isler, Gardiner, & Marum, 2013). For example, a common misconception is when students view the equals sign as marking or calling for a computation, such as interpreting “4+1=5” as “four and one makes five.” This type of operational understanding of the equals sign has been found to be stable in middle school students (Alibali, Knuth, Hattikudur, McNeil, & Stephens, 2007) and is associated with difficulty in equation solving, even when controlled for grade level and standardized mathematics test scores (Knuth, et al., 2006). A robust perspective, in contrast, is what Stephens and colleagues (2013) describe as a *relational-structural* view of the equals sign - understanding that the equals sign expresses an equivalence relation between the two expressions on either side.

Much of the research on students’ understanding of equivalence is embedded in work on students’ understanding of the equals sign (Blanton, Stephens, Knuth, Gardiner, Isler, 2015; Kieran, 1981, 1992, 2007; Knuth, et al., 2006; Leavy, Hourigan, & McMahon, 2013; Rittle-Johnson, Matthews, Taylor, & McEldoon, 2011). As a case in point, Rittle-Johnson and colleagues (2011) talk about the distinction between numerical equivalence - the ability to match sets of objects on the basis of quantity, and mathematical equivalence - “understanding that the values on either side of the equal[s] sign are the same,” (Rittle-Johnson et al., 2011, p. 86). Kieran describes this formally as “the symmetric and transitive character of equality,” and informally as “the ‘left-right equivalence’ of the equal[s] sign,” (1992, p. 398).

Our work takes a different tactic by having students work through what we call *goal state* activities using the dynamic notation tool, Graspable Math (GM). In GM, algebraic objects (terms, expressions, and equations) operate according to the mechanics of symbolic notation. For example, permissible moves (such as adding $2x$ and $4x$) transform an expression or equation, while impermissible moves (such as adding $2x$ and $4y$) do not. From a perceptual learning perspective, students' experience of moving algebraic objects on the screen, reinforced by the visual feedback of transformed expressions, helps students to generalize notation mechanics and attend to relevant details. In goal state activities, students are presented with two equivalent states: an initial state and an equivalent goal state (See Figure 1). The students' task is to transform the initial expression into the goal state. These transformations can only happen through algebraically permissible actions. Thus, students are finding the road map of equivalence between the initial state and the goal state.

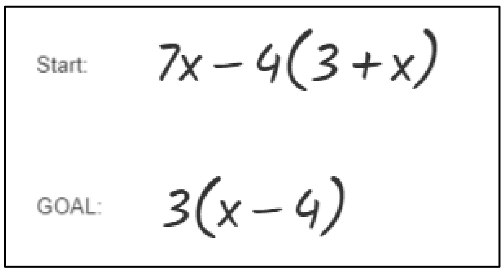

Figure 1: An Example of a Goal State Activity in Graspable Math

In fall of 2019, we will run a randomized control trial of 1500 students in 2 intervention conditions: a GM goal state condition and an online problem set control. Student performance in pre- and post-tests will be compared to explore how GM goal state work might be associated with students' performance on equivalence-related tasks to address the research question, "How does experience with a transformation-based intervention affect student performance on equivalence items compared to a control of traditional online problem set?"

Theoretical Framework

The operational-relational dichotomy in students' perspectives of the equals sign is well-documented in the literature (Blanton et al., 2015; Carpenter, Franke, & Levi, 2003; Kieran, 1992; Knuth et al., 2006; Leavy, Hourigan, & McMahon, 2013; McNeil, Grandau, Knuth, Alibali, Stephens, Krill, 2006). Stephens, et al. (2013) adds nuance to that discussion by differentiating between a *relational-computational* view, where students understand that two sides of the equals sign calculate to the same value, and a *relational-structural* view, where students understand that the equals sign expresses an equivalence relation between the two expressions on either side of the equals sign. This subtle difference is tied to a structural understanding of algebra (Kieran, 2007), where "[students] can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects" (CCSS.Math.Practice.MP7 http://www.corestandards.org/Math/Practice/).

Landy, Allen, and Zednik (2014) proposed that sense making of symbolic notation happens through perceptual and sensorimotor systems. If the capacity for symbolic reasoning is in part the ability to "perceptually group, detect symmetry in, and otherwise perceptually organize symbolic notations," (Landy, Allen, & Zednik, 2014, p. 1), part of the algebraic objectification described above could be perceptual. In other words, since "what students notice mathematically has consequences for their subsequent reasoning” (Lobato, Hohensee, & Rhodehamel, 2013, p. 809), perception of notation is part and parcel of symbolic reasoning (Goldstone, Marghetis, Weitnauer, Ottmar & Landy, 2017). Mathematical fluency, therefore, derives not only from understanding the content, but also a heightened focus on relevant perceptual details. Perceptual learning theory suggests that training one’s perceptual and sensorimotor systems in symbolic notation may result in more effective reasoning about the relationships represented by the symbols (Kellman, Massey, & Son, 2010).

Grounded in this theory, GM creates a learning environment where algebraic objects behave according to the mechanics of symbol manipulation in a virtual environment. By making moves on expressions or equations and observing the system response, generalized mathematical properties such as commutativity, distribution, and order of operations, and simplifying expressions or solving equations becomes experiential for students.

For example, imagine a situation where a student is working with the expression “3 + 8 + 4x” (Figure 2). In the first row, the student moves the “8” to the right and drops it past the “4x,” experiencing the commutative property by seeing the resulting “3 + 4x + 8.” In the second row, the student moves the “8” to the left and drops it on top of the “3.” The terms combine, resulting in the expression “11 + 4x.” Thus, the student experiences addition. In third row, the student drops the “8” on top of “4x,” not a permissible operation since “8 + 4x” is already in simplest form, and the “8” snaps back to its initial location. In each case, the program responds to student actions, and the student has immediate feedback on the impact of their actions. Other gestures beyond selecting and moving allow users to enact most forms of symbolic manipulation including each of the four basic operations, decomposition, distribution, and factoring.

Possible actions on “3+8+4*x*”	Final State in GM	Mathematical Description
move “8” to the right	$3+4x+8$	Commute 8 and 4*x*
move “8” on top of “3”	$11+4x$	Add 3 and 8
move “8” on top of “4*x*”	$3+8+4x$	Not a permissible move (“8+4*x*” shakes)

Figure 2: GM Actions and Response on the Expression “3+8+4*x*”

Research Methods

For the purposes of the 2019 PME North American conference, we will be presenting data from a classroom study to be run in fall of 2019. This study involves a student-level randomized trial of 2 conditions, using the ASSISTments platform (Heffernan & Heffernan, 2014) to maintain condition assignments for each student to either self-paced GM goal state problems or online problem set. The study itself is 3 hours of content over 6 weekly sessions covering the four operations, fractions, and order of operations. Students in the GM tutorials will solve goal state items, while the problem set control will work through similar content items compiled from three open-source mathematics curricula: Engage NY (2014), Utah Math Project (2016), and Illustrative Math (2017). A sample of questions from the control condition is in Figure 3.

Student participants in the pilot will include approximately 1500 middle school students from 50 to 100 classrooms across a large, urban district in the southwestern United States. Participant schools have student populations comprised of 20% to 50% English Language Learners and over

75% identified high needs students, including but not limited to such characteristics of low income, limited English proficiency, and identified learning disabilities. Mathematical performance assessment items are drawn from the Rittle-Johnson, Matthews, Taylor, and McEldoon (2011) and Rittle-Johnson, Star, & Durkin, (2016) assessments of mathematical achievement.

1. Fill in the blank. ____ + $\boldsymbol{g} - \boldsymbol{g} = \boldsymbol{k}$ (adapted from Engage New York)
2. Select **all** the expressions that are equivalent to 4b. a) b+b+b+b b) b+4 c) 2b+2b d) b*b*b*b e) b÷4 (adapted from Illustrative Math)
3. Without solving completely, determine the number of solutions of the equation $3(m - 3) = 3m - 9$ a) No solution b) one solution c) infinitely many solutions (adapted from Utah Middle School Math Project)

Figure 3: Sample Items from Online Problem Set Control

Approach to Analysis

In line with research question, "How does experience with transformation-based intervention affect student performance on equivalence items compared to a control of traditional online problem set?" we will compare post-performance on mathematical equivalence items from the two conditions. We hypothesize that the immediate, experiential feedback provided in GM heightens student awareness of the mechanics of algebraic notation and is an aide to students' generalizing those mechanics. The primary analysis testing GM for improving math achievement will use linear regressions to estimate mean posttest equivalence scores between students in the GM and the control condition, controlling for pretest scores. We will examine whether GM is more effective for students with lower prior knowledge by testing the condition × pretest interaction. We expect that GM will improve students' knowledge of equivalency more than the control condition, and the effects of condition will emerge in procedural fluency, conceptual understanding, and mathematical flexibility items. This effect is hypothesized to be larger for lower performing students.

Conclusion

Despite major efforts in research, curricula development, and policy, students still struggle with understanding equivalence and the equals sign. Our session presents a novel student experience with equivalence: transforming expressions and equations dynamically and explicitly within a notation environment. Positive results from prior work using a similar arrangement of multiple sessions using a gamified version of GM has shown that it may be effective for decreasing notation errors and improving mathematical understanding (Daigle, Harrison, Braith, Ottmar, Hulse, & Manzo, 2019; Ottmar & Landy, 2017; Ottmar, Landy, Goldstone, Weitnauer, 2015). What will be unique in the analyses is the focus on mathematical equivalence and comparison against a control condition.

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