Chapter 12
Grasping Patterns of Algebraic Understanding: Dynamic Technology Facilitates Learning, Research, and Teaching in Mathematics Education

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Abstract Prior work has established that cognitive and perceptual processes influence students' attention to notational structures in mathematical expressions, which in turn affects their problem-solving approaches and performance. Advances in educational technology provide opportunities to further investigate these processes, improve student learning, and inform classroom instruction. Chapter 12 presents Graspable Math (GM), an online dynamic algebra notation system designed based on the research of cognitive, perceptual, and affective processes to support student learning. The log data recorded in GM offer a window into students' mathematical cognition, perceptual processes, and problem-solving strategies that can inform both research and instructional practice. First, we review the evidence of using GM to support algebra learning with elementary and middle school students. Next, we describe how log data from GM provide opportunities to research students' problem-solving processes and their uses of mathematical strategies. To conclude, we discuss how this work can inform classroom instruction and future research by providing teachers and researchers with in-depth feedback on students' use of mathematical strategies and understanding.

Keywords Educational technology · Algebra learning · Dynamic notations · Perceptual learning · Graspable Math

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As students transition from arithmetic to algebra, learning how to attend to notational structures is an important component of developing algebraic thinking that can later impact students' performance (Kieran, 1989). In addition to cognitive and affective factors, perceptual features, such as the color and spacing of symbols, influence students' attention to notational structures in mathematical expressions, consequently impacting students' problem-solving approaches and performance (Alibali et al., 2018; Kirshner & Awtry, 2004; Landy & Goldstone, 2010; Marghetis et al., 2016). For instance, students may be inclined to solve problems from left to right, which sometimes violates mathematical rules, such as the order of operations (e.g., $3 + 4 \times 5$). Perceptual features, such as spatial proximity between symbols, can direct students' attention to important elements of the notation that guide their problem-solving approach (e.g., $3\ +\ 4{\times}5$; the spacing of symbols provides grouping that is congruent with the order of operations). To that end, technology-based learning tools that leverage visual, auditory, and/or sensory features of instructional materials to direct students' attention towards key patterns in the notational structure of mathematical expressions and equations may positively impact students' development of mathematical thinking.

Tapping into perceptual motor systems during algebra practice may provide unique opportunities for students to explore the structures of algebra both physically and visually. The key to designing successful perceptual practice for algebra relies on tools that conceptually embody mathematical rules. Over the past several years, members of our team have developed a digital learning platform called Graspable Math (GM; activities.graspablemath.com). GM is a dynamic algebra notation system in which numbers and mathematical symbols can be physically moved and rearranged through specified gesture-actions (i.e., mouse or touch screen actions, such as dragging, shaking, and tapping symbols) that result in fluid, real-time transformations on the screen. These gesture-actions were developed as analogies to the dynamics of algebraic problem solving, providing students with "gestural congruency" between the gesture-actions in GM and the mathematical meaning of the notation (Lindgren & Johnson-Glenberg, 2013; Segal, 2011) to explore algebraic structures (Ottmar et al., 2015).

Beyond supporting student learning and engagement (Landy & Goldstone, 2007, 2010; Ottmar et al., 2015; Ottmar & Landy, 2017; Weitnauer et al., 2016), the log data recorded within GM (e.g., mouse clicks and actions, the timestamp for each action) enable a granular examination of learning by providing a window into students' cognitive processes and the underlying mechanisms of learning. The data can also inform instruction by providing teachers with detailed information about their students' problem-solving processes and behaviors during practice activities.

In this chapter, we present GM as a technology-based pedagogical tool for elementary and middle school students, and as a research and teaching tool that provides rich information on students' mathematical cognition, perceptual strategies, and problem-solving processes. We synthesize theoretical and empirical work on GM, describe how researchers can use this tool to unpack underlying mechanisms of mathematics learning, and discuss ways for practitioners to incorporate GM in classroom instruction.

12.1 Theories of Perceptual Learning and Embodied Cognition

Perceptual learning theory suggests that reasoning and learning about mathematics are inherently perceptual; the way that students perceive visual, auditory, and sensory information guides the way that they process materials and learn (Goldstone et al., 2017; Jacob & Hochstein, 2008; Kellman et al., 2010; Kirshner & Awtry, 2004; Patsenko & Altmann, 2010). Perceptual learning allows students to make extensive use of perceptual-motor routines and motion-based metaphors when solving problems and equations (Goldstone et al., 2017). For example, the spatial proximity between terms can help learners consistently follow the order of operations when solving equations. Students are more likely to solve equations correctly when the spacing between symbols strategically highlights the grouping of the symbols and the operation that should be completed first (e.g., $6 + 2{\times}9$; Landy & Goldstone, 2010). Regardless of a student's level of mathematical knowledge, the tendency to use perceptual features and groupings in mathematics notation is somewhat automatic (Harrison et al., 2020; Marghetis et al., 2016), and has implications for the ways in which individuals interpret, compute, and produce mathematics notation.

Researchers have found that the visual features of abstract mathematical structures influence students' ability to learn appropriate rules. For instance, when the rules are visually salient in notation (e.g., $2(x - y) = 2x - 2y$), middle school students are better able to remember and recognize these rules compared to when the rules are not visually salient (e.g., $x^2 - y^2 = (x - y)(x + y)$; Kirshner & Awtry, 2004). Further, ample evidence suggests that the visual presentation of notation impacts how students reason, process, understand, and learn mathematics (e.g., Braithwaite et al., 2016; Harrison et al., 2020; Landy & Goldstone, 2010). As an example, using perceptual features, such as color, can direct students' attention to relevant information (e.g., highlighting the equal sign in red within an equation, $4 + 7 = 13 - $ __, to support reasoning of equivalence) and help adapt their perceptual experiences to support high-level cognition (Alibali et al., 2018; Gibson, 1969; Goldstone et al., 2017). With this understanding, researchers, developers, and teachers can leverage perceptual features in instructional materials to support student learning by directing their attention to important visual cues in notation during problem solving.

Beyond shaping students' thinking processes, perceptual features also impact students' actions which reflect and further influence their learning. Embodied cognition theories contend that students' physical experiences in the world impact their cognitive processes, including thinking and reasoning in mathematics (Abrahamson et al., 2020; Foglia & Wilson, 2013; Nathan et al., 2014; Shapiro, 2010; Wilson, 2002). Specifically, Alibali and Nathan (2012) posit that mathematical cognition is "based in perception and action, and it is grounded in the physical environment" (p. 247). In other words, students' learning environments influence the way they perceive instructional materials which, in turn, informs their cognitive processes, learning, and problem solving.

12.2 Leveraging Perceptual and Embodied Learning Within Graspable Math

GM uses perceptual features to direct students' attention to notational structures; it also embeds embodied features to allow dynamic manipulation of symbols. Among other design choices, GM supports perceptual and embodied learning through three distinct features: (1) the visual presentation of spacing between terms and operands in mathematical notation, (2) students' ability to manipulate notation on the screen, and (3) the fluid transformations that provide immediate feedback on students' actions.

12.2.1 Perceptual Features Guide Students' Attention to Notational Structures

Gestalt principles of grouping posit that we tend to perceive groups of objects as a whole rather than individual objects (Hartmann, 1935). For instance, when viewing "×", we perceive one symbol rather than four intersecting lines. Following the Gestalt principle of grouping by spatial proximity, GM intentionally displays notations in a way that encourages students to group terms in the order of operations—terms surrounding higher-precedence operations (e.g., ×, ÷) are closer together than terms surrounding lower-precedence operations (e.g., +, −). Spatial proximity has been shown to impact mathematical reasoning during problem solving (e.g., Landy & Goldstone, 2010). For example, people tend to perform operations that are physically spaced closer together, even if those operations conflict with the order of operations (e.g., incorrectly simplifying $6+2 \times 9$ to 72 by performing addition before multiplication; Harrison et al., 2020). This phenomenon supports the notion that people rely on perceptual systems to process symbolic notations and are influenced by spatial properties of mathematical notation (Goldstone et al., 2017; Wagemans et al., 2012). Such research suggests that perception plays a key role in mathematical thinking. Further, strategically leveraging spatial proximity when presenting mathematical notations in digital learning tools may be an effective approach to support student learning. By systematically varying the spatial proximity between terms following the order of operations, GM aims to help direct students' attention to the correct groupings of terms and operands, which in turn may support their development of perceptual-motor routines for transforming algebraic notation.

12.2.2 Transforming Abstract Symbols into Objects Makes Algebra Concrete for Learners

GM is a dynamic algebra notation system where all symbols are individual objects that can be manipulated by dragging and dropping each object with a mouse or on a touch screen. By treating symbols like objects on a screen, students can work through problems by moving symbols to transform expressions and equations, resulting in a tangible learning experience. This design reflects embodied cognition theories which posit that thinking does not occur internally; instead, it is a process grounded in our physical experiences with tactile or imaginary objects (Abrahamson et al., 2020; Nathan et al., 2014). For instance, aligned with Melcer and Isbister's (2016) Embodied Learning Games and Simulations Framework, GM utilizes object-centered embodiment by having students manipulate numbers and symbols as objects on the screen. Students can tap, drag, and manipulate symbols in a physical-to-digital format of embodiment where students' physical gesture-actions (i.e., mouse movement or touchscreen activity) result in changes to digital objects on the screen, allowing students to connect their actions with mathematical transformations. Given that individuals tend to treat abstract symbols as physical objects distributed in space (De Lima & Tall, 2008; Dörfler, 2003; Landy & Goldstone, 2009, 2010), GM provides a digital playground for students to explore mathematical symbols as manipulatable objects. Through dynamic manipulations that result in fluid visualizations of algebraic principles, students can learn which actions are appropriate and valid in particular mathematical contexts.

12.2.3 Immediate Visual Feedback Informs Students' Problem Solving

GM provides a fluid visualization that allows students to see the transformation process of algebraic expressions and equations. When students complete a valid gesture-action, GM responds with a fluid visualization of an expression or equation transformation in real-time, so students receive immediate visual feedback on their actions. For example, students can tap the "+" in "7 + 3" and see the two numbers combined into "10" (Fig. 12.1a); they can also drag and drop the "3" from right to left in "2 + 3" to change the expression to "3 + 2" (Fig. 12.1b). In GM, students are able to learn patterns of problem-solving behavior and algebraic principles through interacting with symbols and viewing the fluid visualizations that provide automatic feedback. Compared to solving equations using paper and pencil, the fluid visualizations may help direct students' attention towards structural patterns in algebraic notation by offloading the cognitive demands of calculations onto the system and shifting students' focus to the problem-solving process as a whole. By developing students' perceptual-motor routines of algebraic equation solving, students can encounter, discover, and practice mathematical principles in action (Nathan et al., 2016, 2017).

These features of GM have been intentionally designed to support students' perception and action that influence mathematical thinking and learning. By developing a system based on theories of perceptual and embodied learning, GM is uniquely situated to advance learning theories by using log data to address research questions on how students learn algebra, how students' problem-solving strategies and behavior develop over time, and how systems like GM can support and inform classroom

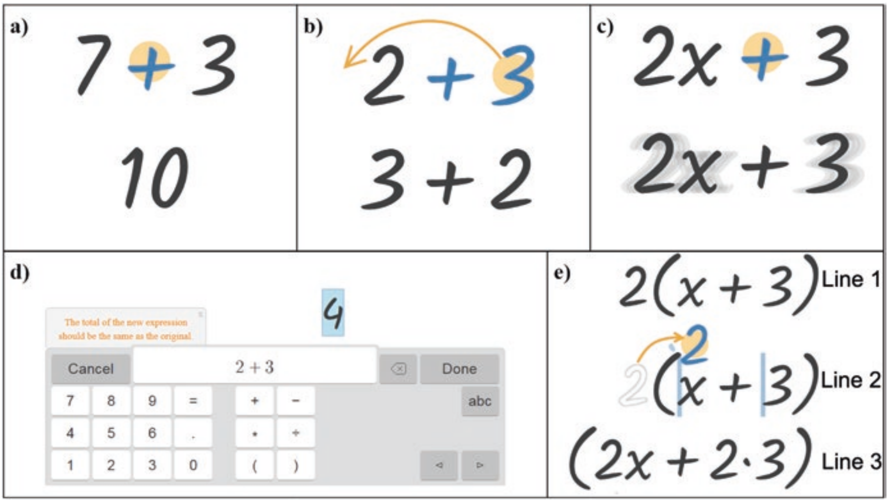

Fig. 12.1 Mathematical expressions are digital objects in Graspable Math. Students can use gesture-actions in GM to (**a**) add numbers together, and (**b**) commute terms. GM also provides visual feedback on invalid gesture-actions through (**c**) shaking and (**d**) an error message. (**e**) GM records a history of students' actions within the system

instruction. Next, we describe GM and the ongoing research efforts to understand the development of mathematical cognition and student learning in GM.

12.3 Graspable Math: A Tool to Advance Theory, Research, and Practice

Developed based on the tenets of perceptual learning and embodied cognition, GM is an interactive algebra notation system that allows students to pick up and transform mathematical expressions and equations (Weitnauer et al., 2016). As students transform expressions, GM provides immediate feedback and fluid visualizations of their mathematically valid gesture-actions (Fig. 12.1a, b). It is important to note that GM only enacts valid mathematical actions. When students attempt mathematically invalid actions, GM provides visual feedback to students. For example, if a student attempts to combine "2*x*" and "3" by tapping the addition sign, the expression shakes and remains as "2*x* + 3", indicating that the action is invalid (Fig. 12.1c). If a student tries to substitute a number (e.g., 4) with a non-equivalent expression (e.g., 2 + 3), the system does not enact the incorrect substitution. Instead, a message appears on the screen informing the student that "the total of the new expression should be the same as the original" (Fig. 12.1d). By allowing students to manipulate and transform notation with immediate visual feedback, they can explore mathematical properties and concepts, such as commutativity, associativity, distributivity, and equivalence (Chan et al., 2022a; Knuth et al., 2006; Prather & Alibali, 2009), and experience the consequences of valid and invalid transformations through perception and action.

GM also has an extensive data logging system that records all of students' actions as they interact with the system. In GM, each mouse click, movement, error, and moment-by-moment problem-solving process is recorded and time-stamped. For instance, as a student picks up and drags "2″ into "($x + 3$)", the system records the initial (i.e., $2(x + 3)$) and end (i.e., $(2x + 2 \cdot 3)$) states of the expression and time-stamps the actions of dragging and dropping the "2″ (Fig. 12.1e). As such, the data in GM reveal students' steps, errors, and the timing of these actions. These detailed logs of students' actions allow researchers and teachers to study or monitor the microstructure of students' problem-solving processes *during* mathematical tasks. Further, by analyzing these actions across problems, we can gain insights into students' behavioral patterns, and their approaches to problem solving. We can also utilize these data to identify common misconceptions or behavioral strategies shared by multiple students within a class.

GM aims to promote students' intuitive, efficient, and mathematically valid perceptual-motor routines while they engage with and explore algebraic concepts in a dynamic environment. Specifically, by leveraging the dynamic capabilities of technology tools, designing activities that target common gaps in student knowledge (NGA Center & CCSSO, 2010), and providing feedback on students' problem-solving processes, tools like GM may help students develop appropriate "structural intuition" (Kellman et al., 2010, p. 299) and perceptual-motor routines (Goldstone et al., 2017) of algebraic notations. Building upon the GM approach and cognitive theories, our team has been designing practical tools, available on activities.graspablemath.com, for classroom uses. These tools include discovery puzzle-based games (e.g., *From Here to There!*), an interactive whiteboard for in-class demonstrations (*Graspable Math Canvas*), and a series of activities that leverage the dynamic notation system for algebra learning (*Graspable Math Activities*). We are also designing dashboards that allow teachers to identify students' potential misconceptions, tailor classroom instruction to students' needs, and respond in real time when students are struggling by seeing automatic updates on students' progress within the activities.

Further, we have begun to explore how digital tools can be used for students to explore the interconnections between multiple representations of algebraic equations, and for teachers to demonstrate these connections. Research has shown that students struggle to make connections between representations (e.g., Bernardo & Okagaki, 1994; Clement et al., 1981; Landy et al., 2014; Martin & Bassok, 2005), hindering student learning and understanding of algebraic symbols (Koedinger & Nathan, 2004). By integrating Geogebra (i.e., a dynamic geometry tool) within GM, teachers and students can link the algebraic equations with coordinate graphs (Fig. 12.2). As teachers or students apply gesture-actions on one representation, such as dragging a line up or down to change the slope in the graph, they can see the corresponding changes to the slope value in the equation. The synchronous changes between an equation and its corresponding graph can help demonstrate the relations

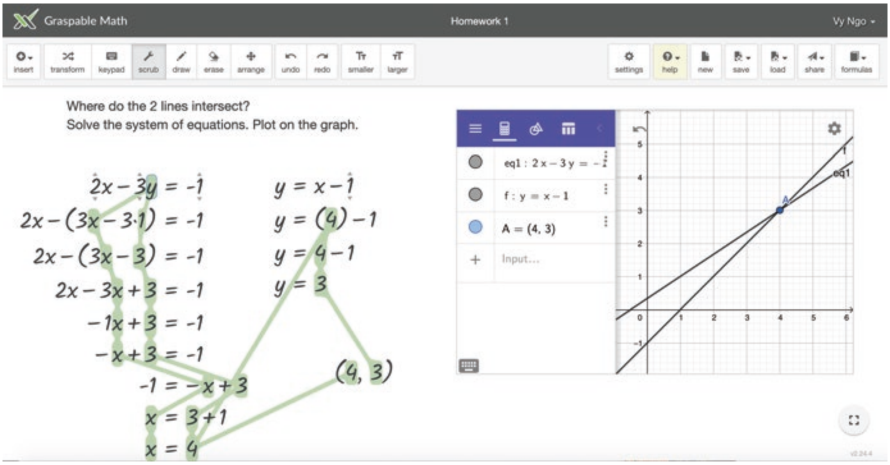

Fig. 12.2 A sample task in Graspable Math with Geogebra integration where students solve and graph a system of equations

between these two representations, as well as the connections between each element of the representations. Further, the green paths connecting the system of equations trace *x* and *y* through the history derivation, allowing students to follow the transformation process for each term.

In summary, tools like GM allow students to experience fluid visualizations of expressions and the interconnections of algebraic representations, as well as allowing teachers to model and discuss mathematical principles using gesture-actions on expressions. The log data collected through these tools also provide researchers a window into students' thinking processes. In these ways, GM acts as an instructional tool to support teachers and students and as a research platform to advance our understanding of how perceptual features and embodied actions impact students' behavior and learning.

12.4 Research on Mathematical Cognition and Student Learning

12.4.1 Evidence of Student Learning in Graspable Math and From Here to There!

The effectiveness of GM has been examined across several studies over the past decade. Early studies demonstrated the usability of GM as well as student benefits of using the tool, such as being able to work through algebra problems more efficiently and with fewer errors than using paper and pencil (Ottmar et al., 2012; Weitnauer et al., 2016). Since then, several classroom-based studies have demonstrated the impact of GM on student learning through playing a gamified version of GM, *From Here to There*!. These studies test GM's effects on learning compared to other educational technologies (Chan et al., 2022a; Decker-Woodrow et al., in press), the potential predictors of these effects (Hulse et al., 2019), the underlying mechanisms (Chan et al., 2023; Ottmar et al., 2015), and ways in which GM can be effectively incorporated in classroom instruction (Ottmar & Landy, 2017).

From Here To There! (FH2T) is an interactive mathematics puzzle game that leverages the GM technology and has been developed through iterative design and testing cycles (Ottmar et al., 2015). The game presents problems that challenge students to transform starting expressions and equations in the center of the screen into a specified goal state (located in the white box; Fig. 12.3)—an expression or equation that is mathematically equivalent to, but visually different from, the starting expression or equation. While the starting expression and the goal state are not connected by an equal sign, the transformation process demonstrates and provides students practice with mathematical equivalence, grounding abstract concepts in physical movements (Abrahamson et al., 2020). Different from other instantiations of GM, problems in FH2T are presented with gamified elements, such as challenges and rewards. For instance, students can retry problems and receive up to three clovers when they solve the problem in the most efficient way. The clovers act as points that help students monitor their performance and serve as an extrinsic motivator for efficient problem solving (Liu et al., 2022; von Ahn, 2013). In FH2T, students can also request hints as needed to receive support so the problems can be challenging without eliciting excessive frustration (Aleven & Koedinger, 2002). Through this design, FH2T integrates GM technology with engaging features to create a playful learning environment for students to practice algebraic skills.

To measure the effectiveness of FH2T compared to other learning tools, we conducted a randomized controlled trial with 475 middle school students (Chan et al., 2022a). Students were randomly assigned to play FH2T or complete online problem sets adapted from open-source curricula. Students completed four 30-minute

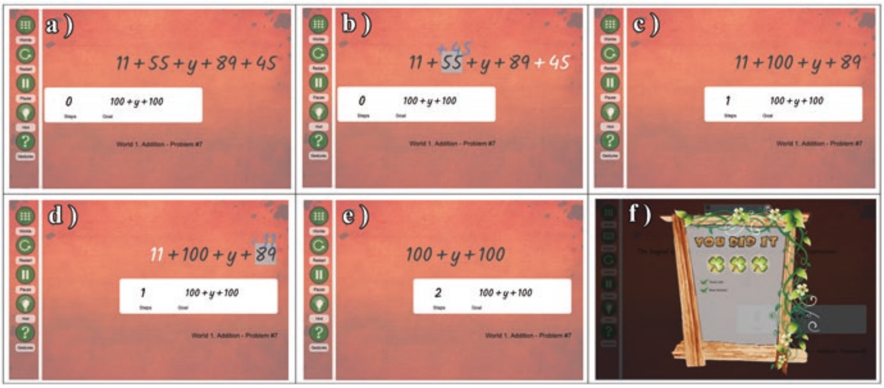

Fig. 12.3 (**a**) A sample problem in From Here to There! and (**b**, **c**, **d**) a potential transformation process involving two steps to (**e**) reach the goal state and (**f**) gain rewards

sessions (a total of 2 hours) of mathematics problem solving using their assigned technology. It is important to note that the online problem sets provided hints and correctness feedback during problem solving, and these supports were previoiusly shown to improve middle school students' mathematical learning compared to traditional paper-and-pencil homework (Mendicino et al., 2009). Results indicated that students, regardless of condition, improved their understanding of mathematical equivalence from pretest (M = 63.33% [percentage of correct answers], 3.80 out of 6 points) to posttest (M = 69.00%, 4.14 points). Further, students in the FH2T condition (M = 71.67%, 4.30 points) scored 5% (0.30 points) higher on the posttest compared to their counterparts in the online problem set condition (M = 66.67%, 4.00 points), Hedge's g = 0.16 and improvement index = 6.4. While the effect size might seem small, practically speaking, the benefit of FH2T did emerge after only a 2-hour intervention in comparison to an established and effective educational technology. Further, the What Works Clearinghouse (2020) improvement index of 6.4 suggests that an average student at the 50th percentile may improve to 56.4th percentile after a two-hour intervention of FH2T compared to completing online problem sets. These findings suggest that FH2T may be effective at improving middle school students' understanding of mathematical equivalence above and beyond traditional online problem sets.

The Elementary version of FH2T (FH2T:E) has also been shown to be effective for classroom use. In FH2T:E, the problems are designed to promote understanding of early mathematical concepts among elementary students. With 185 second graders, we found that completing more problems within FH2T:E was associated with higher posttest scores and this effect was significant above and beyond students' prior knowledge (Hulse et al., 2019). Specifically, for every one standard deviation increase in the number of completed problems within FH2T:E, students scored an average of 3.07% (0.46 out of 15 points) higher on the posttest (M = 74.20%, 11.13 points) when controlling for pretest (M = 65.93%, 9.89 points). In summary, the two studies (Chan et al., 2022a; Hulse et al., 2019) demonstrate that the FH2T games can improve mathematics performance in both elementary and middle school students, showing promise as a digital tool to promote mathematical learning across grade levels.

We have further investigated potential mechanisms of learning behind FH2T and how playing the game leads to gains in students' notation fluency by comparing two versions of FH2T—fluid visualization and retrieval practice (Ottmar et al., 2015). Students in the *fluid visualization* condition used gesture-actions to dynamically manipulate terms and saw the expression automatically transformed on the screen. Students in the *retrieval practice* condition also used gesture-actions, but entered the resulting expression instead of viewing the automatic transformation (e.g., tapping the addition sign in 7 + 3 then typing in 10). Results showed that, after four 30-minute intervention sessions, students in the fluid visualization condition (pretest: M = 33.07%, 9.92 out of 30 points; posttest: M = 36.27%, 10.88 points) showed a 3.20% (0.96 points) increase in their equation-solving performance whereas the students in the retrieval practice condition (pretest: M = 36.67%, 11.00

points; posttest: $M = 34.87\%$, 10.46 points) did not (average gain: −1.80%, −0.54 points). One interpretation of the findings is that rather than requiring students to focus on computations and typing in answers at each step, the fluid visualization liberates students to focus on the overall transformation process of the algebraic expressions. Thus, fluid visualization may help alleviate the cognitive demands of computations, providing opportunities for students to practice and improve their fluency in the perceptual-motor routines of algebra (Goldstone et al., 2010, 2017; Landy & Goldstone, 2007). Further work is needed to understand how these findings relate to prior work on the benefit of practicing arithmetic fact retrievals (Ashcraft & Christy, 1995; McNamara, 1995).

Given that the fluid visualization within FH2T improves learning, we conducted a study to examine *when*, in the instructional sequence, dynamic manipulation combined with fluid visualization is effective for learning (Ottmar & Landy, 2017). In that study, seventh graders who had little knowledge of algebraic equation solving (pretest: $M = 12.61\%$, 2.27 out of 18 points) used GM to transform and solve equations for one hour either *before* or *after* a one-hour lesson of equation solving using paper and pencil. Students who practiced equation solving using GM first scored higher on the immediate algebra posttest ($M = 87.28\%$, 15.71 points) and the retention test one month later ($M = 84.78\%$, 15.26 points) compared to the students who received the traditional instruction with paper and pencil first (posttest: $M = 74.61\%$, 13.43; retention: $M = 77.89\%$, 14.02 points). These findings suggest that using GM early in algebra lessons may support and prepare students for future learning.

In summary, this body of work has demonstrated that GM and FH2T can be powerful tools to support student learning in algebra. Specifically, it provides evidence for the positive impacts of FH2T compared to traditional online problem sets and the influence of progress on this positive impact. It also suggests that coupling dynamic manipulations of mathematical symbols with fluid visualizations of expression transformations may potentially be more beneficial for learning than having students practice mental calculations. Further, providing students the opportunity to dynamically interact with abstract symbols prior to, instead of after, explicit instruction may better prepare students for learning. Beyond their effects on student learning, GM and FH2T also collect rich data that allow researchers to investigate the cognitive processes underlying students' problem solving.

12.4.2 Analyzing Students' Problem-Solving Processes in Graspable Math

Beyond an instructional tool, GM is a research tool that can provide insights into students' problem-solving processes for researchers and teachers. It logs all student actions and mouse-movements, allowing researchers to examine, analyze, and visualize students' problem-solving processes as well as their mathematical errors at scale. By doing so, teachers and researchers can go beyond the correctness of student responses to investigate *how* students solve problems and *what* mathematical misconceptions students may hold. For instance, the log data may show that a student repeatedly tries to add an integer with a variable (e.g., $2 + x$), suggesting that the student may have misconceptions about operations with unlike terms. In short, GM makes student thinking visible for both researchers and teachers.

By leveraging the log data within GM and applying methods and approaches from different fields, we have conducted a number of studies that expand the literature on perceptual learning, student engagement, and problem solving. For example, we have examined students' behavioral engagements (Lee et al., 2022a), problem-solving errors (Bye et al., 2022), and steps in the problem-solving processes (Chan et al., 2022c). Here, we review the findings on the variability and productivity of problem-solving steps as well as predictors of strategy efficiency in middle school students. Efficient and flexible problem solving is a primary goal in mathematics education (NGA Center & CCSSO, 2010), and by understanding *how* students solve problems and *what* influences their problem-solving behaviors, we can inform researchers and educators to design instruction that better support students' problem solving and mathematical learning. By reviewing these findings, we aim to provide examples of the ways in which analyzing the rich log data can offer critical insights into students' problem-solving processes and different learning theories. Further, by situating this work within the larger context of related work, we aim to demonstrate the unique affordances of GM for research and practice.

Observing individual and aggregate visualizations of all student actions in GM and FH2T has revealed notable variation in students' problem-solving approaches and factors that impact their approaches. To visualize students' problem-solving processes across the entire sample, we created Sankey diagrams of paths in students' problem-solving processes to see the variability and frequencies of how students move from step to step (See Fig. 12.4a; Lee et al., 2022c). For example, in the problem of transforming $9 \cdot 4$ into $3 \cdot 6 \cdot 2$, we found remarkable variations in the number of steps that students took to reach the goal state, the sequence of transformations, and the mathematical strategies and properties they used. In this particular example, 64% of the students (the two blue paths at the upper left of Fig. 12.4a) made a productive first step that brought them closer to the goal state, and these students tended to solve the problem using an efficient strategy that involved the fewest number of steps. In contrast, the remaining 36% of the students who made a non-productive first step (the remaining red paths at the lower left of Fig. 12.4a) tended to solve the problem in suboptimal ways that involved more steps than necessary. These findings demonstrate that the log data within GM can provide valuable insights into students' thinking process and decisions as they solve problems. Specifically, students vary in their problem-solving approaches and their first steps on a problem may have important implications on their strategy efficiency. Additionally, these visualizations can help teachers identify patterns of problem-solving behavior to discuss during instruction.

To investigate the factors that impact students' first steps, we have examined how different features of problems influence the productivity of students' solution strategies. Prior work has suggested that students use proximity as a perceptual cue to

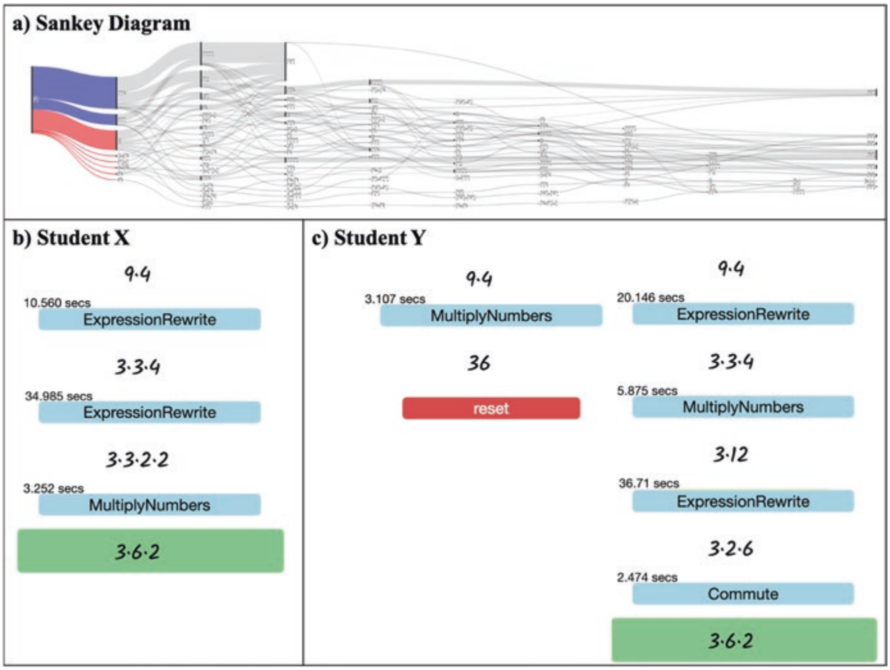

Fig. 12.4 Visualizations of students' problem-solving process. (**a**) A Sankey Diagram showing variations of solution strategies among 343 students on one problem (adapted from Lee, Stalin, et al., 2022). Individual student examples show (**b**) efficient and (**c**) inefficient solution strategies on problem 9 · 4
Note: For readability and interpretability, Fig. 12.4a was truncated to the first 10 steps

group symbols aligning with the order of operations (Landy & Goldstone, 2010), and that mathematics standards tend to focus on base 10 numbers, such as 10 or 100 (NGA Center & CCSSO, 2010). To examine how these factors would interact to influence students' solution strategies, we designed problems that varied in whether the numbers to be combined were adjacent (e.g., 47 + 53 + *b* → 100 + *b*) or non-adjacent to each other (e.g., 47 + *b* + 53 → 100 + *b*) and whether the problem involved 100 or non-100 numbers (e.g., 47 + 52 + *b* → 99 + *b*; Lee et al., 2022b). Using the log data within GM, we coded whether students' first steps were productive or non-productive, and found that students' first steps were more likely to be productive when the numbers to be combined were adjacent versus non-adjacent to each other and when the goal was to make 100 versus non-100 numbers. These findings extend prior work demonstrating the effects of problem features on problem solving. Through the log data, we see that the structure and presentation of problems impact students' first step on a problem, and consequently students' problem-solving process and performance.

In addition to providing information on *what* steps students take to solve problems, the log data also reveal *when* actions are taken. For example, to transform 9 · 4 into 3 · 6 · 2, Student X first paused for 10.56 seconds, made a productive first step by factoring 9, then reached the goal state in three steps—the fewest steps possible to complete this problem (Fig. 12.4b). Making a non-productive first step, Student Y first multiplied 9 and 4 to make 36, reset the problem, then reached the goal state after another four steps, taking a total of 68 seconds (Fig. 12.4c). As shown in these examples, these data visualizations demonstrate that students vary in the amount of time they take between each step while problem solving, and they take different series of mathematically allowable steps to link two states of an expression.

To further explore the microstructure of students' problem-solving processes, we have examined the role of students' pause time (i.e., time paused before first action / total problem-solving time) on their strategy efficiency (i.e., the total number of steps taken to solve problems) in FH2T (Chan et al., 2022b). We focused on pause time because previous studies have shown a positive relation between pausing and mathematical performance, and it has been used as a behavioral indicator of thinking and planning (e.g., Gobert et al., 2015; Paquette et al., 2014). Analyses of the log data in GM have revealed that students with longer pause time use more efficient strategies involving fewer steps, and that pause time remains a strong and significant predictor even when accounting for students' algebraic knowledge, mathematics anxiety, and mathematics self-efficacy. The results extend previous findings in the algebraic problem-solving literature (e.g., Ramirez et al., 2016; Star & Rittle-Johnson, 2008) by suggesting that pause time is a unique predictor of strategy efficiency above and beyond prior knowledge and affective factors. Further, they provide evidence for the importance of examining students' problem-solving processes in digital learning platforms. In particular, the log data offer unique opportunities to examine the relations between students' behavioral patterns and their solution strategies.

In summary, analyzing log data collected in educational technologies like GM allows researchers to efficiently examine students' problem-solving processes at a fine-grained level and effectively visualize the variability of solution strategies in problem-solving contexts across a large group of students. Using a number of analytics techniques, we have created visualizations that reveal students' problem-solving processes, identified ways in which problem features impact students' solution strategies, and found behavioral indicators that predict students' strategy efficiency. This work extends prior literature on mathematics problem solving, provides implications for instruction, and shows promise in using log data to examine potential mechanisms through which educational technologies may improve learning.

12.5 Implications for Research and Education

GM and the larger theoretical framework of this research have several implications. First, the findings show that subtle design choices in the presentation of instructional materials may have consequential impacts on students' thinking and reasoning of mathematics. For students, GM is a tool to actively explore algebraic concepts and develop fluency in mathematical reasoning through dynamic interactions with mathematical symbols and notation. GM allows students to discover relational properties of arithmetic and algebra through exploration and play. For educators, GM is an instructional tool that grounds mathematics teaching and learning in perception and embodiment. GM can facilitate instruction by serving as a formative assessment tool to gain insights into students' understanding and misconceptions. The variety of GM tools, such as the Canvas and Activities, can also supplement classroom instruction and provide teachers with a dynamic platform to design activities for their students. For researchers, GM is a platform to collect fine-grained data about student behavior and performance during problem solving that can inform classroom practice. Further, the iterative design cycle we undertake can serve as a guidepost for researchers to develop theory-driven educational technologies. Looking ahead, future work on GM and similar educational technologies should leverage the log data within the systems to further understand students' thinking process, develop tools that efficiently identify students' misconceptions, and design instructional resources that effectively improves students' mathematical learning.

12.6 Conclusion

Developed based on theories of perceptual learning and embodied cognition, GM is an interactive algebra notation system designed to leverage cognitive, perceptual, and affective processes during learning and instruction. To date, multiple studies have revealed the benefits of using GM on mathematical learning in elementary and middle school students. Further, researchers can utilize the log data in GM to explore students' mathematical cognition, perceptual processes, and problem-solving strategies. This work has advanced our understanding of the mechanisms underlying mathematical learning and informed both research and instructional practice. In conclusion, GM and its extensions allow teachers and students to experience algebraic concepts through dynamic manipulation of symbols. Further, the research on GM can inform classroom instruction and future research by providing teachers and researchers with in-depth, actionable feedback on students' knowledge and use of mathematical strategies.

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References

Abrahamson, D., Nathan, M. J., Williams-Pierce, C., Walkington, C., Ottmar, E. R., Soto, H., & Alibali, M. W. (2020). The future of embodied design for mathematics teaching and learning. *Frontiers in Education, 5*, 147.

Aleven, V., & Koedinger, K. (2002). An effective metacognitive strategy: Learning by doing and explaining with a computer-based cognitive tutor. *Cognitive Science, 26*, 147–179.

Alibali, M. W., & Nathan, M. J. (2012). Embodiment in mathematics teaching and learning: Evidence from learners' and teachers' gestures. *Journal of the Learning Sciences, 21*(2), 247–286. https://doi.org/10.1080/10508406.2011.611446

Alibali, M. W., Crooks, N. M., & McNeil, N. M. (2018). Perceptual support promotes strategy generation: Evidence from equation solving. *British Journal of Developmental Psychology, 36*, 153–168. https://doi.org/10.1111/bjdp.12203

Ashcraft, M. H., & Christy, K. S. (1995). The frequency of arithmetic facts in elementary texts: Addition and multiplication in grades 1–6. *Journal for Research in Mathematics Education, 26*(5), 396–421. https://doi.org/10.2307/749430

Bernardo, A. B. I., & Okagaki, L. (1994). Roles of symbolic knowledge and problem-information context in solving word problems. *Journal of Educational Psychology, 86*(2), 212–220. https://doi.org/10.1037/0022-0663.86.2.212

Braithwaite, D. W., Goldstone, R. L., van der Maas, H. L. J., & Landy, D. H. (2016). Non-formal mechanisms in mathematical cognitive development: The case of arithmetic. *Cognition, 149*, 40–55. https://doi.org/10.1016/j.cognition.2016.01.004

Bye, J., Lee, J.-E., Chan, J. Y.-C., Closser, A. H., Shaw, S., & Ottmar, E. (2022, April 21–26) *Perceiving precedence: Order of operations errors are predicted by perception of equivalent expressions* [Poster]. The 2022 Annual meeting of the American Educational Research Association (AERA).

Chan, J. Y.-C., Lee, J.-E., Mason, C. A., Sawrey, K., & Ottmar, E. (2022a). From Here to There! A dynamic algebraic notation system improves understanding of equivalence in middle-school students. *Journal of Educational Psychology, 114*(1), 56–71. https://doi.org/10.1037/edu0000596

Chan, J. Y.-C., Ottmar, E., & Lee, J. E. (2022b). Longer pre-solving pause time relates to higher strategy efficiency. *Learning and Individual Differences, 93*(102), 109. https://doi.org/10.1016/j.lindif.2021.102109

Chan, J. Y.-C., Ottmar, E. R., Smith, H., & Closser, A. H. (2022c). Variables versus numbers: Effects of symbols and algebraic knowledge on students' problem-solving strategies. *Contemporary Educational Psychology, 71*(102), 114. https://doi.org/10.1016/j.cedpsych.2022.102114

Chan, J. Y.-C., Closser, A. H., Ngo, V., Smith, H., Liu, A., & Ottmar, E. (2023). Examining shifts in conceptual knowledge, procedural knowledge, and procedural flexibility in the context of two game-based technologies. *Journal of Computer Assisted Learning*. https://doi.org/10.1111/jcal.12798

Clement, J., Lochhead, J., & Monk, G. (1981). Translation difficulties in learning mathematics. *The American Mathematical Monthly, 88*(4), 286–290.

De Lima, R. N., & Tall, D. (2008). Procedural embodiment and magic in linear equations. *Educational Studies in Mathematics, 67*, 3–18. https://doi.org/10.1007/s10649-007-9086-0

Decker-Woodrow, L., Mason, C. A., Lee, J. E., Chan, J. Y.-C., Sales, A., Liu, A., & Tu, S. (in press). The impacts of three educational technologies on algebraic understanding in the context of COVID-19. *AERA Open*.

Dörfler, W. (2003). Mathematics and mathematics education: Content and people, relation and difference. *Educational Studies in Mathematics, 54*, 147–170.

Foglia, L., & Wilson, R. A. (2013). Embodied cognition. *Wiley Interdisciplinary Reviews: Cognitive Science, 4*(3), 319–325. https://doi.org/10.1002/wcs.1226

Gibson, E. J. (1969). *Principles of perceptual learning and development*. Appleton-Century-Crofts.

Gobert, J. D., Baker, R. S., & Wixon, M. B. (2015). Operationalizing and detecting disengagement within online science microworlds. *Educational Psychologist, 50*(1), 43–57. https://doi.org/10.1080/00461520.2014.999919

Goldstone, R. L., Landy, D., & Son, J. Y. (2010). The education of perception. *Topics in Cognitive Science, 2*(2), 265–284. https://doi.org/10.1111/j.1756-8765.2009.01055.x

Goldstone, R. L., Marghetis, T., Weitnauer, E., Ottmar, E. R., & Landy, D. (2017). Adapting perception, action, and technology for mathematical reasoning. *Current Directions in Psychological Science, 26*(5), 434–441. https://doi.org/10.1177/0963721417704888

Harrison, A., Smith, H., Hulse, T., & Ottmar, E. R. (2020). Spacing out! Manipulating spatial features in mathematical expressions affects performance. *Journal of Numerical Cognition, 6*(2), 186–203. https://doi.org/10.5964/jnc.v6i2.243

Hartmann, G. W. (1935). *Gestalt psychology: A survey of facts and principles*. Ronald Press. https://doi.org/10.1037/11497-000

Hulse, T., Daigle, M., Manzo, D., Braith, L., Harrison, A., & Ottmar, E. (2019). From here to there! Elementary: A game-based approach to developing number sense and early algebraic understanding. *Educational Technology Research and Development, 67*(2), 423–441. https://doi.org/10.1007/s11423-019-09653-8

Jacob, M., & Hochstein, S. (2008). Set recognition as a window to perceptual and cognitive processes. *Perception and Psychophysics, 70*(7), 1165–1184. https://doi.org/10.3758/PP.70.7.1165

Kellman, P. J., Massey, C. M., & Son, J. Y. (2010). Perceptual learning modules in mathematics: Enhancing students' pattern recognition, structure extraction, and fluency. *Topics in Cognitive Science, 2*(2), 285–305. https://doi.org/10.1111/j.1756-8765.2009.01053.x

Kieran, C. (1989). The early learning of algebra: A structural perspective. In S. Wagner & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra: The research agenda for mathematics education, volume 4* (pp. 33–56). NCTM and Lawrence Erlbaum Associates. https://doi.org/10.4324/9781315044378-4

Kirshner, D., & Awtry, T. (2004). Visual salience of algebraic transformations. *Journal for Research in Mathematics Education, 35*(4), 224–257. https://doi.org/10.2307/30034809

Knuth, E. J., Stephens, A. C., McNeil, N. M., & Alibali, M. W. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education, 37*(4), 297–312. https://doi.org/10.2307/30034852

Koedinger, K. R., & Nathan, M. J. (2004). The real story behind story problems: Effects of representations on quantitative reasoning. *Journal of the Learning Sciences, 13*(2), 129–164. https://doi.org/10.1207/s15327809jls1302_1

Landy, D., & Goldstone, R. L. (2007). Formal notations are diagrams: Evidence from a production task. *Memory and Cognition, 35*(8), 2033–2040. https://doi.org/10.3758/BF03192935

Landy, D, & Goldstone, R. (2009). How much of symbolic manipulation is just symbol pushing? In *Proceedings of the 31st annual conference of the cognitive science society* (pp. 1318–1323). Retrieved from http://csjarchive.cogsci.rpi.edu/Proceedings/2009/papers/253/paper253.pdf

Landy, D., & Goldstone, R. L. (2010). Proximity and precedence in arithmetic. *Quarterly Journal of Experimental Psychology, 63*(10), 1953–1968. https://doi.org/10.1080/17470211003787619

Landy, D., Brookes, D., & Smout, R. (2014). Abstract numeric relations and the visual structure of algebra. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 40*(5), 1404–1418. https://doi.org/10.1037/a0036823

Lee, J.-E., Chan, J. Y.-C., Botelho, A., & Ottmar, E. (2022a). Does slow and steady win the race?: Clustering patterns of students' behaviors in an interactive online mathematics game. *Education Technology Research and Development, 70*, 1575–1599. https://doi.org/10.1007/s11423-022-10,138-4

Lee, J.-E., Hornburg, C. B., Chan, J. Y.-C., & Ottmar, E. (2022b). Perceptual and number effects on students' solution strategies in an interactive online mathematics game. *Journal of Numerical Cognition, 8*(1), 166–182. https://doi.org/10.5964/jnc.8323

Lee, J.-E., Stalin, A., Ngo, V., Drzewiecki, K., Trac, C., & Ottmar, E. (2022c). Show the flow: Visualizing students' problem-solving processes in a dynamic algebraic notation tool. *Journal of interactive Learning Research, 33*(2), 97–126.

Lindgren, R., & Johnson-Glenberg, M. (2013). Emboldened by embodiment: Six precepts for research on embodied learning and mixed reality. *Educational Researcher, 42*(8), 445–452. https://doi.org/10.3102/0013189X13511661

Liu, A. S., Vanacore, K., & Ottmar, E. (2022). How reward- and error-based feedback systems create micro-failures to support learning strategies. In C. Chinn, E. Tan, C. Chan, & Y. Kali (Eds.), *Proceedings of the 16th international conference of the learning sciences - ICLS 2022* (pp. 1633–1636). International Society of the Learning Sciences.

Marghetis, T., Landy, D., & Goldstone, R. L. (2016). Mastering algebra retrains the visual system to perceive hierarchical structure in equations. *Cognitive Research: Principles and Implications, 1*, 1–10. https://doi.org/10.1186/s41235-016-0020-9

Martin, S. A., & Bassok, M. (2005). Effects of semantic cues on mathematical modeling: Evidence from word-problem solving and equation construction tasks. *Memory and Cognition, 33*(3), 471–478. https://doi.org/10.3758/BF03193064

McNamara, D. S. (1995). Effects of prior knowledge on the generation advantage: Calculators versus calculation. *Journal of Educational Psychology, 87*, 307–318.

Melcer, E., & Isbister, K. (2016). Bridging the physical learning divides: A design framework for embodied learning game and simulations. In *Proceedings of the 2016 CHI conference extended abstracts on human factors in computing systems* (pp. 2225–2233).

Mendicino, M., Razzaq, L., & Heffernan, N. T. (2009). A comparison of traditional homework to computer-supported homework. *Journal of Research on Technology in Education, 41*(3), 331–359.

Nathan, M. J., Walkington, C., Boncoddo, R., Pier, E., Williams, C. C., & Alibali, M. W. (2014). Actions speak louder with words: The roles of action and pedagogical language for grounding mathematical proof. *Learning and Instruction, 33*, 182–193. https://doi.org/10.1016/j.learninstruc.2014.07.001

Nathan, M. J., Williams-Pierce, C., Ottmar, E. R., Walkington, C., & Nemirovsky, R. (2016). Embodied mathematical imagination and cognition (EMIC) working group. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th annual meeting of the North-American chapter of the international group for the psychology of mathematics education (PME-NA)* (pp. 1690–1697). University of Arizona.

Nathan, M. J., Williams-Pierce, C., Abrahamson, D., Ottmar, E. R., Landy, D., Smith, C., & Boncoddo, R. (2017). Embodied mathematical imagination and cognition (EMIC) working group. In E. Galindo & J. Newton (Eds.), *Proceedings of the 39th annual meeting of the North American chapter of the international group for the psychology of mathematics education (PME-NA)* (pp. 1497–1506). Hoosier Association of Mathematics Teacher Educators.

National Governors Association Center for Best Practices (NGA Center) and Council of Chief State School Officers (CCSSO). (2010). *Common core state standards for mathematics*. Retrieved from: http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

Ottmar, E. R., & Landy, D. (2017). Concreteness fading of algebraic instruction: Effects on learning. *Journal of the Learning Sciences, 26*(1), 51–78. https://doi.org/10.1080/10508406.2016.1250212

Ottmar, E. R., Landy, D., & Goldstone, R. L. (2012). Teaching the perceptual structure of algebraic expressions: Preliminary findings from the pushing symbols intervention. *Proceedings of the 34th Annual Conference of the Cognitive Science Society, 34*, 2156–2161. Retrieved from http://www.indiana.edu/~pcl/papers/pushingsymbols.pdf

Ottmar, E. R., Landy, D., Goldstone, R., & Weitnauer, E. (2015). Getting From Here to There! : Testing the effectiveness of an interactive mathematics intervention embedding perceptual learning. *Proceedings of the Annual Conference of the Cognitive Science Society*, 1793–1798.

Paquette, L., de Carvalho, A., & Baker, R. S. (2014). Towards understanding expert coding of student disengagement in online learning. In *36th annual cognitive science conference* (pp. 1126–1131).

Patsenko, E. G., & Altmann, E. M. (2010). How planful is routine behavior? A selective-attention model of performance in the Tower of Hanoi. *Journal of Experimental Psychology: General, 139*(1), 95–116. https://doi.org/10.1037/a0036823

Prather, R. W., & Alibali, M. W. (2009). The development of arithmetic principle knowledge: How do we know what learners know? *Developmental Review, 29*, 221–248. https://doi.org/10.1016/j.dr.2009.09.001

Ramirez, G., Chang, H., Maloney, E. A., Levine, S. C., & Beilock, S. L. (2016). On the relationship between math anxiety and math achievement in early elementary school: The role of problem solving strategies. *Journal of Experimental Child Psychology, 141*, 83–100. https://doi.org/10.1016/j.jecp.2015.07.014

Segal, A. (2011). *Do gestural interfaces promote thinking? Embodied interaction: Congruent gestures and direct touch promote performance in math*. Columbia University.

Shapiro, L. (2010). *Embodied cognition*. Routledge Press.

Star, J. R., & Rittle-Johnson, B. (2008). Flexibility in problem solving: The case of equation solving. *Learning and Instruction, 18*(6), 565–579. https://doi.org/10.1016/j.learninstruc.2007.09.018

von Ahn, L. (2013). Duolingo: Learn a language for free while helping to translate the web. In *Proceedings of the 2013 international conference on intelligent user interfaces* (pp. 1–2). ACM.

Wagemans, J., Elder, J. H., Kubovy, M., Palmer, S. E., Peterson, M. A., Singh, M., & von der Heydt, R. (2012). A century of Gestalt psychology in visual perception I. Perceptual grouping and figure-ground organization. *Psychological Bulletin, 138*(6), 1172–1217. https://doi.org/10.1037/a0029333.A

Weitnauer, E., Landy, D., & Ottmar, E. R. (2016). Graspable Math: Towards dynamic algebra notations that support learners better than paper. In *Future technologies conference* (pp. 406–414).

What Works Clearinghouse. (2020). *What works Clearinghouse standards handbook, Version 4.1*. U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance. Available at https://ies.ed.gov/ncee/wwc/handbook

Wilson, M. (2002). Six views of embodied cognition. *Psychonomic Bulletin and Review, 9*(4), 625–636. https://doi.org/10.3758/BF03196322