Big Ideas in Mathematics: Exploring the Dimensionality of Big Ideas in School Mathematics

Jahangeer Mohamed Jahabar
National Institute of Education
jahangeer.jahabar@nie.edu.sg

Toh Tin Lam
National Institute of Education
tinlam.toh@nie.edu.sg

Tay Eng Guan
National Institute of Education
engguan.tay@nie.edu.sg

Tong Cherng Luen
National Institute of Education
chernglun.tong@nie.edu.sg

Big Ideas in school mathematics can be seen as overarching concepts that occur in various mathematical topics in a syllabus. For teachers, this knowledge can be used to help students develop a better understanding of mathematics by making visible the central ideas, and connection across topics and across levels. For students, this knowledge can further be helpful affectively by engendering an appreciation of mathematics as a subject that is coherent and comprehensible. Although there has been much interest recently in the understanding of Big Ideas, there is little research done in the assessment of Big Ideas thinking. In this paper, we discuss our development of an instrument to measure the Big Ideas of equivalence and proportionality. Our analysis of some pilot items suggests that Big Idea thinking is a multidimensional construct within most school environments.

We concur with Charles' (2005) definition of a Big Idea in mathematics as "a statement of an idea that is central to the learning of mathematics, one that links numerous mathematical understandings into a coherent whole" (p.10). Each Big Idea connects various concepts and understanding across topics, strands and levels. The notion of Big Ideas became prominent in the teaching and learning of mathematics when it was highlighted by the National Council of Teachers in Mathematics in 2000, where it stated: "Teachers need to understand the Big Ideas of mathematics and be able to represent mathematics as a coherent and connected enterprise. Their decisions and their actions in the classroom—all of which affect how well their students learn mathematics—should be based on this knowledge." (NCTM, 2000, p17)

While there has been an interest in the understanding and teaching with the understanding of Big Ideas, to date, there has been little research done on how the knowledge and understanding of Big Ideas can be assessed. One reason can be attributed to the different classifications and definitions of the Big Ideas by various educational bodies and researchers. Charles (2005) listed twenty-one Big Ideas in mathematics. Niemi (2006) identified a list of twenty Big Ideas for school algebra alone and wrote about the development of an assessment instrument to measure the Big Ideas based on his list. Hurst (2019) had further categorized his list of Big Ideas into three categories: sequential Big Ideas, umbrella Big Ideas and process Big ideas. The Singapore Ministry of Education (MOE) syllabus identified six Big Ideas for the Primary grade levels in Singapore (Grades 1 to 6), and an additional two for the Secondary grade levels (Grades 7 to 10) (MOE, 2018a, 2018b).

Another possible reason for the scarcity of Big Ideas assessment instruments could be the lack of clarity on the intent of the assessment in the first place. Educational institutions may not rate it important and pressing enough to know how well their students are able to think along the Big Ideas to incorporate specific items into their class and grade assessments. With a proper instrument available at the national level to measure Big Ideas thinking, professional development can be provided to educational institutions on how to assess Big Ideas thinking and enactment in their environments. Teachers could also use this information for curriculum development and design, resource design and formative assessment. It would seem then that any instrument designed to assess Big Ideas should be an addition to the current suite of assessments already put in place and, in particular, to be done outside of high stakes mathematics assessments. Assessment instruments

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 355–362). Newcastle: MERGA.

developed to assess Big Ideas thinking will have to factor in the testing load on the students as well as the amount of time taken away from curriculum to administer the instrument.

A third reason could be that it is difficult to create items that assess Big Ideas thinking because Big Ideas link numerous mathematical understandings and cuts across topics and grade levels. Thus, a major consideration for items created to assess Big Idea thinking is that they must be able to test specifically such cross-topic thinking and not be confounded by specific topical familiarity and expertise. Another aspect to consider, akin to Carroll's three-stratum theory of cognitive abilities (e.g., Carroll, 1993; 1997; Warne & Burningham, 2019) is when to see Big Ideas as a unidimensional construct and when as a multi-dimensional construct.

This paper reports the initial stage of developing an instrument, to measure Equivalence and Proportionality, two out of the eight Big Ideas in the mathematics curriculum document (MOE, 2018a, p5). The pilot instrument was administered to students in two primary schools and two secondary schools in Singapore. We took into consideration the need to create an item with questions that test across different topics. The items were analysed to explore if Big Ideas thinking in schools is a unidimensional construct or if it is 2-dimensional along the two Big Ideas to be assessed.

Equivalence and Proportionality

There are no generally accepted formal definitions yet on any of the Big Ideas. For example with regard to Equivalence, Warren & Cooper (2009), in their EATP project, highlighted 5 key aspects: equations as equivalence; the balance principle; sign systems for unknowns; identity and inverses; and finding solutions and generalisations. On the other hand, Fyfe et al. (2018), restricted equivalence to the idea in which both sides of an equation are equal and interchangeable. While understanding the broad nature of the construct, their study was focused on the symbolic understanding of equivalence: the understanding of the equal sign.

For this study, we will specify our understanding of two Big Ideas, Equivalence and Proportionality, and use them in developing our instrument. We look at two Big Ideas common to the list proposed by Charles (2005) and the list of Big Ideas developed by MOE: Equivalence and Proportionality.

Charles (2005) described Equivalence as when "any number, measure, numerical expression, algebraic expression, or equation can be represented in an infinite number of ways that have the same value". MOE described equivalence as: "A relationship that expresses the equality of two mathematical objects that may be represented in two different forms. In every statement about equivalence, there is a mathematical object (e.g., a number, an expression or an equation) and an equivalence criterion (e.g., value(s), part-whole relationships)." (MOE, 2018b, p.15) The former defines equivalence focusing on representations having the same value while the latter goes beyond the equivalence of values. Consider the mathematical problem as follows:

> Example 1: Ali had $220 and Colin had $310. After each of them bought an identical T-shirt, Colin had thrice as much money as Ali had left. How much does each T-shirt cost?

In the mathematical problem stated, the difference in money Ali and Colin had at the beginning is the same as the difference in money at the end. The two equivalent entities are the difference in money before and after having bought identical T-shirts.

When defining the Big Idea of proportionality, Charles (2005) explained that "when two quantities vary proportionally, that relationship can be represented as a linear function". MOE describes proportionality as a "relationship between two quantities that allows one quantity to be computed from the other based on multiplicative reasoning" (MOE, 2018b, p.16). We operationalise proportionality by making explicit the two entities that vary in direct proportion. This can be seen in the following example:

> Example 2: 60 workers can make 180 chairs in one day. All workers work at the same rate. How many chairs can 36 workers make in one day?

It can be clearly seen from this example that the number of chairs made in one day varies proportionally to the number of workers.

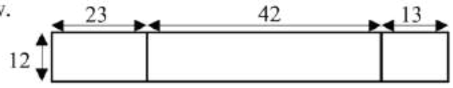
Methodology

Our design of the items for the instruments is guided by the three characteristics of Big Ideas as detailed by Hsu et al. (2007). The items that we develop serve to (a) connect different parts of the curriculum under the umbrella Big Idea; (b) be a basis for understanding other topics; and (c) make choices about curriculum. Objectives (a) and (b) are meant for the instruments for both teachers and students, and (c) is for the teachers. In this paper, we shall only consider the instrument for the students. We shall call this instrument Big Ideas in School Mathematics, BISM for short.

Part 1: The diagram below shows three rectangles joined together to form a bigger rectangle.

You may use the diagram to fill in the blanks below.

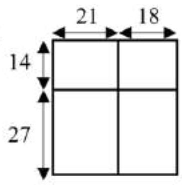
$23 \times 12 + 42 \times 12 + 13 \times 12 = ? \times 12$



Part 2: The diagram below shows four rectangles joined together to form a bigger rectangle.

Fill in the missing numbers below.

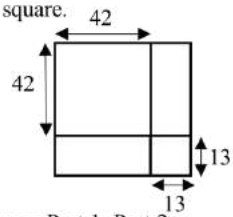
$14 \times 21 + \Delta \times 18 + 27 \times 21 + 27 \times 18 = \heartsuit \times 39$



Part 3: The diagram below shows 2 squares and 2 rectangles joined together to form a bigger square.

Fill in the missing numbers below.

$55 \times 55 = 42 \times 42 + 13 \times 13 + 13 \times \Delta \times \heartsuit$



Part 4: Which of these following statements best describes the common mathematical idea across Part 1, Part 2 and Part 3?

a) I used diagrams for the parts

b) I used Equivalence for the parts

c) I used Guess and Check for the parts

d) I used Proportionality for the parts

e) Others (Please elaborate)

Part 5: The shaded area of the figure below can be used to show a mathematical statement. Which of the following statements matches the shaded part of the figure?

a) $(1 + 6) + (2 + 5) + (3 + 4) \ldots + (6 + 1) = 6 \times 7$

b) $1 + 2 + 3 + \ldots + 6 = (6 \times 7) \div 2$

c) $1 + 2 + 3 + \ldots + 7 = (7 \times 8) \div 2$

d) $1 + 2 + 3 + \ldots + 8 = (8 \times 9) \div 2$

e) $1 + 2 + 3 + \ldots + 7 = (7 \times 8)$

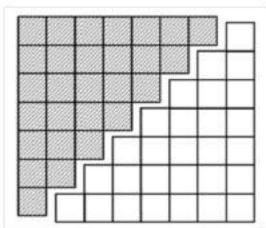


Figure 1. An item on the *Big Idea of Equivalence* consisting of five parts.

Each item in BISM tests only one Big Idea of either Equivalence or Proportionality and has five parts. Part 1 to Part 3 each consists of a selected response question focusing on the same Big Idea and are from the same topic, for example 'Area of rectangles'. Ability to correctly answer these parts could be due to topical and procedural knowledge and/or knowledge of the Big Idea under focus. To facilitate thinking beyond the topical and procedural knowledge, Part 4 seeks to assist participants to look for the link connecting the three parts. Part 4 also seeks to trigger students' Big Idea concepts, if any. Participants then attempt Part 5, a question that focuses on the same Big Idea but based on a different topic. Figure 1 shows one of the items under the Big Idea of Equivalence.

Each of the first three parts can be solved by seeing the equivalence between the area of the large rectangle and the sum of the areas of its component parts. Part 5 needs also the Big Idea of Equivalence but not exactly in terms of areas of rectangles. The focus is on the equivalence of two mathematical forms, a diagram and its equivalent mathematical equation. It can be subsumed within the strand of Whole Numbers. Part 5 was refined later to dissociate it from the idea of Areas with changes in the figure as well as the phrasing as shown in Figure 2.

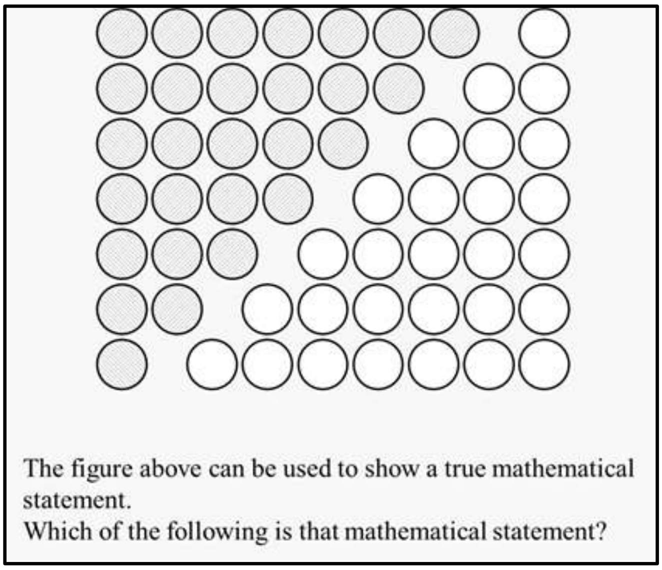


Figure 2. Refined part 5.

Figure 3 shows Part 5 of an item on the Big Idea of proportionality. Students at this level have not learnt about areas of circles but are expected to have knowledge of areas of rectangles as well as that the sum of angles at a point is 360°. The problem can be solved by seeing that the area of the "slice" varies proportionally with the angle at the centre.

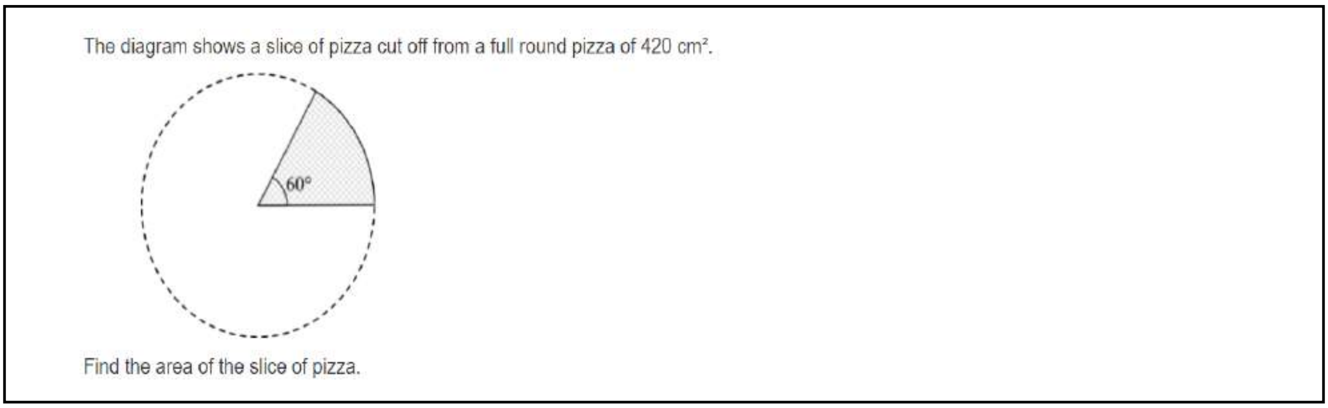


Figure 3. Part 5 of an item on the *Big Idea of Proportionality*.

Analysis of Instrument and Item Performance

The items have been validated by three content specialists at the National Institute of Education, with a doctorate degree in Mathematics, Educational Assessment and Mathematics Education, respectively. To validate the items further, the instrument was analyzed using Rasch analysis via the Winsteps software.

Firstly, the instrument was analysed to check if the items collectively are unidimensional, i.e., that it measures only one latent trait, Big Ideas thinking.

The items were administered to 360 students from two participating schools at grade levels five and six. Each student is given two complete sets of items, one each from the different Big Ideas of Equivalence and Proportionality. The items given to various students overlapped and interlaced to allow for item equating to be done. Each student was given 45 minutes to complete the items.

Figure 4 shows the analysis report from Winsteps on the standardized residual variance based on the students' responses.

T1_Std_Pri_Dimen_NoPart4 - Notepad

File Edit Format View Help

TABLE 23.0 All_data_Testpoint 1 trial.xlsx ZOU827WS.TXT Mar 1 2022 9: 3

INPUT: 360 PERSON 8 ITEM REPORTED: 360 PERSON 8 ITEM 2 CATS WINSTEPS 5.2.0.0

Table of STANDARDIZED RESIDUAL variance in Eigenvalue units = ITEM information units

		Eigenvalue	Observed		Expected
Total raw variance in observations	=	10.7998	100.0%		100.0%
Raw variance explained by measures	=	2.7998	25.9%		26.4%
Raw variance explained by persons	=	1.4784	13.7%		13.9%
Raw Variance explained by items	=	1.3214	12.2%		12.4%
Raw unexplained variance (total)	=	8.0000	74.1%	100.0%	73.6%
Unexplned variance in 1st contrast	=	1.4077	13.0%	17.6%	
Unexplned variance in 2nd contrast	=	1.3068	12.1%	16.3%	
Unexplned variance in 3rd contrast	=	1.1508	10.7%	14.4%	
Unexplned variance in 4th contrast	=	1.1369	10.5%	14.2%	
Unexplned variance in 5th contrast	=	1.0840	10.0%	13.6%	

Figure 4. Standardized residual variance report.

If the items are unidimensional, the variance explained by the items is expected to be considerably larger than the unexplained variance in the first contrast. As can be seen from Figure 4, the variance explained by the items are about equal in magnitude with all the unexplained variances in the other contrasts, signaling that the items may not be unidimensional. A closer analysis is needed to understand how the items function. We shall proceed to analyse the first two contrasts. Figure 5 shows the Principal Component Analysis report for the first contrast, Contrast 1.

CONTRAST 1 FROM PRINCIPAL COMPONENT ANALYSIS

STANDARDIZED RESIDUAL LOADINGS FOR ITEM (SORTED BY LOADING)

CON-TRAST	LOADING	MEASURE	INFIT MNSQ	OUTFIT MNSQ	ENTRY NUMBER	ITEM	LOADING	MEASURE	INFIT MNSQ	OUTFIT MNSQ	ENTRY NUMBER	ITEM
1	.60	-.46	.93	.82	1	1 E-P-013	-.53	-.66	.93	.80	6	6 P-P-032
1	.45	1.06	1.10	1.09	4	4 E-P-016	-.41	.48	1.20	1.23	5	5 P-P-031
1	.35	.54	1.02	1.04	7	7 P-P-034	-.38	-.38	.88	.78	3	3 E-P-015
1	.16	-.44	1.07	1.19	8	8 P-P-035	-.32	-.14	.88	.84	2	2 E-P-014

Figure 5. Principal component analysis for contrast 1.

From the report, the contrasting questions are identified based on their polarizing loading. The two most positively loaded questions are those belonging to items testing for Equivalence, recognizable from the 'E-P' prefix (the second 'P' refers to the Primary track) on the question identifications while the most negatively loaded questions are those belonging to items testing for the Big Idea of proportionality, recognizable from the 'P-P' prefix on the question identifications. Figure 6 shows the Principal Component Analysis report for the second contrast, Contrast 2.

CONTRAST 2 FROM PRINCIPAL COMPONENT ANALYSIS
STANDARDIZED RESIDUAL LOADINGS FOR ITEM (SORTED BY LOADING)

CON-TRAST	LOADING	MEASURE	INFIT MNSQ	OUTFIT MNSQ	ENTRY NUMBER	ITEM	LOADING	MEASURE	INFIT MNSQ	OUTFIT MNSQ	ENTRY NUMBER	ITEM
2	.60	.48	1.20	1.23	5	5 P-P-031	-.51	-.14	.88	.84	2	2 E-P-014
2	.46	-.44	1.07	1.19	8	8 P-P-035	-.38	1.06	1.10	1.09	4	4 E-P-016
2	.38	.54	1.02	1.04	7	7 P-P-034	-.28	-.46	.93	.82	1	1 E-P-013
							-.24	-.38	.88	.78	3	3 E-P-015
							-.23	-.66	.93	.80	6	6 P-P-032

Figure 6. Principal component analysis for contrast 2.

Similar to the first contrast, the two most positively and negatively loaded questions are from the items testing the Big Ideas of Proportionality and Equivalence respectively. From these two PCA analyses, it can be preliminarily concluded that the two Big Ideas of Equivalence and Proportionality may require very different skillsets and that Big Ideas thinking may be multidimensional in nature.

LARGEST STANDARDIZED RESIDUAL CORRELATIONS
USED TO IDENTIFY DEPENDENT ITEM

CORREL-ATION	ENTRY NUMBER ITEM	ENTRY NUMBER ITEM
-.24	4 E-P-016	5 P-P-031
-.24	1 E-P-013	5 P-P-031
-.24	1 E-P-013	6 P-P-032
-.22	2 E-P-014	7 P-P-034
-.19	3 E-P-015	8 P-P-035
-.18	4 E-P-016	7 P-P-034
-.18	2 E-P-014	4 E-P-016
-.17	4 E-P-016	8 P-P-035
-.17	3 E-P-015	4 E-P-016
-.17	6 P-P-032	7 P-P-034
-.16	2 E-P-014	5 P-P-031
-.16	5 P-P-031	7 P-P-034
-.16	2 E-P-014	8 P-P-035
-.14	3 E-P-015	7 P-P-034
-.14	6 P-P-032	8 P-P-035
-.13	4 E-P-016	6 P-P-032
-.13	1 E-P-013	3 E-P-015
-.13	7 P-P-034	8 P-P-035
-.12	5 P-P-031	6 P-P-032
-.11	5 P-P-031	8 P-P-035

Figure 7. Item correlation report.

The items are further analysed by examining how the questions are correlated to each other. A positive correlation of the questions will reinforce the unidimensionality of the latent trait. Figure 7 shows the item correlation report.

From the report, the largest negatively correlated items are between Proportionality and Equivalence items. We do see items within the same Big Ideas also being negatively correlated, as seen from the 7th entry in the table as well as some more pairs of items down the table. A closer study of the items showed that these items are from different topics. Both item E-P-014 and E-P-016 are shown as Part 2 and Part 5 respectively in the earlier Figure 1. Part 2 is from the measurement topic of area while Part 5 is a counting question under the strand of Whole Numbers. Item P-P-031 under the topic of whole numbers is shown earlier as Example 2 while P-P-034, under the measurement strand is shown as Part 5 in Figure 3. In both these pairs of questions, students may have solved them using topical knowledge instead of knowledge of equivalence or proportionality. Hence, they may be able to do an item from a topic but not another item from another topic with the same Big Idea.

Findings and Discussion

At this point in the research, we have tested some items at both the primary and secondary levels. Only two complete sets of items were tested for each Big Idea of Equivalence and Proportionality at the primary level. The findings and discussions are based on this small number of items that was tested at the primary level. From the standardized residuals and principal component analysis, it can be concluded preliminarily that it may not be feasible to construct just one instrument that measures the latent trait of knowledge of Big Ideas. Big ideas in mathematics may likely be multidimensional. At this juncture, it may seem more appropriate that two separate instruments be created, one for each Big Idea. Thus, the final BISM instrument may consist of a battery of subtests. At the next data collection stage, a detailed analysis will be made for the separate instruments to check once again for the unidimensionality of the items for each instrument first before making further analysis on students' performance.

For some of the items we have created, especially those in the first three parts of an item, participants may have solved them using topical content knowledge and not because they have the knowledge of Big Ideas. These earlier parts tend to be easier to solve and thus could be solved well by students equipped with sound content knowledge. Part 5 of each item was administered after the participants were asked to reflect and find similarities for each of the first three parts, guiding them to think beyond the topical content and more towards Big Ideas thinking. We have decided to analyse Big Ideas thinking through the participants' performance on Part 5 of the items only.

Finally, this study is part of a larger project that includes the professional learning development (PLD) of teachers in their knowledge of Big Ideas. The instrument that we are developing will also be used in the formative assessment of the teachers in the PLD and subsequently for the students in their classes.

Acknowledgements

The study reported here has been funded by the Singapore Ministry of Education under the research project OER/31/19BK which is managed by the National Institute of Education, Singapore.

References

Carroll, J. B. (1993). *Human cognitive abilities: A survey of factor-analytic studies*. NY: Cambridge University Press.

Carroll, J. B. (1997). The three-stratum theory of cognitive abilities. In D. P. Flanagan, J. L. Genshaft et al., *Contemporary intellectual assessment: Theories, tests, and issues* (pp. 122–130). NY: Guilford Press.

Charles, R. (2005). Big ideas and understandings as the foundation for elementary and middle school mathematics. *Journal of Mathematics Education Leadership, 7*(3), 9–21.

Fyfe, Matthews, P. G., Amsel, E., McEldoon, K. L., & McNeil, N. M. (2018). Assessing formal knowledge of math equivalence among algebra and pre-algebra students. *Journal of Educational Psychology, 110*(1), 87–101.

Hsu, E., Kysh, J., Ramage, K., & Resek, D. (2007). Seeking Big Ideas in algebra: The evolution of a task. *Journal of Mathematics Teacher Education, 10*, 325–332.

Hurst, C. (2019). Big Ideas of Primary Mathematics: It's all about connections, in Toh, T. and Yeo, J. (Eds.), *Big ideas in mathematics: Yearbook 2019, Association of Mathematics Educators* (pp. 71–93). Singapore: World Scientific.

Ministry of Education. (2018a). *2020 Secondary mathematics syllabuses* (draft). Singapore: Author.

Ministry of Education (2018a). *2021 Primary mathematics syllabuses* (draft). Singapore: Author.

National Council of Teachers of Mathematics (NCTM). (2000). *Curriculum and evaluation standards for school mathematics*. Reston, VA: NCTM.

Niemi, D., Vallone, J., & Vendlinski, T. (2006). *The power of big ideas in mathematics education: Development and pilot testing of POWERSOURCE assessments*. CSE Report 697, National Center for Research on Evaluation, Standards, and Student Testing (CRESST).

Warren, E., & Cooper, T. J. (2009). Developing mathematics understanding and abstraction: The case of equivalence in the elementary years. *Mathematics Education Research Journal, 21*(2), 76–95. https://doi-org.libproxy.nie.edu.sg/10.1007/BF03217546

Warne, R. T., & Burningham, C. (2019). Spearman's g found in 31 non-Western nations: Strong evidence that g is a universal phenomenon. *Psychological Bulletin, 145*(3), 237–272. https://doi.org/10.1037/bul0000184.