IDENTIFYING PERSISTENT UNCONVENTIONAL UNDERSTANDINGS OF ALGEBRA: A CASE STUDY OF AN ADULT WITH DYSCALCULIA

Katherine E. Lewis
University of Washington
kelewis2@uw.edu

Gwen Sweeney
University of Washington
gwens@uw.edu

Grace M. Thompson
University of Washington
graciemt@uw.edu

Rebecca Adler
Vanderbilt University
rebecca.adler@vanderbilt.edu

Kawla Alhamad
Imam Abdulrahman bin Faisal University
kawlaalhamad@gmail.com

Research on dyscalculia has focused almost exclusively on elementary-aged students' deficits in speed and accuracy in arithmetic calculation. This case study expands our understanding of dyscalculia by documenting how one college student with dyscalculia understood algebra during a one-on-one design experiment. A detailed case study of 19 video recorded sessions revealed that she relied upon unconventional understandings of algebraic quantities and notation, which led to persistent difficulties. This exploratory case study provides new insights into the character of difficulties that arose and persisted for one student with dyscalculia in the context of algebra and suggests the utility of documenting the persistent understandings that students with dyscalculia rely upon, particularly in understudied mathematical domains, like algebra.

Keywords: Students with Disabilities, Algebra and Algebraic Thinking, Design Experiments

Although many students may have difficulties with mathematics, the 6% of students with dyscalculia (Shalev, 2007) have a neurological difference in how their brains process quantity (Butterworth, 2010). Research on dyscalculia has identified that students have difficulty processing both symbolic (e.g., 5) and pictorial (e.g., *****) representations of quantity (Butterworth, 2010). This neurological difference in number processing may render standard mathematical tools, like symbols or representations, less accessible for students with dyscalculia (Lewis, 2014; 2017). Currently, research on dyscalculia has predominantly focused on elementary-aged students engaged in basic arithmetic (Lewis & Fisher, 2016). It remains largely unknown what kinds of difficulties students may experience when encountering more complex mathematics, like algebra. This is a critical omission because algebraic reasoning is qualitatively different than arithmetic (e.g., Carraher & Schliemann, 2007; Kaput, 2008; Kaput et al., 2008; Kieran, 1992; Stephens et al., 2013), quantities are represented abstractly in a variety of forms (Kaput et al., 2008; Kieran, 1992), and failure to pass algebra can limit students' academic and career opportunities (Adelman, 2006).

Large-scale studies of students with dyscalculia in algebra are not currently feasible because of difficulties in accurately identifying students with dyscalculia. Researchers emphasize the importance of differentiating between students with dyscalculia and students who have mathematical *difficulties* that are due to environmental, language, instructional, or affective factors (Lewis & Fisher, 2016; Mazzocco, 2007; Mazzocco & Myers, 2003). Researchers also argue that it is essential to differentiate students with dyscalculia from other disabilities (e.g., dyslexia) who may struggle with math, because these students have different cognitive profiles (Lyon et al., 2003) and conflating these groups of learners may mask unique characteristics of each (Mazzocco & Myers, 2003). To establish whether students' low mathematics achievement is due to cognitive or noncognitive factors, researchers often use longitudinal designs (e.g., Geary et al., 2012; Mazzocco & Myers, 2003; Mazzocco et al., 2013) or work with adult learners (e.g., Lewis, 2014; Lewis & Lynn, 2018). For example, in the context of fractions, longitudinal research has found that the difficulties experienced by students with dyscalculia are qualitatively different than low achieving students (Mazzocco & Devlin, 2008), and that these difficulties have been found to persist over years (Mazzocco et al., 2013). Detailed analyses of adults with dyscalculia have demonstrated that these difficulties may be due to persistent, unconventional understanding and use of standard mathematical tools, which suggests that all mathematical tools are not equally accessible for students with dyscalculia (Lewis, 2014; 2016; 2017; Lewis et al., 2020). Although both studies of adults with dyscalculia and those with a longitudinal design have identified characteristic patterns of reasoning students with dyscalculia demonstrate in fractions (Lewis 2016; Lewis et al., 2022; Mazzocco et al., 2013), no similar studies have been conducted in algebra.

To extend work on dyscalculia into algebra, we conducted a detailed analysis of an adult learner with dyscalculia ("Melissa") as she engaged in a weekly videorecorded one-on-one design experiment focused on algebra. We adopt an anti-deficit theoretical orientation to disability (Vygotsky 1929/1993), and we identify the understandings she relied upon rather than interpreting her data through a deficit frame. A detailed analysis of 19 weekly hour-long videorecorded sessions suggests that the student relied upon unconventional understandings of algebraic symbols. This exploratory case study provides new insights into the character of difficulties that arose and persisted for one student with dyscalculia in the context of algebra and suggests the utility of documenting the unconventional understandings that students with dyscalculia persistently rely upon.

In this section we review research on algebra teaching and learning which has established both the common misconceptions experienced by all students when learning algebra, as well as instructional approaches intended to address these issues. We then present our theoretical framework – grounded in an anti-deficit Vygotskian framing of disability. We conclude by considering how this framing influenced the design decisions for our one-on-one learning environment.

Prior Research on Algebra

In this study, we aimed to extend research on dyscalculia to the mathematical topic of algebra. Algebra is a particularly appropriate content area to explore dyscalculia because algebra is representationally and conceptually far more complex and abstract than arithmetic (Kaput, 2008). Kaput (2008) defines algebraic reasoning as generalizations within a conventional symbol system and syntactically guided action on those symbols. Because students with dyscalculia have difficulty both using symbols to represent quantities and manipulating those quantities in arithmetic (Piazza et al., 2010) – it is critical that we begin to explore how these difficulties emerge in algebra when symbol use and manipulation is core to the mathematical activity. Fortunately, research with *nondisabled* students offers considerable insight into the nature of common student difficulties and a wealth of instructional approaches for addressing these difficulties (e.g., Carraher & Schlieman, 2007). For example, using real world problems, manipulatives, and two-sided scale models have been recommended to support students' understanding of unknowns, equality, and algebra (e.g., Common Core State Standards, National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010; van de Walle et al., 2016). Although common difficulties and effective instructional approaches have been identified for nondisabled students, it is unclear what kinds of unique difficulties students with dyscalculia may experience, as well as which mathematical representations and tools may be inaccessible.

Theoretical Perspective – Reconceptualizing Dyscalculia as Difference

Although dyscalculia is typically conceptualized in terms of cognitive *deficits* (e.g., Geary, 2010), we argue that it is more productive to conceptualize dyscalculia in terms of cognitive *difference.* Our perspective is derived from a Vygotskian perspective of disability (Vygotsky, 1929/1993). Vygotsky argued that mediational signs and tools (e.g., language, symbols), which developed over the course of human history, were often incompatible with the biological development of children with disabilities (Vygotsky, 1929/1993). For example, the mediational tool of *spoken* language is not accessible to a Deaf child, and therefore does not serve the same role in supporting the child's development of language as it would for a hearing child. In the case of students with dyscalculia, it is possible that standard mathematical mediational tools (e.g., numerals, graphs, equations), which support the mathematical development of most students, may be incompatible with how a student with dyscalculia cognitively processes numerical information (e.g., Piazza et al., 2010). Students may have difficulties accessing and using these standard tools and may understand representations and symbols in unconventional ways. Although all students may use standard tools in unconventional ways as they are first learning a topic, we propose that students with dyscalculia may experience *persistent* incommensurability because of the inaccessibility of these mathematical mediators. Therefore, in this study, we identify unconventional use or understanding of standard mathematical tools that persist across problems and contexts – we term these *persistent understandings*.

Disability Through the Lens of a Design Experiment

In this study, we capture the student's attempts to learn during a design experiment (Cobb et al., 2003). A design experiment involves engineering learning environments and systematically studying the forms of learning (Cobb et al., 2003). In this design experiment not only do we capture the student's unconventional understandings as she is engaged in attempts to learn, but we attempt to design instructional approaches which address her difficulties. It is through the iterative cycles of design, enactment, and analysis that we can understand both the student's unconventional understandings and what instructional approaches are accessible for the student. The outcome of this design experiment is not a recommendation for a particular sequence of instructional activities or tools. Instead, the design experiment serves as the context through which we are able to better understand the kinds of inaccessibility this student with dyscalculia experienced in mathematics and what kinds of tools were more accessible. In this paper we focus on identifying the kinds of unconventional understandings of mathematical mediators that the student persistently relied upon during the design experiment (persistent understandings).

Methods

Case Study Participant History

Melissa was a 31-year-old, woman, native English speaker, who identified as half Black and half White. We recruited Melissa from her pre-college mathematics class at community college. All students in the pre-college mathematics class were given a written fractions assessment (Lewis et al., 2022), which has been shown to identify students who demonstrate unconventional understandings, characteristic of dyscalculia (Lewis et al., 2022). Melissa demonstrated unconventional understandings on this assessment and was invited to participate in an interview, formal assessment, and design experiment focused on algebraic concepts. On the Woodcock-Johnson Test of Achievement IV (WJ-IV; Schrank et al., 2014), Melissa composite math score was at the 19^{th} percentile – which is below the 25^{th} percentile – the most commonly used cutoff for determining dyscalculia eligibility (Lewis & Fisher, 2016). An interview revealed that

Melissa had a long history of difficulties with mathematics through school, despite having sufficient resources (e.g., private tutor). She had repeatedly failed pre-college mathematics classes at the community college. She reported doing all her homework and practicing problems "over and over and over again," but she still struggled to understand the content. She did not pass this class. She explained, "how my mind processes it, is quite different than the average person. It seems easy for other people, but for me you have to explain it in a different way." She explained that she did well in all her other classes, "it's just math that gets me." Given Melissa's unconventional fraction understandings, low mathematics achievement score, and her history of continued mathematics failure despite sufficient resources, Melissa meets the dyscalculia criteria.

One-on-One Design Experiment

We conducted 19 videotaped design experiment sessions with Melissa. This design experiment involved iterative microcycles of design, enactment, and analysis (Gravemeijer & Cobb, 2006). Each microcycle involved designing and enacting an individual, one-on-one instructional session and then analyzing a video of the session in order to design the subsequent session. The first and third authors participated in the design microcycles and the first author ("Kelly") was the tutor for all sessions. The goal of these sessions was to identify the ways in which Melissa used mathematical tools in unconventional and problematic ways and to provide Melissa with alternative mediational tools to support her understanding (for more details about this iterative approach to design see Lewis et al., 2020). In designing alternative mathematical mediators we (a) drew upon prior research on the teaching and learning of algebra with nondisabled students (e.g., Kieran, 2007), (b) leveraged instructional recommendations offered by an adult with dyscalculia who developed ways of compensating (Lewis & Lynn, 2018) and (c) built upon Melissa's intuitive notations about mathematics and what she reported was more or less effective for her. In our design we aimed to provide Melissa with mediators that would help support a conventional understanding of algebra, specifically solving for an unknown, which is a core algebraic concept.

Retrospective Analysis

After the conclusion of data collection, we began our retrospective analysis. We transcribed all video recordings and scanned all written artifacts. We parsed each transcript into individual problem instances, which began with a question and ended with a student answer. The first and second author iteratively reviewed videos of each of the sessions and generated and refined analytic categories that captured the nature of the student's understanding. We produced operational definitions for persistent understandings, specifying inclusion criteria and identifying prototypical examples of each. Five persistent understandings were related to Melissa's understanding of algebra (see Lewis et al., 2020 for a description of persistent understandings associated with integer operations). Three coders (first, fourth, and fifth authors) systematically coded each problem instance for correctness, problem type, instructional approach, mediational tools, and any persistent understandings. Each problem instance was coded by at least 2 coders. Reliability for the coding of the 5 algebraic persistent understandings was 95.4%. Any discrepancies in coding were resolved during our weekly research team meetings by rewatching the video and discussing whether there was sufficient evidence to warrant the attribution of that operational definition (for a similar approach see Schoenfeld et al., 1993; Lewis, 2014).

Findings

The detailed analysis of video recordings revealed a collection of five persistent algebraic understandings that reoccurred, were unconventional, and led to difficulties. These persistent understandings were related to (1) the value of unknowns, (2) the equal sign, (3) coefficients, (4) the meaning of "x=", and (5) the value of zero. We begin by providing a high-level overview of the unconventional understandings and data about their prevalence throughout the sessions, which indicate that these unconventional understandings were often associated with an incorrect answer. We then dive into detail for the first unconventional understanding to demonstrate how this unconventional understanding emerged in the data. We then present a problem instance taken from the first session which shows how multiple unconventional understandings sometimes occurred in tandem and led to significant difficulties.

We identified five persistent unconventional understandings based on the detailed analysis of Melissa's video recorded data, these include:

1. **Expansive and static view of unknowns** - When working with unknowns/variables, Melissa asserted that any non-numeral symbol (e.g., +, =) was a variable, that x could be different values in the same problem, and that x was a static value equal to 1.
2. **Equal sign as a bridge** - When working with equations, Melissa treated the equal sign as a symbol which indicated the result of a calculation, so she often used intermediate equal signs between solution steps or had equations with more than 1 equal sign. She often made invalid transformations of the equations moving terms from one side to the other across "the bridge", and did not object to invalid equalities (e.g., -4=5).
3. **Unconventional manipulation of coefficients** - When working with coefficients, Melissa often assumed an additive relationship between the coefficient and the unknown and would subtract the coefficient away from the unknown (e.g., 6x-6=x). At other times, she would resolve coefficients by dividing by the entire term, rather than the coefficient value (e.g., to solve 6x=12, diving by 6x rather than 6).
4. **x = the answer** - When working with unknowns, Melissa understood x to equal the answer, so would often ignore the location of x in an equation and tack "x=" in front of the answer she calculated, regardless of whether it represented the value of the unknown.
5. **Zero is not a value** - When solving algebra problems, Melissa treated the value 0 as if it were not a quantity. At times she treated like any other constant, (subtracting it from both sides of the equation) and at other times she argued that it was not a valid value.

Within these 19 individual sessions, 427 problem instances involved algebra content. Of the 427 problem instances, 186 were coded as incorrect (44%) and 90% of these incorrect answers were associated with an unconventional persistent understanding. Indeed, in problems where Melissa relied upon these unconventional persistent understandings, she often produced an incorrect answer (see Table 1).

Table 1: Prevalence of Persistent Understandings and Correctness of Problem Instances

Persistent Understanding	Number of problem instances	Associated with an incorrect answer
1. Expansive and static view of unknowns	80	80%
2. Equal sign as a bridge	110	59%
3. Unconventional manipulation of coefficients	70	74%
4. x = the answer	46	74%
5. Zero is not a value	13	61%

These five persistent understandings (along with the integer persistent understandings, Lewis et al., 2020) provided a relatively comprehensive explanatory frame for the difficulties that the student experienced. We now provide details for one persistent understanding, to illustrate how this emerged over the course of the sessions.

Persistent Understanding #1 - Expansive and Static View of Unknowns

The first persistent understanding – ***expansive and static view of unknowns*** – involved an unconventional understanding of algebraic unknowns and their values, that was both overly expansive and overly static. Melissa was overly *expansive* in her definition of unknowns, in that she used the term "unknown" or "variable" to refer to any non-numeral mathematical symbol. She explained, "a variable is a… an unknown number," and that a variable could be an "addition or subtraction problem" and then identified a whole range of different mathematical symbols (e.g., +, π, $[, x, m, \div, <, =$) as variables. She explained, "[The equal sign] is a variable, as well as an *x* is a variable, or a plus is a variable." In addition to treating all symbols as variables, she was also overly expansive about her understanding of unknowns in that she believed that an unknown (e.g., *x*) could be different values within the same problem (e.g., 2*x*+4=3*x*), and argued that even after solving for x, that that unknown could still be anything.

Although she was often overly expansive in her understanding of unknowns, she also demonstrated a *static* view of unknowns, and often asserted that unknowns were equal to 1. She explained, "The rule of x is 1, that's most common unknown, in other words, for x to be 1." For example, during one session Kelly asked her to write a value that was greater than x. She incorrectly determined that 2 was larger than x and explained, "because I look at x, or a letter, as 1. And 2 is just bigger."

Believing that unknowns were static values equal to 1 was sometimes problematic when she attempted to solve for *x*. For example, when asked to solve 12=*x*+5, she replaced the x with a 1 simplifying the equation to 12=6, then divided both sides by 6 to get an answer of 2. Melissa's static understanding of x, being equal to 1, was used in this example to create an invalid equation 12=6. She did not find this to be problematic and continued to procedurally manipulate the values, as if she was still solving for x, determining that the answer was 2.

In another example, when she solved the problem x+3=8, she correctly determined that x=5, but when Kelly asked her what it meant that x=5 (a standard question), she explained that "one equaled five [*writes 1=5; see Figure 1*] because I see x as 1." In this instance, even though she had just determined that x was equal to 5, her understanding that x=1 emerged. This resulted in her creating an invalid equality 1=5, which she did not find problematic.

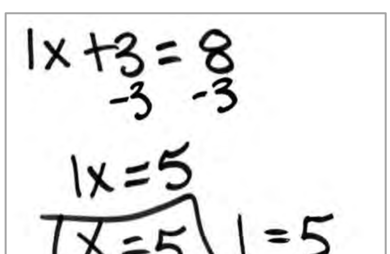

Figure 1. Melissa's written work to solve the problem x+3=8

Both Melissa's understanding of x as a static value, equal to 1, and her overly expansive understanding of unknowns, which involved believing that x could be any value, even after determining the value of x, led to her unconventional use of unknowns and errors across the sessions. This persistent understanding was evident in 80 problem instances across the sessions, and 80% of the time was associated with an incorrect answer. A similar pattern of prevalence and errors were found for the other 4 persistent understandings (see Table 1).

Persistent Understandings Occurring in Tandem

To illustrate how these persistent understandings often appeared together in the same problem, we illustrate how Melissa relied upon several persistent understandings as she solved the problem x/2+7=10. This episode was taken from the first instructional session as Kelly tried to assess her existing strategies for solving for x. The prevalence of these persistent understandings in this first section suggests that Melissa came to this design experiment with these persistent understandings. In Figure 2 we present the student's work, along with the persistent understanding identified and the rationale for that attribution. This example illustrates how Melissa's persistent understandings often occurred together and resulted in difficulties.

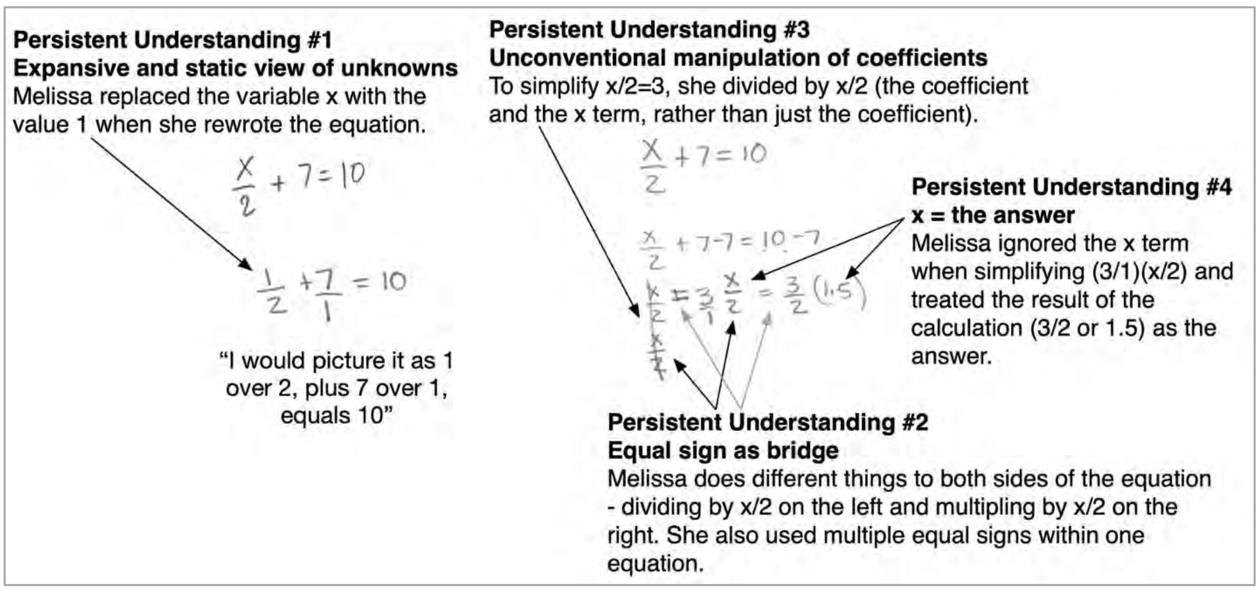

Figure 2. Melissa's written work, and illustration of multiple persistent understandings within one solution process.

Discussion

This detailed case study of Melissa, an adult student with dyscalculia, found that she had persistent unconventional understanding of standard mathematical symbols (e.g., unknowns, the equal sign, coefficients, x=, and zero). Unlike the kinds of difficulties that all students experience when first learning a topic, these unconventional understandings were persistent. We argue that the persistence of these unconventional understandings suggests that these standard mathematical tools – used to represent quantities and relationships between quantities – were at least partially inaccessible to the student. In other work (Lewis et al., in press) we explore how Melissa's persistent understandings interacted with tools specifically designed to provide her with increased access. Here we note that the persistent understandings continued to emerge as Melissa engaged with new tools, but reorientation to the tools enabled Melissa to reason in more conventional ways.

This detailed case study extends prior research on dyscalculia in several important ways. First, this research demonstrates how number processing difficulties found in younger students with dyscalculia (Landerl, 2013; Rousselle & Noël, 2007), occur in older students engaged in algebraic reasoning. Just as prior research has demonstrated that students with dyscalculia are slower and more error prone when asked to compare or manipulate *arithmetic* quantities (e.g., Desoete et al., 2012), Melissa often made errors (44% of algebra problems were incorrect) and she experienced persistent difficulties understanding, comparing, representing, and manipulating *algebraic* quantities. This study, therefore, extends findings that have been documented in students with dyscalculia, and begins to identify how these difficulties would emerge in an algebraic context. This kind of detailed case study can enable researchers to begin to explore mathematical topic domains beyond basic arithmetic, and provides much needed insight into dyscalculia across mathematical topic domains.

Second, this study offers an anti-deficit framing of dyscalculia by providing a detailed depiction of a student engaged in the process of learning and doing mathematics, rather than describing the student's performance from a deficit frame in terms of speed and accuracy. Unlike prior research on dyscalculia which infers learning difficulties based on patterns of errors on outcome measures (e.g., Bouck et al., 2016; Mazzocco et al., 2008), this study explored the student's reasoning underlying these errors. This study documented the ways in which Melissa was understanding, representing, and manipulating quantities, while engaged in problem solving. This anti-deficit framing is critical for making progress in the field towards accurate identification of dyscalculia and offers new avenues to explore for re-mediation.

Future research is needed to determine whether these understandings identified in this study are unique to Melissa, or if they are typical of students with dyscalculia. Building from case study research to large scale studies to examine the prevalence of these characteristics and to develop screening measures to accurately screen for dyscalculia has been demonstrated in the domain of fractions (Lewis et al., 2022).

Conclusion

This exploratory case study provides new insights into the character of difficulties that arose and persisted for one student with dyscalculia in the context of algebra. Findings suggests the utility of documenting the persistent understandings that students with dyscalculia rely upon to understand how dyscalculia may impact students learning of algebra. Beginning to understand dyscalculia in algebra is critical, as algebra often acts as a gate keeper, like it did for Melissa, limiting students' academic and career opportunities.

Acknowledgments

This research was supported by a post-doctoral grant from the National Academy of Education / Spencer Foundation. The opinions expressed are those of the authors and do not represent views of the National Academy of Education / Spencer Foundation.

References

Adelman, C. (2006). *The toolbox revisited: Paths to degree completion from high school through college*. Washington, DC: U.S. Department of Education.

Booth, L. R. (1984). *Algebra: Children's strategies and errors: A report of the strategies and errors in secondary mathematics project*. NFER-Nelson Publishing Co.

Bouck, E. C., Bouck, M. K., Joshi, G. S., & Johnson, L. (2016). Understanding problem-solving errors by students with learning disabilities in standards-based and traditional curricula. *Learning Disabilities: A Multidisciplinary Journal*, *21*(1), 14–26.

Butterworth, B. (2010). Foundational numerical capacities and the origins of dyscalculia. *Trends in Cognitive Sciences*, *14*(12), 534–541. https://doi.org/10.1016/j.tics.2010.09.007

Carraher, D. W., & Schliemann, A. D. (2007). Early algebra and algebraic reasoning. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol. 2, pp. 669–705). Reston, VA: National Council of Teachers of Mathematics.

Cobb, P., Confrey, J., & diSessa, A. (2003). Design experiments in educational research. *Educational Researcher*, *32*(1), 9–13. https://doi.org/10.3102/0013189X032001009

Cobb, P., Jackson, K., & Dunlap, C. (2017). Conducting design studies to investigate and support mathematics students' and teachers' learning. In J.Cai (Ed.), *Compendium for research in mathematics education (pp. 208-236)*. Reston, VA: National Council of Teachers of Mathematics.

Desoete, A., Ceulemans, A., De Weerdt, F., & Pieters, S. (2012). Can we predict mathematical learning disabilities from symbolic and non-symbolic comparison tasks in kindergarten? Findings from a longitudinal study. *British Journal of Educational Psychology*, *82*(1), 64–81. https://doi.org/10.1348/2044-8279.002002

Duane, D. D. (1979). Toward a definition of dyslexia: A summary of views. *Bulletin of the Orton Society*, *29*, 56–64. https://doi.org/10.1007/BF02653733

Filloy, E., & Rojano, T. (1989). Solving equations: The transition from arithmetic to algebra. *For the Learning of Mathematics*, *9*(2), 19–25.

Geary, D. C. (2010). Mathematical disabilities: Reflections on cognitive, neuropsychological, and genetic components. *Learning and Individual Differences, 20*(2), 130–133. https://doi.org/10.1016/j.lindif.2009.10.008

Geary, D. C., Hoard, M. K., Nugent, L., & Bailey, D. H. (2012). Mathematical cognition deficits in children with learning disabilities and persistent low achievement: A five year prospective study. *Journal of Educational Psychology, 104(1)*, 206–223. https://doi.org/10.1037/a0025398

Gravemeijer, K., & Cobb, P. (2006). Design research from a learning design perspective. In J. Van den Akker, K. Gravemeijer, S. McKenney & N. Nieveen (Eds.), *Educational design research* (pp. 17-51). London: Routledge.

Grünke, M., Hord, C., & Macdonald, S. J. (2021). A rationale for the use of case reports in special education: The significance of detailed descriptions of assessment and intervention scenarios for bridging the research-to-practice gap. *Insights into Learning Disabilities*, *18*(1), 91–101.

Kaput, J. J. (2008). What is algebra? What is algebraic reasoning? In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 5–17). New York, NY: Erlbaum.

Kaput, J. J., Blanton, M. L., & Moreno, L. (2008). Algebra from a symbolization point of view. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 19–55). New York, NY: Erlbaum.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational studies in Mathematics*, *12*(3), 317-326.

Kieran, C. (1992). The learning and teaching of school algebra. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390–419). New York, NY: Macmillan.

Kieran, C. (2006). Research on the learning and teaching of algebra: A broadening of sources of meaning. In A. Gutiérrez, P. Boero, A. Gutiérrez (Ed), & P. Boero (Ed) (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future.* (pp. 11–49). Rotterdam, Netherlands: Sense Publishers.

Kieran, C. (2007). Learning and teaching algebra at the middle school through college levels: Building meaning for symbols and their manipulation. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol. 2, pp. 669–705). Reston, VA: National Council of Teachers of Mathematics.

Landerl, K. (2013). Development of numerical processing in children with typical and dyscalculic arithmetic skills—A longitudinal study. *Frontiers in Psychology*, *4*. doi: 10.3389/fpsyg.2013.00459

Lange, K. W., Reichl, S., Lange, K. M., Tucha, L., & Tucha, O. (2010). The history of attention deficit hyperactivity disorder. *ADHD Attention Deficit and Hyperactivity Disorders*, *2*(4), 241–255. https://doi.org/10.1007/s12402-010-0045-8

Lewis, K. E. (2014). Difference not deficit: Reconceptualizing mathematical learning disabilities. *Journal for Research in Mathematics Education*, *45*(3), 351–396. https://doi.org/10.5951/jresematheduc.45.3.0351

Lewis, K. E. (2016). Beyond error patterns: A sociocultural view of fraction comparison error patterns in students with mathematical learning disabilities. *Learning Disability Quarterly 39*(4), 199-212. https://doi.org/10.1177/0731948716658063

Lewis, K. E. (2017). Designing a bridging discourse: Re-mediation of a mathematical learning disability. *Journal of the Learning Sciences*, *26*(2), 320–365. https://doi.org/10.1080/10508406.2016.1256810

Lewis, K. E., & Lynn, D. M. (2018). Access through compensation: Emancipatory view of a mathematics learning disability. *Cognition & Instruction*, *36*(4), 424–459. https://doi.org/10.1080/07370008.2018.1491581

Lewis, K. E., Sweeney, G., Thompson, G. M., & Adler, R. (2020). Integer number sense and notation: A case study of a student with a mathematics learning disability. *Journal of Mathematical Behavior, 59, 1-27.* doi: 10.1016/j.jmathb.2020.100797

Lewis, K. E., Sweeney, G., Thompson, G. M., Adler, R. M., and Alhamad, K. (in press). Dyscalculia in algebra: A case study. *Insights into Learning Disabilities.*

Lewis, K. E., Thompson, G. M. & Arvey, S. A. (2022). Screening for characteristics of dyscalculia: Identifying unconventional fraction understandings. *International Electronic Journal of Elementary Education 14*(3), 243-267. doi: 10.26822/iejee.2022.242

Lyon, G. R., Shaywitz, S. E., & Shaywitz, B. A. (2003). A definition of dyslexia. *Annals of Dyslexia*, *53*, 1–14.

Mazzocco, M. M. M. (2007). Defining and differentiating mathematical learning disabilities and difficulties. In D. B. Berch, & M. M. M. Mazzocco (Eds.), *Why is math so hard for some children? The nature and origins of mathematical learning difficulties and disabilities* (pp. 29–48). Baltimore, MD: Paul H. Brookes.

Mazzocco, M. M. M., Devlin, K. T., & McKenney, S. J. (2008). Is it a fact? Timed arithmetic performance of children with mathematical learning disabilities (MLD) varies as a function of how MLD is defined. *Developmental Neuropsychology, 33*(3), 318–344. https://doi.org/10.1080/87565640801982403

Mazzocco, M. M. M., & Myers, G. F. (2003). Complexities in identifying and defining mathematics learning disability in the primary school-age years. *Annals of Dyslexia*, *53*, 218–253. https://doi.org/10.1007/s11881-003-0011-7

Mazzocco, M. M. M., Myers, G. F., Lewis, K. E., Hanich, L. B., & Murphy, M. M. (2013). Limited knowledge of fraction representations differentiates middle school students with mathematics learning disability (dyscalculia) versus low mathematics achievement. *Journal of Experimental Child Psychology, 115*(2)*,* 371–387. https://doi.org/10.1016/j.jecp.2013.01.005

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). Common Core State Standards for Mathematics. Washington, DC: Author. Retrieved from http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

Piazza, M., Facoetti, A., Trussardi, A. N., Berteletti, I., Conte, S., Lucangeli, D., … Zorzi, M. (2010). Developmental trajectory of number acuity reveals a severe impairment in developmental dyscalculia. *Cognition*, *116*(1), 33–41. https://doi.org/10.1016/j.cognition.2010.03.012

Rousselle, L., & Noël, M.-P. (2007). Basic numerical skills in children with mathematics learning disabilities: A comparison of symbolic vs non-symbolic number magnitude processing. *Cognition*, *102*(3), 361–395. https://doi.org/10.1016/j.cognition.2006.01.005

Schoenfeld, A. H., Smith, J. P., III, & Arcavi, A. (1993). Learning: The microgenetic analysis of one student's evolving understanding of a complex subject matter domain. In R. Glaser (Ed.), *Advances in instructional psychology* (pp. 55–175). Hillsdale, NJ: Lawrence Erlbaum Associates.

Schrank, F. A., Mather, N., & McGrew, K. S. (2014). Woodcock-Johnson IV tests of achievement. *Rolling Meadows, IL: Riverside*.

Shalev, R. S. (2007). Prevalence of developmental dyscalculia. In D. B. Berch & M. M. M. Mazzocco (Eds.), *Why is math so hard for some children? The nature and origins of mathematical learning difficulties and disabilities* (pp. 49–60). Baltimore, MD: Paul H. Brookes.

Stephens, A. C., Knuth, E. J., Blanton, M. L., Isler, I., Gardiner, A. M., & Marum, T. (2013). Equation structure and the meaning of the equal sign: The impact of task selection in eliciting elementary students' understandings. *Journal of Mathematical Behavior*, *32*(2), 173–182. https://doi.org/10.1016/j.jmathb.2013.02.001

Van de Walle, J., Karp, K., & Bay-Williams, J. (2016). *Elementary and Middle School Mathematics: Teaching Developmentally* (9th ed.). Boston, MA: Pearson.

Verhoeff, B. (2013). Autism in flux: A history of the concept from Leo Kanner to DSM-5. *History of Psychiatry*, *24*(4), 442–458. https://doi.org/10.1177/0957154X13500584

Vlassis, J. (2002). The balance model: Hindrance or support for the solving of linear equations with one unknown. *Educational Studies in Mathematics*, *49*(3), 341–359.

Vygotsky, L. S. (1993). Introduction: Fundamental problems of defectology. In R. W. Rieber & A. S. Carton (Eds.), *The collected works of L. S. Vygotsky: Vol. 2. The fundamentals of defectology (Abnormal psychology and learning disabilities)* (J. E. Knox & C. Stevens, Trans., pp. 29-51). New York, NY: Plenum Press. (Original work published 1929)

Wolff, S. (2004). The history of autism. *European Child & Adolescent Psychiatry*, *13*(4), 201–208. https://doi.org/10.1007/s00787-004-0363-5