PLAYFUL MATH: MODELING STUDENTS' ENGAGEMENT IN PLAY-BASED ALGEBRA ACTIVITIES

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Interest-driven activities, such as mathematical play, can support student agency, motivation, and engagement, and can foster dispositions that reflect authentic disciplinary engagement. However, the bulk of research on mathematical play investigates the mathematics that emerges in young children's natural play or in informal spaces such as video games. We introduce the term "playful math" to highlight the potential of playifying classroom-based activities, and we explore the nature of students' activity when engaged in playful math tasks in a teaching experiment. Our findings show that playful math tasks increased students' agency, authority, investment, and goal selection, as well as encouraged the development of creative, challenging ideas. We present a case of two students' playful engagement in the form of an Explore-Strategize Cycle and discuss implications of playful math for student engagement.

Keywords: Algebra and Algebraic Thinking, Cognition, Problem-Based Learning

Motivation and engagement are critical factors in supporting students' abilities to understand and persist in mathematics (e.g., Durksen et al., 2017). Students' experiences of self-efficacy can foster motivation, which in turn can predict persistence in STEM fields (Simon et al., 2015). There remain, however, ongoing challenges with student motivation and engagement in mathematics (Martin & Marsh, 2006). These challenges are particularly salient in algebra, which represents a critical transition to secondary mathematics, and can act as a gatekeeper, with algebra performance often serving as the main criterion to determine a student's readiness for more advanced courses (Riegle-Crumb, 2006).

One contributing factor to these challenges is that the mathematics taught in algebra can emphasize routinized procedures, resulting in students reporting decreased motivation and engagement (Herzig, 2004). In contrast, focus on interest-driven activities such as mathematical play invites student agency and can increase equitable access to algebra (Widman et al., 2019). In fact, much of authentic disciplinary engagement involves many of the same features as play (Gresalfi et al., 2018; Jasien & Horn, 2018), and professional mathematicians have been shown to engage in mathematical play as part of their disciplinary practice (Lockhart, 2009).

Mathematical play has the potential to support a productive environment for conjecturing and exploring by centering student voices and by offering opportunities to investigate novel ideas (Gresalfi et al., 2018). It can also offer important engagement, motivation, and conceptual benefits, with studies suggesting positive effects for enjoyment, attitudes, and learning outcomes (e.g., Barab et al., 2010; Plass et al., 2013; Wager & Parks, 2014). However, the bulk of existing research on mathematical play is situated either in early childhood learning, or in informal spaces such as video games. Mathematical play can certainly occur in classroom settings, but less is understood about how to incorporate play into the school mathematics that students and teachers navigate in classroom settings, particularly for adolescents. Therefore, this study investigates what occurs when incorporating play-based elements in algebra problem-solving tasks. We use the term "playful math", rather than "mathematical play", to highlight the potential of "playifying" classroom mathematical activity. In particular, we address the following research questions: (1) What characterizes students' mathematical activity when investigating rates of change within play-based activities? (2) How does playifying mathematical tasks affect the nature of students' engagement, if at all?

Background Literature: The Potential Benefits of Playful Math

Mathematics is an area for which interest-driven engagement is not always a consideration in pedagogical design, and yet, focus on interest-driven activities can invite student agency and create spaces for students to bring in more of their whole identities (Widman et al., 2019). Playful math can offer multiple points of access into mathematical ideas, providing opportunities for students to create new challenges for themselves and experiment with ideas that go beyond familiar operations and connections (Featherstone, 2000). Playful engagement can also free students from the stigma of traditional assessment, encouraging a disposition of exploration and innovation (Barab et al., 2010). Research investigating children's and adolescents' playful math suggests both affective and conceptual benefits. Playful math has been shown to support increased enjoyment (Plass et al., 2013), increased engagement (Barab et al., 2010), positive social interaction and communication (Edo et al., 2009), and to engender positive attitudes towards mathematics (Holton et al., 2001). Playful math has also been shown to offer some cognitive benefits, supporting children's geometric thinking (Levine et al., 2005), spatial skills (Casey et al., 2008; Levine et al., 2012), and number development and numeracy (Siegler & Ramani, 2008; Wang & Hung, 2010). Some studies even suggest increased learning efficacy in play-based environments (Barab & Gresalif, 2010; Bodrova, 2008).

Some benefits afforded by play may occur because playful math offers avenues for students to "conjecture and explore in disciplinarily authentic ways" (Jasien & Horn, 2018, p. 624). In fact, there are a number of features of play that mirror the forms of engagement seen in the work of mathematicians, including open exploration, the use of imagination, being voluntary, being ordered and rule governed, and exercising personal agency to determine and pursue goals (Gresalfi et al., 2018; Featherstone, 2000). These parallels point to the importance of intellectual play in learning mathematics, and suggest a need for research examining ways to make mathematics tasks more playful. In particular, the field needs more research exploring playful math with older students in middle-school, high-school, and college, particularly in terms of incorporating playful elements into classroom mathematics in critical domains such as algebra.

Theoretical Frameworks: Defining Playful Math and Quantitative Reasoning

Definitions of mathematical play vary, but all emphasize students' agency in exploration, self-selection of goals, and self-direction in how to accomplish them (Jasien & Horn, 2018). For instance, Williams-Pierce (2019) defines mathematical play as "voluntary engagement in cycles of mathematical hypotheses with occurrences of failure" (p. 591), and Holton et al. (2001) describe mathematical play as the playful exploration that emerges when learners find themselves in mathematical contexts with an open goal. One challenge in characterizing play is that it describes both a form of activity (such as playing a video game) and a stance or orientation towards an activity (Malaby, 2009). We address the second aspect to define playful math as a particular form of engagement in mathematics, one that entails (a) agency in exploration, (b) self-selection or investment in mathematical goals, (c) self-direction in goal accomplishment, and (d) a state of immersion and/or enjoyment.

For the purposes of this study, we leveraged situations involving covarying quantities that students can manipulate and investigate in order to reason flexibly about function as a representation of dynamically changing events (Carlson et al., 2002). By quantities, we mean schemes composed of a person's conception of an object or event, such as a rectangle; a quality of the object or event, such as the rectangle's length or area; an appropriate unit, such as centimeters or square centimeters; and a process for assigning a numerical value to the quality (Thompson, 1994). By covariation, we refer to the visualization of two quantities' values simultaneously, uniting the quantities' magnitudes in order to understand that at every instance, both quantities have a corresponding value (Saldanha & Thompson, 1998; Thompson & Carlson, 2017). Researchers argue that students naturally attend to coordinated changes (e.g., Blanton & Kaput, 2011), which is supported by studies showing that students typically first analyze functional situations from a coordinated change perspective (e.g., Confrey & Smith, 1995; Madison et al., 2015). Furthermore, we hypothesized that the visualization opportunities afforded by a covariation approach could potentially support the type of open exploration and agency that occurs in playful engagement. As we describe in the next section, we built on students' natural ways of reasoning to modify a set of existing covariation activities in order to playify them.

Methods

Setting and Participants

We conducted two teaching experiments (TEs) (Steffe & Thompson, 2000), which both met weekly for 60-75 minutes a session for 5 consecutive weeks. The first author was the teacher-researcher (TR). Both teaching experiments were video and audio recorded, with the exception of Day 2 for each, when a technical error prevented video recording. After each session, we collected all student written work. The first teaching experiment was a paired TE with two rising 7th-grade students, Stewie and GJ. The second TE was with three participants, Artemis (a rising 7th-grade student) and Apollo and Francis (rising 6th grade students). (Participants chose their own pseudonyms.) Due to schooling interruptions from Covid-19, the students' experiences with graphing and functions was largely limited to plotting points. The rising 7th graders also had some experience with graphing lines, but none of the students had experienced a unit on linear functions. For the purposes of this paper, we report on data from the first TE with Stewie and GJ.

Task Design Principles

Our aim was to playify existing covariation problems in order to investigate students' mathematical activity when exploring rates of change within play-based tasks. By "playify", we mean increasing the potential for playful math engagement, while also acknowledging that a task cannot dictate how students will engage with it. We drew on a set of established research-based activities to support understanding of linear and quadratic growth with continuously covarying quantities (Ellis et al., 2020; Matthews & Ellis, 2018). In these activities, students investigate dynamically growing shapes, determining the rate of change of a shape's area compared to its changing length as it sweeps out from left to right, and then graph that relationship (Figure 1).

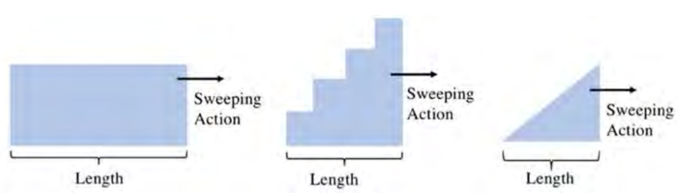

Figure 1: Growing Rectangle, Stairstep, and Triangle

We followed four design principles to playify the tasks (Plaxco et al., 2021): (1) allow for free exploration within constraints; (2) allow the student to act as both designer and player; (3) engender anticipating within the task; and (4) provide a method for authentic feedback. These led to the *Guess My Shape* activity, in which the students created secret shapes of their choice (design principles #1, #2), constructed graphs comparing length and area (design principles #2, #3), and then challenged one another or the TR to determine the shape based on the graph alone (design principles #2, #3, #4). We hypothesized that the playified tasks would encourage playful engagement, without assuming that they would guarantee such engagement.

For Session 1, we only used the extant covariation tasks (Figure 1), which we call standard tasks. For Sessions 2 - 5, we used both the standard tasks and the playified tasks. For each of those sessions, we spent the first two-thirds of the session working with the standard tasks and the last one-third implementing the playified tasks. Figure 2 shows an example of the types of shapes the students encountered and their associated graphs.

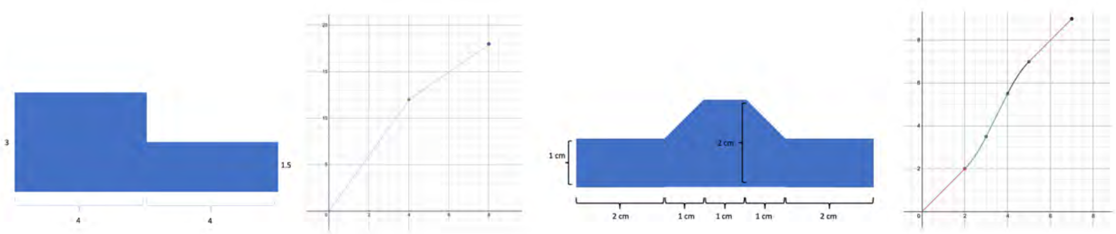

Figure 2: Shapes and associated length-area graphs

Analysis

We employed retrospective analysis (Steffe & Thompson, 2000) in order to characterize the students' conceptions throughout the teaching experiment. We first transcribed each session and then produced a set of enhanced transcripts that included verbal utterances, images of student work, and descriptions of non-verbal actions. We then analyzed the data with two lenses. With the first lens, we identified students' conceptions of graphs, covariation, and rates of change, drawing on Thompson and Carlson's (2017) framework of variational and covariational reasoning, Ellis et al.'s (in press) conceptual acts for constructing linear and quadratic growth through covariation, and Moore and colleagues' graphing actions (Liang & Moore, 2020; Moore et al., 2019; Tasova & Moore, 2020). With the second lens, we identified aspects of playful mathematics, relying on a combination of a priori codes from our definition of playful math, as well as emergent codes that occurred during the coding process through the constant comparative method (Strauss & Corbin, 1990). For the purposes of this paper, we focus on the analysis conducted through the second lens of playful math, and we describe those codes in the next section. The first three authors coded each transcript independently, and then the project team met weekly to refine and adjust codes and resolve discrepancies; this iterative process continued through eight rounds of code adjustments, until all codes had stabilized.

Results: The Explore / Strategize Cycle

Recall that our second research question considered whether and how playifying tasks may affect the nature of students' mathematical engagement. In addressing this question, we turned to our playful math codes to assess whether there was any difference in code frequencies across the two task types. We developed six playful math codes: (a) self-selection or investment in mathematical goals, (b) agency in exploration, (c) investment and/or enjoyment, (d) taking on authority, (e) creative / unusual, and (f) harder math. The first three codes were drawn from our definition of mathematical play. *Self-selection of goals* occurred when students chose their own goals or showed evidence of being invested in a goal. *Agency in exploration* refers to instances of students demonstrating agency in how they explored a task or its goals, and *investment and/or enjoyment* occurred when students showed evidence of investment or immersion, and/or demonstrated enjoyment of their activity. The remaining three codes were emergent. *Taking on authority* refers to instances of students demonstrating mathematical authority with a task or in their engagement with the TR. *Creative/unusual* applies when students have developed an idea, representation, or task that we perceived as creative or novel, and *harder math* occurred when students created a goal or task that introduced challenging mathematics that surpassed the TR's intended set of topics or concepts.

Across the five sessions we implemented 18 standard and 5 playified tasks. In our initial rounds of coding, we applied the relevant code any time we observed a student demonstrating evidence of playful engagement, resulting in a total of 130 code occurrences. We then revisited the codes to determine how many times they occurred during the standard tasks compared to the playified math tasks. We found that our first three codes occurred across both tasks, but with greater frequency in the playified tasks, supporting our hypothesis that the playified tasks would in fact elicit more playful engagement. The last three codes only occurred during playified tasks (Figure 3). The difference in code frequencies is striking given that 78% of the tasks enacted during the teaching experiment were standard tasks.

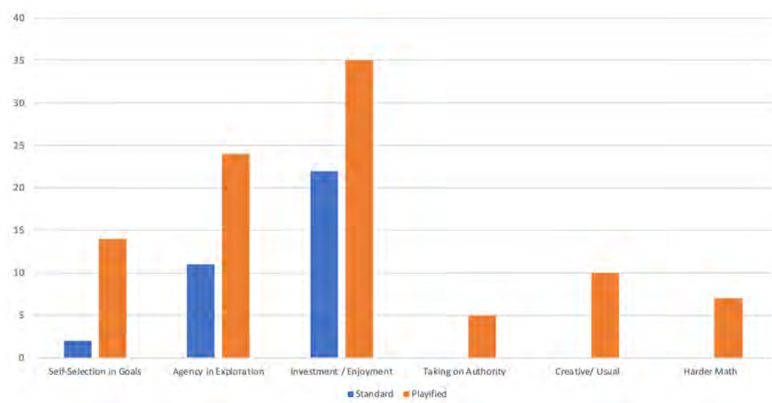

Figure 3: Playful math codes across standard and playified tasks

The difference in code frequencies led us to consider in more detail the nature of students' activity within the playified tasks, addressing our first research question. We found that during those tasks, the students demonstrated a novel form of engagement, which we call the *Explore-Strategize Cycle* (Figure 4). Within this cycle, students shift back and forth between exploration and strategizing. Exploration occurs when students either (a) engage in an action with little or no anticipation of an associated outcome, or (b) notice properties of outcomes and wonder about the connection between those outcomes and the actions that led to them. For example, in the Guess My Shape game, students engaged in exploration when they invented new shapes or when they plotted points arbitrarily to try to develop an unusual graph. In contrast, strategizing occurs when students engage in actions tied to an anticipated outcome. This could include modifying a shape to create a graph that will have a specific property, such as symmetry, or recognizing that a particular shape will be difficult to graph. Once we identified the Explore-Strategize Cycle in students' activity, we revisited the entire data corpus to determine whether this cycle occurred across both task types. We found no instances of the cycle in the standard tasks; instead, during those tasks, students engaged in repeated acts of strategizing but without associated exploration.

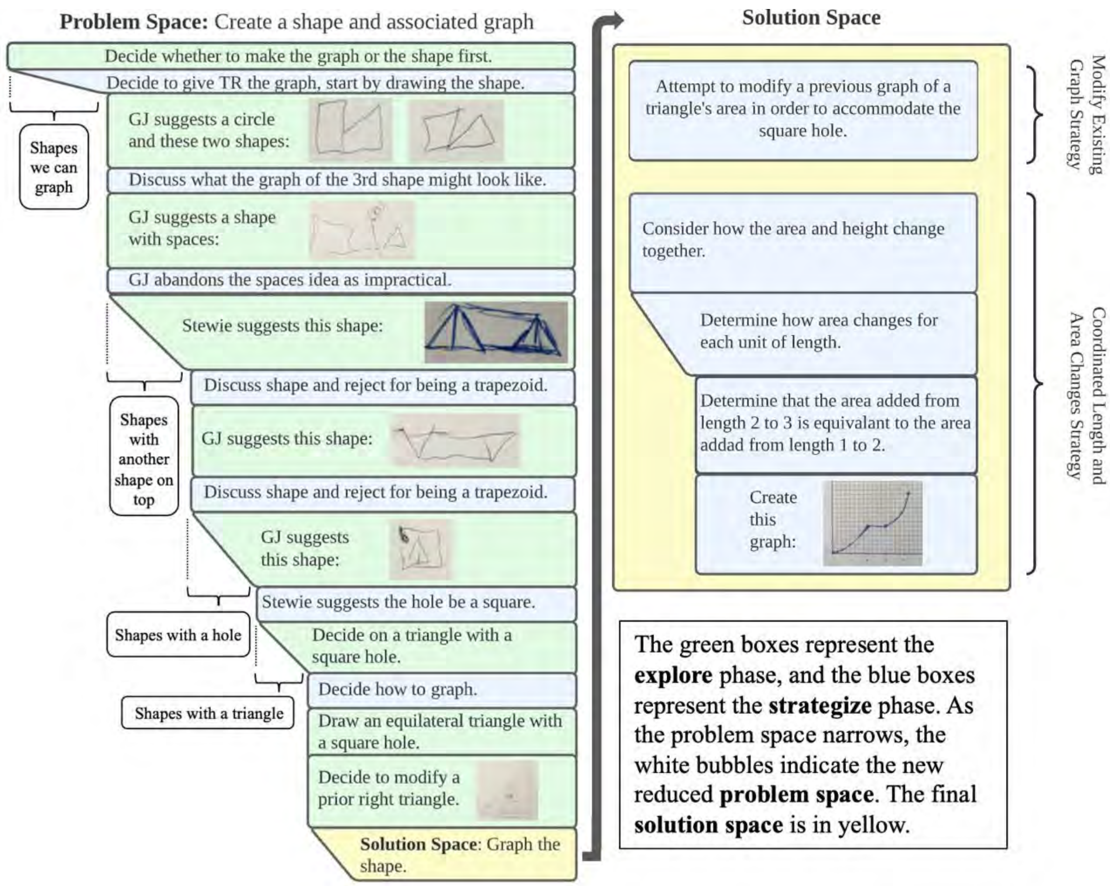

Figure 4: The Explore-Strategize Cycle for A Triangle with a Square Hole

In order to exemplify the nature of students' playful engagement, we present one Explore-Strategize cycle, in which Stewie and GJ collaborated on the Guess My Shape game to create a task that would stump the TR. Within any cycle, there are both problem spaces and solution spaces. As students engage in exploring and strategizing, they narrow the problem space. For instance, when playing Guess My Shape, the set of possible problems are determined by the shapes that the students draw and graph. As they consider what shape to create and how to graph it, they progressively narrow the problem space by exploring characteristics of their invented shape (such as symmetry, the inclusion of a hole, and so forth), or by strategically selecting aspects of the graph or shape to examine. In doing so, the students exclude shapes that do not meet the desired criteria. This can be seen in Figure 4 in the narrowing of the initial problem space, "create a shape and associated graph", down to the final problem they chose to solve, "graph the area and length of this right triangle with a square hole". Once the students have narrowed down to the final problem, they shift into the solution space.

As GJ and Stewie narrowed the problem space, they set implicit and explicit goals to guide their actions. The first decision they had to make was whether to start with a shape or a graph. Stewie suggested that they begin with the graph, "because that would be harder", but GJ wanted to begin by drawing the shape. Together, the students had an implicit goal of creating a shape that was both feasible for them to construct and still challenging for the TR. Stewie strategically negotiated these goals by suggesting that they create the shape first but give the TR the graph. In doing so, he anticipated that determining the shape from the graph would be more difficult than determining the graph from the shape. From this strategizing action, the students have now narrowed the problem space to shapes within their ability to graph.

GJ and Stewie then began to explore a series of potential shapes. Each time, their exploration phase was followed by a strategizing phase in which they tried to anticipate what the associated area-length graph would be. During this process, they rejected multiple shapes as being too difficult to graph, including a circle, composite shapes, and shapes with spaces. They also rejected shapes as being too simple, such as composite shapes that would have area-length graphs identical to trapezoids. GJ then suggested a square with a triangular hole. However, in shifting to considering the associated graph (strategizing phase), the students decided that it would be easier to graph a shape with a square hole. They then explored a triangle shape with a square hole, and then when strategizing how to graph it, anticipated that they could create a graph by partitioning their chosen shape into squares. In exploring final shape options, GJ and Stewie settled on a right triangle, similar to a prior shape they had graphed, but this time with a square hole.

Once the students reached the solution space, they used two different strategies. Their first strategy was to modify a prior graph they had made of a similar triangle without a hole. Stewie said, referring to the constantly-increasing area, "It's going to be a curved line the whole way, and the triangle, but then, what do you do?" Stewie was anticipating that the graph would look different where the hole occurred, but he trailed off, becoming uncertain about how to account for the hole. He then shifted to the second strategy, suggesting that they consider how the area and length change together. In doing so, he quantified the length and area segments, writing "4 in." for both the length and the height of the triangle. He then suggested they determine the area and its change for each unit of length: "At the first point, it was half, all right. For the second one, was one and a half." As he spoke, Stewie began to graph a curve for the first two points. In determining the area from $x = 2$ in. to $x = 3$ in., the portion of the triangle with a 1-in.^{2} hole, both students decided that the area added would be the same as the area added for the prior increment, from $x = 1$ in. to x = 2 in:

Stewie:	So, third one is the same as the second one pretty much.
GJ:	It is.
Stewie:	So, so it would just be like, here. [Draws a horizontal line segment.]

Stewie and GJ anticipated, incorrectly, that the graph would be horizontal if the amount of area added from $x = 2$ to $x = 3$ was the same as the amount of area added from $x = 1$ to $x = 2$. They were correct that the amount of area added was the same across the two increments, but they conflated total accumulated area with added area. When next considering the final increment, from $x = 3$ in. to x = 4 in., Stewie returned to a curve, recognizing that the amount of area constantly increased throughout the increment. It was later in their conversation with the TR that the students realized that the portion of the graph from $x = 2$ in. to $x = 3$ in. would be another curve, rather than a horizontal line segment.

The students demonstrated all six playful math codes during this Explore/Strategize Cycle. They self-selected multiple goals in deciding what types of shapes to pursue, and they also demonstrated agency in exploring those shapes and their associated graphs, freely shifting across different potential shapes. In suggesting novel shapes such as circles, composite shapes, and shapes with spaces and holes, the students demonstrated ideas that were creative and unusual; many of their shapes were markedly different from those that had been previously introduced.

Furthermore, their decision to introduce a hole was not only creative, it also represented a new challenge (harder math), one that took the TR by surprise and brought up a set of mathematical ideas that the TR had not anticipated addressing. In particular, once holes are allowed, area-length graphs no longer uniquely determine a shape, which raises new questions about the set of shapes with holes that can be determined by an area-length graph.

The students also demonstrated immersion, investment, and enjoyment in the task. Their engagement was sustained; they spent 27 minutes on this task alone, and they considered and abandoned many shapes before ultimately settling on a final version. When bringing the TR in to determine the shape from their graph, they actually changed their minds about the accuracy of their graph in the moment. In doing so, the students decided to send the TR away again so that they could re-evaluate. Stewie told the TR, "I think I have an idea. I'm, I'm not totally confident. Yeah, go away." In this moment, the students evidenced investment in their graph, as well as mathematical authority, deciding not only what the graph should be, but also dictating how the process of sharing their work should proceed.

Discussion

Playifying tasks did result in more playful engagement during those tasks; in fact, three forms of playful engagement (taking on authority, creative / unusual, and harder math) occurred only during the playified tasks. However, we still found evidence of playful engagement in the standard tasks, which underscores that playful math as a construct describes a form of engagement, rather than a type of task. Furthermore, although we only observed explore-strategize cycles during the playified tasks, this may be due to the fact that the playified tasks were also more open-ended tasks. More research is needed to tease out the effects of playifying tasks that begin as open-ended tasks. The presence of playful engagement across all tasks suggests that playifying mathematics might support a more playful disposition in general. In our data, we found that one form of playful engagement, investment and/or enjoyment, increased throughout the sessions (occurring six times across the first two days, compared to 34 times on the last two days). It may be that sustained engagement in playful math supported students' increased investment in subsequent tasks, regardless of their form.

One affordance of playful math is that it has the potential to center student voices, particularly those voices less likely to be taken up in more traditional classroom settings. Two of our design principles were particularly important in this role: allow free exploration within constraints, and allow the student to act as both player and designer. By shifting the student's role to designer and encouraging exploration in designing, we opened a space for students to introduce their own ideas in ways that were less constrained by pre-determined topics. Their resulting ideas were novel, creative, and mathematically challenging, and the students demonstrated autonomy and authority. We know that mathematicians, in their own disciplinary practice, also experience this type of autonomy and shift back and forth between exploration and strategizing in a manner similar to what we observed in our Explore / Strategize Cycles (e.g., Lockwood et al., 2016). By playifying mathematics, we set up spaces for more students to adopt productive mathematical dispositions. But more importantly, playful math has the potential to increase overall engagement and enjoyment of mathematics, particularly for students who may not experience traditional mathematics as engaging or enjoyable.

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