UNDERGRADUATE PERSPECTIVES ON THE NATURE OF MATHEMATICS THAT ARISE THROUGH EXPLORATION OF UNSOLVED CONJECTURES

Jeffrey Pair
California State University Long Beach
Jeffrey.Pair@csulb.edu

Gabe Calva
California State University Long Beach
Gabe.Calva@student.csulb.edu

We report on a research study conducted within a transition-to-proof course for mathematics majors at a large public university. Within the course, students explored famously unproven conjectures and reflected on how their perspectives of mathematics changed through this exploration (if at all). In this report we share students' takeaways from the project. For instance, some students experienced mathematics as a creative subject for the first time, as they tried their own methods to solve the conjectures; other students reflected on developing a greater understanding of the behind the scenes work of mathematicians that goes into mathematical creation. We also report on the subjective emotional experiences of the students, which ranged from frustration from being unable to find patterns to enjoyable exploration.

Keywords: Affect, Emotion, Beliefs, and Attitudes; Reasoning and Proof; Undergraduate Education

Within mathematics education scholarship, authors have lamented that students do not experience mathematics in ways similar to those experienced by practicing mathematicians (Boaler 2015; Lampert, 1990; Lockhart, 2009; Hersh, 1997). After describing the creative activity of research mathematicians, Boaler (2016) wrote, "I strongly believe that if school math classrooms presented the true nature of the discipline, we would not have this nationwide dislike of math and widespread math underachievement" (pp. 22-23). What can a mathematics instructor do so that students understand the nature of the discipline of mathematics that Boaler refers to?

If mathematics classrooms are to provide a space for students to experience mathematics in ways similar to mathematicians, then mathematics teacher education programs must provide space for future teachers to experience mathematics in novel ways. But curriculum change is a slow-moving process—the curriculum of the undergraduate mathematics major has remained relatively constant for decades. Mathematics majors frequently take Calculus, Introduction-to-Proof, Number Theory, Geometry, Abstract Algebra, Analysis, etc… How can we improve mathematics instruction in these existing courses so that future teachers are better prepared to support their students in engaging in authentic mathematical practices? One approach that has been used in content courses is to design novel curriculum so that future teachers can collaboratively develop their mathematical knowledge for teaching (Lischka, Lai, Strayer, & Anhalt, 2020). These approaches have been successful in courses for geometry, algebra, statistics, and modeling. But what about traditional proof courses? Inquiry-based and collaborative pedagogical approaches within proof courses are another way to provide future teachers opportunities to experience mathematics in ways more aligned to the discipline (Bleiler-Baxter & Pair 2017, Bleiler-Baxter, Pair, & Reed 2021, Pair & Calva, 2021). In this paper we report on a new approach aimed at enriching students' perspectives of mathematics within a transition-to-proof course—engaging students in the exploration of unsolved mathematical conjectures.

Purpose

Our goal for this research project was to engage students in a novel activity—the exploration of unsolved mathematical conjectures. Unbeknownst to the general human population, mathematics is a dynamic field in which new results are constantly being discovered and reported (Pair, 2017). Unanswered questions and unsolved problems drive mathematical research. Two such unsolved problems, the Twin Primes Conjecture and the Collatz Conjecture have remained unsolved for decades (Bairrington & Okano, 2019; Rezgui, 2017). Our goal with this project was to expose these conjectures to students and provide them the opportunity to develop their own methods to try and prove the conjectures and then reflect on what they learned about mathematics through the activity. Our hypothesis was that engaging in these activities would provide the undergraduates novel opportunities to enrich their perspectives on the nature of mathematics. Our research question is: How are students' perspectives on the nature of mathematics enriched through their exploration of unsolved conjectures (if at all)?

Theoretical Perspective

We designed the study and corresponding unsolved conjectures project in efforts to promote the humanistic vision of mathematics as captured by the four characteristics of mathematics as expressed in the *IDEA* Framework for the Nature of Mathematics (Pair, 2017). This framework was developed through a dissertation study that involved immersive experiences in mathematics (collaboration with a research mathematician, teaching a transition-to-proof course). The purpose of the framework is to provide a list of possible goals for students' understandings of the nature of mathematics. This framework posits that, in order for students to possess a refined humanistic view of mathematics, they should understand that:

I - Mathematical ideas are part of our *I*dentity.
D - Mathematical knowledge is *D*ynamic and ever-changing.
E - Doing mathematics involves an *E*motional *E*xploration of ideas.
A - Mathematical knowledge is socially validated through *A*rgumentation.

In order to foster an understanding of these characteristics of mathematics, students must be engaged in learning activities for which these characteristics of mathematics become apparent. We hypothesized that by engaging in an activity that research mathematicians also engage in (trying to prove an unsolved conjecture), students would have the opportunity to reflect on mathematics in ways that align with these four characteristics.

Methods

Participants, Context and Data Collection

The research took place at a large public university in the United States. The university is a Hispanic-serving institution, and has very diverse student body. This diversity is represented in the students who participated in this study, which was approved by our university's institutional review board. Participants included volunteers enrolled in a transition-to-proof course. There were 16 female participants and 9 male participants. All names in this paper are pseudonyms.

As an added component of the course, students kept a mathematician's notebook in which they were scaffolded into exploring unsolved conjectures and reflecting on the process. These notebooks served as the primary source of data for the project. This exploration was conducted through several assignments over the course of the semester. For the first assignment, students reflected on questions such as 1) What is mathematics all about? And 2) What is mathematical proof and how is it used by mathematicians? Students were then introduced to both the Twin Primes Conjectures and the Collatz Conjectures. After doing some relatively simple explorations, they described how the assignment was similar to or different from their previous mathematical activity. For subsequent assignments, students were assigned to research teams to work on either conjecture according to their preferences. For the second assignment, students were scaffolded into relatively difficult tasks to explore the conjectures—for instance looking for patterns in the twin primes or Collatz sequences; or articulating what it would mean to attempt a proof by contradiction to prove the conjectures. For the third assignment, students invented their own methods and approaches to try and solve the conjectures; there was then time for in-class sharing of approaches and methods. For the fourth and final assignment, students continued to use their own invented approaches to try and prove the conjectures; they also watched videos in which mathematicians described their own work on the conjectures; and finally, they reflected upon the process—this reflection included revisiting the questions: 1) What is mathematics all about? And 2) What is mathematical proof and how is it used by mathematicians? Additionally, students also responded to two more questions: 3) How has your thinking regarding mathematics and mathematical proof developed and changed during this semester? Which changes were the result of studying the Twin Primes and Collatz conjectures? And 4) What were the challenges and successes of your experience with mathematical research this semester?

Data Analysis

The researchers analyzed the notebooks through a thematic analysis process (Ryan & Bernard, 2003). This analysis process took place through two main stages. During the first stage, two researchers individually read through each participants' notebook, one at a time, making note of 1) Student quotations that aligned with the nature of mathematics as expressed in the IDEA framework[1]; 2) Evidence of change in perceptions of the nature of mathematics; 3) Other substantial reflections about mathematics and the research process; 4) Interesting mathematical ideas. Subsequently, the researcher's discussed their findings from each notebook, describing their observations. After this task was completed for the individual student notebooks, each researcher created a holistic summary of what they saw as the major trends in student reflections. The researchers also used student quotations as evidence for the trends they observed. This process resulted in a list of several recurring themes in student reflections (e.g. seeing mathematics creatively for the first time; frustration in being unable to find patterns). During the second stage of analysis, the researchers took the list of recurring themes developed in phase 1, and then reviewed each notebook again, making note of which ideas were expressed by each individual student. We then tallied these recurrences, resulting in a quantitative summary of the ideas expressed by the students. For a more detailed description of the methodology, see (Pair & Calva, 2021).

Results

Overall, we found that exploring conjectures provided an opportunity for students to experience and see mathematics in new ways. Students recognized that they had the opportunity to engage in a process similar to mathematicians, and felt they had a better understanding of the process of how mathematical knowledge is developed. They also experienced first-hand the emotional aspects of mathematics, and recorded a wide variety of emotions from frustration to joy. We now present our results, generally organized according to the four categories of the IDEA Framework.

[1] We also used de Villiers (1990) roles of proof framework in our analysis. Those results are shared in (Pair & Calva, 2020).

Identity

In terms of identity, Figure 1 shows the tallies of students whose notebook reflections expressed related ideas. We identified ten total students whose reflections showed that they either took ownership of their ideas using personal (I/my) language, showed that they grew in their own mathematical identity, or identified with mathematicians to a greater degree than they did before (the total is more than ten as some students were categorized in more than one category).

Identity	10 (total)
Take owernship of ideas (using I/my language)	9
Grew in own mathematical identity (right major choice).	4
Identify with mathematicians	4

Figure 1: Recurring Themes Related to Identity

For instance, we can see that Isabella uses the language “my work” and “things I figured out” in this excerpt from her notebook as she references a concept she invented, “pink numbers”:

> Overall, I liked this process of communicating with the class and working together. I got excited when I figured ‘the first pink number’ out […] I’m even more excited to figure stuff out, or simply just share my work with my classmates. My most important work are on pages 12 and 18 because they show the 2 things I figured out.

See Figure 2 for an image that shows an excerpt from Isabella’s work that she referred to:

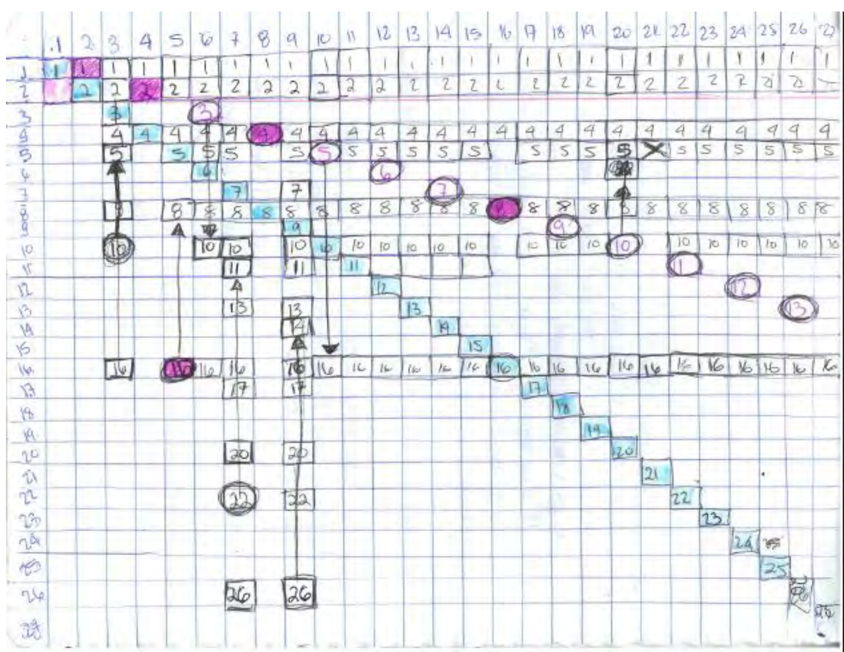

Figure 2: Excerpt from Isabella’s Notebook

Four students grew in their mathematical identity as the conjecture exploration helped them see that they enjoyed proofs and/or mathematical research. For instance, Brent wrote,

> This notebook assignment has made me realize that even the simplest arithmetic can lead to problems that even modern mathematics isn’t ready for. But it’s also grown my passion for

> proofs and the subject itself. I want to dive deeper into this and see what I can learn! Oh, it also made me realize that I made the right choice in majoring in math since I love proofs so much ☺.

Four students wrote that they believed the assignment helped them to identify with mathematicians as they got to engage in the same activities as mathematicians. Vincent wrote,

> I had a lot of fun working on this second assignment because we are working on an unsolved proof like many other mathematicians. It is pretty cool to try and give one's own spin and perspective on the twin primes conjecture and if not prove it, at least come to the understanding of the conjecture and have some hands-on experience.

Dynamic

In terms of the dynamic nature of mathematical knowledge, see Figure 2 for the tallies of how many students reflected upon a particular theme. We found that eleven students either described how they learned about conjectures for the first time, understood better how mathematical knowledge is built, conjectured that new mathematics may be required to solve conjectures, and/or referenced discovery (some students were identified for more than one category).

Dynamic	11 (total)
Learning about conjectures for first time	5
Described building up of knowledge in mathematics	5
Believed new mathematics required to solve conjectures	4
Reference Discovery	6

Figure 3: Recurring Themes Related to Dynamic Nature of Mathematics

Vanessa and at least four other students learned about conjectures for the first time through the assignments. Early in the semester she wrote, "I enjoyed learning about conjectures. I didn't know what conjectures were, but now I do." Near the end of the semester, after watching a video of a mathematician describing their work on their conjectures, Vanessa wrote "Mathematicians use these proofs to help prove other conjectures. As Maynard's proof was influenced by Zhang's proof. Eventually, Maynard's proof will be used to help prove other conjectures."

We found that four students hypothesized that new mathematics may be required to solve the conjectures. Lorenzo wrote, "The Collatz conjecture is unprovable… for now. There has to be some future mathematics that will prove it. However right now, in 2019, the Collatz conjecture remains unsolved." Five students referenced mathematical discovery in their final reflections. Daniel concluded, "Proofs are used by mathematicians to assist them in creating other proofs to eventually have a breakthrough that is groundbreaking in mathematics as well as the world."

Emotional Exploration

Almost every student in the course reflected on significant aspects of mathematical exploration. Five students experienced mathematics creatively for the first time and seven recognized the need to think outside the box in exploring the conjecture. Eight students reflected that the process allowed them to get a glimpse of the discipline of mathematics and what goes into the making of mathematics. Students also described a range of emotions from frustration to joy. We found that five students described frustrations but no positive emotions, 9 students described both frustration and positive emotions, whereas seven students described only positive emotions. See Figure 4 for a summary of themes related to exploration (some students were counted for multiple categories).

Exploration	24 (total)
View of Math as creative	5
Need to think outside the box or think critically	7
Looking at conjecture from different perspectives	5
Getting a glimpse of the discipline.	8
Contrasted conjectures with prior math experiences	11
Frustrated	14
Enjoyed/Excited/rewarding (positive emotions)	16

Figure 4: Recurring Themes Related to Exploration

Students were frustrated when they could not find patterns, or when they were at a loss for how to proceed in working on the conjectures. For example, consider this excerpt from Charity: "I feel lost. Working on this assignment feels like a road with no end. I don't feel confident in what I'm doing unlike when I'm working on what we're working on in class." Other students described similar challenges, as the assignment was different from their previous mathematical experiences. Cassandra found that she needed to look at math more creatively to meet the challenge of the assignment.

> My experience working on this conjecture thus far has been challenging. It is difficult to shift how you approach mathematics from just learning and applying formulas and theorems that define what you do to then try and figure it out yourself. This has made me look at math more creatively than I have before. This conjecture has also made me try and see more patterns in number sequences and how they relate to one another.

After a period of frustration, most students productively engaged in the conjecture by trying a variety of approaches. The quote from Carlos below illustrates how he found enjoyment in the experience and also developed a new perspective on mathematics.

> For the most part, I was able to get a solid foundation about number theory, primes, "3n+1" and all its uses, and also going down the rabbit hole of several given solutions on Collatz or twin primes. This assignment was fun. Really fun. Being able to think of math in a more analytical sense was like a breath of fresh air. Being able to break down, and to be able to show each part of a statement to be true or false. It's kind of eye opening that this is how to "read math" and "talk math."

In addition to considering how to read and talk math, other students described how they got a picture of what goes into the making of mathematical knowledge. Jennifer wrote,

> This assignment was very interesting to me because it had me learn more about why Collatz sequences have not been proven yet. This was a different experience for me compared to my prior mathematical experiences because usually I work with problems that have already been proven. So it was definitely interesting getting a glimpse of what professional mathematicians struggle with when trying to prove something for its first time.

Argumentation

While we have evidence that students had opportunities to enrich their perspectives on the nature of mathematics in ways that aligned with the first three categories of the IDEA framework, we did not find much evidence that students reflected on the fourth category—argumentation. Perhaps this is because the students only had one day in class for in-person sharing of our findings related to the conjecture. That day was a show and tell experience rather than a critiquing experience—the students did not have the opportunity for social validation of mathematical knowledge through the conjecture assignment.

Discussion

We are encouraged by the results of our study. We have some evidence that providing students a scaffolded opportunity to explore unsolved conjectures, and time to reflect on this activity, is a productive means to enrich students mathematical experiences/perspectives. While we chose the Twin Primes and Collatz conjectures due to our familiarity with them, we believe that any accessible conjecture could be used fruitfully with undergraduate students. This activity was implemented in a transition-to-proof course, but a similar activity could be used as a project in courses on analysis, number theory, abstract algebra, etc… It is especially important for future teachers in such courses to have opportunities to experience mathematics in non-traditional, collaborative activities that more closely align with the work of practicing mathematicians. By doing so, they will be better positioned to provide their future students with opportunities for authentic mathematical practice.

In this exploratory study, we used the IDEA framework as a conceptual framework in data analysis for the first time. In our analysis, we carefully looked for student reflections that expressed ideas that aligned with the categories of the IDEA framework. This was a lengthy qualitative process. Future studies may improve upon this methodology, perhaps creating quantitative assessments that can be used to measure the changes in students' perspectives of the nature of mathematics through novel classroom interventions.

We note that of the four categories in the IDEA framework, students had the opportunity to learn about each aspect of the nature of mathematics except that mathematics is socially validated through *A*rgumentation. Future research studies are needed to discover classroom interventions that help students experience and understand this aspect of the nature of mathematics.

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