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# MANIPULATIVES, AFFORDANCE AND THE LEARNING OF FRACTION: THE TWELFTH TASK

## MATÉRIEL DE MANIPULATION, AFFORDANCE ET APPRENTISSAGE DES FRACTIONS: LA TÂCHE DOUZIÈME

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*This study investigates the use of manipulatives by elementary students working on a fraction task. Extending previous work on the role played by the manipulatives in students' activity, we aim at describing how the choices made for the task design disrupt students' activity, creating opportunities to learn. The theoretical underpinnings allow envisioning the students' activity through the concept of routine and the manipulatives through the concept of affordance. The analysis of the students' mathematical activity allows us to better understand how manipulatives can serve as breaching elements, leading students to modify their mathematical activity, and thus, creating opportunities to learn.*

Keywords: Elementary School Education, Rational Numbers, Instructional Activities and Practices, Mathematical Representation

In this paper, we draw on our previous research (Jeannotte & Corriveau, 2020; Corriveau et al, forthcoming) about the role played by manipulatives in students' activity when solving arithmetical tasks. More specifically, we are studying the relation between didactic choices linked to manipulatives and the students' mathematical activities in a task about fractions. We aim to describe how those choices disrupt students' activity, creating opportunities to learn. To do so, we first describe why it is still important to research the use of manipulative in mathematics classrooms. We then define mathematical activity from a commognitive perspective. The concepts of routines and affordance are theoretically explicated to frame our methodological and analytical work. The result section presents the main routines that emerged from our analysis of a mathematical task called the "Twelfth task". We illustrate with examples how those routines are bound to the manipulatives used in this specific task and how different didactic choices may create different opportunities to learn.

### Manipulatives and Mathematical Learning

The use of manipulatives for teaching arithmetic at the elementary level is encouraged and even prescribed by several curricula around the world (OCDE, 2019). Thus, it is often seen as an efficient teaching practice and is rarely questioned (authors, 2020). In particular, the learning of fractions, considered as one of the most difficult areas in elementary school mathematics, seems to benefit from this use (Carbonneau et al., 2013). However, research on manipulatives mainly focuses on whether manipulatives support student achievement in mathematics (e.g., Moyer, 2001), without articulating the uses of manipulatives and the different choices made when designing the educational setting. Some research sheds light to a certain extent on some didactic variables. For example, the use of familiar manipulatives may mislead students (McNeil & Jarvin, 2007) or, for the same task, two different manipulatives can lead students to engage in different mathematical activities (Authors, 2020). Familiarity and types of manipulatives are two examples of variables among others that can shape the students' mathematical activity. In this

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paper, we present a fraction task that students solve using pattern blocks. We aim to show how different choices, especially the choice of limiting the number of Pattern blocks available to students, introduce breaching elements that can disrupt the students' activity and thus create opportunities to learn.

## Conceptual Framework

From a commognitive point of view, mathematics is defined as a particular form of communication, with its vocabulary, its visual mediators, its distinct routines, and its narratives generally accepted by a mathematical community (Sfard, 2008). By positioning ourselves in this sociocultural and non-mentalist perspective, manipulatives, and therefore their use, become part of the mathematical activity developed in the culture of the classroom. The use of manipulatives, when defined in commognitive terms, is viewed as participation in the mathematical discourse of the classroom through the manipulation of physical apparatus. In other words, using manipulatives is not a concrete version of mathematical activity, but a mathematical activity with its specific vocabulary, visual mediators, and narratives generally accepted by the classroom. For example, we often hear students say, "I am breaking down the tens into units" (referring to base-ten blocks) instead of "I am dividing the tens into units."

### The Concept of Routine

Sfard (2008) defines a mathematical routine as a repetitive pattern of discourse discernable by the vocabulary and visual mediator used, and the narratives produced. Moreover, a routine is linked to the concept of 'task situation.' A task situation is "any setting in which a person considers herself bound to act—to do something" (Lavie et al., 2019, p. 159). For example, a 4th-grade student may refer to number fact to calculate mentally twelve divided by three to answer the question: "What is the third of twelve?". Another may draw a rectangle, divide the rectangle into four horizontal lines and three vertical lines to form twelve equal parts, and color four of them. Thus, "a routine performed in a given task situation by a given person *is the task, as seen by the performer, together with the procedure she executed to perform the task*" (Lavie et al., 2019, p. 161, italic added). Manipulatives can play distinct roles in routines. We can recognize manipulatives through vocabulary. Indeed, when using Pattern blocks, students will refer to a kind of block by its color or its shape. We can also observe students act upon the manipulatives, by moving them, stacking them, regrouping them in certain ways. Manipulatives can also play the role of visual mediator, representing mathematical objects.

Learning is defined as a change of discourse, and more specifically as a change of routine (Lavie et al., 2019). Discourse, therefore, progresses in an individualization-(re)communication direction (i.e., learning) through changes in routines, thus in vocabulary, visual mediators, and accepted narratives, what Lavie et al. (2019) name routinization. Therefore, because we aim to describe how manipulatives can create opportunities to learn, we are particularly interested in moments when routines seem disrupted, i.e., when students are bound to act differently.

### The Concept of Affordances

We call affordances the relational properties of an object within a certain environment that are bonded to certain actions. "An affordance relates attributes of something in the environment to an interactive activity by an agent who has some ability, and an ability relates attributes of an agent to an interactive activity with something in the environment that has some affordance" (Greeno, 1994, p. 338). For example, a pen will generally be seen as an object to write with, but it could be seen as a tool to unlock a door. For some people with long and thick hair, it could be seen as an object to tie hair. Thus, "[w]hether or not the affordance is perceived or attended to

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will change as the need of the observer changes, but being invariant, it is always there to be perceived" (Gibson, 1977, as cited in Brown et al., 2004, p. 120).

In a given task situation, the use of manipulatives by students generates different affordances. In other words, the affordance realized in a task situation relies not only on the tool itself but rather on the interactions between the manipulatives, the students, and the teachers, in a particular educational setting (Drijvers, 2003, in Brown et al., 2004).

To try to disrupt students' routine, we can play with didactic variables to subtly change the affordance, thus the task situation by adding some breaching elements. Didactic variables are parameters, linked to the design of a teaching situation, which can vary according to the teacher's choices, and which can lead to a change in the students' mathematical activity (Brousseau, 1981). The concept of breaching (from ethnomethodology) refers to a subtle breakage of what is familiar. By varying several aspects related to the use of manipulatives, it is possible to place students in conditions familiar enough for them to engage in a task but sufficiently unfamiliar that it resonates as a conflictual use of the manipulatives. In a way, it plays into how students perceive the affordances of the manipulative and leads them to explicit their usual routines and renew them.

## Methodology

In collaboration with teachers from grades 4 to 6, we created a task (titled "Twelfth") that aims to develop different routines to represent and compare fractions using Pattern blocks. Pattern blocks were familiar manipulatives for the students as they already had worked with them before the experimentation. We experimented "Twelfth" into two fourth-, two fifth-, and one sixth-grade classes of about twenty students each. The task was implemented with different choices according to teachers' rationales (mainly based on students' abilities and familiarity with fractions) during a 60-minute period.

### The Task "Twelfth" and the Didactic Choices

"Twelfth" is, at first sight, a common task in which students are asked to represent a certain fraction with pattern blocks. We chose to place students in pairs. Each pair received a kit of limited Pattern blocks (Figure 1) and were asked to do several subtasks. These can be divided into three categories:

1. The given fraction is a twelfth and students must represent a third (the piece representing one-twelfth varies): e.g., the green triangle (blue rhombus, red trapezoid, etc.) is worth 1/12, we want 1/3 of the same whole.
2. The given fraction is one-twelfth and students must represent other varied fractions (unit and non-unit): e.g., the blue rhombus is worth 1/12, we want 1/24 of the same whole; the red trapezoid is worth 1/12, we want 5/6 of the same whole.
3. The given fraction varies, and students must represent varied fractions: e.g., the yellow hexagon is worth 2/5, we want 1/15 of the same whole.

**Figure 1: kit of pattern blocks available to each pair of students**

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The subtasks were projected on the whiteboard, one or two at a time. There was a whole class discussion after each subtask. In the two 4th grade classes, the students were allowed to use scrap paper to take notes. In one 5th grade class, the students were not allowed to use any paper at all. In the two other classes (5th and 6th grade), the students were given a document to work on. On each sheet, they had two subtasks with a space to draw and record their answers.

### Data Analysis

In each class, a camera with a primary focus on the front of the class filmed the whole group and other cameras filmed pairs of students with a primary focus on their hands (10 in 4th grade, 14 in 5th grade, and 7 in 6th grade). For this paper, we focus our analysis on the resolution of the first subtask by the students working in pairs: "Find what is 1/3 of a whole, if the triangle is 1/12." Following Powell, Francisco, and Maher (2003), each video excerpt was watched multiple times and described; critical events were identified. We transcribed, coded, constructed the storyline, and composed the narratives. Descriptions of the videos and codes helped us compose the narrative to report the results. Conversely, composing the narrative forced us to revisit some critical events. Critical events are moments when students' routines seem disrupted, i.e., when they seemed to doubt, when the routine implemented did not lead to solving the task and was challenged by someone, or when they suddenly changed their routines.

## Routines and Affordances

By analyzing the data, we were able to highlight three intertwined routines that could be linked to the interpretation of the subtask: 1) 'comparing-and-naming', 2) 'forming and partitioning' a whole, and 3) 'covering'. Those routines had distinct functions throughout the resolution of the task but were all bound to certain affordances of the manipulatives in this specific setting.

### The Routine of Comparing-and-Naming

Several pairs used a routine we called "comparing-and-naming" at different moments during their resolution. They attributed a name to each kind of block, or group of blocks by comparing pieces and relying on number facts. Some used fraction-names (e.g., a hexagon is ½ [of the whole], or a rhombus is 1/3 [of a hexagon]) and other whole number names (e.g., a rhombus is 2 [triangles]). Adam and Phil, a pair of 4th graders in Heloise's class, help us illustrate how this "comparing-and-naming" routine was bound to the manipulatives, the didactic variables, and the students' discourse. Right at the beginning, Adam associated some blocks with a fraction-name by referring to the given value of different blocks: "yellow is ½ because 6+6", "red is ¼ because it's half of the half." Like 1/3, those unit fractions are familiar to the students as they start working with them in 1st grade. However, because of our didactic choices, it was impossible to associate one and only one block to 1/3. Indeed, 6 triangles are equivalent to a hexagon and a hexagon to two trapezoids, but 4 triangles could not be replaced by a single block. Phil proposed to name the rhombus 1/3 of a hexagon. This proposition combined with the accepted narrative that the whole was not the hexagon led them to realize that more than one piece was requisite. If we had chosen 1/6 as the given value of the triangle, their routine could have led them to solve the subtask. Here, the given value combined with how the Patterns blocks are designed created a need to change the routine, and thus created an opportunity to develop a new discourse about fractions, that is to accept a visual mediator of a unit fraction composed of more than one piece, a necessary narrative to solve this task.

### The Routine of Forming-Then-Partitioning a Whole

The routine of forming-then-partitioning a whole appeared in three different variations linked to the function of the routine: solving the subtask, validating the calculation, and justifying the

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answer prompted by a request from an adult or a peer. This routine was recognizable by the action of composing a whole with a certain number of blocks followed by its partitioning in a certain number of equivalent (mostly identical) groups. In a way, this routine was the expected one. Indeed, the curriculum (MEQ, 2001) prescribes the learning of fractions through manipulatives and schemas use. Thus, we expected them to use this routine. The blocks available to students for this first subtask allowed them to compose a whole, but not with twelve identical blocks, leading to the impossibility to partition it into three identical groups of four triangles. Figure 2 presents possibilities of different wholes with their partitioning. Except for the last example (Figure 2d) where each group is formed of one rhombus and two triangles, the students needed to form three equivalent but not identical groups. The shape of the groups is the same, but neither the orientation nor the kind of blocks (colors and shapes) are the same.


**Figure 2: Partitioning of the whole with three equivalent groups**

**Solving the subtask.** John and Owen, a pair of 5th graders in Christophe's class, engaged in the routine of forming-then partitioning the whole to solve the task, after reformulating the question as "1/3 of twelve is what?":

Owen: We say the whole is that [taking one hexagon in his hand]
John: No, this is six! […]
Owen: Oh, ok, so how many there are [counting the triangles]. Ok, six, so this [pointing a hexagon] is six. So, that [sliding two hexagons on the side], means, it's our whole. And this [sliding a triangle beside the two hexagons, Figure 3a] is one-twelfth.
John: No! This [the triangle] is one-sixth. We have to find out the half of it. Impossible... Oh, wait, wait, I know what it is! Look, [regrouping six triangles]
Owen: it's the half.
John: [regrouping a hexagon to the six triangles]. This is twelve. So, this [separating one triangle, Figure 3b] is one-twelfth.
Owen: It's what I've been saying since earlier!
John: we've found the twelfth, we have to divide by four... by three. [Separating 3 triangles from the whole and covering it with a trapezoid, Figure 3c].

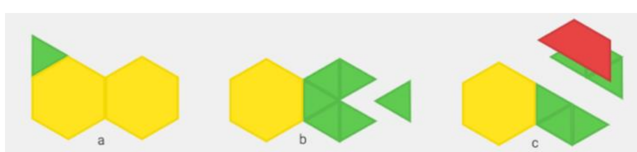


**Figure 3: Forming-then-Partitioning a Whole**

In this short excerpt, we observe how Owen and John struggle to make sense of all the information with the manipulatives at their disposition, juggling between one hexagon as the whole/one triangle as one-sixth and two hexagons as the whole/one triangle as one-twelfth,

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suggesting that the routine of "naming" interfered with the routine of "forming-then-partitioning" the whole. Moreover, after convincing themselves that they succeeded at representing the whole, they intended to partition it. But the interpretation of the division led them to the wrong partitioning.

**Validating.** Some pairs used manipulatives to validate an answer obtained through calculation. Indeed, some students interpreted the task as one of mental calculation. At some point, usually early in the resolution, they calculated 12 divided by 3 or searched for the number that multiplied by 3 gave 12. Tim and Nick, two 5$^{th}$ graders, used this method. Tim seemed to calculate mentally and then added two rhombi on top of the two hexagons. Nick wanted to validate. To do so, he counted "one-third, two-thirds, and three-thirds" pointing with his fingers holding imaginary blocks over the whole but without moving any pieces. Tim and Nick were able to perceive the twelve triangles inside the two hexagons and the three groups formed of four triangles. The Pattern Blocks design facilitated this identification for Tim and Nick.

**Justifying the answer.** Adults and peers sometimes challenged an answer by asking why it was the good one. Tim and Nick had already found the solution through calculation and validated through forming-then-partitioning the whole. When an adult asked them to explain their solution, they argued that one-twelfth means there are twelve triangles in the whole. "The third of twelve is four." Therefore, the answer is four triangles. Their explanation relied on calculation, but they also overlaid four triangles on top of the two hexagons (see Figure 4). They answered that four triangles "fit" three times in two hexagons. They completed their justification by effectively making three-thirds "fit" on top of the two hexagons pointing fingers to show the three distinct groups as Nick did when validating the answer earlier. For Tim and Nick, a calculation was not an acceptable justification. They used the manipulative to visually back their claim in the context of the task, showing the whole, the part, and the fact that it fits three times.

If the students did not use the wanted whole, the teacher could use the manipulatives to challenge not only the answer but the students' routines. For example, Mike and Edith, two 5th graders, attributed the name 1/3 to the hexagon and then formed a whole with three of them. Incidentally, they had placed the manipulatives as positioned in figure 1 in front of them when they received the bag, the three hexagons were a bit separate from the other blocks. They reproduced three hexagons on a sheet of paper and wrote 1/3 in one of them. An adult asked them to explain their answer: "[After they answer that 1/3 was a hexagon]. Ok, if this [showing a triangle] is one twelfth, can you show me the whole in which this triangle represents one twelfth?" By reintroducing the triangle and its value in the conversation, the students were able to determine the wanted whole [two hexagons] and modified their routine accordingly:

Mike: Look, it can be those two [touching two rhombi]
Edith: Ha, yes, it works... mmm, but you cannot split it in three.
…
Mike: Yes, look in this [taking the two hexagons], there are six like that [covering the hexagons with two rhombi]
Edith: but we are looking for the third
Mike: exactly! [splitting the whole in three by pointing rhythmically]

After the adult intervention, Mike and Edith were able to modify their routine by using new visual mediators (the whole and the part) that considered the given value of the triangle (one-twelfth) and Mike convinced Edith that two rhombi were an adequate answer by simulating a partitioning through gesture.

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**The (Sub)-Routine of Covering.**

As illustrated in the examples presented above, we observed students in each grade using a "covering routine" (see Figure 4). This routine has at least two functions. First, it allows comparing blocks or groups of blocks to determine that they fit exactly on top of each other. For example, by covering the trapezoid with three triangles, we can figure out that a trapezoid is equivalent to three triangles. However, as said before, it was impossible to cover two hexagons with twelve triangles, forcing students to rely on imaginary blocks, as Tim and Nick did, or to use equivalent combinations as in Figure 2. The breaching element brought to bear by the limited manipulatives pushed the students to slightly modify their routine.

Second, it allows showing not only the part representing one-third but the whole in which it represents one-third. This second function was used more in 5th and 6th-grade classes, leading us to hypothesize that it was part of the classes' discursive rules, i.e., that, in those specific classes, students were expected to justify their answers. Indeed, to be able to judge the validity of the answer, you need to refer to the whole.

**Figure 4: Example of an answer obtained through the covering routine**

## Interpretations

Through the lens of routine and affordance, we aim to describe how different choices disrupt students' activity, creating opportunities to learn. The analysis exposes how the students' mathematical activity is bound to the manipulatives used in this task and how different didactic choices may create different opportunities to learn. First, we hypothesize that the manipulatives led students to different routines than a paper-and-pencil task would. With no manipulatives, students may have reproduced the triangle twelve times (as a set or a whole) and circled four of them, or circled three groups of four. The comparing-and-naming nor the covering routine would have been needed.

Second, the limited quantity of pattern blocks is an important breaching element that created different opportunities to learn, here, through three routines. In this task situation, students were bound to represent different fractions with manipulatives. They could have represented the given fraction, the whole, the sought fraction, or other specific fractions. The 'representation-able' quality of the manipulative is linked to one of the main affordances. Indeed, the diverse ways that pattern blocks can be used as visual mediators  are specific to this type of manipulatives, leading to breaches of routines.

A first breach was observed when students were not able to associate one-third to only one block, this permitted us to note a breach in their routine of 'comparing-and-naming'. This choice of limiting the blocks used created an opportunity to enrich visual mediators associated with unit fractions and to challenge the rule that "a unit fraction should be represented by one and only one block." A second breach was observed for the routine of 'forming-then-partitioning' a whole. Students could not represent the whole with twelve triangles. They had to adapt their routine. One way to do so was to use two hexagons. This breaching element led to work on equivalent surfaces. The third breach is also linked to this routine and to the importance of equivalence. Indeed, using only one kind of block to represent a fraction was a common routine for students.

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The limited manipulatives introduced a breaching element that created opportunities to develop new narratives about partitioning a whole, focusing on equivalence rather than on the identity of the parts. A fourth breach was observed when students lost sight of the given value one-twelfth, leading them to associate the whole with one or three hexagons. In both cases, the manipulatives available led to the perception of a whole partitioned into three identical parts. Clearly for some students, in this task situation, one-third was associated with the narrative "one of three identical parts." Thus, they were looking for a whole constituted of three identical parts.

Even though it was not the focus, this analysis also put to light other affordances linked to the qualities of the manipulative. The 'representation-able' quality of the manipulatives did not only depend on the number of blocks available but on the design of the pattern blocks. Indeed, depending on the fraction given, Pattern blocks do not allow the representation of any fractions. A second affordance can be linked to the "move-able" quality of manipulatives. Students can move the blocks, stack them, group them, separate them. This quality helps students perceive and validate the equivalence of distinct groups, an element that cannot be avoided in this task partly because the manipulatives are limited. A third affordance can be linked to the "communication-able" quality of Pattern Blocks. Indeed, these manipulatives are tied to certain keywords as each piece has a specific color and a specific form. Other layers of analysis are required to push further the interpretation here.

## Conclusion

The limited manipulatives, as a breaching element, created opportunities to learn by introducing nuances in what is mathematically important when working with fractions (e.g., the use of identical parts when partitioning a whole to represent a fraction is not necessary, the use of equivalent parts is). The didactic choices made helped us identify shared narratives and routines in those specific communities of students and how those choices created opportunities to learn, i.e., created changes in routine. Referring to Nachlieli and Tabach (2019), we could define those opportunities to learn as exploration-requiring opportunities to learn, i.e., to complete the task or meet the teacher's expectations, students cannot apply an already familiar single procedure. They needed to create a new combination of actions, a new routine. While Nachlieli and Tabach (2019) linked the creation of those opportunities to the kind of question asked (e.g., presenting request with words like what, why, find, explain), we add that those breaching elements can also play a key role.

Those findings also point out some implications for the classroom. Because breaching elements are possible only when disrupting familiar ways of doing (Corriveau, 2013), introducing such elements in tasks is not trivial. To go further than the question of whether to use manipulatives or not, one must attend to the implicit culture of the class. In other words, it is to be able to realize what is usually taken for granted and articulate it with the targeted mathematical learning to make clever didactic choices. It also brings us back to the fact that the use of manipulatives is not a magic solution (Ball, 1992).

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