MATHEMATICIANS' LANGUAGE FOR ISOMORPHISM AND HOMOMORPHISM

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Isomorphism and homomorphism appear throughout abstract algebra, yet how algebraists characterize these concepts, especially homomorphism, remains understudied. Based on interviews with nine research-active mathematicians, we highlight new sameness-based conceptual metaphors and three new clusters of metaphors: sameness/formal definition, changing perspectives, and generalizations beyond algebra. Implications include a way to articulate a conceptual purpose for homomorphism beyond its relationship to isomorphism: namely, as a tool for changing perspectives when problem-solving.

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Isomorphism and homomorphism are widely recognized as central to introductory abstract algebra coursework—not only do research-based curricula expressly focus on their development (Larsen, 2013) but they also serve crucial roles in fundamental theorems of algebra (e.g., the First Isomorphism Theorem (Gallian, 2009), also referred to as the Fundamental Homomorphism Theorem (Fraleigh, 2003)). Furthermore, isomorphism can be viewed as a type of sameness that expands on students' prior experiences with equality and congruence (Rupnow & Johnson, 2021) and serves an important classification role for objects like groups and rings (Randazzo & Rupnow, 2021). Nevertheless, limited research has focused explicitly on how mathematicians understand homomorphism's utility in abstract algebra.

To address this gap, we examine nine research-active mathematicians' language for isomorphism and homomorphism. In so doing, we expand on Rupnow (2021)'s characterization of instructors' language, especially with respect to homomorphism. Furthermore, we provide insight into conceptual purposes for homomorphism beyond relating it to isomorphism.

Literature Review and Theoretical Perspective

Researchers have long been interested in understanding students' views of isomorphism and enhancing students' problem-solving around isomorphism. Early research examined how students approached determining whether groups were isomorphic (Dubinsky et al., 1994; Leron et al., 1995) or approached proving theorems related to isomorphism (Weber & Alcock, 2004; Weber, 2002). More recent work has created a local instructional theory for isomorphism (Larsen, 2013), built on students' approaches to proofs using isomorphism (Melhuish, 2018), and delved into the function-nature of isomorphism (Melhuish et al., 2020).

Researchers have also begun examining students' views of homomorphism. Though many early studies examined conceptions of homomorphism in service of examining students' understanding of isomorphism (e.g., Larsen et al., 2013; Weber, 2001), more recent work has examined conceptions of homomorphism independently. Hausberger (2017) highlighted homomorphism's description as a "structure-preserving function" in textbooks, as well as what students took away from tasks intended to help students abstract the notion of ring homomorphism from their experiences with groups. Like with isomorphism, recent attention has often focused on how students coordinate the homomorphism concept and their function knowledge (Melhuish et al., 2020) as well as metaphors students use for homomorphism (Melhuish & Fagan, 2018; Rupnow, 2017).

In contrast to the many student-focused studies, limited research has been conducted on mathematicians' views of isomorphism or homomorphism. Weber and Alcock (2004) examined mathematicians' understandings in the context of proof while Ioannou and Nardi (2010) studied mathematicians' use of images in the classroom. Recent work has focused more on instructors' conceptual metaphors (described below) for isomorphism and homomorphism. Rupnow (2021) interviewed and observed instruction of two abstract algebra instructors, neither of whom did research in abstract algebra, and found four clusters of isomorphism and homomorphism metaphors: sameness, sameness/mapping, mapping, and the formal definition. Rupnow and Sassman (2021) built upon this by examining survey results from 197 mathematicians and observed examples of metaphors from each of these four clusters. To extend this work, we interviewed nine mathematicians with research specialties related to abstract algebra or category theory to see whether a larger population and a more research-oriented group of mathematicians invoked other types of conceptual metaphors for isomorphism or, especially, homomorphism.

Conceptual metaphors are a theoretical lens aimed at revealing individuals' structure of thought based on their choices of language (e.g., Lakoff & Johnson 1980; Lakoff & Núñez, 1997). Specifically, cross-domain conceptual mappings are used to connect one's cognitive structure for a target concept (e.g., isomorphism, homomorphism) to one's more developed thoughts in source domains (e.g., sameness, structure-preservation). For instance, "A homomorphism is a method for gaining information" is a conceptual metaphor that gives information about a target domain (homomorphism) by relating it to a more developed source domain with which people have had other experiences (a method for gaining information). We acknowledge that this theoretical perspective imposes the researchers' view on the mathematicians' statements; that is, we do not claim that the mathematicians intended to speak metaphorically in their responses (Steen, 2001). Nevertheless, we believe this lens does permit insight into how understandings can be clustered and the types of reasoning that can be employed when thinking about or using isomorphism and homomorphism.

Conceptual metaphors have influenced examinations of students' understandings of bases in linear algebra (Adiredja & Zandieh, 2020) as well as been used as a framework for examining college students' beliefs about mathematics (Olsen et al., 2020). More closely tied to this study, conceptual metaphors have been used to examine understandings of functions in high school and linear algebra (Zandieh et al., 2016) and in abstract algebra (Melhuish et al., 2020; Rupnow, 2017), as well as mathematicians' views of isomorphism and homomorphism (Rupnow, 2021; Rupnow & Sassman, 2021). Here we aim to extend the framework proposed in Rupnow (2021) to incorporate new clusters of metaphors based on new ideas raised by the mathematicians in this study. We thereby answer two research questions: What conceptual metaphors do research mathematicians employ to characterize (1) isomorphism and (2) homomorphism?

Methods

Data were collected from Zoom interviews conducted with nine mathematicians, given gender-neutral pseudonyms in this paper. These mathematicians had previously completed a survey about sameness in mathematics and were selected from those who had provided clear responses to the survey and had characterized themselves as research-active in abstract algebra, category theory, or a field interacting with abstract algebra. All participants had taught abstract algebra and/or category theory at least once (one once, three 2-5 times, four 6-10 times, and one 11+ times). The interviews focused on how participants characterized isomorphism and homomorphism for research, teaching, and laypeople (e.g., "How, if at all, does isomorphism play a role in your research?", "How would you describe a homomorphism to a layperson?").

Two researchers coded the interviews separately using the conceptual metaphors for isomorphism and homomorphism in Rupnow (2021) as codes (e.g., operation-preserving, journey) and then met to discuss coding and reach consensus. Each talk-turn was coded with all metaphors that appeared within that talk-turn. Although each phrase could only be associated with one metaphor, a talk-turn could contain multiple sentences and multiple metaphors. When new metaphors arose that were not clearly aligned with existing codes, these were added to the codebook, and interviews were iteratively reexamined in light of the new codes. This process aligns with codebook thematic analysis (Braun et al., 2019) in which an a priori codebook provides structure to the analysis, but space is made for emergent themes to be incorporated.

Results

We found that the original four metaphor clusters in Rupnow (2021) (sameness, sameness/mapping, mapping, and formal definition) were represented in this data as well, with some new codes being added to the sameness and mapping clusters. Additionally, three new clusters (sameness/formal definition, changing perspectives, and generalizations) were added to the framework to accommodate the new language used by participants in this study.

Sameness

This cluster contains codes referring to isomorphism or homomorphism as encoding information about the sameness of objects. This includes codes originally from Rupnow (2021), such as *generic sameness*, where participants described isomorphic objects as objects that are (essentially) the same, and *same properties*, where participants described isomorphic objects as those that have the same properties or invariants (e.g., cardinality). Several new codes were also included in this cluster and are discussed below.

Two participants were given the code *indistinguishable* when they described isomorphic objects as exact copies or not distinct from one another. For example, Avery said:

> I'm trying to teach [students] that isomorphism is measuring sameness so that they actually start to think of isomorphic objects as no longer being distinct… I might try to get them to stop distinguishing between isomorphic objects, and therefore, we can talk about *the* dihedral group of order 8 as opposed to different models of that group.

While Avery appears to be encouraging students to avoid viewing different manifestations of the dihedral group of order 8 as different, this identity-focused interpretation was not necessary to be coded as *indistinguishable*. Note that participants here were still thinking of isomorphic objects as being "the same," but used more specific language than those coded with *generic sameness*.

The code *embedding* was used when participants described homomorphisms as an embedding or a mapping into part of a larger object. This code also included sameness of homomorphic images in terms of the structure of one object appearing in another object or a copy of one object "sitting inside" another object. For instance, Indy remarked:

> [S]o I say, an isomorphism is an exact copy. These things are exactly the same. But a homomorphism,…maybe we don't have this bijection anymore. But somehow some of this structure is appearing in this other object. And the ways that that could happen—one of the ways is maybe—I have an exact copy sitting inside this larger object.

This quote was also given the code *indistinguishable* due to the comment about an isomorphism being an exact copy, which highlights the similarity between these two ideas: *indistinguishable* refers to isomorphisms as producing exact copies, whereas *embedding* refers to homomorphisms producing copies inside a larger object.

The code *logical equivalence* was given to participants who described an isomorphism or homomorphism as a transfer of knowledge in which truth values or facts about objects are preserved. When discussing different ways of writing isomorphic groups, Blair remarked:

> And so it's okay if we pick different ways of writing down what ultimately results in the same structure…At the end of the day, they're going to have the same structure because they're isomorphic in the logical sense as well. Any statement of group theory that is true about one of those structures will be true if and only if it's true in the other structure.

Others talked about isomorphism in terms of classification problems and were given the code *classification*. For instance, Hayden said:

> So I think the example of the classification of finite simple groups in the 20th century is just one of the tools of 20th century algebra. But that statement properly understood suggests that we've written sometimes in infinite families…what all the finite simple groups are up to isomorphism, not what they all are but up to isomorphism…

Notice they specify these classification problems are not listing all possible objects, just those "up to isomorphism", which relates to Avery's focus on "*the* dihedral group of order 8" above—these participants seem to view isomorphism as the "sameness" that matters in abstract algebra.

Sameness/Mapping

This cluster contains codes about isomorphism or homomorphism that showcase sameness via mappings, which are all included in Rupnow (2021). In particular, five participants spoke of isomorphism as a *renaming/relabeling*, or described isomorphic objects as the same, except for their names. Others were coded with *matching* when they talked about matching or pairing up elements between isomorphic objects in some way. The code *equivalence classes* was given to homomorphism-focused responses that explicitly mentioned equivalence classes or cosets. This code also included responses talking about homomorphisms in terms of an "orderly collapsing where things line up," or the process of "stacking [elements] into the same bins" (Indy). These latter types of responses make it clear why this code was included in this category, as they emphasize the sameness of a grouping of elements under a homomorphism. Note here that the idea of "lining things up" is what is implying the existence of *equivalence classes*; the word "collapsing" is coded with *structure loss* in the changing perspectives cluster below.

Mapping

This category includes responses that talk about isomorphisms and homomorphisms as functions or maps between mathematical objects, focusing on the map or process of mapping rather than the objects. The code *generic mapping* was given to participants who described isomorphism or homomorphism generally in terms of a function, morphism, arrow, map, or correspondence, whereas the code *journey* was given to participants who were explicit about the directionality of the map or used some sort of movement metaphor (e.g., elements being "sent to" one another). Both of these codes are included in Rupnow (2021).

The new code *invertibility* was given to four participants who highlighted the necessity of an isomorphism being reversible or comprised of maps that compose to the identity. Greer provides a clear example of the former:

> I mean, all the things I really think about when I think about the way isomorphisms would occur in the non-mathematical world maybe, really are sort of reversible processes. So now I was just thinking about cyphers and codes,…and that's a really concrete example of the way

> an isomorphism would work. It's even within the English language, but it's turning all messages in English to other messages in English in a completely reversible process.

Notice Greer's explanation of isomorphism for a non-mathematician focuses on the reversibility of isomorphism. Finley provides an example of the identity-focused version of invertibility:

> An isomorphism between groups is a homomorphism from G to H together with a homomorphism from H to G so that the composites are identities. But then, you could also say an isomorphism is a bijective homomorphism because that's a theorem that in the category of groups, the categorical isomorphisms coincide with the bijective homomorphisms. So I usually will present those two different ways.

Observe Finley distinguishes between bijective homomorphisms and homomorphisms with inverses that are also homomorphisms—although the definitions coincide in abstract algebra, they do not when generalizing to other contexts (e.g., continuous bijections need not have continuous inverses in topology).

The new code *transformation* was given to two participants who talked about isomorphism or homomorphism as a process that morphs or transforms one object into another (similar to Zandieh et al., 2016). For instance, Greer said:

> I use the word mechanism like an isomorphism really is a process to me. It's a process of turning one object into another in some sense. It's not turning the objects into each other, it's reframing your thinking from…one object to another object, I would say. Objects themselves are distinct. Completely distinct to me, and they're identified by isomorphisms.

Notice that Greer characterizes isomorphic objects as being distinct, in contrast to the examples given the *indistinguishable* code above.

Formal Definition

These are codes given to responses that utilized the formal definition to reason about isomorphism or homomorphism. All of these codes are included in Rupnow (2021). The code *literal formal definition* includes instances of describing an isomorphism as a bijection, often with special properties, or describing isomorphic objects as simply objects that have an isomorphism between them. While this code included responses using the literal string of symbols in the homomorphism property in Rupnow (2021), these types of responses did not exist in this study, likely because the mathematicians were not asked to engage in problem-solving.

Several participants also defined isomorphism based on homomorphism, and vice versa. The code *special homomorphism* was given to responses describing isomorphism in terms of homomorphism, either formally (e.g., "an isomorphism is a bijective homomorphism", Finley) or informally (e.g., "I think of isomorphism as homomorphism plus extra things", Greer). Similarly, the code *isomorphism without bijectivity* was used for responses that focused on describing homomorphism by relating to isomorphism (e.g., "[Homomorphism is] 'Sort of an isomorphism' is what comes to my mind. So we still want to preserve the structure, but maybe we don't insist on one-to-oneness anymore or one-to-one correspondence", Cameron).

Sameness/Formal Definition

Codes here include crucial parts of the formal definition (i.e., structure/operation preservation), but stated in an informal way. These two codes were originally included in the formal definition category in Rupnow (2021) because they were generally used as unexplained stand-ins for the homomorphism property/homomorphism by those participants. However, we now view them as a separate cluster because participants here seemed to use them to explain what they meant by sameness (e.g., "preserving the algebraic structure, same algebraic structure", Dallas). Cameron illustrates both *structure-preserving* and *operation-preserving*:

> And so when I cite preservation of structure, I mean that all these [mathematical sub-]fields have a notion of isomorphism in there. And you're usually referring to a bijection which preserves the structure of whatever it is you're looking at. So in topology, it's like a bijection that preserves continuity in both directions. And in abstract algebra, it's a bijection that preserves your multiplication, addition operations, whatever. In graph theory, it's a bijection that preserves adjacency.

We note here that the ideas of operation-preservation and structure-preservation are very similar, and Cameron seems to be using them interchangeably in the context of abstract algebra, though the notion of structure-preservation could carry over to other mathematical sub-fields as well.

Changing Perspectives

The metaphors in this category are all new codes involving responses that emphasize how homomorphisms force a change of perspective (*quotient group construction, structure loss*), or that explicitly mention using isomorphisms or homomorphisms to change perspective for the purpose of aiding mathematical research *(information gain*).

Two participants were given the code *quotient group construction* when they mentioned that homomorphisms arise from quotient groups or vice versa. Blair talked about viewing these two concepts interchangeably, even though students may see them as distinct initially.

> And a quotient object is exactly how you make rigorous this notion of collapsing down and the most important result… is that homomorphisms are the same things as quotients… a really important idea for students is that a group homomorphism between two groups is *the same thing* as a certain type of equivalence relation, a certain type of quotient group as well.

This quote mentions the idea of "collapsing" in relation to homomorphism, so was given the code *structure loss* as well, which involved responses talking about homomorphisms in terms of collapsing or similar ideas such as simplifying, losing, or ignoring structure. For example, Emerson describes homomorphism in the following way: "[T]aking a homomorphism is preserving some structure but losing something along the way. Hopefully, something that you are trying to ignore or that you don't care as much about as the stuff you're trying to preserve." The similar concept of getting only partial or limited information from a homomorphism was also included here. Greer observed:

> I think describing a homomorphism to a layperson… I want to say that it's about maybe collapsing and simplifying structure in mathematics, but I think it's actually a pretty foreign idea to the real world, this idea that you can take something you care about and record only partial information, and still learn something about whatever you're studying, but maybe not.

Thus, while only partial information is retained from a homomorphism, Greer believes that homomorphisms are still helpful in learning something about the relevant objects.

Four participants used metaphors related to *information gain.* This code captures the idea that homomorphisms are used to gain understanding about one or both of the objects involved. For example, Blair gave the following reason for using homomorphisms in their research but was relatively vague on the details about what sort of information is gained from this.

> And a great example of that would be like group actions. A group acts on a metric space. And even if you—a group action is a homomorphism from a group into the isometry group

> of this metric space. And even if you didn't understand that much about the metric—or you knew some things about the metric space, you knew some things about the groups, the information can go both ways. You can use things you learn about the metric space to learn things about the group, and you can use algebra that you can actually compute in the group to learn things about the metric space. So, you can gain knowledge in both directions.

Others, like Hayden, referred to this information gain as part of their mathematical tool set:

> It's our basic tool in moving around between algebraic objects [we] want to sort of exhibit. Often when you're trying to find out something about some object, you will apply some homomorphism to it to understand it, maybe in a simpler context. That's part of the grammar of doing research in algebra.

Isomorphisms were similarly mentioned as useful for shifting perspectives to gain information (e.g., "isomorphisms are used in my research, I would say, to build bridges between two different ways of thinking," Greer).

Generalizations

Here we include generalizations and analogues of isomorphism and homomorphism across different branches of mathematics. Five participants noted that isomorphisms are an example of an *equivalence relation*. Blair uses this idea to talk about isomorphisms broadly in any category.

> So *everything* I do when I talk about two things being equal or the same, there's always some explicit or implicit notion of an equivalence relation. It's up to something. And every equivalence relation gives rise to some notion of isomorphism in the right category.

Blair seems to be speaking about isomorphism in the category theoretical sense here, which includes the algebraic notion.

In a similar vein, all nine participants brought up *other branch analogues* to isomorphism in abstract algebra. Some of these were in response to being asked whether they view isomorphism in a specific context or more broadly. For example, Dallas made connections to analogous concepts in topology: "But the concept of isomorphism… extends kind of beyond just algebra.… So I think of concepts like homeomorphism, diffeomorphism, or homotopy equivalence even as being analogous to isomorphism." Finley also talked about isomorphism existing outside of algebra: "I think most mathematicians will think of [isomorphism] as a general thing across mathematics. But for me, the reason… is because it's a thing in category theory, and then you can apply it in any category." Again, we see the idea of category theory being a way to talk about these concepts in a more general way.

Some *other branch analogues* were also discussed in response to the final interview questions, which specifically asked about these analogues: "Some people answering the survey saw connections between isomorphism/homomorphism and equivalent fractions or congruence/similarity in geometry. Do you agree that there is some level of similarity between these contexts? Do you think it would be helpful to highlight these similarities with students?" Indy compared equivalent fractions to the relabeling conception of isomorphism: "I think equivalent fractions is kind of an interesting notion in the sense that they're the same number, but they're written differently. So it is kind of this idea of, we have different names for the same object." However, they didn't feel like similar triangles were a strong enough analogy to use for homomorphism: "The similar triangles… that one I don't like as much because… you're only zooming in and out. You're not even kind of like folding it… I feel like the similar triangles would give an impression of too much rigidity." This comment brings up the importance of being careful how analogues are used in the classroom, in order to avoid encouraging too narrow or loose conceptions about isomorphism or homomorphism.

Discussion

Isomorphism and homomorphism have a variety of conceptual facets to them. As previously observed, they can be interpreted formally through their definitions, through the lens of sameness, as mappings, and as sameness-focused mappings (Rupnow, 2021). One way we add to prior work is by noting new ways that sameness (e.g., *indistinguishable*) and mapping (e.g., *invertible*, *transformation*) metaphors can manifest. While these metaphors were only used by a few mathematicians, we note that they reveal opposite views of how much sameness is conveyed by isomorphism. For *indistinguishable*, objects linked by the isomorphism are not worth viewing as different—in all important ways they are the same. In contrast, *transformation* emphasizes differences still exist, even for isomorphic objects. Although these perspectives are in tension, they provide complimentary views depending on what is important for a specific context.

Furthermore, even the four original clusters do not fully capture ways in which isomorphism and homomorphism are understood. Here we see formal definitions conveying a type of sameness (formal definition/sameness cluster) through operation-preservation and structure-preservation. Though these metaphors can be used as stand-ins for the homomorphism property, the preservation aspect also highlights the sameness of elements' interactions with each other. We also see ways in which isomorphism and homomorphism are part of a broader system of interconnected ideas, permitting connections to similar concepts in other parts of mathematics (generalizations cluster). These connections can be viewed thematically (*equivalence relations*) or as specific other instantiations (*other branch analogues* like homeomorphism). Finally, the changing perspectives cluster highlights a route for viewing problems in new ways when problem solving. Specifically, these mathematicians working in or near algebra/category theory provide homomorphism a purpose of its own rather than viewing homomorphism as important only for its relationship to isomorphism or for having a tenuous connection to sameness.

Furthermore, the changing perspectives cluster highlights potential routes for future research. For instance, considering these homomorphism purposes did not arise in the prior study and were only noted by four mathematicians here, how prevalent are these notions? Similarly, would math instructors who teach but do not research algebra benefit from explicit conversations and professional development on this topic to make their teaching more relevant? Alternatively, do students find the changing perspectives cluster relevant if they are not interested in pursuing higher level math courses? Further explorations of such connections between instructors' understandings and teaching as well as teaching and students' understandings seem justified.

Finally, this examination of experts' language highlights desirable conceptions for students. Prior work has carefully examined students' use of properties and approaches to determining whether groups are isomorphic (e.g., Dubinsky et al., 1994; Leron et al., 1995) as well as focused on the function nature of isomorphism and homomorphism (e.g., Melhuish et al., 2020). Here we highlight a framework that permits and structures simultaneous examination of both while connecting to analogous topics in other mathematical subfields. Future research could examine the benefits of using particular clusters of metaphors, contexts in which different clusters are optimal, and ways to foster explicit connections among these metaphor clusters in the classroom.

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