UNDERGRADUATE STUDENTS' CONCEPTIONS ABOUT COMPLEX NUMBERS: A TRAJECTORY OF THEIR MENTAL STRUCTURES

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This paper describes the conceptions about complex numbers that a group of university students has, these were built from the application of an activity sequence centered on these numbers. This sequence is based on the APOS theory, some aspects of semiotic representation theory, and the use of digital technology. Particularly, both the general results of a pretest and a posttest are shown and compared. Additionally, the example of a student is analyzed to show evidence of how the mental structures and mechanisms that define the students' conceptions are built through the implementation of the sequence. The results show how the activity sequence allowed students to coordinate algebraic and geometric processes on complex numbers to improve their conceptions.

Keywords: Complex Numbers, APOS Theory, Conceptions, Undergraduate.

Introduction

Researchers who have taken the teaching-learning of complex numbers as an object of study have reported various difficulties in students and trends in the teaching of this concept (Panaoura et al., 2006; Pardo and Gómez, 2007; Randolph and Parraguez, 2019). In teaching, they have identified an emphasis on the algebraic register, which results in a loss of essential characteristics of complex numbers (Distéfano et al., 2012). This teaching approach limits students in the construction of mental structures and mechanisms related to the geometric representation of complex numbers in their different forms. Therefore, the need to include the geometric register in teaching is established to favor the conversion and coordination of processes (Aznar et al., 2010) for the characterization of the elements of the concept under study. Derived from this problem, the research objective was to improve the conceptions about the complex numbers of students who start their university studies from the design and implementation of a teaching sequence based on the APOS theory, the use of digital technology, and with a focus on the geometric representation of those numbers. In this way, we provide an answer to the following question: how do students' conceptions of complex numbers change when they participate in a teaching sequence with the aforementioned characteristics?

Theoretical Framework

In order to become the objective of this research, we consider the elements of the APOS theory, initially developed by Dubinsky and a group of researchers (Arnon et al., 2014), as well as some elements of the theory of semiotic representations (Duval, 2006). From the APOS theory, a student's conceptions are developed in a context when he is faced with solving problems that involve the use of mental structures and mechanisms (Dubinsky, 2014). The structures are actions, processes, objects, and schemes that are transformed from the use of mechanisms such as: internalization, coordination, reversion, encapsulation, decapsulation, among others. These transformations and their relationships are shown in Figure 1, which is adapted from the work of Arnon et al. (2014).

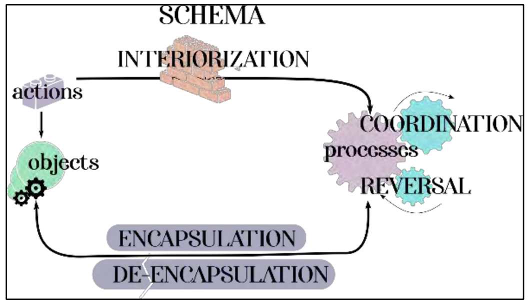

Figure 1: Mental structures and mechanisms cycle

The cycle begins with actions on previously constructed mental objects. In this conception, the student needs external help, which is why they require explicit steps for execution. When the actions are reflected, they are internalized and result in a process. Another way to build processes is through the coordination and reversal of two or more previous processes. In this conception, the student can explain the generality of the concept and describe its characteristics. To favor the development of this mental structure, we consider, like Duval (2006), that mathematical activity needs the articulation between the different semiotic representation systems, and this is achieved when the student can make treatments and conversions, these transformations allow the construction of processes and consequently when these processes are encapsulated, mental objects are achieved.

The object is established when the process can be seen in its entirety, that is, the student can perform actions on this object through encapsulation and it can be returned to the processes that made up the object through decapsulation. Thus, if the set of these structures is consistent, it allows the construction of the scheme. This cycle of structures and mechanisms constitutes the theoretical aspect of the APOE theory, but the same theory contemplates methodological elements, which is called the research cycle.

Methodology

For the development of the research, the research cycle proposed from the APOE theory (Asiala et al., 1996) was followed. This cycle is made up of three elements (Figure 2). The first element is the theoretical analysis, which is developed from the epistemology of the mathematical concept and results in the construction of a genetic decomposition. Arnon et al. (2014) defines genetic decomposition as a hypothetical model that describes the structures and mechanisms that an ideal student requires to understand a concept. The second element is the design and implementation of the instruction, this design addresses the characteristics of genetic decomposition, and leads and transforms the previous conceptions of the students, which is explained from the cycle of mental structures and mechanisms. The third element is the collection and analysis of data, this enables the comparison of the preliminary genetic decomposition and the developments obtained in the analysis. It is worth mentioning that this cycle is repeated with the intention of perfecting the genetic decomposition.

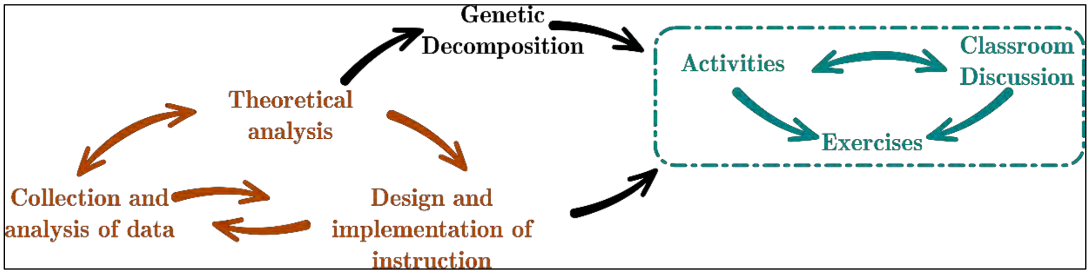

Figure 2: Research cycle and ACE Teaching Cycle

The implementation of the activities was developed from the ACE teaching cycle (Arnon et al., 2014). The acronym for this cycle (Figure 2) presents its elements: (A) activities, (C) class discussion and (E) exercises. In the first moment of the cycle, the activities are guided and elaborated tasks to favor the construction of mental structures arranged in the genetic decomposition, from this moment it is suggested to incorporate teamwork. The second moment is the discussion in class, here it is possible for the students to work in small groups, and that the discussions are guided by the teacher and complemented through worksheets and technological tools. The third moment considers the exercises, whose purpose is to help to consolidate and evaluate the conceptions that have been developed in the activities and the discussion in class. Therefore, the activities of this research were designed taking into account the genetic decomposition and the ACE cycle, for reasons of space and the focus of this writing, no details are given about the design, but they can be consulted at the link: https://www.geogebra.org/m/zx5ww8bf.

Population and method

The investigation began with the application of a pretest, later the sequence of activities was implemented considering the ACE teaching cycle, and finally a posttest was applied. The questionnaires have the same characteristics and are made up of two parts, their application was made individually. The sequence has ten activities and has two teaching cycles. The first cycle comprised of activity 1 to activity 6 with discussions every two or three activities, as required by the group, and ended with activity 7. The second cycle is made up of activities 8 and 9, a general discussion, and ended with activity 10. Activities 7 and 10 were considered exercises. The activities were carried out in teams. The general study was developed with a group of 15 first-semester students of the degree in electrical mechanics from a public university in Mexico. The application of the questionnaires and the sequence of activities was carried out in the hybrid modality.

To carry out the analysis, initially a general comparison was made between the pre- and post-test results of the entire group. Subsequently, and to account for how the conceptions of the students changed throughout the implementation of the sequence, one student (E1) was chosen. According to the teacher's criteria, E1 was characterized by having a performance below the group average, but was responsible, participative, showed interest in learning mathematics and, in addition, showed interest during the implementation of the sequence. Given that E1 teamed up with another student who was characterized by having an above-average performance (E2), this last student will also be mentioned in the results.

Result

In the first part of the pretest, students were inquired about previous objects related to complex numbers. That is, to solve a quadratic equation whose solutions are complex, and perform two operations in which the imaginary unit had to be recognized. In the general results, the students did not perform actions on the imaginary unit object from the square root of a negative number, so there is no imbalance in their mental structures. This is because 9 of 15 students recognized the square root of a negative number as a solution to the equation (Figure 3a). Another 4 of 15 students recognized the imaginary unit, but had errors in the algebraic treatment (Figure 3b).

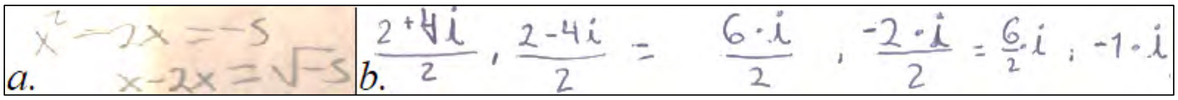

Figure 3: students' responses of the first part of the pretest

The second part of the pretest includes explicit elements of complex numbers. 3 reagents were proposed, in the first and second it was asked to determine the module and the argument of a complex number: first from its representation in the complex plane, and then from its binomial form. In question 3, it was asked to determine the real part and the imaginary part of a complex number given its module and argument. The results show that 13 of 15 students did not answer that part of the questionnaire (Figure 4a) and only 2 of the 15 presented actions on the object prior to distance to establish the value of the module (Figure 4b). In general, most of the students did not present previous conceptions related to complex numbers.

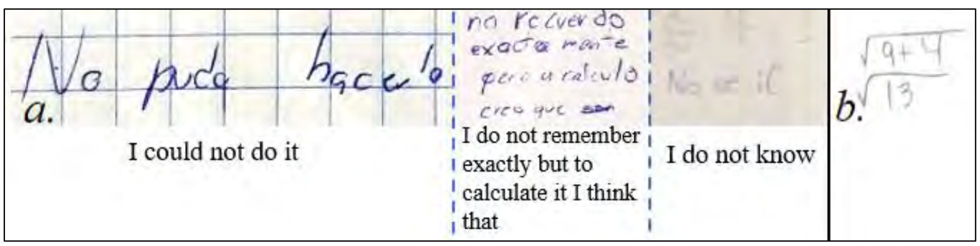

Figure 4: students' responses of the second part of the pretest

In general, the results in the pretest allowed us to conclude that the students had difficulties and did not associate their previous objects with the complex numbers, and therefore did not perform actions on them. On the other hand, in the post-test there is evidence of the construction of new mental structures such as actions, processes and objects. In the first part, 11 out of 15 students presented approximations to the imaginary unit more frequently than in the pretest (Figure 5). There are processes on complex numbers (Figure 5a); however, in the first question, 8 of 15 students showed difficulties in operating with fractions associated with the imaginary part of the complex number (Figure 5b). In the operative part, second question, 6 of 15 students presented difficulties in the properties to operate with the imaginary unit (Figure 5c).

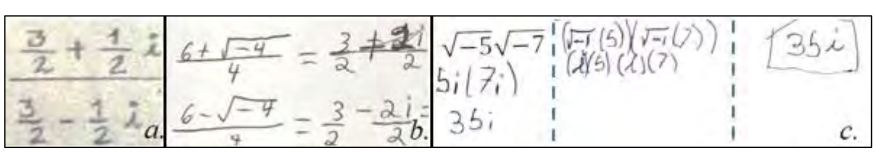

Figure 5: students' responses of the first part of the postest

In contrast, in the second part of the post-test, 14 of 15 students presented an important evolution in the construction of new mental structures through the characteristics of complex numbers in their algebraic and geometric registers. Geometric and algebraic actions are evidenced on objects such as: coordinates, angle, distance between two points, trigonometric ratios, among others (Figure 6a). The students identified the explicit elements of each notation and used the different complex number notations in their treatments, in particular ordered pair, polar and trigonometric (Figure 6b).

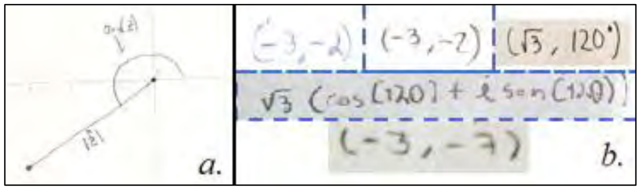

Figure 6: students' responses of the first part of the postest

In the second part of the post-test there is evidence of algebraic and geometric processes and their combinations that allowed the students (13 out of 15) to develop generalities for the representations of complex numbers. It is worth mentioning that the group discussion and teamwork guaranteed these generalities, since in the post-test questionnaire the students worked with specific cases. The geometric processes allowed the students to represent the elements of the complex number in the different quadrants of the complex plane, as well as to convert to the algebraic register with the use of the trigonometric ratios and the distance between two points. The students performed treatments on the algebraic register to find the values of the module and argument. They used generality to find the argument from the arctangent with the real part and the imaginary part, the students added 180° when considering that the complex number is located in the second or third quadrant (Figure 7a). Similarly, these processes are also explicit in the item in which the module and the main argument are explicitly given to obtain the real and imaginary part and represent the number in the complex plane (Figure 7b).

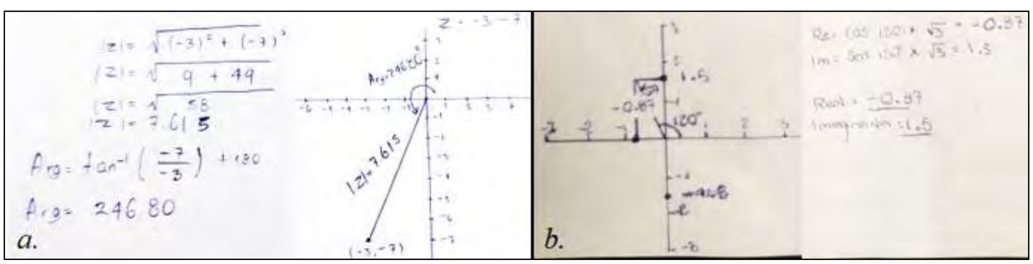

Figure 7: Algebraic and geometric processes in student's responses

The results allow us to establish that, in general, the students built new mental structures from the development of the sequence of activities. To detail these structures, the case of a student (E1) is exemplified, for this the trajectory of it is established and a description of its conceptions is made. In the results of the first part of the pretest, the student did not present actions on previous objects of complex numbers and presented difficulties on objects such as solutions of quadratic equations and properties of real numbers. E1 did not answer the second part of the pretest.

As mentioned, in the development of the sequence of activities E1 worked as a team with E2. In activities 1 and 2, the students built actions on previous objects such as: angle, distance between two points, coordinates and trigonometric ratios. They identified in the complex plane each one of the elements of the complex numbers, these actions were established when they represented different complex numbers in the quadrants of the plane through the applet. An example of these algebraic actions is shown in Figure 8, in which the students worked in the second quadrant. Also shown are the worksheets in which they determined the modulus and argument values for a specific complex number.

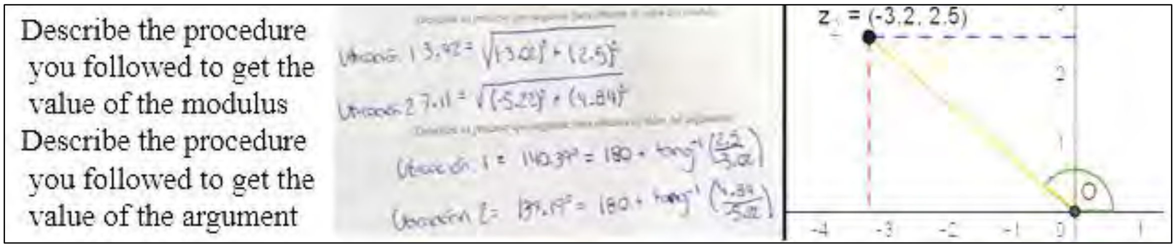

Figure 8: E1 and E2's response in activity 2 of the sequence

The geometric actions of the students considered specific complex numbers within the complex plane, identified the different elements of the complex numbers, and found their values by means of the trigonometric ratios and the distance between two points (Figure 8). In particular, E1 participated in the team discussion mentioning the differences of the complex numbers when they are in each quadrant. From the reflection of the actions in each of the quadrants, the students established the generality to find the module, an argument (Figure 9), the real part and the imaginary part, which gave way to algebraic and geometric processes.

¿Cómo hallaste los argumentos? Explica tu desarrollo

[sic]

como el valor de x es 0 en ambos tenemos que tomar el valor de y. y así podemos opservar sus angulos. si es positivo son 90 grados, si es negativo son 270 grados.

How did you find the arguments? Explain your procedure
since the value of x is 0 in both we must take the value of y, and thus we can observe their angles. If it is positive, it is 90 degrees and if it is negative, it is 270 degrees.

Figure 9. E1 and E2's response in activity 3 of the sequence

Algebraic and geometric processes helped students build and relate the elements of complex numbers in each of these representations. The students started from the geometric register in its polar form and made the conversion to the algebraic register, this allowed them to characterize and find the real part and the imaginary part. These constructions were based on their previous knowledge about trigonometric ratios, in this way E1 and E2 found the generality through the reflection of examples that they chose with the applet. E1 systematized the particular cases raised by E2, this allowed E1 to raise the generality, which he expressed in the discussion with E2 (Figure 10). This generality was achieved from the coordination of geometric and algebraic processes. Regarding the processes related to the module and an argument, the students worked in activities 5 and 6 with sets of complex numbers. The students were able to describe different sets of complex numbers with their elements in the cases of a constant modulus and argument.

PARTE REAL= COS ARGUMENTO X MODULO	PARTE IMAGINARIA = SENO ARGUMENTO X MODULO
PARTE REAL = COS ALFA X R	PARTE IMAGINARIA = SENO ALFA X R
Real part= cos argument x modulus	Imaginary part= sine argument x modulus

Figure 10. E1 and E2's response in activity 4 of the sequence

In activity 7, E1 worked individually, evidencing processes that involve reversal, since in previous activities they usually started with the geometric register of complex numbers, but here they started with the algebraic register. E1 presented in the development of the questions algebraic and geometric processes, when using and describing the generalization of the characteristics of the complex number. Additionally, E1 presented a characterization of the representations, in particular the ordered pair and trigonometric form, and performed conversion and treatment to determine the values of the module and the argument of a number (Figure 11).

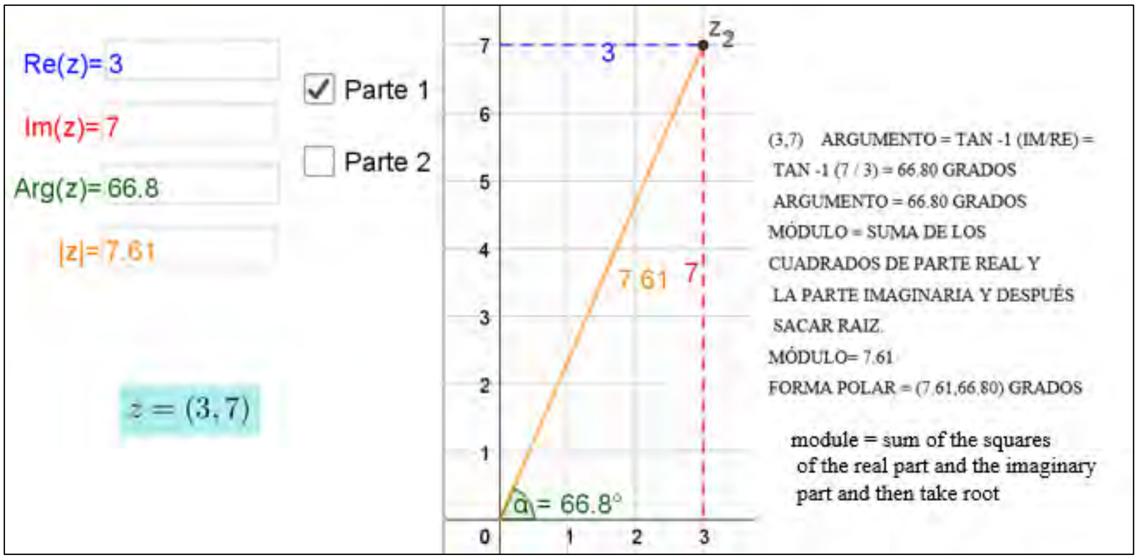

Figure 11: E1 and E2's response in activity 7 of the sequence

The second cycle of teaching is made up of activities 8 and 9. These activities are focused on the arguments and the imaginary unit. In activity 8, students E1 and E2 presented difficulties in establishing the generality of the main argument when they worked as a team. However, in the general discussion some students presented different results, and in this way students E1 and E2 reflected and were able to generalize the arguments from the main argument, and from additional questions asked by the teacher. In this sense, Oktaç et al. (2019) affirm that there is no division between the teacher and the researcher since they have the common goal of enabling the construction of mental structures. Additionally, the authors establish that the construction of mental structures does not always occur in a linear fashion.

In activity 9 the students established the relationship between the modules and the arguments from the geometric register, and the students understood the binomial and trigonometric notations of complex numbers. This cycle concluded with activity 10, which was worked on individually and is called an exercise. Here, E1 coordinated previous processes to build new algebraic and geometric processes that allowed him to characterize elements of the new representations (Figure 12a), as in the post-test questionnaire where the construction of these mental structures in the student was evidenced, he identified and related those elements of the complex number, in addition, used and understood the different representations and notations of complex numbers (Figure 12b).

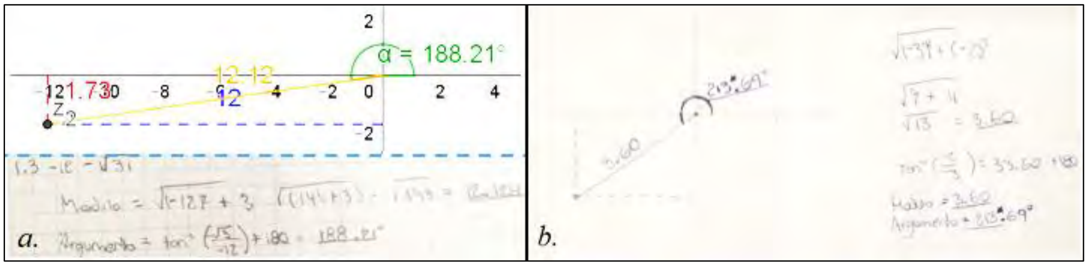

Figure 12. E1 and E2's response in activity 10 of the sequence

Conclusions

The results show that most of the students built mental structures related to complex numbers from the development of the proposed sequence of activities, thus improving their conceptions. The sequence guided students to start from previous objects that led them to activate their mechanisms and articulate the different representations of complex numbers. This allowed generating and coordinating algebraic and geometric processes that in some students can be characterized as objects due to their argumentation in the discussions and their development in the worksheets.

An important element in the design of the sequence of activities is the approach that was given to the registry of geometric representation of complex numbers to later make a conversion to the algebraic registry, and in this regard, evidence was given of how the conversion and treatment in this record allowed students to reach higher level mental structures. In addition, the interaction that the students had with the applets that make up the sequence gave rise to the first actions that led the students to internalize them to build processes on complex numbers. Similarly, the general discussions contemplated in the teaching cycle allowed the students to present their results and argue based on the work with their classmates. According to the results, it is essential that the teaching of complex numbers involves different representations where students have the possibility of expressing generalities about the different elements of these numbers.

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