TOWARDS AN ELABORATION OF CONCRETENESS FADING: REFLECTIONS ON A CONSTRUCTIVIST TEACHING EXPERIMENT

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Over half a century has passed since Bruner suggested his three-stage enactive-iconic-symbolic model of instruction. In more recent research, predominantly in educational psychology, Bruner's model has been reformulated into the theory of instruction known as concreteness fading (CF). In a recent constructivist teaching experiment investigating two undergraduate students' combinatorial reasoning, we utilized an instructional approach that maintains the enactive-iconic-symbolic stages of CF, but through a gradual and much elaborated process. We found that our theory of levels of abstraction explicated the "fading" effect that is central to CF. In this theoretical report, we discuss how CF can be elaborated by our instructional approach and theoretical perspective.

Keywords: Learning Theory; Instructional Activities and Practices; Advanced Mathematical Thinking; Mathematical Representations

Introduction

Substantial research points to the potential affordances of using manipulatives in mathematics teaching (Bouck & Park, 2018; Carbonneau et al., 2013; Domino, 2010; Moyer-Packenham & Westenskow, 2013; Peltier et al., 2020). However, *how* should manipulatives be used to benefit student learning? From a constructivist perspective, manipulatives—by which we mean physical or virtual objects on which sensory-motor actions may be performed—do not carry inherent mathematical meaning (Ball, 1992; Wheatley, 1992). Students must *construct* the meanings that they come to associate with representational forms, even when, from a more knowledgeable person's perspective, those forms "look like" the concepts they are intended to represent. Thus, the shift from using manipulatives to using formal *symbols* for a given concept needs to be given careful consideration (Clements & McMillen, 1996; Fennema, 1972; Resnick & Omanson, 1987).

Research suggests one effective means for introducing abstract symbols and ways of operating on them in ways that are meaningful to students is the three-stage enactive-iconic-symbolic "concreteness fading" (CF) instructional model, originally proposed by Bruner (1966), and similarly the concrete-representational-abstract (CRA) model. Within the CF model, a concept is first represented using "concrete" materials on which students may perform sensory-motor actions, followed by "iconic" representations which may include graphic or pictorial forms, and lastly "symbolic" representations such as words or letters for the concept.

The purpose of this theoretical paper is to suggest a potential elaboration of the CF model using a theory of levels of abstraction (Battista, 2007), and an elaboration that emerged from a teaching experiment investigating two preservice teachers' combinatorial reasoning (Antonides & Battista, under review). Our instructional approach utilized concrete/enactive tasks in that students were asked to enumerate permutations represented as "towers" by constructing towers using physical, multi-colored connecting cubes (cf. Maher et al., 2011). Our students used these manipulatives to enumerate towers 3-cubes, 4-cubes, and 5-cubes-high, all the while constructing numerical symbols and computational expressions that were explicitly linked to their tower constructions. Gradually, the students' reasoning shifted from operating on towers to operating primarily on symbolic representations, which they could later use to reason in novel situations. This theoretical report seeks to establish two claims: (a) that our instructional approach represents a case of a much-elaborated instantiation of CF, and (b) that our theoretical framework focusing on students' levels of abstraction serves to explicate the "fading" effect that is central to CF.

Concreteness Fading and Related Perspectives

Bruner argued for a theory of instruction that includes three broad representational forms: enactive, iconic, and symbolic. According to Bruner (1964), "Their appearance in the life of the child is in that order, each depending upon the previous one for its development, yet all of them remaining more or less intact throughout life" (p. 2). *Enactive representations* are characterized by sensory-motor actions within experiential situations; Bruner suggests examples of riding a bicycle, tying knots, or driving a car. *Iconic representations* "[summarize] events by the selective organization of percepts and of images, by the spatial, temporal, and qualitative structures of the perceptual field and their transformed images" (p. 2). To reason about an experience, such as riding a bicycle or tying a knot, a student can call forth internalized mental representations as material on which to operate. *Symbolic representations* include, in particular, words that are used to point to particular conceptual referents. Symbols, unlike icons, typically do not bear a perceptual resemblance to the objects that they represent.

Goldstone and Son (2005) introduced the term "concreteness fading" to refer to the process of successively decreasing the level of concreteness of a simulation for a scientific concept, with the eventual goal of "attaining a relatively idealized and decontextualized representation that is still clearly connected to the physical situation that it models" (p. 70). While Goldstone and Son related CF to Bruner's theory of instruction, their formulation of CF did not specify a three-stage representational sequence.

McNeil and Fyfe (2012) formulated CF as a three-step gradual fading process, similar to Bruner's recommended model. They conducted the first study to experimentally test the benefits of such a three-stage progression, specifically within the context of undergraduate students' learning and transfer of the properties of an abelian group of order three (associativity and commutativity, and the existence of inverse elements and an identity element). The students were randomly placed into one of three instructional conditions: generic, which used abstract symbols (two-dimensional shapes); concrete, which used iconic representations of measuring cups; and fading, which used a concrete-to-generic approach with an explicit linking of the two representations through an intermediary, Roman numeral-based representation. Their results showed that students in the fading condition performed significantly better than students in the generic or concrete conditions.

Notably, McNeil and Fyfe's "concrete" condition did not involve perceptual materials on which sensory-motor actions were performed—a distinction from Bruner's characterization of the enactive. However, Bruner's focus in his original formulation of the three-stage model seemed to be on *children's* intellectual development, whereas McNeil and Fyfe's study focused on undergraduates. This raises important questions about the nature of "concrete" and "enactive" representations. What do these terms mean? Fyfe and Nathan (2019) "use the term concrete representation to refer to any external representation" (p. 410), which they suggest may vary along at least two dimensions: physicality and perceptual richness. The physicality of a concrete representation refers to whether it is two-dimensional (such as a drawing, consistent with Bruner's term iconic) or three-dimensional (such as physical cubes). Perceptual richness typically refers to the visual features maintained by a representation, such as colors, patterns, or texture. Fyfe and Nathan defined CF "as the three-step progression by which a concrete representation of a concept is explicitly faded into a generic, idealised representation of that same concept," which they suggested is accomplished "by removing perceptual and conceptual information either within or across lessons until one arrives at the representation that involves the least effort to infer and generalise the invariant relation" (p. 411). Fyfe et al. (2014) also suggested "fading" may occur "by encouraging structural recognition and alignment" (p. 19).

CF is related to additional theories of instruction and learning. For instance, CF is a form of the more general notion of progressive formalization (PF), a model of instruction in which students first gain experience about a concept through concrete materials, then transition to operating on symbols that are conceptually grounded in these initial concrete experiences (Nathan, 2012). According to Nathan, the PF model combines advantages of both concrete and abstract representations:

> Concrete entities are meaningful to learners early on and so provide accessible entry points, abstractions transcend the applicability of the representations and rules from any one context, and *grounded abstractions* support learners understanding of what the formalisms "say" and how they apply widely to new application areas. (p. 139)

CF is also related to the types of mathematical activity suggested by Gravemeijer (1999; see also Gravemeijer, 2002). Students operating at the level of referential activity use models that are conceptually grounded in experientially real task settings. At the level of general activity, students transcend from operating with "models of" to operating with "models for," meaning "students' reasoning loses its dependency on situation-specific imagery," which Gravemeijer suggests "can be seen as a process of reification" (p. 164).

Context of Our Proposed Elaboration

We conducted one-on-one constructivist teaching experiments (Steffe & Thompson, 2000) with two preservice middle school teachers with the goal of developing second-order models of the students' concepts and actions/operations for enumerating permutations. To help make our students' reasoning salient, and to provide sensory-motor experiences that we hypothesized would serve to conceptually ground our students' developing mathematical meanings, the tasks included in our study generally represented permutations of n objects as n-cube "towers," each tower comprised of n different colors of cubes connected together with a vertical spatial orientation (see also Maher et al., 2011).

As noted in the Introduction, we claim that our instructional approach represents an instantiation of CF. Indeed, both students (DC and NK, neither of whom had studied combinatorics previously) initially counted tower possibilities by constructing towers 1-by-1 using the available perceptual materials—consistent with Bruner's descriptions of *enactive* representations, as well as Fyfe and Nathan's (2019) definition of the broader term, *concrete*. As the tasks became increasingly complex, DC and NK transitioned from relying on 1-by-1 construction techniques alone to constructing partial sets of towers before generalizing multiplicatively. They could mentally imagine towers that they had not yet constructed, consistent with Bruner's meaning of the term *iconic* representation. Each student progressed to constructing multiplicative operations without needing to first operate on towers, but justified using (and thus conceptually grounded in) this sensory-motor cube-towers context. This is consistent with Bruner's characterization of *symbolic* representations, as well as Nathan's (2012) description of abstract representations in the PF model of instruction. Ultimately, each student was able to construct, through a process of guided reinvention (Gravemeijer, 1999), a generalized formula for counting permutations. The instructional sequence through which our students constructed their formulas, and a brief summary of each student's reasoning, are provided in Table 1. Note that students were provided with physical cubes to construct 3-, 4-, and 5-cube towers, and more than enough cubes were available to construct all 3- and 4-square towers (but not 5-square towers, of which there were 120 possibilities).

Table 1: Instructional Sequence and Summary of Students' Reasoning

Task	Summary of DC's Reasoning	Summary of NK's Reasoning
1. Counting 3-cube towers each containing 3 colors of cubes	Constructed towers systematically one-by-one using physical cubes	Constructed towers systematically one-by-one using physical cubes
2. Counting 4-cube towers each containing 4 colors of cubes	Constructed 12 towers one-by-one, organized by base color, then multiplied $6\times4=24$	Reasoned there are six 3-cube towers for any 3 colors, and 4 possible top-cube colors, so there are $6\times4=24$ towers
3. Counting 5-cube towers each containing 5 colors of cubes	Used enactive processes to enumerate 4-cube permutations with a fixed cube in the fifth position, then multiplied 24×5	Reasoned there are 24 4-cube towers for any 4 colors, and 5 possible top-cube colors, so $24\times5=120$ towers
4. Counting 6-cube towers each containing 6 colors of cubes	Multiplied 120×6	Multiplied 120×6
5. Counting 9-Cube towers each containing 9 colors of cubes	Multiplied $720\times7\times8\times9$	Multiplied $720\times7\times8\times9$
6. Counting 20-cube towers each containing 20 colors of cubes	Multiplied $362{,}880\times10\times\ldots\times19\times20$	Described the multiplication $20\times19\times\ldots\times2\times1$
7. Counting 100-cube towers each containing 100 colors of cubes	Described the multiplication $1\times2\times\ldots\times99\times100$	Described the multiplication $100\times99\times\ldots\times2\times1$
8. Counting *n*-cube towers each containing *n* colors of cubes	Described the formula $1\times2\times\ldots\times n$	Described the formula $n\times(n-1)\times(n-2)\times\ldots\times2\times1$

Theoretical Perspective

We adopted a psychological constructivist view of mathematical knowing and learning (cf. Piaget, 1970; von Glasersfeld, 1995). Within this view, a *concept* is a mental representation of a phenomenon that is stable enough to be re-presented (e.g., visualized or described) in the absence of relevant sensory-motor input (von Glasersfeld, 1991). An *action* refers to either a physical transformation on perceptual material or a mental action on imagined/re-presented material. An action constitutes an *operation* when (a) it is internalized (so that it can be performed mentally), (b) it is reversible, and (c) it can be composed with other mental actions (Piaget, 1963). A *scheme* is a way of operating under certain situations. It consists of an assimilatory mechanism for recognizing situations along with an integrated set of abstractions used to mentally represent the situation at hand (i.e., a *mental model*); a sequence of actions/operations associated with the assimilated situation; and a set of expectations or anticipations about possible results from those actions (Battista, 1999; see also von Glasersfeld, 1995).

Levels of Abstraction

To frame and analyze our students' conceptual progressions, we used Battista's (2007) theory of levels of abstraction, a reformulation of Piaget's theory of abstraction via Steffe and von Glasersfeld (Steffe, 1998; Steffe et al., 1988; von Glasersfeld, 1995). Battista's original levels of abstraction were generally stated and could be used to analyze and interpret student understanding of concepts and operations across a wide range of mathematical domains. At the *perceptual/recognition level*, an item from one's experiential flow is isolated and entered into working memory. Sensory properties necessary to recognize future instantiations of the item are empirically abstracted. At the *internalized level*, abstracted object-concepts may be re-presented (i.e., visualized) in the absence of relevant perceptual input, or abstracted actions and action sequences may be re-enacted in the absence of relevant kinesthetic signals prompting the abstracted action sequence. At the *interiorized level*, the student's understanding of the abstracted item becomes generalized in that they can apply the item to reason in novel situations. It is at this level that structures, patterns, and abstract forms are abstracted from particular sensory-motor contexts (Steffe et al., 1988). In particular, it is at the interiorized level that actions can be come operations, as the action can be performed in thought, is reversible, and can be composed with other mental actions. At the *second interiorized level*, the student constructs symbols as conceptual "pointers" to interiorized material, and these symbols are used as substitutes for the originally abstracted material in reasoning. At the *third interiorized level*, more complex operations can be performed on these symbols, such as curtailing sequences of symbolic operations into more condensed forms.

We used the original levels to explicate students' progressively abstract conceptualizations of, and operations on, spatial material within permutation enumeration contexts. However, through our data analysis, we found a need to elaborate the theory by developing an additional, empirically supported set of levels of abstraction. This new set of levels served to explain students' abstractions of *computational operation schemes* (COS), and we found the original set of levels occurred in parallel with this new set of levels. When a sequence of symbolic computations has been perceptually abstracted, the student can re-enact the action sequence step-by-step, but only with external cues such as instructions or a formula. The student can activate this COS when they recognize a situation as being similar to the original one in which they abstracted the sequence of computations. Once internalized, the sequence of computations can be performed, step-by-step, without such external cues. At the interiorized level, a student's understanding of a COS shifts from its step-by-step performance to analyzing the meanings and results of each computational operation, enabling them to apply meaningful deviations from the original and to adapt their COS to reason about novel situations. At the second interiorized level, the student can treat the sequence of operations of their COS as a *conceptual symbol*, meaning they can operate on the COS itself. For instance, they could reverse the sequence of operations, or they could decompose its constituent components and rewrite each computation in terms of this decomposition. At the third interiorized level, the student can algebraically generalize their COS using variable expressions. At the fourth interiorized level, the student can construct and conceptually operate on written/verbal symbols as pointers to their algebraically-generalized computations (e.g., $n!$ as a symbol for $n\times(n-1)\times(n-2)\times\ldots\times 2\times 1$).

To summarize, the levels of abstraction sequence occurs first on actions/operations on objects, then reappears as an individual transitions to numerical, algebraic, and other forms of symbolic mathematical reasoning. The inter-relationship between the original set of levels for actions/operations on objects and the levels for computational operations is illustrated in Figure 1.

Figure 1: The Two Sets of Levels of Abstraction, Linked via S*NLS

Links between the two sets are made via S*-numerical linked structuring (S*NLS). S*-structuring is the mental process of constructing an organization or form for sequences of physical/mental actions on perceptual/imagined material; numerical (or N-) structuring is the process of constructing an organization or form for a set of numerical/algebraic symbols; and S*NLS is a form of reasoning that coordinates S*-structuring and N-structuring consistent with a learner's understanding of the relevant spatial/numerical properties (Antonides & Battista, 2022).

Discussion

As noted in the Introduction, we believe that our instructional approach and theoretical perspective provide an elaborated description of both the cognitive mechanisms that underlie CF and its instructional implementation—two sites in which Fyfe et al. (2014) called for additional research. First, incorporating multiple levels of abstraction greatly elaborates the notion of concreteness "fading," with each abstraction level fading some of the "concreteness" of the previous level, as illustrated in Figure 2.

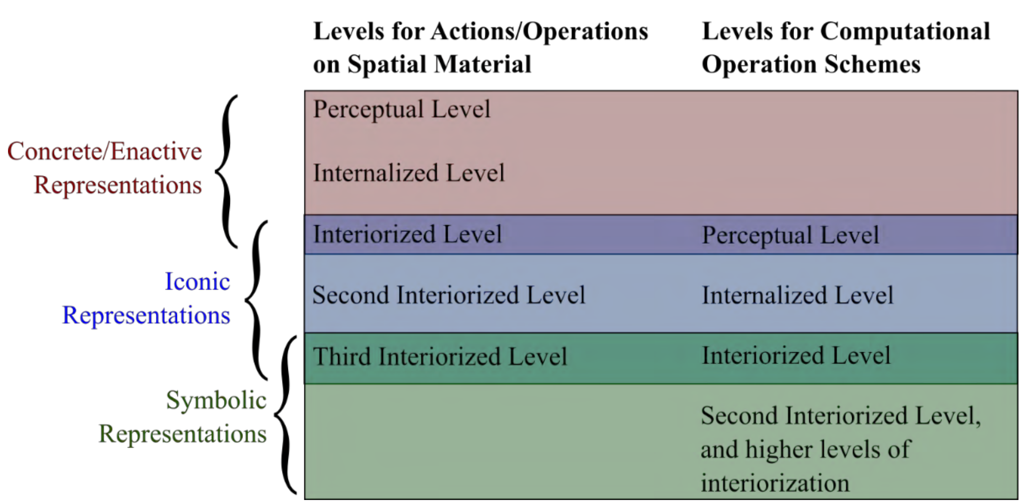

Figure 2: Elaborating Concreteness Fading Using Levels of Abstraction

At the perceptual and internalized levels of abstraction, student reasoning about a given mathematical concept is constrained to the original perceptual context in which the concept was initially encountered, with relevant figurative (or "concrete") materials perceptually available. We hypothesize that even in the shift from perceptual abstraction to internalization, many of the sensory properties initially registered into memory become "faded." Upon reaching the interiorized level, mathematical ideas can be extended beyond this initial sensory-motor context, and more abstract representations, such as drawings and motor-kinesthetic items (e.g., counting using fingers), become available to the student. Interiorization affords the construction of a more generalized structure from a student's actions on sensory-motor material, enabling a shift from concrete/enactive to iconic/enactive representations. For instance, DC and NK interiorized the structure of a 3-cube tower; for DC, this was reflected in his spatial structuring [base cube] + [reversible 2-cube tower]. He used gestures and verbal descriptions to describe aspects of his reasoning, but still strongly linked to the cube-towers context. At the second interiorized level, DC and NK constructed and operated on symbols, such as NK's enumeration of 4-cube towers by multiplying 4×6 *without* needing to construct towers one-by-one.

At the third interiorized level, students' focus of attention begins to shift from actions/operations on the context-specific spatial material to their computational processes themselves, thus emerging upon the symbolic stage within CF theory. DC and NK, at this level, performed more complex symbolic computations, such as enumerating 9-cube towers by multiplying $720 \times 7 \times 8 \times 9$. Our students' reasoning then focused entirely on their computational operation schemes, without reference to cube-towers (though they could, if asked, explain their reasoning in terms of towers).

Second, our theory uses S*NLS (Antonides & Battista, 2022) to elaborate Fyfe et al.'s (2014) discussion of structuring and linking to elaborate the connection between concrete and formal symbolic representations. This connection is illustrated by the levels of abstraction occurring in parallel in Figure 2, with concrete representations on the left and formal symbolic representations on the right, linked via S*NLS as illustrated in Figure 1. Our S*-structuring perspective emphasizes the importance of providing students with opportunities to act on sensory materials when forming their initial combinatorial conceptualizations and reasoning; for us, S*-structuring represents a cognition-based elaboration of Lockwood's (2014) set-oriented perspective, as students draw on their concrete/enactive S*-structuring experiences to conceptually ground symbolic computational formulas and expressions.

Lastly, our investigation and theoretical framework which draws on mental models elaborates Fyfe et al.'s claim that "the concrete stage enables learners to acquire a store of images that can be used when abstract symbols are forgotten or disconnected from the underlying concept" (p. 13). In fact, while mental images may be a constituent part (such as stored mental representations of particular towers or sets of towers), it is *mental models* that enable students to conceptually operate with the combinatorial composites and symbolic manipulations (Battista, 2007).

Furthermore, returning to Fyfe et al.'s (2014) call for describing ways to optimize the fading technique, using the theory of levels of abstraction to guide CF seems to have great potential for optimizing its instructional use. As opposed to the representations used by Braithwaite and Goldstone (2013) in their concreteness-fading study of combinatorics with undergraduates (specifically, letter sequences followed by arithmetic explanations for factorials), we provide a very different and much elaborated interpretation of concrete versus abstract representation, as well as CF, for combinatorial reasoning. Indeed, our "fading" is accomplished by starting with small-number permutations of physical cubes and incrementally increasing this number so that actual manipulation of cubes becomes impractical and necessarily needs to fade into the background as symbolic representations are abstracted and come to the fore. At this point, students start operating on symbolic representations, first with simple arithmetic operations, then with pre-algebraic to algebraic operations, with the fading being accomplished by increasing levels of abstraction, generalization, and consequent symbolic representation. All-the-while during our fading, the students were encouraged and supported, via S*NLS, to directly connect reasoning on each new problem to reasoning on the previous problem using the powerful mathematical reasoning of recursion.

Finally, we view the process of generalization to be a critical component of CF. Drawing on Ellis et al.'s (2021) research synthesis of generalizing actions, CF often involves "deriving broader results from particular cases to form general relationships, rules, concepts, or connections" as well as "extending one's reasoning beyond the range in which it originated" (p. 2), which the authors connected to the process of abstraction. Thus, in our teaching experiments, an essential component of our CF is progressive formalization (Nathan, 2012) in which students *incrementally generalized* their reasoning, again via S*NLS, as they moved from considering small-number to large-number towers. Consistent with our instructional approach, we hypothesize that the enactive stage of instruction may be more effective when it becomes impractical or impossible for the student to completely model the task situation using concrete materials, which would create an intellectual need for a transition to a different symbol system. This hypothesis is one potential avenue for future research. Fyfe et al. (2014) similarly argued that CF enables concepts to be "generalized in a manner that promotes transfer" (p. 12). Consistent with this claim, in later teaching-experiment sessions, DC and NK transferred their cube-based reasoning about permutations to permutation tasks not involving cubes, with both students referring back to their reasoning about permutations of cubes. This suggests the perceptual context of constructing and enumerating permutations of cubes provides powerful mental models for applying and transferring one's reasoning about permutations.

Conclusion

In this theoretical report, we have suggested a potential elaboration of CF theory using our recent teaching-experiment research and theoretical perspective. Our elaborated theory of levels of abstraction explicates the fading mechanism central to CF, with multiple levels of abstraction occurring within enactive-iconic-symbolic representational stages and with specific levels of abstraction at the transition between stages. Our investigation provides a case of implementing a much-elaborated instantiation of CF in the context of one-on-one instruction. We acknowledge that the one-on-one nature of our investigation is a limitation of our study, and it could explain our positive findings. However, significant research has found CF to be a powerful instructional method for supporting transfer of mathematical and scientific concepts (Bouck & Park, 2018; Fyfe et al., 2014). In future research, we intend to investigate how our instructional approach (with appropriate adaptations) may support student learning in classroom-based contexts.

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