INTENTIONALITY IN USING LEARNING TRAJECTORIES TO "REFRAME" TEACHER NOTICINGS TOWARDS ANTI-DEFICIT AND ASSET-BASED INSTRUCTION

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Learning Trajectories have the potential to be used as a tool to advance equity by explicitly connecting to anti-deficit framing and asset-based instruction. This plenary paper highlights research on three use cases for learning trajectories (LT) with an intentionality around promoting equity: 1) the use of LT based Lesson Study with vertical teams of teachers to position students as capable and teachers as knowledgeable, 2) the use of LT coupled with anti-deficit framing in curriculum design research to provide students with access to rigorous educational resources and asset based instruction, 3) the use of LT with formative assessment to develop preservice teachers' equitable teaching practices to advance students understanding. The presenter invites the PMENA community to consider how learning trajectories can be coupled with powerful equity-focused research and frameworks to disrupt the status quo, broaden the notion of learning mathematics, eliminate labeling, and dismantle inequitable structures and hierarchy in the mathematics classroom.

Keywords: Learning Trajectories and Progressions, Anti-deficit Orientation, Asset-based Instruction, Teacher Noticing, Professional Development, Preservice Teacher Education

Introduction

In this paper, I discuss how learning trajectory (LT) research should attend to equity by providing access to rigorous educational resources, positioning students as capable and teachers as knowledgeable, and questioning the curriculum and high stakes assessment practices. I do this work by engaging and privileging the voices of teachers and coaches as co-designers and researchers in Lesson Study and Curriculum Design Research with the focus on rehumanizing mathematics for students (Gutiérrez, 2018).

I am a first generation Korean American mathematics education scholar, who attended my formative years of elementary schools in the 70s between two countries, experiencing Korean as a second language and also what was at the time called, English as a second language. I experienced first-hand differential learning experiences (NCTM, 2020; Jong et al., 2020) where some students were centered and others marginalized. Experiencing schooling in two countries, I also noticed the differential treatment of the teaching profession, one where it is a revered and noble profession and the other where the teacher's professional judgment is constantly questioned and viewed where anyone can teach. This has motivated me to focus my work on elevating the voices of teachers and the teaching profession in the US and building on students' mathematics strengths, particularly those who are marginalized in the mathematics classroom. My research is informed by a commitment to equity and culturally sustaining pedagogy in mathematics education. I work mostly with schools that are racially, culturally, and linguistically diverse and receive title 1 funding. My research has focused on LT use in Lesson Study and Community-based Math Modeling to connect mathematics to students' lived experiences, attending to both cognitive and socio-cultural perspectives. I lean on the work of Aguirre et al.'s (2013) centering their definition of equity where,

> All students in light of their humanity – personal experiences, backgrounds, histories, languages, physical and emotional well-being–must have the opportunity and support to learn rich mathematics that fosters meaning-making, empowers decision-making, and critiques, challenges and transforms inequities/injustices. Equity demands responsive instruction that promotes equitable access, attainment, and advancement for all students" (p. 9).

Given my research orientation, my plenary paper focuses on two questions-1) How do we use LT with teachers and coaches as a tool to deepen teacher knowledge and promote asset-based instruction, especially for students who have been historically marginalized? 2) How might we use LT to "reFrame" teacher noticings towards an anti-deficit orientation?

Shifting from Deficit to Anti-deficit Orientation by "reFraming" Teacher Noticings

This PMENA Plenary event in 2022 marks a significant time in our society, where we experienced the struggles and pain due to the Pandemic as well as systemic racism and escalation of racial tension leading to the Black Lives Matter movement. The pandemic unleashed hate, xenophobia and scapegoating leading to AAPI Hate with racist rhetoric. Labeling the "Asian" community as a monolith with an erasure of individual identity and the myth of the model minority or that "Asians are good at math" perpetuates a stereotype that is racist and dehumanizing (Shah, 2019), ignoring the huge diversity of linguistic, socio-economic, political and cultural backgrounds. It also masks the issues that different communities may need different supports in the school setting to succeed and excel. The Pandemic magnified the inequities that have long-existed in our society, education and communities. Deficit framed discourse streamed the media with outcries of "learning loss", while educational organizations worked hard to fight against this harmful language and discourse (i.e., Where is Manuel? A rejection of 'Learning Loss' TODOS, 2020). In addition, the danger in the discourse that marks the achievement of marginalized students being "more behind" in their learning, again perpetuates a pernicious mindset of achievement gap that our community has worked tirelessly to move away from (Gutierrez, 2008). Our professional organizations showed solidarity in fighting against systemic racism and this deficit framing and advocated for the "Mo(ve)ment to Prioritize Antiracist Mathematics: Planning for This and Every School Year" (TODOS, 2020), and AMTE's (2022) statement on "Equitable and Inclusive Mathematics Teaching and Learning" and the press release on systemic racism advocating for practices that draw on students' mathematical, cultural, and linguistic resources/strengths, and challenge policies and practices grounded in deficit- based thinking. The voices from our leading mathematics educational organizations (NCSM, NCTM & ASSM, 2020, 2021) "In Continuing the Journey: Mathematics Learning 2021 and Beyond", and "In Centering Our Humanity: Addressing Social and Emotional Needs in Schools and Mathematics Classrooms" (TODOS, 2020) advocated for math educators and school leaders to keep our focus on teachers, families and students well being, during this contentious socio political climate and Pandemic.

Rather than returning to the pre-pandemic status quo, Ladson-Billings (2021) argued for a "hard re-set" for a new "post-pandemic pedagogy" stating,

> In a re-set school environment, we will begin a school year with an accurate assessment of what students already know. The school year will have varied and regular formative assessments to determine how well students are understanding what they are taught, and an end of the year assessment would be keyed to what was actually taught in their classrooms. Assessment would no longer be a punitive tool to "catch" students but rather a diagnostic and

> developmental tool that will tell teachers and schools how to adjust their curriculum and pedagogy (p. 74).

And yet, we know the opposite is happening where teachers are again being pressured to "catch students up" so that they can once again administer high-stakes tests. This is problematic as we know as scholars like Louie et al. (2021) describe the danger when a teacher with a framing around "closing the racial achievement gap" implicitly frames Black, Hispanic, and Indigenous students as mathematically lacking and White students' achievement as the standard by which they should be measured (Gutiérrez 2008; Martin 2009). This framing makes one more likely to attend closely to Black, Hispanic, and Indigenous students' errors without attending to their knowledge or strengths, to interpret these errors as evidence of misconceptions and failures, leading to deficit noticing.

Instead, we should be focused on varied and regular formative assessments to holistically determine how well students are understanding what they are taught and focus on asset based pedagogy like Complex Instruction (Horn, 2012; Cohen et al., 1999; Eli & Wood, 2016) where we develop teachers skills in assigning competence in student work. Research from Cohen et al.'s (1999) work on Complex Instruction showed that when teachers praised low-status students publicly for a task-related accomplishment, those students' participation increased; their status differences were mitigated or eliminated; and ultimately, their achievement increased.

When teachers better understand the learning trajectory continuum and anticipate a broader range of strategies, teachers can spot the strength of students along the LT continuum who may typically not get highlighted (Suh et al., 2018). In fact, Empson (2011) reflected on the 2010 PMENA and noted how LT impacts teacher professional noticings, "As teachers interact with students and decide how to proceed, there are many types of decisions to be made – how to gather information about children's thinking, how to respond to it appropriately in the moment, how to design tasks that extend it, and even what to pay attention to" (p. 587).

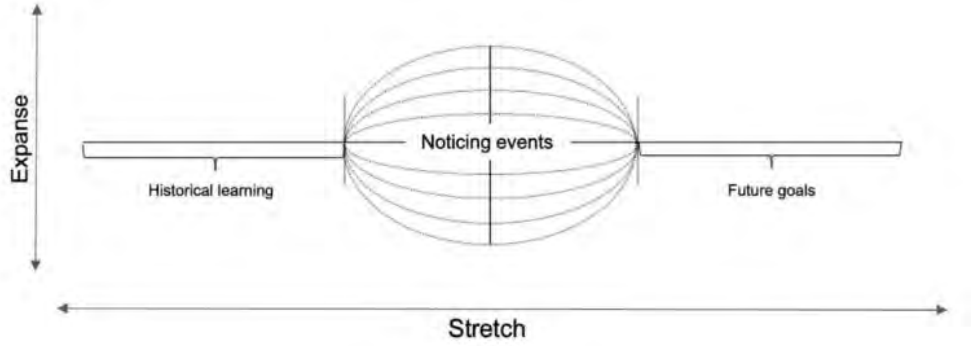


Figure 1: Reflecting Multi-dimensional Noticing for Equity (van Es et al., 2022)

Developing teachers' ability to assign competencies requires specialized skills of equitable noticing (Jacobs et al., 2010; Jacobs & Spangler, 2017; Kalinec-Craig et al., 2021; Jilk, 2016; Jong, 2017; Wager 2014). More recently, van Es et al. (2022) described a framework for Multidimensional Noticing for Equity, a system of noticing to disrupt inequities. Their framing towards a more multidimensional noticing for equity include the perspective of "taking account for how the histories and cultures of teachers, learners, and mathematics —and the broader historical, cultural, and political contexts in which they exist—are at play in moment-to-moment classroom interactions" (van Es et al, 2022, p.115 citing Louie, 2018; Mendoza et al., 2021; Shah & Coles, 2020). In their more expansive framework, the multidimensional noticing was used to

interpret teachers' enactment of culturally sustaining instructional practices organized around a) Stretch, which captures the relation of teachers' noticing to both their own and students' past and futures, and (b) Expanse, which reflects the breadth and range of what teachers identify as noteworthy in moments of classroom interaction teaching and teacher noticing (see Figure 1).

Louie et al. (2021) discuss the importance of framing as a way to challenge deficit discourses about marginalized students that devalue the knowledge and abilities of students of color in classrooms in the US. They note,

> Deficit discourses may give rise to deficit noticing, wherein teachers attend almost obsessively to the errors and shortcomings of students of color; interpret errors and shortcomings as evidence of deficiencies in students, their families, or their cultures; erase students' assets; and disregard schooling practices and social structures that limit students' opportunities to learn and thrive. (p. 96)

Louie et al.'s (2021) most recent work framed anti-deficit noticing explicitly emphasizing how **F**raming is critically important in the ways teachers **A**ttend, **I**nterpret and decide to **R**espond (see Figure 2)

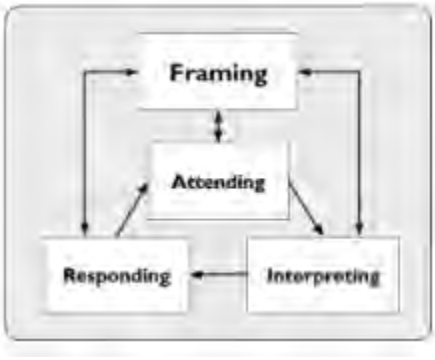


Figure 2: FAIR framework for Anti-deficit Noticing (Louie et al., 2021)

These two frameworks helped our research team think more broadly as we worked with our teachers in how we needed to frame anti-deficit orientation and asset-based instruction when working with learning trajectories. An important implication that we gleaned from the Multi-dimensional Noticing for Equity (van Es et al., 2022) is that although noticing captures the moment-to-moment events, what teachers notice and attune to is multidimensional in that it takes into account historical knowledge of the student, class and context and attends to the complexity of the instruction at play (i.e. Culturally Sustaining Pedagogy). The FAIR framework for Anti-deficit Noticing (Louie et al., 2021) noted the importance of being intentional and explicit when framing toward anti-deficit orientation to shift teachers noticing from deficit to anti-deficit noticing (See Figure 1). In particular, this FAIR framework's emphasis on framing allowed our team to pay close attention to *"how we notice"* students as full human beings with many mathematical strengths and resources, framing math learning as a creative exploration of ideas and framing interaction and interpersonal relationship as integral to learning (Louie et al., 2021).

Expanding the Notion of Math Competence using LT and Students' Assets

We emphasize in our LT PD work that teachers bring these multiple aspects and framing into view as they engage in multidimensional noticing. We focus on encouraging narrative that emphasizes the strengths students bring to the classroom including their mathematical thinking as well as their disposition and mathematics practices. LT has a long history in mathematics

education research (Battista, 2011; Blanton et al., 2015; Clements & Sarama, 2004, 2009; Confrey et al., 2009, 2011; Ellis, 2014; Hackenberg, 2013; Ebby et al., 2020; Petit et al., 2020; Simon, 1995; Steffe & Olive, 2010; and others cited in the synthesis by Lobato & Walters, 2017) and well featured in past PMENA proceedings and has many metaphors and descriptors including a climbing wall (Confrey et al., 2021), in a conceptual corridor (Confrey, 2006);and levels of sophistication plateaus (Battista, 2004), just to name a few. Confrey and Maloney (2010) describe learning trajectories as

> a researcher-conjectured, empirically-supported description of the ordered network of constructs a student encounters through instruction (i.e. activities, tasks, tools, forms, of interaction and methods of evaluation), in order to move from informal ideas, through successive refinements of representation, articulation, and reflection, towards increasingly complex concepts over time. (p. 968)

Clements and Samara (2004) describe learning trajectories for early childhood mathematics for narrow sequences of topics as "a conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking" (p. 83).

The affordance of using LT deepens teachers' understanding of the progression of student learning-drawing upon their knowledge of the learning trajectories to make instructional decisions. More specifically, LTs have been used with teachers and researchers to better understand how students come to understand concepts (Battista 2004; Hackenberg & Tillema 2009) and to use "the learning goal, the learning activities, and the thinking and learning in which the students might engage" (Simon 1995, p. 133) to provide direction for teachers as they plan learning activities and predict the potential reasoning, misconceptions, and learning of students. Hypothetical learning trajectories (Simon 1995) have also been used in professional development settings to enhance instructional practices. Wilson et al. (2015) reported on a study using professional development where LTs bridged "guidelines for student-centered instruction with domain-specific understandings of students' thinking for teachers" (p. 227).

According to Sztajn et al. (2012) existing research on teachers' use of learning trajectories "shows that as teachers make sense of trajectories, these trajectories can support growth in mathematical knowledge, selection of instructional tasks, interactions with students in classroom contexts, and use of students' responses to further learning" (p. 149). Research on Learning Trajectory based Instruction (LTBI, Sztajn et al., 2017) and the specific design decisions (Sztajn, 2010) the team attended to revealed the importance of setting discursive norms to focus on student thinking and teaching from a strength-based perspective particularly "at a time when deficit perspectives and language of differentiated instruction (such as having "high", "medium", and "low" children) to express ideas about student learning have been normalized." (Sztajn et al., 2017, p. 30). With the many descriptors that include terms like "trajectory", "progression", "increasingly complex", "levels of sophistication" (Battista, 2010), some caution that translated LT research can be misused as hierarchical levels that teachers use to sort students and dangerously place labels on students as being high or low. In fact, Myers et al. (2014) concluded that learning about LTs without additional support was insufficient to challenge deeply rooted ideas about student abilities. Even with attention to the design of PD, Myers (2014) found that teachers with severe deficit orientation used LTs to talk about what students could not do as opposed to thinking about moving students forward. Through discourse intervention, teachers

started to use ability as a temporary descriptor to present students' current mathematical performance and used language from the LT to support these claims. The explicit attention to having teachers refer to the LT language instead of labels for students demonstrated the potential of LTs to support equitable instruction.

Celedòn-Pattichis et al. (2018) asset-based approaches to mathematics education are a conscious way to move away from deficit perspectives by teaching in ways that view students' language and culture as well as families', and communities' ways of knowing (Civil 2007; Bartell et al., 2017) as intellectual resources to engage with mathematics in the classroom. Asset based approaches offer a more humanizing view of student thinking that extends beyond school mathematics and recognizes that mathematics thinking and learning happens at home and in communities but is often unrecognized in school settings. Opening up learning trajectories to be able to recognize other forms of math thinking and experiences is key. Celedòn-Pattichis et al. (2018) also cautioned the community to recognize that not all communities and families focus on counting and operations in the specific way that Cognitively Guided Instruction (CGI) has described. Studies that combined CGI with culturally responsive instruction improved the mathematics performance of Native American students with learning disabilities (Hankes, et al., 2013) and other studies with culturally and linguistically diverse students engaged in complex CGI problem solving where teachers drew from language and culture as intellectual resources showed positive outcomes (Celedón-Pattichis et al., 2010; Turner et al., 2008).

Below I share use cases with LT PD and curriculum design research where coupling LT with asset-based approaches yielded anti-deficit teacher noticings. I detail how the use of vertical lesson study teams and other PD structures focused on learning trajectories and multidimensional noticing supported the development of anti-deficit professional noticings. I will refer to my research team as "we" in the case studies to represent the collaborative efforts of multiple researchers and doctoral students from the VDOE projects called TRANSITIONS and Bridging for Math Strength and an NSF project called IMMERSION.

Case #1: Synthesizing Previous Research on using Learning Trajectory-based Lesson Study- Appreciating Students' Robust Understanding

For over a decade, my colleagues and I used Learning Trajectory based Lesson Study (LTLS, Suh et al., 2021; Suh et al., 2019a; Suh et al., 2019b; Suh et al., 2018; Suh et al., 2017; Suh & Seshaiyer, 2014) in a series of multi-year state funded project called TRANSITIONS, where we worked with vertical teams of K-8 teachers, studying, planning, implementing and reflecting on teaching through rich tasks. In the Study phase of the Lesson Study (Lewis, 2012) instead of focusing on grade level standards, we used Confrey's (2012), five elements to unpack the LTs and to plan and anticipate strategies for a rich task starting with discussing: 1) the conceptual principles and the development of the ideas underlying a concept; 2) strategies, representations, and "conceptions"; 3) meaningful distinctions, definitions and multiple models; 4) recognizing coherent structure or pattern in the development of progressively complex mathematical ideas ; and 5) bridging standards or identifying the underlying concepts.

In Suh et al.'s article (2019a) we detailed a LTLS with a group of teachers ranging from Kindergarten to six grade, at an elementary school near a military base with the highest mobility rate of 33% in the district. With this transient population, the LTLS team wanted to use LT to bridge coherence in their curriculum. The team chose the often used submarine sandwich sharing task related to equipartitioning (Confrey et al., 2021; Confrey et al. 2009) typically used in Grades 3 or 4. They decided to launch the first lesson iteration in the Kindergarten classroom to study how very young students might approach this task. We framed the LTLS using elements

from the Teaching for Robust Understanding (TRU) Framework (Schoenfeld & The Teaching for Robust Understanding Project, 2016), particularly focusing on the dimensions of *equitable access to content* using the LT and *agency, ownership, and identity* described as " the extent to which students are provided opportunities to contribute to conversations about disciplinary ideas, to build on others' ideas and have others build on theirs—in ways that contribute to their development of agency (the willingness to engage), their ownership over the content, and the development of positive identities as thinkers and learners" (p. 9).

Though teachers studied the Equipartitioning LT during the Study phase of the LTLS, seeing how students approached equipartitioning through the Lesson Study surprised the educators and brought the LT to life. They noticed the strength in Kindergarten students being able to halve equal sized parts and share among friends, and understand the context of fair share. One group split the 6 sandwiches into 12 halves and made sure everyone of the 8 friends had equal sized parts (halves) and decided to remove 4 halves not eaten. Another group contributed the idea that there were some extras and wanted to use up the whole set of sandwiches and gestured cutting the halves into another half (fourths). Knowing how students approached this task allowed teachers to think about how to advance students to exhaust the whole without leaving any part of the sandwich unused. We used the LT "look fors" to help us better understand criteria for equipartitioning:

1. Having the correct number of parts
2. Exhausting the whole, leaving no parts unused
3. Having equal-size parts

(LT TurnonCC website, Confrey et al., n.d.).

Through lesson study, this vertical team saw this same lesson enacted in second, fourth and sixth grade classrooms. In our debrief, instead of focusing on grade level standards, the focus of the conversation came from observing how the students responded to the task related to the equipartitioning LT 'look fors' as well as using language from like non-anticipatory sharing, additive coordination to multiplicative coordination (Empson & Levi, 2011) as shown in Figure 3.

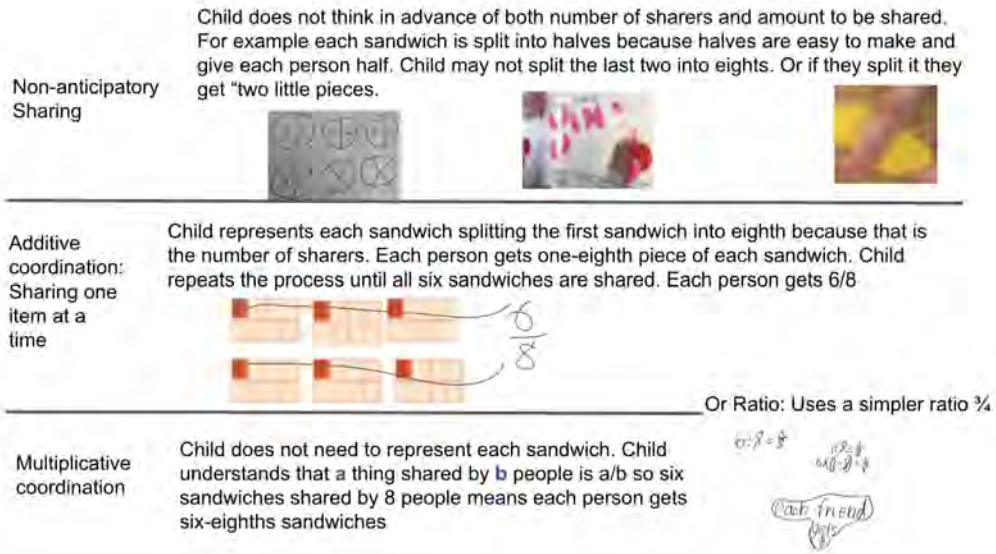


Figure 3: Analysis of Student Thinking using Equipartitioning LT on Sharing Tasks

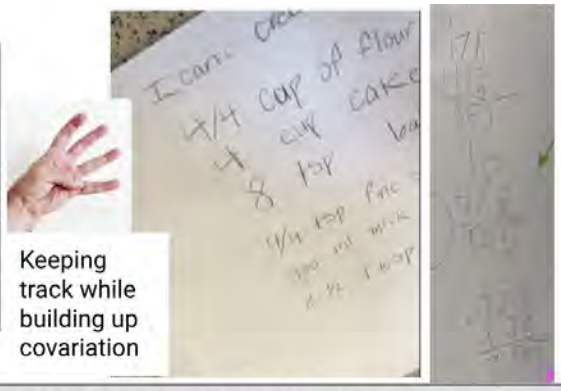
The kindergarten teacher was proud to share the brilliance of her students and the LTLS team

acknowledged and appreciated the informal understandings that emerged in earlier grades as valuable prerequisites for building more complex ideas.

In another vertical lesson study through an NSF project called IMMERSION (Suh et al., 2022), a team of 3rd through 6th grade teachers wanted to celebrate the Lunar New Year by making mooncakes with the many Asian students who celebrated the holiday. This school with its culturally and linguistically diverse student population identified 71% Latinx, 14% Asian, 11% White, 2 % Black and 3% others with 73% qualifying for free and reduced fee, embraced culturally sustaining pedagogy viewing students' home and community cultural practices as resources "to honor, explore, and extend" (Paris, 2012, p. 94). Scaling up a recipe is a rich task, typically classified as a middle grade proportional reasoning task but because there was a real need to scale up a recipe that was set for 6 servings, the teachers chose this task and decided that the first iteration would be launched in a 3rd grade classroom. As teachers discussed the learning trajectory continuum, they identified the skills that students have already developed like skip counting, as well as emerging skills like repeated addition and multiplication, connecting to future learning goals like scaling up using a ratio table. They discussed the connection to students' assets in terms of cultural funds of knowledge (Moll et al., 1992) and family practices in cooking and emphasized bringing in realia for measurements as well as the ingredients to connect to students' multiple knowledge bases (Turner et al., 2013). They anticipated some students using manipulatives/realia to make sense of figuring out how many times their recipe might have to be scaled up based on the serving size, in addition to repeated addition, using multiplication facts as well as using as manipulatives and tables. Just like the previous task with the sharing sandwich, students in third grade used their multiple knowledge bases (Turner et al., 2013) to figure out how much of the ingredients they would need to make enough mooncakes. Teachers noted how students used manipulatives to figure out how many batches they would need (the scale factor) and noted how they used their fingers or notes on paper as they scaled up the ingredients. This coordination through iterative skip counting is the precursor to recognizing the covariation nature of early proportional reasoning (Steinthorsdottir & Sriraman, 2014).

With this in mind, teachers recognized that these emergent ways of keeping track with their fingers and scaling up was a brilliant way of thinking and reflected formative strategies to show covariation. Teachers were seeing the LTs in action as they observed student work and thinking (see Figure 4).

batch	servings
1	6
2	12
3	18
4	24



Build-Up (repeated addition)	The given ratio of first-measure-space quantity to second-measure-space quantity is repeated and combined using addition or multiplication as a form of repeated addition.	
Multiplicative	A multiplicative relationship is explicitly used.	
	Between-Measure Space	The given between-measure-space ratio is maintained in the target ratio.
	Within-Measure Space	The scale factor is determined and applied within each measure space, a reduced rate is scaled, or a unit rate is scaled.

Figure 4: Analysis of Student Thinking using Early Proportional LT on Scaling up a Recipe with LT Chart from Riehl & Steinthorsdottir, 2014

Through both of these LTLS, we found that,

1. The observers as well as the host teacher acknowledged and appreciated the brilliance and quality of students' fraction and proportional reasoning and positioned students' conceptions and multiple strategies as strengths.
2. The coach facilitated a productive debrief with the participating teachers to verify, validate and sometimes dispute the hypothetical learning trajectories based on their observations noting that they saw some attributes of more advanced thinking with earlier formative strategies.
3. Bridging the learning trajectories through a rich task across multiple grade levels allowed teachers to better focus on LT and talk less of grade level standards. Seeing the students work vertically across grades k-6 allowed teachers to appreciate informal understandings as valuable prerequisites for building more complex ideas.
4. Teachers can play an active role in validating LTs with researchers and at times disrupt notions of traditional sequencing of mathematics prescribed by standards.
5. Rich tasks can go beyond the realm of standards and provide a low floor and high ceiling where teachers can use their knowledge of LT to highlight students' strengths and position students as capable along the LT.

Lessons learned from LTLS informed our most recent work below with LT based professional development and curricular design activities that highlight the ideas of using LT as a tool to build on students' assets and promote anti-deficit framing in our work.

Case #2: Using LT with Formative Assessment to Build on Math Strength- Multi-dimensional Professional Noticing focused around Anti-deficit Framing

This case study began in 2020 at the start of the pandemic when mathematics leaders in the state department of education approached our team to create curriculum resources for teachers to

support deep conceptual learning around essential concepts. One of the problems of practice presented was that teachers were skillful at finding what gaps students had in their understanding but did not always know where to go next to advance student thinking. In reflecting on this problem of practice, I thought back to Shaun Harper's (2010) paper called *An Anti-Deficit Achievement Framework for Research on Students of Color in STEM*, where he states that the kinds of questions we ask can focus on failure and not successes. For example, we can reframe the question- "Why do so few pursue STEM majors?" (Deficit-Oriented Questions) to "What stimulates and sustains students' interest in attaining degrees in STEM fields?" (Anti-deficit reframing). He notes that it is both important to unearth systemic inequities and barriers as well as identify structures and strategies that support students of color to thrive. In this same vein, we wanted to reorient this problem of practice from, "How do we work with students once we know the gaps in their understanding?" to "How do we spot students' strengths and use that to advance their learning in mathematics?"

In our design research, we invited teachers and coaches as our co-designers, tapping into the geniuses in our schools (Wiseman et al., 2013) to build an LT based curricular resource site for educators. The key design components for our design institute included asset-based instruction, knowledge and integration of learning trajectories as our teacher designers created formative assessment with bridging activities. Bridging for Math Strength design work engaged teachers to unpack LT, use formative assessment to articulate the 'look-fors' for building on math strength and purposeful questions to advance student learning through a designed set of learning activities along that continuum. The participants in our Math Strength Design team included twenty-seven K-8 teachers and coaches working as teacher designers in teams of three. The summer design institute took place in June of 2020 with Implementation Cycles in Fall 2021 and Spring 2022. We used a rapid prototyping method with iterative design and refinement through implementation cycles.

Using a variety of research-based strength building strategies in our Design Institute, we equipped our teachers with the knowledge and research on Learning Trajectories and concrete strategies to support asset-based thinking. This included:

1) Locating relevant usable LT research translated for practitioners – (https://www.learningtrajectories.org/, http://www.ogapmath.com/, https://www.sudds.co/ Blanton, 2008; Empson & Levi, 2011; Ebby et al., 2020; Hackenberg, 2013; Petit, st al., 2020; Steffe & Olive, 2010)
2) Broadening the notion of math competence and smartness with Complex Instruction (Kobett, & Karp, 2020; NCTM, 2020; Jilk, 2016; Kalinec-Craig, et al., 2021; Horn, 2012; Lotan, 2003; Cohen et al., 1999; Featherstone et al, 2011)
3) Anti-deficit & Multi-dimensional Noticing frameworks (Louie et al., 2021; van Es et al., 2022)
4) Lesson Study to collectively learn about students' brilliance in thinking and strategies competence (Lewis, 2022; Suh & Seshaiyer, 2014)
5) Evaluating and sequencing learning activities to advance students' thinking

We used the data sources including the designed module, implementation narrated through Flipgrid, a recording platform, debrief webinars about the implementation cycles and interviews with teacher designers.

Our research questions focused on 1) How can we use LT with teachers and coaches as a tool to deepen teacher knowledge and promote asset-based instruction for students? 2) How might we use LT to "reFrame" teacher noticings towards anti-deficit orientation?

Each of the fall follow- up sessions with teacher designers included debriefs around implementing the modules and analyzing student work. Using Flipgrid, the teacher-designers narrated their implementation so other teachers could follow their process. They started by sharing the formative assessment they chose, and then outlined the sequence of activities picked based on the student work on the formative assessments. Knowledge of LT helped them sequence learning activities for students. These voices centered teachers and coaches in the LT and focused the lesson implementation to elevate the strengths of students.

Below Kelly shares her implementation of the LT based curricular resource with some of her students. The excerpt brings to life how a coach might use LT as a tool to build on students' strengths. In her noticings (see Figure 5), she honored this student's funds of knowledge when he shared with her, "Well I like 7s because I watch a lot of football so I'm good at counting by 7s". She interprets his strengths then decides that she would respond by asking questions to see how he might recognize patterns and apply reasoning strategies. She planned for questions such as, "What do you notice about the relationship of 7x2, 7x4, and 7x8? You mentioned you felt comfortable with 7s. What other numbers feel friendly to you? How could you use them to solve more challenging facts?" In deciding on rich educational learning experiences to strengthen his reasoning, she proposed a center activity called Strive to Derive which is a game that shows arrays that students can break apart to rehearse the strategy of using known derived facts or distributive property. In addition, she proposed visual number strings to build on patterns and relationships for multiplication.

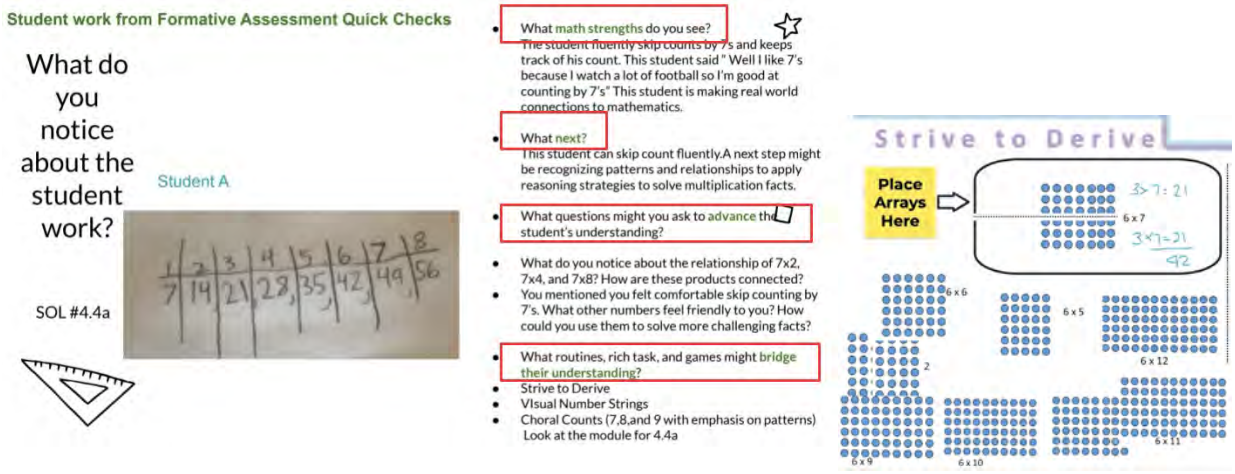


Figure 5: Providing question prompts to support asset based/anti-deficit noticings

In our analysis we found that using formative assessment and LT based bridging activities as a curricular resource during the pandemic revealed the power of removing high stakes testing and letting teachers use formative data in meaningful and humanizing ways. One of the teachers shared, "It was *liberating* because I didn't think so much about grade level standards and the state assessment. Instead, I focused on the LT and where students showed strength and built on their strength through routines, rich tasks and games." The LT-based curricular structure supported teachers and coaches in reframing how they view student learning beyond grade levels- moving away from language like "below grade level" and "at risk". Sienna, an instructional coach, found that the LT structure helped teachers see their students' learning as a progression and consider next steps rather than visualizing a gap between students' current understanding and the "final goal of the standard". Kara had a similar epiphany in her second

grade classroom. She found that the LT structure allowed her to focus on conceptual understanding in her second grade students' work rather than simply quantifying the number of incorrect answers, stating "When you go to grade something or check it over, I'm not necessarily looking at 'Oh, they got 15 out of 20. They're missing a bunch.' ...I'm really zoning and honing in on what patterns I can find. I feel like that's what this cycle has taught me is that there are patterns in student work." Kara's attention to patterns in her students' work allowed her to identify their position on the LT and plan targeted instruction to support their learning as well as identify strengths and growth areas.

Case #3:PST Teachers Use of LT to Assess Student Thinking and Design Sequence of Activities

This next case illustrates ways LTs can be used with preservice teachers to prepare them as competent mathematics teachers. The term "learning trajector(ies)" appears at least sixty-three times in the document for the Standards for Preparing Teachers of Mathematics (SPTM) published by the Association of Mathematics Teacher (AMTE, 2017). For example, Standard EC3 and EC7 emphasizes the importance of LT in curricular knowledge as well as for assessment.

> EC.3. Mathematics Learning Trajectories: Paths for Excellence and Equity: Well-prepared beginning teachers of mathematics at the early childhood level understand learning trajectories for key mathematical topics, including how these learning trajectories connect to foundational knowledge, curriculum, and assessment frameworks. [Elaboration of C.1.4]
>
> EC.7. Seeing Mathematics Through Children's Eyes: Well-prepared beginning teachers of mathematics at the early childhood level are conversant in the developmental progressions that are the core components of learning trajectories and strive to see mathematical situations through children's eyes. [Elaboration of C.3.1]

Supporting PSTs on how to use LT and formative assessment to strengthen student thinking is a priority in a mathematics methods course. Focusing on these two standards, I designed an assignment called *Learning Trajectory-based Formative Assessment & Sequenced Digital Math Activities.* In this assignment, PSTs planned and enacted asset-based solutions that included digital tools as learning activities. The goal of the tasks was to re-engage students in mathematics as the students worked to develop their sense of agency, identity, and ownership in their mathematical learning. This case focuses on how PSTs used learning trajectory research to analyze formative assessment data and then sequenced bridging activities using digital technologies to enrich mathematics instruction for individual and collective learning. PSTs also learned to use digital technology teacher dashboards, which allowed them to be responsive by providing ease of analysis of student proficiency, facilitating immediate feedback, and providing information to form targeted small groups to support student learning. The *Learning Trajectory-based Formative Assessment and Digital Math Activities* (Suh et al., 2022) assignment was designed using two important frameworks, Learning Trajectory Based Instruction (LTBI, Sztajn et al., 2012) and Technological Pedagogical Content Knowledge (TPACK, Mishra, P., & Koehler, M. J., 2006))

Sara mapped out her learning trajectory concept map with the core idea of 'unitizing' (Lamon 1994, Steffe 1992, 1994) as "operating with singleton units to coordinating composite units" (Singh 2000, p. 273) highlighting the array as part of the development of spatial and numeric composite (Battista, 2012) necessary for multiplicative reasoning.

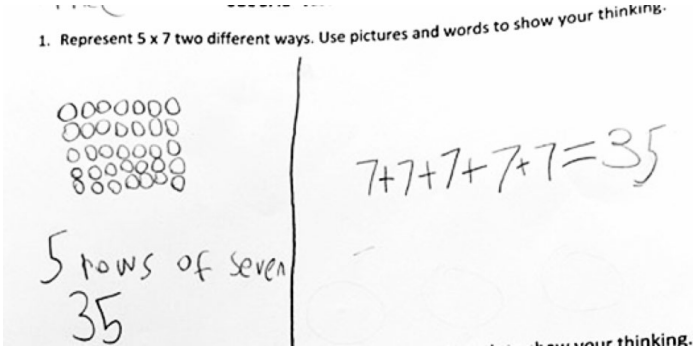


Figure 6: Sample Formative Assessment that Puzzled the Preservice Teacher

Sara, had been working with a student named Selena who could draw rows and columns with dots to model multiplication, and use repeated addition (see Figure 6), but oftentimes she would be off by 1 or 2 with her final answer. Sara was curious how to support Selena. She watched her during a formative assessment and noticed that for every problem, she would count every dot. Knowing that she wanted to move the student from counting all to seeing "many as one" as a composite unit, she found a technology tool called Bunny Times (see Figure 7) that worked with an array model with an added feature. Her analysis of the tool highlighted the affordance of the "fog feature" actually helped Selena advance to other strategies that were more efficient like skip counting or adding on from a known fact. She liked this applet because the visual helps students make connections between rows and columns. The game can be scaled in the size of math facts. Additionally, 'fog' can settle over the field obscuring some of the answers disallowing counting. Facts can be differentiated when starting the game.

Figure 7: Learning activity that supports unitizing and skip counting

Sara stated in her assessment report,

> For my target student, I plan to use Bunny Times math after working with her on skip counting. Bunny Times allows for multiplication facts to be scaled to learner readiness. Additionally, it can be played with all rows and columns visible or with some hidden under a layer of fog. For my student, with practice on unitizing, skip counting, and counting on, I hope that she will be able to complete problems using the "fog" feature.

Through the assignment, Sara learned that she could lean on her clinical faculty, a math coach in the school as well as her course instructor to assess where her student was in the multiplicative learning trajectory. She reflected on this practice-based assignment stating,

> I think this assessment will prove to be one the most important things which happened to this

> student in 3rd grade. Because of her fabulous math disposition and other areas of proficiency, it is likely that her struggles with unitizing and counting would have been masked and not observed. This project allowed for data collection, meeting with a math specialist, testing, and ultimately transferring that into specific intervention. -Sara, the PST

This PST noticed many strengths in this child's mathematical understanding related to multiplication and the intervention built on those strengths to stretch the student to be a stronger mathematician.

Implications for Mathematics Teacher Educators and Researchers

In reflecting on PMENA 2022's Theme on Dissonance and Harmony, I share some concluding thoughts and implications for mathematics teacher educators and researchers. First, we need to take the notion of a "hard reset" (Ladson-Billings, 2021) seriously to dismantle inequitable structures and practices that exist in mathematics teaching and learning and challenge the status quo. We found in our LTLS research how teachers can play an active role in validating researcher-conjectured LTs and at times challenge the traditional sequencing of mathematics prescribed by standards. We have viewed LT as a tool to help reframe teachers thinking about what students are capable of doing and finding a strength based asset orientation to instruction (Bartell et al., 2017). Building on the work of LTBI and our previous work around LTLS (Suh et al., 2018), the project that began during the pandemic called "Bridging for Math Strength" with the professional development and design research study continues to go through iterations of refinement with our model and products to support teachers teaching and student learning. With this work, we focused on changing the narrative and mindset of teachers, moving away from looking at gaps and solely focusing on error patterns (deficit-approach) to finding strengths in student thinking and using the LT to advance student thinking based on strengths and growth areas. Working on explicitly noticing and assigning competence (Gresfali et al., 2009) to shift classroom status and using LT has helped look for the strengths and where to move students forward in their learning. Too often data-talk focuses on looking at gaps using white performance as a standard to show how marginalized students are performing. Bridging the learning trajectories through a rich task across multiple grade levels allowed teachers to better focus on LT and talk less of grade level standards as the final arbiter of learning.

In order to create more socially just contexts for learning and teaching mathematics, we propose a paradigm shift in learning more deeply about the LT so that we can assess student strength and make a path of learning activities through rich tasks, place more emphasis on formative assessment and move away from gap gazing that continues to persist with state assessment (Gutierrez, 2008). Catalyzing Change Early Childhood (NCTM, 2020) - shows that students often marginalized are not given rich tasks instead given more rote learning. We advocate moving away from a "my students can't" narrative and the opportunity gaps that exist from students engaging in rigorous tasks. Continuing with data meetings with state assessments to "close the achievement gap" and catch students up will perpetuate deficit discourses, deficit noticing, obsession over errors and shortcomings of students of color blaming deficiencies in students, their families, or their cultures (Louie et al., 2017). Instead, Celedon-Pattichis et al., (2018) urges researchers and mathematics teachers to embrace asset-based approaches to mathematics education and to consciously move away from deficit perspectives that view students, parents, and communities as lacking in different aspects that enable them to be ready for schooling (Coleman et al., 2016). They encourage the mathematics education community to appreciate the math knowledge/experiences that students bring from home and communities and

by doing so students bring ways of thinking that broaden mathematics beyond what is written in standards or embodied in curriculum.

To impact society more broadly, beyond individual mathematics classrooms and school districts our work must improve learning conditions for each and every mathematics learner. With a hard reset and a focus on asset-based instruction and anti-deficit noting with the intentional use of learning trajectory (LT) with equity focused PD, our work revealed that teachers felt liberated and empowered to open up varied and expansive ways to discuss students' mathematics competencies, name students' strength and position students as capable. LTLS allowed teachers to be researchers and share their expertise and validated research-conjectured LT with real classroom data. This positioned them as knowledgeable and elevated their status as learning scientists. With the Bridging for Math Strength project, the use of LT coupled with anti-deficit framing in curriculum design research provided teachers with a tool and the language to analyze student thinking and plan rigorous educational resources and asset-based instruction for their students.

Lastly, the use of LT with formative assessment in the practice-based assignment for preservice teachers provided a scaffolded learning experience in the field with multiple educators supporting them with LT research and equitable teaching strategies to advance students' understanding. As I close this paper, I invite the PMENA community to consider how learning trajectories can be coupled with powerful equity focused research (Gutierrez, 2007; Celedon-Pattichis et al., 2018; Hand, 2012; Wager, 2014) and frameworks (Aguirre & Zavala, 2013; Bartell et al., 2020; Yeh et al., 2020) to disrupt the status quo, broaden the purposes of learning mathematics (NCTM, 2020), eliminate labeling, and dismantle inequitable structures and hierarchy in the mathematics classroom. In continuing this work, I invite mathematics researchers and math teacher educators to consider border crossing (Silver & Lunsford, 2017) as boundary spanners (AACTE, 2018) to not only translate but engage teachers and researchers in the viewing the relationship between research and practice in education as bidirectional rather than unidirectional so that "research could/should influence/inform practice, but also that practice could/ should influence/inform research" (Silver & Lunsford, 2017, p. 36) while centering the voices of students who are at the margins and attending to the socio-political lens with their LT research.

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