WHAT'S IN A NAME? SEEKING GEOMETRY IN GEOMETRIC SEQUENCES

Andrew Kercher	Anna Marie Bergman	Rina Zazkis
Simon Fraser University	Simon Fraser University	Simon Fraser University
andrew_kercher@sfu.ca	anna_marie_bergman@sfu.ca	zazkis@sfu.ca

Mathematical terminology is sometimes created according to conventions that are not obvious to students who will use the term. When this is the case, investigating the choice of a name can reveal interesting and unforeseen connections among mathematical topics. In this study, we tasked prospective and practicing teachers to consider: What is geometric about geometric sequences? Participants embedded their explanations within a scripted dialogue between teacher- and student-characters in a mathematics classroom, provided commentary on this dialogue, and expanded on its mathematical content. Participants most often leverage the concept of a geometric mean to explain why geometric sequences are named as such. To capture this informal arguments, we built on the work of Toulmin (1958/2003) to conceptualize and develop the Toulmin-Reversed (Toulmin-R) model.

Keywords: Teacher Knowledge, Communication, Professional Development, Preservice Teacher Education

Introduction

Mathematical communication relies on the use of terms that have a particular meaning within the mathematical register. However, confusion may arise when a learner interprets a mathematical term by referencing the use of that word outside of mathematics. For example, the phrase "only a fraction of students participated in the Olympiad" implies that a "fraction" is a small amount and may interfere with a learner's ability to recognize fractions that are greater than one (Hackenberg, 2007). To indicate such fractions within a mathematical conversation, a particular adjective is chosen: *improper* fractions are fractions greater than one.

Such adjectives often resonate with the intuitive interpretation of the words. Consider for example increasing functions, whose graphical appearance is often exactly in accord with students' preconceived notions of what increasing entails. There are also instances of particular adjectives which may at first seem unrelated to the mathematical object, but which are immediately explained by the associated definition. For example, *rational* numbers are those expressed as ratios (under some constraints).

However, there are examples of adjectives where the choice is somehow puzzling even after learning the corresponding definition. Consider for example a *perfect* number, defined as an integer that is equal to the sum of its factors (excluding itself). It is not immediately obvious why "perfect" is an appropriate adjective to describe this particular property. It is only in light of the fact that perfect numbers are usually considered together with *deficient* (smaller than the sum of its factors) and *abundant* (larger than the sum of its factors) numbers that the choice of adjective is given some context; and even so, the adjectives "sufficient" or "balanced" could be seen as more appropriate than associating the property with perfection.

We acknowledge that mathematical convention, including terminology, is sometimes *arbitrary* in the sense that it cannot be deduced using logical principles (Hewitt, 1999). For example, a student simply cannot deduce the arbitrary fact that circles are agreed to have 360 degrees—they must be told as much by an authority. But as Hewitt illustrates, it is sometimes possible to embed a seemingly arbitrary convention within a historical or mathematical context to justify its use. When introducing degree measure, a teacher may appeal to the Babylonian number system in which the degree originated in order to rationalize our modern conventions. Our study is concerned with similar explanations surrounding the arbitrary use of the adjective “geometric” in connection with a geometric sequence. It is stimulated by a student question, “What is geometric about geometric sequences?”, to which we collected responses from prospective and practicing teachers.

Our investigation addresses the following research question: How do teachers justify a convention of mathematical terminology? In particular, what is the most common argument invoked by teachers to explain the adjective “geometric” in a reference to a geometric sequence?

Background: Toulmin’s Model

Toulmin’s model is comprised of six elements which, when taken in relation to each other, outline the structure of an informal argument. The *data* is an assertion from which follows the truth of the *conclusion* (sometimes called the claim: cf. Inglis et al., 2007; Conner et al., 2014). The degree of confidence with which the arguer believes the conclusion follows from the data is inferred from the use of a *modal qualifier* (e.g., “therefore, it is necessary that…” or “so, it is probably the case that…”). Throughout this report, we refer to the combination of data, claim, and modal qualifier as the core argument.

A lower degree of confidence in the conclusion is often caused by *rebuttals*, the existence of which may be known or only anticipated at the time the argument is made. Rebuttals are statements that present contrary evidence to the conclusion by describing how it may not follow from the data. Conversely, a higher degree of confidence could be the result of a convincing *warrant* and any associated *backing*. A warrant is an attempt to support the relationship between the data and conclusion with reasoning or evidence; a backing statement is further evidence in support of the warrant (Toulmin, 1958/2003). These elements are not considered part of the core argument because an argument need not contain rebuttals, warrants, or backings—on the other hand, it may contain multiple.

In mathematics, informal argumentation often serves as a foundation for, or supplement to, logical proof; as such, Toulmin models can be used to visualize the structure of mathematical activity surrounding more rigorous mathematics. For example, Inglis et al. (2007) examined the work of mathematics graduate students who had been tasked with deciding the truth value of certain mathematical statements; the authors used Toulmin models to coordinate instances of intuitive and inductive reasoning as their participants informally developed the ideas of a rigorous proof. Weber et al. (2008) and Conner et al. (2014) used Toulmin models to analyze whole-class discussions of middle and secondary school mathematics classrooms, respectively. These data were used to illustrate the role of explicit warrants in facilitating learning opportunities (Weber et al., 2008) and to generate a framework that categorizes the ways in which teachers can support collective argumentation (Conner et al., 2014).

In this paper, we extend the use of Toulmin models to a situation in which the conclusion is known, but the data used to reach that conclusion is not. It is certainly true that geometric sequences are named as such—but what set of data and accompanying warrants might have led to that conclusion?

Methods

Participants and Setting

A total of 24 participants (referred to here as T-1 through T-24) took part in the study. Of these, 9 were prospective teachers in the last term of their teacher certification program. At the time of data collection they were enrolled in a course that used mathematical problem solving as a lens through which secondary mathematics could be connected to advanced mathematics. The remaining 15 participants were practicing teachers enrolled in a professional development course that provided them with an opportunity to investigate and extend their own mathematical thinking by focusing on the learning and teaching of mathematics. The practicing teachers held bachelor's degrees in mathematics or science; in the latter case, the degree also included a number of mathematics courses sufficient for teaching certification.

Both populations completed several scripting tasks (see next section) as part of their regular coursework. Their responses to one such task, along with their responses to accompanying discussion prompts, serve as the dataset for this report.

The Task

The data analyzed in this report is based on participants' responses to a *scripting task*. Scripting tasks were initially introduced in mathematics teacher education as *lesson plays*, which were envisioned as a more robust form of lesson planning. In a lesson play, the scriptwriter constructs dialogue that captures key interactions between a teacher and student characters (Zazkis et al., 2009; Zazkis, et al., 2013). More recently, the idea of a lesson play has been extended to the activity of writing an imaginary dialogue between interlocuters in any mathematical context. In this expanded scope, scripting tasks can provide multiple affordances not only for teachers but also teacher educators and researchers.

Scripting tasks often begin with a prompt, which serves as the beginning of a dialogue between the scripted characters. In prior research, prompts have introduced a student error (e.g., Zazkis et al., 2013), a student question (e.g., Bergman et al., in press; Marmur & Zazkis, 2018; Zazkis & Kontorovich, 2016), or a disagreement among students (e.g., Marmur et al., 2020; Zazkis & Zazkis, 2014). The scriptwriter responds to a prompt by continuing the dialogue. These dialogues reveal mathematical understanding and, for teachers, demonstrate "awareness-in-action" (Mason, 1998, p. 255). That is, they show an envisioned response in practice, rather than in theory, to students' errors, queries, or unusual ideas.

Part 1 of the geometric sequence task consists of a scripting task, the prompt for which is seen in Figure 2. We refer to the scripted characters as "teacher-characters" and "student-characters" throughout this report. We refer to participants also as respondents and scriptwriters, interchangeably. In Part 2, participants were asked to explain the actions taken by the characters in the script. This included justifying both the explanation(s) chosen by the teacher-character as well as the responses given by the student-characters. In Part 3, participants were asked to elaborate on how their personal understanding of the mathematics might have differed from what was presented in the script; that is, they were given the opportunity to clarify the mathematics using more formal or advanced language appropriate for a colleague rather than a student.

You are starting a unit on geometric sequences. After you have provided several examples, you are faced with a student's question.

Student: What is GEOMETRIC about geometric sequences?
Teacher: What do you mean?
Student: You called these sequences "geometric", but these are just sequences of numbers…
Teacher: …

YOUR TASK is to develop an imaginary dialogue in which the student question is discussed and to justify your course of action.

Figure 2: The prompt for the geometric sequences task

Data Analysis

Analysis of participants' responses began with reflexive thematic analysis (Braun & Clarke, 2006, 2019; Nowell et al., 2017). First, the research team thoroughly familiarized themselves with the data by reading and rereading both the scripts and the accompanying commentary. The coding process began by identifying and classifying within Part 1 how the teacher-character chose to answer the student-character's initial question from the prompt: "What is geometric about geometric sequences?" Explanations that only appeared in Part 2 or 3 were also identified and classified in the same way. During this process, supplementary codes emerged from the data that captured other common aspects of the participants' submissions. These included: ways in which the teacher-characters' explanations were unsatisfactory, either to a student-character or the scriptwriter themselves; how characters chose to define the adjective "geometric" and what mathematical objects should be described as such; and how comparisons to other types of sequences bolstered or diminished the explanatory power of the teacher-characters' justification. In total, the initial codes and supplementary codes combined to form a preliminary codebook.

Upon review of the submissions, the research team recognized that Toulmin's model of informal argument could capture the scripted dialogue. When a student-character opposed the teacher-character's explanation, we perceived this as voicing a rebuttal. When the dialogue attended to what is or is not geometric, the characters were seen to be establishing a warrant or its associated backing. However, the existing Toulmin model needed modification given that the scripted characters were attempting to find data that supported a forgone conclusion. That is, it is certainly the case that geometric sequences are named as such—but one cannot establish that they "should" be named this way through formal, deductive logic, as one might establish that the square root of 2 is an irrational number. To capture this novel dynamic, we developed the Toulmin-Reversed (Toulmin-R) model pictured in Figure 3. The Toulmin-R model contains the same elements as a Toulmin model but reverses the direction of the arrows within the core argument and draws attention to the fact that the conclusion is already known; instead, the data is the subject of the argument.

The codebook was then reorganized and refined in light of the Toulmin-R model. Finally, the research team created Toulmin-R models to visualize the informal arguments as they were used by participants to answer the student-character's question from the prompt.

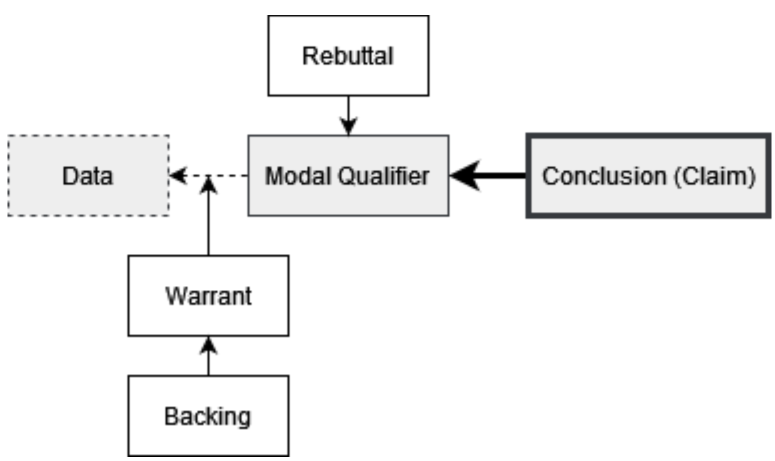

Figure 3: A Toulmin-R diagram template, with the reversed core argument emphasized

Findings

In 13 of the 24 submissions, participants used a connection to the geometric mean as data that could rationalize the naming convention of geometric sequences. This made it the most prevalent source of data employed in arguments.

Scripts that invoked the geometric mean in Part 1 sometimes introduced that concept as an analogue to the arithmetic mean. When this was a purely computational comparison, the student-characters typically responded with skepticism. For example, in T-17's script, an unsatisfied student-character observed that the two means are just "more things called 'arithmetic' and 'geometric' where one's about addition and the other is about multiplication." In these cases the geometric mean was, like the sequence, being introduced and handled entirely with arithmetic—and so its power to explain the choice of "geometric" as an appropriate adjective was limited. A student-character in T-20's script voices this concern explicitly: "If a geometric sequence is geometric because it involves geometric means, then why do we call the geometric mean 'geometric'?"

In response to these student-characters, the teacher-character typically provided one of two geometric explanation that warranted the geometric nature of the mean, and thus, the associated sequence. Both warrants were sometimes accompanied by diagrams, as exemplified in Figure 4.

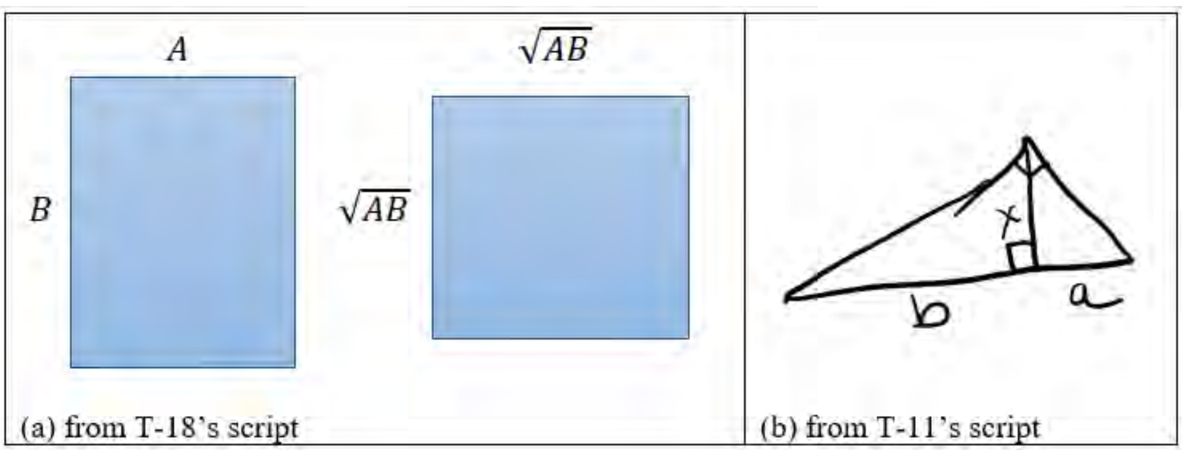

Figure 4: Diagrams used by teacher-characters as warrants

The more common warrant was that if two given numbers were used as the sides of a rectangle, then the geometric mean was the side of a square with the same area as that rectangle. T-18's teacher-character provided the accompanying visualization of this relationship pictured in Figure 4(a). More infrequently, it was explained that the geometric mean could be visualized as the altitude of a right triangle that had been cut into two similar right triangles. The teacher-character in T-11's script drew the diagram seen in Figure 4(b) to illustrate this warrant.

For each of these warrants, an associated backing (sometimes explicit, but usually only implied) was that the polygons in these diagrams were clearly geometric, thereby justifying the use of that adjective for the associated mean and sequence. Figure 5 provides a *composite* Toulmin-R model illustrating these arguments. This model is a composition in the sense that it includes all the rebuttals, warrants, and backings of any submission that argued for the geometric mean as a source of data. We also note that the model in Figure 5 express sequences with notation familiar to a research audience but, as exhibited in the visualizations provided in Figure 4, not typically used by participants in their scripted dialogues.

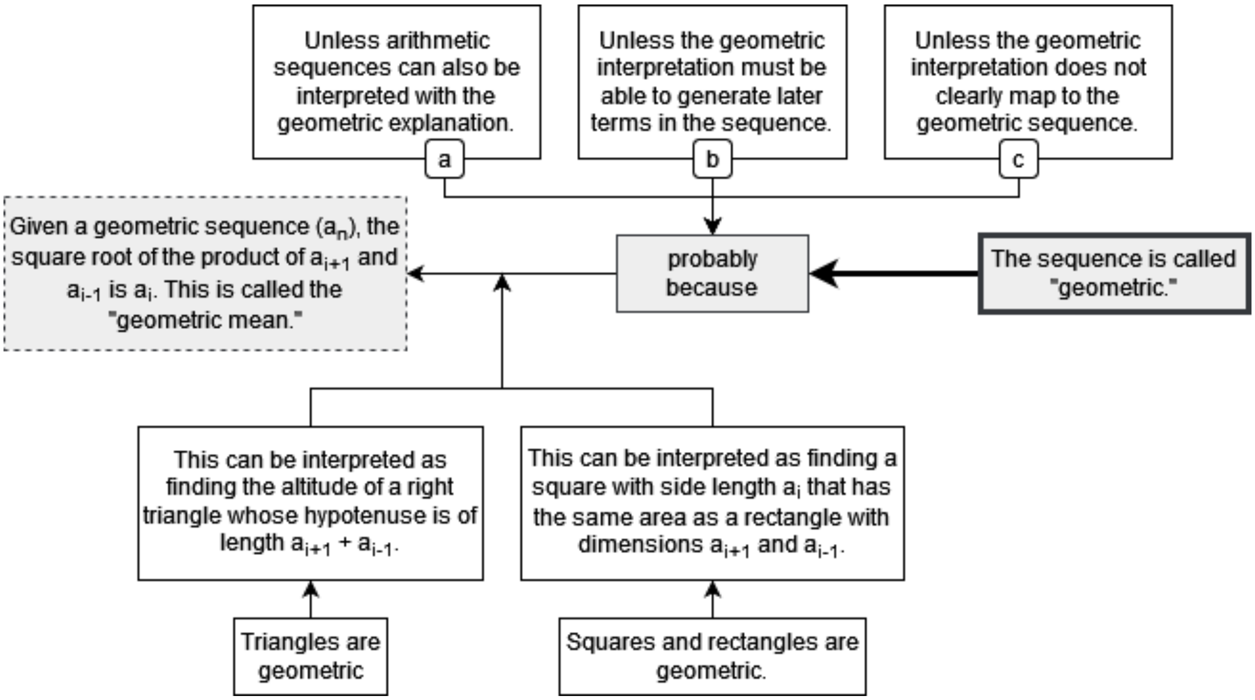

Figure 5: Composite Toulmin-R model featuring geometric mean as a source of data

The selection of the geometric mean as the likely source of data sometimes led to rebuttals, even with the accompanying warrants described above. For example, the teacher-character in T-18's script extended his explanation warranting the geometric nature of the geometric mean to also include a geometric explanation of the arithmetic mean. After drawing the diagram in Figure 4(a), the following dialogue occurs:

Teacher: […] Suppose you have A and B are the lengths of two sides of a rectangle. So the geometric mean $\sqrt{AB}$ is the length of a side of a square having the same area of the rectangle.
Student 1: Okay, then?
Teacher: And the arithmetic mean $\frac{(A+B)}{2}$ is the length of the side of a square having the same perimeter of the rectangle.
Student 1: I still not see your point. Why they are both expressed geometrically?

From Student 1's perspective, accepting the geometric mean explanation as a likely source of data meant that the arithmetic sequence was also similarly geometric. This perspective corresponds to rebuttal-a in Figure 5. Even though the geometric mean does justify that the geometric sequence is in some way geometric, it does not clearly establish why the adjective "geometric" was chosen to apply to one sequence and not the other—thereby calling into question whether it is the correct source of data. Some participants avoided this rebuttal by providing additional historical context. In one such case, T-23's teacher-character explained that "in ancient times" measures of length, such as perimeter, were not considered geometric.

Other rebuttals hinged on the fact that the geometric mean did not immediately lend itself to generating subsequent terms of a geometric sequence, as seen in rebuttal-b in Figure 5. For example, T-17 recognized that the geometric mean could interpolate additional points between two known terms of a geometric sequence, but called into question whether this was a useful property: "Why would we ever be in a position where the nth element is unknown but its 2 neighbours are known?" Similarly, T-14 noted in Part 2 that she was unsatisfied with the geometric mean explanation because it "doesn't nicely describe the progression of numbers […] and it isn't exactly logical what number would come next in the sequence." The characters in T-6's script handled this concern by deriving the common ratio in terms of the geometric mean:

Allen: Uh… A, $\sqrt{AB}$, and B.
Teacher: Exactly! Short rectangle side, then square side, then long rectangle side, which makes A, $\sqrt{AB}$, and B. Now, what's the pattern here? […] To go from A to $\sqrt{AB}$, what do you have to multiply?
Allen: Multiply? Uh… $\sqrt{AB}$/A?
Teacher: Exactly! […] This "pattern" you found is the sequence: the geometric sequence.

Finally, T-11's script contained an example of rebuttal-c in Figure 5. This rebuttal was based on the fact that warranting the geometric mean explanation with a geometric representation actually obscured the underlying sequence. This is illustrated in the following conversation between the characters:

Teacher: […] This is why this progression is called a geometric progression since it's exactly like growing similar triangles.
Student: ………………………….. what is a similar triangle? I forget………………..
Teacher: I'll draw similar triangles for you. Here we have 3 similar triangles: There's the large triangle and the two small triangles.
Student: But where's the geometric sequence?

The visualization drawn by the teacher-character in this excerpt is the right triangle in Figure 4(b). The teacher-character does not answer the student-character's final question in Part 1, but the scriptwriter later explains in Part 3 that she might clarify the construction of this triangle for a mathematically mature colleague.

Discussion

Mathematics teachers must be careful that they do not take language for granted. This is especially true of mathematical names, which are (perhaps naively) expected to encapsulate the essence of the object they denote. An expert in mathematics might take familiar terminology for granted. But before a novice connects mathematical properties to a name, they might associate with it their personal connotations and experiences—which may sometimes cause conflict. It is an awareness of this fact that we hope to explore by assigning the geometric sequences task to teachers.

In analyzing participants' submissions, we considered the following research questions: *How do teachers justify a convention of mathematical terminology? In particular, what is the most common argument invoked by teachers to explain the adjective "geometric" in a reference to a geometric sequence?* Our findings indicate that the most common argument leveraged by teachers to explain why geometric sequences are in fact geometric relies on the relationship between a geometric sequence and the geometric mean. This approach necessitated a warrant justifying that the geometric mean is in fact geometric. Some participants handled this with a geometric diagram: either of a square that preserves the area of a given rectangle, or of a right triangle decomposed into two similar right triangles. In both cases, the fact that these diagrams were made of familiar polygons implied that they were genuinely "geometric."

In addressing our research questions, we considered the ways in which each explanation was considered insufficient to either student-characters or scriptwriters. These observations were captured in the findings as rebuttals. Here, we frame these rebuttals as the inability on the part of the core argument to meet an intellectual need (as in Harel, 2013). Rebuttal-a in Figure 5 does not satisfy the need for *structure*. For student-characters who voiced this rebuttal, the argument from geometric mean did not logically reorganize their understanding of mathematical terminology by meaningfully differentiating between geometric and arithmetic sequences. Rebuttal-b in Figure 5 does not satisfy the need for *computation*. Participants already knew how to compute the next term of a geometric sequence arithmetically by multiplying by the common ratio; the inability of the geometric mean argument to replicate that capability in a geometric context was therefore seen as a shortcoming. Finally, rebuttal-c in Figure 5 captures a need for *communication*. The student-character who voiced this rebuttal could not productively translate between their symbolic understanding of geometric sequences and the visual embedding of that concept in the triangle diagram.

Recasting other elements of an informal argument in light of intellectual need is one direction for future research; for example, how do certain warrants successfully attend to intellectual need? Another direction for additional research is to consider the origin of other mathematical terms. What is linear about linear algebra? What is natural about the natural logarithm? Whether posed to prospective teachers or to other students of mathematics, we anticipate that such questions will provoke meaningful introspection on the nature, structure, and history of the subject. This will, in turn, lead to further exemplification of the role of informal argumentation in mathematical discourse.

Our study contributes to research in mathematics education by developing the Toulmin-R model as a tool for representing informal arguments in which the goal is to establish a likely source of data rather than a meaningful conclusion. We have used this model in the context of explaining mathematical terminology; we suggest it could also be of use when a mathematical effect is observed but its cause is unclear. For example, a novice will eventually prove that the product of any two odd numbers is itself odd. Before their thinking is rigorously expressed by deductive proof, however, the Toulmin-R model might be leveraged to capture their informal arguments as they seek data leading to this conclusion.

This report also contributes to the body of knowledge on methods for mathematics teacher education. In their *Standards for Preparing Teachers of Mathematics*, the Association of Mathematics Teacher Educators recommend that to be “well-prepared,” teachers must “understand that mathematics is a human endeavor” (2017, p. 9). It is empowering for teachers to understand mathematics as the cumulative result of centuries of human effort—even though, as a result, it may sometimes seem disorganized or arbitrary. A well-prepared teacher should recognize that this is not always to the subject’s detriment. It may not be set out clearly why a geometric sequence is geometric, but by attempting to answer the question, teachers are apt to discover unexpected connections between areas of mathematics. Paradoxically, it is often those things that seem inexplicable that ultimately reveal a more cohesive and interrelated subject.

Acknowledgements

On behalf of all authors, the corresponding author states that there is no conflict of interest. This research was supported by a grant from the Social Sciences and Humanities Research Council of Canada. The datasets generated and analyzed during the current study are not publicly available due the fact that they constitute an excerpt of research in progress but are available from the corresponding author on reasonable request.

References

Association of Mathematics Teacher Educators. (2017). *Standards for Preparing Teachers of Mathematics.* Available online at amte.net/standards.

Bergman, A. M., Gallagher, K., & Zazkis, R. (in press). Prospective Teachers' Responses to Students' Dialogue on Fractions: Attribute Substitution and Heuristic Approaches. *Research in Mathematics Education*.

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative research in psychology*, *3*(2), 77-101.

Braun, V., & Clarke, V. (2019). Reflecting on reflexive thematic analysis. *Qualitative Research in Sport, Exercise and Health*, *11*(4), 589-597.

Conner, A., Singletary, L. M., Smith, R. C., Wagner, P. A., & Francisco, R. T. (2014). Teacher support for collective argumentation: A framework for examining how teachers support students' engagement in mathematical activities. *Educational Studies in Mathematics*, *86*(3), 401-429.

Hackenberg, A. J. (2007). Units coordination and the construction of improper fractions: A revision of the splitting hypothesis. *The Journal of Mathematical Behavior*, 26(1), 27-47.

Hewitt, D. (1999). Arbitrary and necessary part 1: A way of viewing the mathematics curriculum. *For the learning of Mathematics*, *19*(3), 2-9.

Inglis, M., Mejia-Ramos, J. P., & Simpson, A. (2007). Modelling mathematical argumentation: The importance of qualification. *Educational Studies in Mathematics*, *66*(1), 3-21.

Kontorovich, I., & Zazkis, R. (2016). Turn vs. shape: Teachers cope with incompatible perspectives on angle. *Educational Studies in Mathematics*, 93(2), 223-243.

Marmur, O., Moutinho, I., & Zazkis, R. (2020). On the density of $\mathbb{Q}$ in $\mathbb{R}$: Imaginary dialogues scripted by undergraduate students. *International Journal of Mathematical Education in Science and Technology*, 1-29.

Marmur, O., & Zazkis, R. (2018). Space of fuzziness: Avoidance of deterministic decisions in the case of the inverse function. *Educational Studies in Mathematics*, 99(3), 261-275.

Mason, J. (1998). Enabling teachers to be real teachers: Necessary levels of awareness and structure of attention. Journal of Mathematics Teacher Education, 1(3), 243-267.

Nowell, L. S., Norris, J. M., White, D. E., & Moules, N. J. (2017). Thematic analysis: Striving to meet the trustworthiness criteria. *International journal of qualitative methods*, *16*(1).

Toulmin, S. E. (1958/2003). *The uses of argument*. Cambridge: Cambridge university press.

Weber, K., Maher, C., Powell, A., & Lee, H. S. (2008). Learning opportunities from group discussions: Warrants become the objects of debate. *Educational Studies in Mathematics*, *68*(3), 247-261.

Zazkis, R., Liljedahl, P., & Sinclair, N. (2009). Lesson plays: Planning teaching versus teaching planning. *For the learning of mathematics*, *29*(1), 40-47.

Zazkis, R., Sinclair, N., & Liljedahl, P. (2013). Lesson play in mathematics education: A tool for research and professional development. New York: Springer.

Zazkis, R., & Zazkis, D. (2014). Script writing in the mathematics classroom: Imaginary conversations on the structure of numbers. *Research in Mathematics Education*, 16(1), 54-70.