PRESERVICE TEACHERS' MATHEMATICAL KNOWLEDGE FOR TEACHING: FOCUS ON LESSON PLANNING AND REFLECTION

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Mathematics teacher educators have suggested that preservice mathematics teachers' (PMTs') practices provide evidence of their Mathematical Knowledge for Teaching (MKT). In an effort to explore connections between MKT and PMTs' practices, we developed a framework that operationalizes Ball et al.'s (2008) six MKT domains in terms of approximations of practice. We then used our framework to investigate which domains were evidenced in eleven PMTs' lesson plans and how PMTs described MKT in their lesson plan reflections. We found Knowledge of Content and Teaching most evidenced and Horizon Content Knowledge least evidenced. Also, PMTs made few instances of Knowledge of Content and Students as they struggled to address students' mathematical thinking in their plans. We propose alternative forms of approximations of practice to optimize PMTs' opportunities to demonstrate and conceptualize MKT.

Keywords: Mathematical knowledge for teaching, Teacher knowledge, Preservice teacher education; Instructional activities and practices

Educators have emphasized that teacher knowledge should be discipline-specific, and evidenced through teacher practices (e.g., Ball et al., 2008; Shulman, 1986; Tchoshanov, 2010). In mathematics education, discipline-specific knowledge is commonly conceptualized as Mathematical Knowledge for Teaching (MKT, Ball et al., 2008), which provides a way to contextualize mathematical knowledge during mathematics teaching (Thomas et al., 2017). Preservice mathematics teachers (PMTs) often engage in approximations of practices (referred to as "approximations") such as peer teaching (Grossman et al., 2009) during their teacher education programs, thus, their MKT could be evidenced in approximations. However, the forms of approximations of practice that are appropriate for PMTs to demonstrate specific aspects of MKT (e.g., anticipating students' mathematical reasoning; formulating meaningful questions to students) are underexplored. In addition, mathematics teacher educators would benefit from opportunities to consider the operationalization of Ball and colleagues' (2008) six MKT domains in terms of associated practices (see the description of the domains in the Perspectives section). In this study, we operationalized MKT domains in terms of peer lesson planning and reflections. First, we synthesized the literature to develop descriptors for the MKT domains. Then, we used this framework to investigate the following questions: (a) How did PMTs demonstrate MKT in their lesson planning? (b) How did PMTs describe the use of MKT domains in their lesson plan reflections? We conceptualized lesson planning (e.g., formulating teacher and student actions) as approximations of practice. As such, PMTs engaged in approximations through peer teaching; thus, the actual practices were altered by changing the context of the practice (i.e., PMTs planned a lesson to implement in a university classroom) and by offering scaffolding such as peer and instructor feedback and a guided lesson plan template (Tyminski et al., 2014).

Perspectives

We framed our study around the concept of discipline-specific knowledge, or MKT, with foundations in Shulman's (1986) interpretation of Pedagogical Content Knowledge (PCK)—

teachers' skills to identify and unpack critical mathematical components that are fundamental for the teaching of mathematics and use those components in a way that are comprehensible to students. PCK was further elaborated by Ball and Bass (2000) and other mathematics educators (e.g., Hill et al., 2005; Wasserman & Stockton, 2013). These educators proposed that teacher knowledge is discipline-specific; suggesting that focusing solely on generic pedagogy, such as classroom management, poses a risk of simplifying the complexities of teaching because generic pedagogy does not include the unique characteristics associated with content or discipline. Moreover, Shulman and recent educators (e.g., Styers et al., 2021) proposed teacher knowledge should be evidenced through their practice as opposed to paper and pencil tests.

Ball et al. (2008) formalized MKT through a framework, which consists of two domains—Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK), each consisting of three subdomains (referred to as "MKT domains"). SMK consists of the following three subdomains: Common Content Knowledge (CCK), Horizon Content Knowledge (HCK), and Specialized Content Knowledge (SCK). CCK comprises the general mathematical knowledge required to solve mathematics problems. HCK is the knowledge of core disciplinary values and major structures of the discipline. SCK refers to teachers' conceptualization of mathematics in nuanced ways that include mathematical reasoning and multiple mathematical representations. PCK consists of Knowledge of Content and Student (KCS), Knowledge of Content and Teaching (KCT), and Knowledge of Content and Curriculum (KCC). KCS includes the understanding of students' mathematical conceptions and reasoning. KCT includes knowledge of effective mathematics teaching strategies. KCC comprises knowledge of learning goals and horizontal and vertical organizations of mathematics across grade levels. Even though the MKT framework provides a way to unpack MKT in terms of six domains, how these domains are operationalized in PMTs' teaching is still underexplored. In addition, teacher education programs have yet to ensure practices for PMTs to conceptualize and demonstrate MKT (Wasserman et al., 2019); which forms of approximations of practice (our second conceptual framework) are appropriate for PMTs to engage with and demonstrate MKT is still underexplored.

As we worked to operationalize Ball et al.'s (2008) six MTK domains in terms of approximations of practice, we looked at how approximations have been defined in research. Approximations of practice are defined as "opportunities for novices to engage in practices that are more or less proximal to the practices of a profession" (Grossman et al., 2009, p. 2058). Approximations are different from actual practices because they often (a) mimic components of teaching practice, (b) include scaffolding, and (c) do not always replicate the complexities of teaching (Campbell et al., 2020; Janssen et al., 2015; Tyminski et al., 2014). Through approximations, teacher educators focus explicitly on the enactment of teaching and aim to develop skilled teachers (Forzani, 2014; Janssen et al., 2015). By using the concept of approximations, we designed instructional activities that provided PMTs with opportunities to rehearse several practices in a setting different than the actual classroom; PMTs rehearsed peer teaching and received feedback from their instructors and peers to improve their teaching. We explored the explicit connections between MKT and approximations of practice— how MKT domains are operationalized in approximations involving lesson planning and reflections.

Methods

Using a single case-study design (Yin, 2017), we identified the MKT domains associated with teacher and student actions described in the PMTs' lesson plans and lesson plan reflections. We considered all PMTs' lesson plans as one case and investigated all MKT domains demonstrated across all PMTs' lesson plans. This merging of all data into one case was appropriate for our study because we were interested, not in individual PMT's demonstration of the MKT domains, but rather in the total instances of MKT domains in all PMTs' lesson plans and the way they describe MKT domains in their reflections.

Context, Participants, and Instructional Activities

Using convenience sampling (Nielsen, et al., 2017), we recruited 11 secondary PMTs enrolled in a secondary mathematics methods course at a large Midwestern University. The first author was a course instructor and developed and implemented instructional activities based on the fundamental concepts of approximations of practice (Grossman et al., 2009). The PMTs had completed content and general education courses prior to taking this course, but this was their first methods course. During this course, the PMTs planned, reflected on, and implemented lessons with their peers. In the first phase, the PMTs worked with a peer to plan a lesson ("Lesson Plan I") for middle school students using the Connected Mathematics curriculum (Connected Mathematics Project, n.d.). Afterward, they individually reflected on their lesson plans with prompts (e.g., Describe how the examples, strategies, and representations that you have listed in the lesson plan help to build on students' understanding of mathematics). We used González et al. (2020) and Özgün-Koca (2020) to develop prompts. In the second phase, the instructor facilitated a discussion of Ball and colleagues' (2008) MKT framework. In the third phase, PMTs individually planned a lesson ("Lesson Plan II") by adapting activities from the College Preparatory Mathematics curriculum (CPM Educational Program, n.d.). For both lessons, PMTs were provided a lesson plan template, which partitioned the lesson into three sections: Launch (helping students understand the problem setting, the mathematical context, and the challenge), Explore (inviting students to explore mathematical ideas), and Summary (inviting students to present and discuss problem-solving strategies).

Data Collection and Analysis

Our primary data was the PMTs' lesson plans. We used the PMTs' reflections on the lesson plans as an additional data source to understand how and why the PMTs selected certain teacher and student actions. We used content analysis methods (Chi, 1997; Schreier, 2012) to analyze the data. We first synthesized the literature to develop descriptors for the MKT domains (Table 1) and then used these descriptors to code lesson plans and reflections.

Table 1: Descriptors of MKT Domains

MKT Domains and Associated Descriptors/Codes
Common Content Knowledge -CCK (Ball et al., 2008; Baumert et al., 2010)
Define mathematical terms
Unpack mathematics concepts and/or symbols
Solve mathematical problems
Specialized Content Knowledge- SCK (Ding & Heffernan, 2018; Hill et al., 2005; Morris & Hiebert, 2017)
Anticipate alternative representations, provide explanations, evaluate unconventional student solution approaches
Identify multiple representations for mathematics problems or concepts
Offer multiple representations of mathematical solutions
Unpack mathematical concepts and concrete models for justifying standard procedures utilized in the process
Horizon Content Knowledge- HCK (Ribeiro et al., 2013; Wasserman & Stockton, 2013)
Address how a topic is connected with the broader disciplinary territory
Include big mathematical ideas that contribute to the teaching of mathematics topics
Knowledge of Content and Curriculum-KCC (Ball et al., 2008)
Address curricular trajectory

Knowledge of Content and Students-KCS (Hill et al., 2005; Özgün-Koca, 2020; Schilling & Hill, 2007)
Anticipate the contextual factors that would support (or impede) the development of students' understanding

MKT Domains and Associated Descriptors/Codes
Anticipate what students are likely to do with a mathematical task and/or concepts
Evaluate the diagnostic potential of tasks or recognize typical student errors
Foresee students' alternative conceptions and plan to address those conceptions
Predict what students will find interesting or motivating or useful
Knowledge of Content and Teaching-KCT (Ding & Heffernan, 2018)
Anticipate an item's difficulty level and plan for mathematical concepts that address rigor demanded by the item
Describe mathematical tasks or procedures that students would be engaged in
Formulate, sequence, pose questions to students
Identify strategies to assess students' mathematical understanding
Identify supplemental resources associated with mathematics topics
Identify what different methods and procedures might afford during instruction
Select and sequence examples or performance tasks that would allow students to understand the topic
Generic-No MKT-domains
Include generic teacher and student actions without referencing mathematics

We also modified the descriptors based on our coding. For example, we identified the code "anticipate the contextual factors that support (or impede) the development of students' understanding" under KCS from PMTs' descriptions of contextual factors associated with students' mathematical understanding. The first author used the data from the pilot study (i.e., four PMTs' lesson plans and their reflections collected in an earlier semester) to test the initial codes and further develop the coding scheme. The first and second author (a mathematics education graduate student and an instructor of the methods course in subsequent semesters) identified new descriptors in the data which we added to the framework. For example, we found that the PMTs planned strategies to assess students' thinking. It was not in the original set of codes, and we added it as a descriptor of KCT because it is related to teachers' knowledge associated with selecting instructional activities to understand students' mathematical thinking.

We coded teacher and student actions described and the assessment strategies noted in the lesson plans; the standards, instructional materials, students' prior knowledge, and vocabulary were articulated while describing the teacher and student actions. For the second data set, we coded all the reflections. We identified each instance in which the MKT domains were evidenced in PMTs' lesson plans as one coding unit. For example, one PMT mentioned that "students also need prior knowledge of probability notation, including that P(x) means that we are looking for the probability of x." We identified this as one coding unit. In addition, we identified instances wherein PMTs described generic teacher and student actions without referencing mathematical content. We added those activities under the new category "Generic-No MKT-domains." In addition, we refined our coding scheme by revising codes. For example, we aimed to distinguish between types of specific questions PMTs planned to pose versus types of tasks PMTs planned to implement while selecting mathematical tasks. Thus, we added the code: "formulate, sequence, and pose questions." We each used the coding scheme to code all lesson plans independently, then discussed our coding processes and codes to resolve any discrepancies. We collaborated to decide which codes or descriptors needed to be added or eliminated.

Findings

We first report the frequency of the MKT domains that appeared in the PMTs' Lesson Plan I and Lesson Plan II. We then elaborate on the context of the frequencies— how we identified these frequencies from PMTs' lesson plans and what these frequencies suggest in terms of

PMTs' MKT; we first report MKT domains that were evidenced the most and the shifts in the frequency from Lesson Plan I to Lesson Plan II. Afterward, we include examples of less evidenced domains in PMTs' lesson plans. We used PMTs' lesson plan reflections to provide the context and description of potential reasons that PMTs included certain activities in their lesson plans. In Table 2, we present the MKT domains that we identified in the PMTs' lesson plans.

Table 2: MKT Domains in PMTs' Lesson Plans

MKT Domains	KCC		KCS		CCK		KCT		HCK		SCK	
Lesson Plan	I	II	I	II	I	II	I	II	I	II	I	II
Total Instances	16	22	14	45	8	7	87	102	4	1	11	21

As indicated in Table 2, KCT was the most highlighted domain in both lesson plans, including 87 instances in Lesson Plan I and 102 instances in Lesson Plan II. KCT was primarily identified when PMTs described the following instructional plans related to teacher actions: (a) selecting and describing mathematical tasks, (b) formulating, sequencing, and posing questions for students, and (c) identifying what different methods or procedures would afford during instruction. PMTs had more instances (102) in their Lesson Plan II associated with teacher actions compared to Lesson Plan I (87 instances), suggesting that PMTs may have begun to think more about mathematics-specific teacher actions. PMTs presented examples of mathematical tasks from textbooks while other times they elaborated on the mathematical tasks from the textbook to describe which mathematical concepts students would be engaged in and which mathematical concepts or procedures that given mathematical tasks would afford. PMTs also discussed how tasks would allow students to engage with certain mathematical concepts. For example, Adam (pseudonym) discussed the following task and procedure in Lesson Plan I:

> This [Factor Game] provides students the chance to improve their understanding of factors. While playing the game, students become familiar with the factors of the numbers from 2 to 30. They also review the multiplication and division of small whole numbers while playing.

In Lesson Plan I, John anticipated that students would be confused by the problems that he planned: "I anticipate that students will have trouble initially figuring out what to do and will tell the students that the solution lies in using the tools they were given, e.g., a ruler and protractor."

In Lesson Plan II, the PMTs began to anticipate students' mathematical thinking; however, their skill of identifying and incorporating students' mathematical thinking was still evolving and they even stated in their reflections they needed to engage with authentic student work. In Lesson Plan I, the PMTs' included generic teacher and student actions (48% of the instances were generic). In Lesson Plan II, PMTs began to include more specific teacher and student actions.

Thus, we identified more instances that were related to both KCT and KCS. For example, Sarah included more specific questions in Lesson Plan II than in Lesson Plan I. In Lesson Plan I, she included the following generic statement: "Pose further questions to keep them thinking and understanding their answers to the questions." In Lesson Plan II, Sarah included specific students' mathematical thinking: "How did you know where to put each variable? If you had a word problem that required three equations, do you think it would be possible to solve that?"

In addition, the PMTs began planning for more mathematics-specific student actions in Lesson Plan II with 45 instances of specific student actions compared to only 14 instances in

Lesson Plan I. Specifically, in Lesson Plan II, the PMTs began describing how their students would be engaged in the mathematical tasks, including how their students might solve mathematical problems or what questions students might ask. For example, Sarah anticipated that her students could have the following response to a task she planned to pose: “[Students] could have an incorrect single equation that doesn’t incorporate all the information from the problem.” However, many PMTs still included generic teacher and student actions in Lesson Plan II. Some of their generic teacher actions included teacher questioning, division of groups in their classrooms, how students would respond to teacher actions. For example, George and Kate included the following generic student action: “[Students will] discuss the process with peers to gain a better understanding and will respond to teacher prompting to dig deeper and prepare responses for class discussion.” PMTs had to use a Lesson Plan template asking them to write teacher and student actions separately. Not surprisingly, KCS was evidenced in the PMTs’ descriptions of student actions. However, there were only a few instances wherein PMTs described which mathematical topics students would find interesting, what would be motivating to students, and what kinds of mathematical problems their students might find difficult to understand. For example, Jacob in his Lesson Plan II included the following: “I anticipate there will be questions and confusion. Students in the past have only worked with equations, so they have the tools they just are unaware of how to implement them into solving the problems.”

We identified more MKT domains from the Launch and Explore sections of the lesson plans than in their Summaries. In these sections, PMTs described mathematical tasks building on their students’ prior knowledge, what the teacher would present to the students, and what actions the teacher and students would perform. We identified only a few instances of the MKT domains in the Summary sections. We noticed that the PMTs struggled to plan for those specific actions because they had to anticipate what their students would explore in the Explore section. As a result, most PMTs planned for generic actions, specifically inviting students to share their ideas. For example, Griffin and Sarah wrote: “Walk around to each group and observe what they are discussing. Pose further questions to keep them thinking. Decide which groups should share.”

KCC and CCK were also evidenced in the PMTs’ lesson plans. KCC was evidenced when PMTs discussed students’ prior mathematical knowledge and how they would build on it in their lessons. For example, in Lesson Plan II, Kate included: “Prior knowledge about calculating area and probability knowledge is needed for this lesson. Students also need prior knowledge of probability notation, including that P(x) means that we are looking for the probability of x.” CCK was evidenced when PMTs presented the specific definitions or solution methods. For example, in her Lesson Plan I, Kiara included her plan for defining a *factor of a number*: “Teacher will review what a factor of a number is by reminding them that it can be thought of as both the numbers multiplied to get a product and the divisor of a number.” This quote indicates Kate’s knowledge of MKT in the CCK domain as she defined factors.

In their reflections, PMTs mostly described KCT, including which mathematical tasks their students were engaged in. For example, in Lesson Plan II reflection, Kate noted:

> I think she wanted the students to think of solutions on their own because there were several ways for the students to solve the problems. Through the student presentations, other students were able to discover different strategies so that they could figure out which method works best for them or what they understand the most.

In this reflection, the notion of students sharing multiple strategies and different thinking allows the teacher to assess their students’ understanding, one descriptor of KCT. In Tania’s

Lesson Plan II reflection, another code of the KCT domain, "Identify what different methods and procedures might afford during instruction," was referenced:

> I encourage each activity to involve some sort of discussion with their peer, and the activities themselves are asking the students to present the information in different ways, which is allowing the students to enhance their conceptual understanding of the material.

While this reflection could be seen as more generic, the mentioning of "enhance their conceptual understanding of the material" points to understanding how students' conceptual knowledge of simplifying exponential expressions was important.

From Table 2, we also notice that there were only a few instances of HCK (four instances in Lesson Plan I and one instance in Lesson Plan II). Recall that HCK includes logical connections of mathematics topics within broader mathematical disciplines when presenting big mathematical ideas. Thus, this domain could have been evidenced when the PMTs presented instructional strategies in a way that made connections with advanced mathematical topics. For example, in the following excerpt, Sarah used her knowledge of "transitivity" to plan her lesson: "If we know that t is equal to two different things, does that mean those two things are equal to each other? (transitivity)."

In addition, SCK, KCC, and CCK were less evidenced in the PMTs' lesson plans. SCK was evidenced in instances where the PMTs were able to offer alternative mathematics solution strategies, multiple mathematical representations, and concepts needed to understand mathematics. For example, Jacob, in Lesson Plan II, mentioned:

> You are looking for something a little bit bigger but will fit in your TV cabinet and you only know the width and height of the TV cabinet. When you get to the store, they do not list the Width and Height of the TVs, instead, they only have the diagonal. What are some possible ways that you could find out the size of the TV you need?

Here, Jacob presented his mathematics task, and it included a real-world problem. Tayra planned to address students' conceptions in Lesson Plan II: "If the students aren't understanding conceptually, then I'll spend more time individually with the students and track specifically what things they're finding difficult to understand." Later, while reflecting on the lesson plan, Tayra mentioned that she anticipated the task to be difficult for students; however, she did not explain how: "we anticipated this task being difficult for many. It involves students to get creative and think outside the classroom." This finding suggested that while PMTs had generic explanations of what their students might find confusing, there were fewer instances wherein PMTs mentioned which mathematical concepts students might find interesting and their anticipation of students' alternative problem-solving strategies (e.g., patterns of student errors).

In their reflections, PMTs noted that they did not anticipate students' mathematical thinking because they wanted to have flexibility about how to modify students' task-solving strategies during lesson plan implementation. For example, Kate reflected that she did not try to anticipate students' thinking because she planned to be open to students' ideas: "I do not feel like we tried to anticipate a lot because I did not want to get caught up in what I think the students will answer so that I am more flexible for when they do not answer the way someone typically would." This finding suggests that PMTs need more opportunities to explore students' authentic work. KCC was evidenced when PMTs described how they built their plans on students' prior knowledge. For example, Jacob planned for addressing students' prior knowledge: "While this is a new chapter, this lesson relates back to measurements, triangles and shapes, and heights and areas."

Discussion and Implications

Our findings indicated that KCS and KCT were the primary MKT domains identified from PMTs' explanations of teacher and student actions. We identified two possible reasons associated with this finding. First, due to the lesson plan template, the PMTs demonstrated some domains more than others. For example, the template did not include an obvious place for PMTs to demonstrate HCK. Each PMT discussed CCK (i.e., addressing students' prior knowledge). Given that the lesson plan template explicitly included the prompts about students' prior knowledge, each PMT included at least one instructional activity that described how they would address students' prior knowledge. Also, the lesson plan template had prompts for the PMTs to describe teacher and student actions and not many options to demonstrate how their advanced mathematical knowledge connected with the teacher and student actions, which could be a reason we identified less evidence of HCK.

Second, when the PMTs had to explain student actions, they demonstrated fewer instances related to mathematics-specific student actions. PMTs noted they could not plan for student actions because they had not had a chance to work with specific students in the field. We identified that our approximations of practice (i.e., peer teaching, reflections) did not contain students' authentic work. To address this limitation, incorporating authentic student work prior to lesson planning could evoke PMTs' conceptions about mathematics-specific student work. Moreover, engaging PMTs with the research about students' possible mathematical thinking and rehearsing how to respond to students' mathematical thinking could be a way to cultivate PMTs' skills to plan for activities to address students' mathematical reasoning. Analysis of student work could assist PMTs in identifying activities to understand and develop students' mathematical reasoning (Álvarez et al., 2020). We propose authentic forms of approximations wherein PMTs first read and discuss students' potential mathematical reasoning, then engage with students' authentic work in some form. Afterward, they will have opportunities to plan instructional activities to develop students' mathematical understanding of a certain topic, including possible ways to address students' existing conceptions. Finally, PMTs could implement those strategies with their peers first and with real students later during their field experiences.

We found minimal instances associated with SCK, indicating alternative representations of mathematics topics and contextual factors associated with the given mathematics topics were less discussed. Prior research suggested textbooks played a significant role to enhance PMTs' MKT; for example, textbooks predominantly promoted CCK (Atanga, 2021). Given that the textbooks that PMTs used in our study focused on conceptual understanding, our findings do not align with Atanga's findings. Thus, exploring the explicit connections between how MKT domains in PMTs' lesson plans were influenced by the textbook should be a question for further research. In addition to textbook content, a section in the lesson plan template about the contextual factors and alternative representations of mathematics could help PMTs explore, demonstrate, and conceptualize how mathematics topics relate to other concepts and real-life.

Our study has implications for mathematics teacher educators and is connected with the conference theme of Critical Dissonance and Resonant Harmony. As opposed to assessing teacher knowledge through paper and pencil tests to identify what teachers know versus do not know, we invite educators to critically reflect on approximations of practice so PMTs have the opportunity to explore how they can contextualize their advanced mathematical knowledge during mathematics teaching. Our MKT domain descriptors provide a way to design instructional activities that engage PMTs in both teacher knowledge and practices.

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