HOW MANY ANGLES DO YOU SEE? PROSPECTIVE TEACHERS' ASSIMILATORY DOMAINS FOR ANGULARITY

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Given the centrality of angle in mathematics curricula and scarcity of research in this area, we investigated 64 PTs' assimilatory domains of angularity by analyzing the angles they indicated when presented with four segments mutually sharing an endpoint. In both interview and written settings, we found PTs were more likely to recognize convex angles than reflex angles. Additionally, they were more likely to assimilate disjoint angles than angles formed via additive angular compositions. In particular, we found that PTs were unlikely to recognize full angles or additive compositions involving reflex angles. We consider future directions and implications.

Keywords: Geometry and Spatial Reasoning, Cognition, Preservice Teacher Education

Teachers play a significant role in students' mathematical learning. Therefore, prospective teachers (PTs) enrolled in undergraduate teacher education programs need opportunities to develop strong mathematical content knowledge to support their future teaching (AMTE, 2017). Geometry and Measurement are critical domains of mathematics, and, within these domains, angle and angle measure are pervasive topics throughout mathematics curricula from elementary school through higher education (Barabash, 2017).

Although many researchers have studied and worked to support prospective and practicing teachers' content knowledge related to other quantities (e.g., length, area, and volume), research on teachers' conceptions of angle and angular measure is scarce (Smith & Barrett, 2017). However, we know from the few extant studies in this area that teachers tend to experience challenges when it comes to angle concepts (Smith & Barrett, 2017). Prior to the study of precalculus mathematics, conceiving of angle measures in degrees is particularly important given current conventions and curricular standards. For example, in the Common Core State Standards for Mathematics (CCSSM), angle measure is explicitly introduced in Grade 4:

> An angle is measured with reference to a circle with its center at the common endpoint of the rays, by considering the fraction of the circular arc between the points where the two rays intersect the circle. An angle that turns through 1/360 of a circle is called a "one-degree angle," and can be used to measure angles. (NGA Center and CCSSO, 2010, p. 31)

In subsequent grade levels, robust understandings for angles and their measures are essential for topics including the analysis of geometric figures and relationships, constructions, transformations, proof, and trigonometry. Given this, more research is needed to understand teachers' conceptions of angles and their measures. Working toward this goal, we focus this report on a fundamental question: What angles do prospective teachers recognize when presented with multiple line segments mutually sharing an endpoint? We view this question as fundamental because PTs' conceptions of angles as objects will necessarily impact their understanding of angular measure.

Theoretical Components

The study reported here was informed by principles of quantitative reasoning (Thompson, 1994; 2011). A quantity is an individuals' conception of a measurable attribute of an object or phenomenon. Quantities are conceptual entities existing in the minds of individuals. Quantities vary both within an individual over time and across individuals. Characterizing a quantity involves attending to an individuals' conception of three interrelated components—an object of interest, a particular attribute of the object, and a conceived measurement process (i.e., a quantification). Although all three components are important, we focus primarily on the object of interest in this report. Specifically, we are interested in PTs' conceptions of angles as mathematical objects and variation in these conceptions.

To investigate these conceptions, we leverage a radical constructivist (von Glasersfeld, 1995) perspective. In this perspective, knowledge is not seen in relation to a "true" understanding of the real world (or a mathematical one); instead, knowledge consists of conceptual structures an individual has produced to organize experiential constraints (von Glasersfeld, 1997). Within an individual, conceptual structures are constructed and modified over time, and the structures that persist do so because they remain viable. In other words, the existence of an individual's knowledge implies that the knowledge is useful (for the individual).

Radical constructivists often offer nuanced characterizations of individuals' conceptual structures via scheme theory, which has roots in Piaget's (1970) genetic epistemology. According to von Glasersfeld (1995), schemes consist of three parts (a) an individual's recognition of a prior situation in a present experience, (b) an executable activity associated with the recognized situation, and (c) an expected or beneficial result. In this report, we focus in particular on the first component, which is often referred to as assimilation. Regarding assimilation, von Glasersfeld remarked, "The mind primarily assimilates, that is it perceives and categorizes experience in terms that are already known" (1997, p. 301). Specifically, we are interested in specifying the perceptual material PTs assimilate as angle models. In other words, we are interested in what PTs recognize as angles and the assimilatory operations of recognition, which are crucial steps toward better understanding how individuals quantify angularity.

Methods

The findings reported here come from a larger, design-based research (DBRC, 2003; Cobb et al., 2003) project. In this project, we designed and iteratively refined task-based lessons for elementary and middle-grades PTs enrolled in geometry content courses in order to foster critical ways of reasoning about angularity. For more information on the project and lessons, see Hardison and Lee (2019). In this report, we focus on a single task we used in two different contexts: (a) follow-up interviews conducted approximately one year after eight PTs' enrollment in the course (Method 1) and (b) written responses from 58 different PTs prior to a subsequent implementation of the lessons (Method 2).

Method 1: Follow-up Interviews (Spring & Summer 2020)

In Fall 2018, we implemented task-based lessons with three sections of elementary and middle-grades PTs enrolled in a geometry course. We have argued elsewhere that these lessons were successful in engendering productive quantifications of angularity during the course (Hardison & Lee, 2019). However, we wanted to investigate whether participants sustained these ways of reasoning about angle measure beyond the course. Therefore, we contacted the PTs previously enrolled during Fall 2018 to solicit participants for follow-up interviews. Eight PTs volunteered, and we conducted interviews with these eight participants during Spring and Summer 2020, a little more than one year after the PTs' original enrollment in the course.

Interviews, task, and protocols. The one-on-one, task-based clinical interviews (Clement, 2000; Goldin, 2000) with PTs were semi-structured and conducted by their previous course instructor, which was one of the first two authors. Each interview was approximately 60 minutes in length. Due to restrictions associated with Covid-19, each interview occurred remotely via Zoom. Microsoft OneNote was used to pose tasks and for PTs to record written work. Each interview was video-recorded. Data sources for subsequent analyses included interview recordings, PTs' digitally written work, interview transcripts, and researcher notes. Each interview consisted of 6–8 tasks on angle or angle measure, which were either tasks previously presented during the geometry course, variations on these tasks, or entirely original tasks.

Here, we focus on a single original task, *How Many Angles*, which is shown in Figure 1. We designed this task to investigate what perceptual material PTs would assimilate as an angle as well as the assimilatory domains for angularity (e.g., whether PTs considered reflex angles to be angles in this context). In this task, PTs were asked how many angles they could identify in the provided configuration of line segments. If a PT did not spontaneously draw to indicate the angles they counted, the interviewer asked the PT to indicate the angles they counted using the drawing function. Once a PT committed to a final number of angles, the interviewer pressed by asking whether it would be possible to find any more angles. This aspect of the interview protocol was repeated until the PT indicated it was not possible to find any more angles.

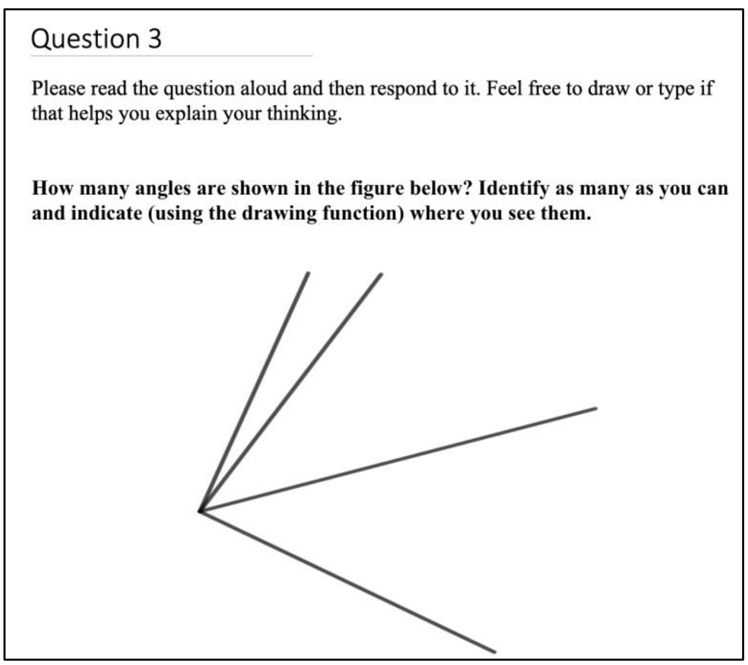

Figure 1: How Many Angles Task

Coding and analysis. Procedures for analyzing the data were established after the interviews were conducted. In order to systematically track the angles PTs identified, we created a reference and notation for the angles participants considered in the interviews (Figure 2). In Figure 2, convex angles are designated C1–C6 and reflex angles R1–R6 with matching enumeration for conjugate angles, (e.g., C6 and R6 are conjugate angles in that they share the same sides but have different interiors). Designations for full, closed, and straight angles were F, O, and S, respectively. We tracked the angles PTs indicated regardless of the indication method. Indication methods included usage of arcs, shading, and labels. As shown in Figure 3, we tracked the angles

PTs identified, the order in which angles were indicated, and the number of angles indicated. If PTs identified more angles after an interviewer press (e.g., "Would it be possible to identify any more?"), we continued additional rounds of coding. Each interview was coded by three of the authors, codes were compared, and discrepancies were discussed until consensus was achieved.

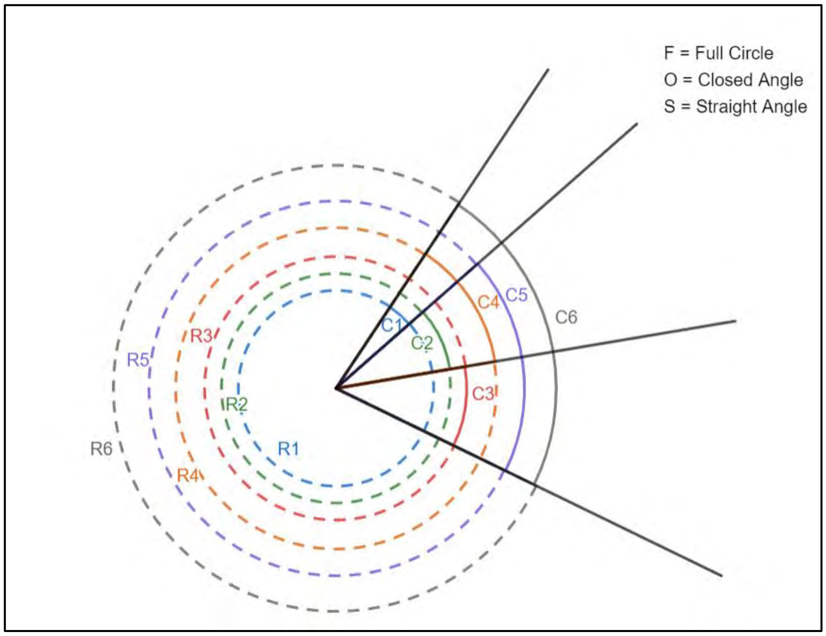

Figure 2: Reference for Angle Names

				How Many Angles			
Student	Round 1	Number 1	Round 2	Number 2	Round 3	Number 3	Round 4	Number 4
Example	C1, C2, C3	3	C4, C5, C6	6	R6	7		

Figure 3: Sample Coding Spreadsheet

Method 2: Written Responses (Fall 2021)

In Fall 2021, the first author taught two sections of the same geometry content course for elementary and middle-grades PTs. Written data was collected throughout the course from 58 PTs, none of whom were participants in the previous study. The How Many Angles task, with minor modification in phrasing (see Figure 4 below), was posed in a written journal prompt, which PTs submitted electronically, during the second week of class. This occurred prior to any lessons on angle or angle measure. Given the written format of the task, we were limited to the responses PTs submitted. So, in contrast to the interview task, we were unable to request clarification, track the order in which PTs identified angles, or press for finding additional angles. As a consequence, we tracked only the angles PTs indicated and the number of angles reported. These responses were coded by two of the authors, and discrepancies in coding were discussed until resolution was achieved. We noted the angles indicated using the coding shown in Figure 2, with one significant modification described in the paragraph below.

In their written responses, some PTs indicated a pair of sides without clearly indicating an interior (e.g., Figure 4). In these cases, since we were unable to determine whether the PT intended to indicate a convex angle or a reflex angle, we adopted the notation P1–P6 to indicate the pair of sides that was indicated. For example, we coded the response shown in Figure 4 below as "P1, P2, P3, P4, P5, P6" since all six pairs of sides were designated in different colors.

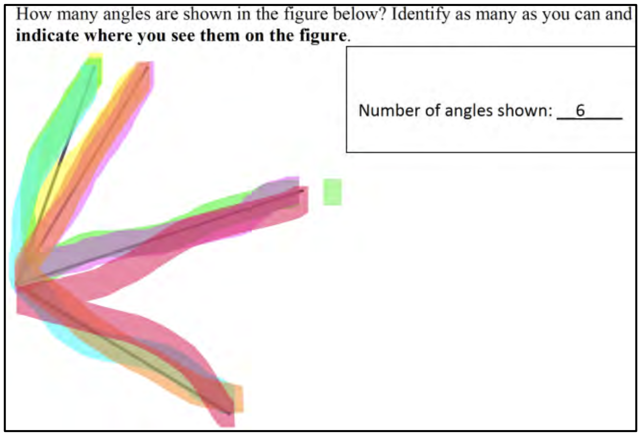

Figure 4: Indicating Pairs of Sides Without Interiors

Findings

We organize the findings in two sections according to the methods described above. We first present findings from follow-up interviews and then present findings from written responses.

Findings from Method 1: Follow-up Interviews (Spring & Summer 2020)

We begin by discussing the angles PTs identified prior to any interviewer press for additional angles (i.e., Round 1). In Round 1, PTs' numbers of stated angles ranged from four to seven, with 50% of the eight PTs indicating six angles were shown in the task (see Table 1).

Table 1: Distribution of Number of Angles Stated in Interviews in Round 1

Number Stated	4	6	7	**Total**
Number of PTs (% of 8 PTs)	2 (25)	4 (50)	2 (25)	58 (100)

The distribution of the angles PTs indicated is shown in Table 2. PTs were more likely to indicate convex angles than reflex angles in Round 1 (100% vs. 38%). Thus, PTs were more likely to assimilate convex angles than reflex angles. Additionally, we noted that disjoint angles (i.e., angles with no perceptual segments penetrating their interiors; C1–C3) were more frequently indicated than composite convex angles (i.e., angles that could be formed through additive compositions of C1–C3, namely C4–C6). Furthermore, the only reflex angle indicated in Round 1 was R6, which was indicated by three PTs. We hypothesize that other reflex angles were not indicated since R6 is the only disjoint reflex angle that might be assimilated without additive composition. Thus, PTs' assimilatory domains for angularity appeared not to spontaneously include additive compositions involving reflex angles (i.e., R1–R5).

Table 2: Angles Indicated by PTs in Round 1 of Interviews

Angle Indicated	C1	C2	C3	C4	C5	C6	R6
Number of PTs (% of 8 PTs)	8 (100)	8 (100)	8 (100)	6 (75)	6 (75)	7 (88)	3 (38)

At the end of Round 1, each PT was pressed by the interviewer for whether it would be possible to find any more angles in the figure. Five of the eight PTs indicated that it was not possible; one subsequently entertained R6 as a possibility and discarded it; leaving only two PTs who ultimately changed their final number of angles after Round 1. We discuss the responses of these two PTs (Lori and Taneem) in further detail in the subsequent sections.

Lori's subsequent rounds. At the conclusion of Round 1, Lori had identified 4 angles (C1–C3 and R6). That is to say, Lori initially assimilated all possible disjoint angles (both reflex and convex). In Round 2, she considered whether additive compositions of angles were permitted:

Lori: If you can use, like any lines, like in any way that you wanted, you know, like I'm not sure if that's allowed, but if you could there's tons of possibilities of angles.

Int: Oh yeah? So, what would, what other possibilities would that introduce if that was allowed?

…

Lori: Okay, does it go from here to here, here to here, this one to here, this one *all the way back around* that'd be 1, 2, 3, 4. And this one would have four – oh, would it be 16?

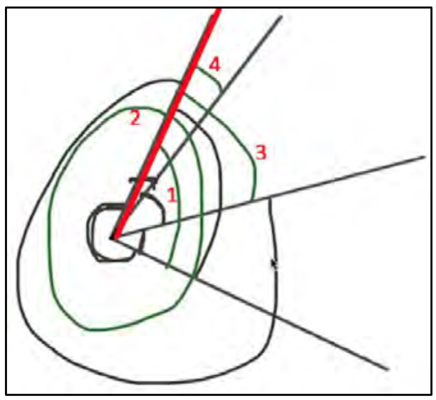

Figure 5: Lori's indication of 4 green angles initiating clockwise from the red segment.

Lori's consideration of using lines "like in any way" indicated that she began to consider all possible pairings of the line segments as a way to determine the number of angles shown. Lori's last line of transcript indicates her picking a single initial segment and considering pairing it with each of the four segments (including the initial segment she selected) and concluding that four possible pairings would be possible. Lori's conclusion that it might be 16 angles was indicative of her awareness that the enumeration of four pairings could be repeated taking any of the four segments as the initial segment. Because Lori's gestures were always enacted in a clockwise fashion from the initial segment, she did not double count any of the 12 convex or reflex angles (i.e., C1–C6, R1–R6). The additional four angles in Lori's final count of 16 were four full angles, each of which went "all the way back around" and initiated at a different segment. Her illustration of the four possible clockwise pairings (in green) with a single initial segment (in red) is shown in Figure 5.

Taneem's subsequent rounds. At the conclusion of Round 1, Taneem had identified 7 angles (C1–C6 and R6). That is to say, Taneem initially assimilated all possible convex angles (disjoint and additive compositions) and the only disjoint reflex angle. In Rounds 2 and 3, she continued considering additive compositions until she had identified all remaining possible reflex angles (i.e., R1–R5) and concluded she saw 12 angles in the task. When pressed again (Round 4), Taneem considered whether full angles should be counted and how many.

Taneem: So, like, if you were just using like this the line [*highlighting a line segment in pink in Figure 6*], I'm thinking about and then, if you just counted that as like, if that was, like the only line here then this would be like a, hold on. [*Drawing the circle in green*] Around there, like a 360-degree angle. I don't know.

Int: Okay. If you, if you counted the one that you just drew, how many would there be then?

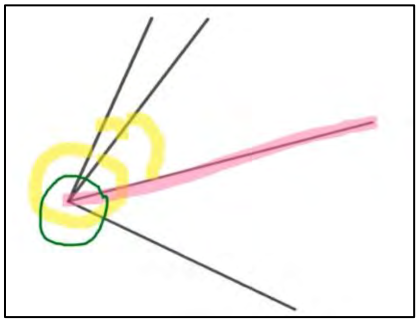

Figure 6: Taneem indicates a full angle initiating and terminating at the pink segment.

Subsequently, Taneem questioned whether each of the four segments could be used in the generation of a distinct full angle remarking, "Well, actually I think that will all be the same for all these lines because they all meet up with the same point. So just be like one more angle possibly…so it'd be like the same 360-degree angle for all of them." Taneem ultimately determined that if she counted full angles, then only one should be counted, bringing her total angle count to 13. Taneem's decision to count only one full angle, rather than four, was rooted in that the four angular candidates shared the same vertex and interior. Ultimately, she decided her final count for angles in the figure was 12 as she did not wish to count any full angles remarking, "So, I'm not sure if that counts an angle because it's like 360-degrees. So, it's like a circle."

Findings from Method 2: Written Responses (Fall 2021)

We now shift our focus to an analysis of 58 PTs written responses to the same task in Fall 2021. In these responses, the number of angles PTs stated ranged from 3 to 12, with a plurality of PTs (47%) stating six angles were shown (Table 3). Only three PTs stated that more than six angles were shown, with two PTs stating 7 shown angles and one PT stating 12.

Table 3: Distribution of Number of Angles Stated for How Many Angles Task (Written)

Number Written	3	4	5	6	7	12	**Total**
Number of PTs (% of 58 PTs)	14 (24)	11 (19)	3 (5)	27 (47)	2 (3)	1 (2)	58 (100)

Regarding the angles PTs indicated, we begin first by noting two findings different from the interview setting. First, six PTs (10%) stated the number of angles shown but did not indicate any angles on the figure. Second, another nine PTs (16%) did not indicate an interior for any angle. We attribute these findings to a constraint in our second research method, namely that we could not press for clarification in the written setting. An alternative interpretation is that, for some of the PTs, assimilating an angle did not necessarily mean assimilating an angular interior.

We restrict our remaining findings to analyses of the 43 PTs who indicated the interior of at least one angle, either reflex or convex. In other words, we are excluding from subsequent analyses the six PTs who did not indicate any angles on the figure and the nine PTs who indicated only pairs of sides without indicating an interior for any angle. All of the remaining 43 PTs indicated at least one convex angle. In other words, convex angles were in the assimilatory domain of angularity for each of the 43 PTs. As shown in Table 4, all 43 PTs indicated C1, C2, and C3, while only about half of PTs indicated C4, C5, and C6. We attribute these differences in percentages to the disjoint nature of C1–C3 and the composite nature of C4, C5, and C6 in the figure. This indicates additive compositions of convex angles are outside of some PTs' assimilatory domain of angularity.

Table 4: Convex Angles Indicated by PTs (Written)

Convex Angle Indicated	C1	C2	C3	C4	C5	C6
Number of PTs (% of 43 PTs)	43 (100)	43 (100)	43 (100)	21 (49)	22 (51)	23 (53)

Reflex angles were indicated less often than their convex counterparts with only 10 of 43 PTs (23%) indicating a reflex angle. This suggests that reflex angles are outside of the assimilatory domains of the majority of PTs in our study. Moreover, R6 was the only reflex angle indicted by these 10 PTs. In other words, none of the 43 PTs indicated R1, R2, R3, R4, or R5 on the provided figure. We suggest that the disjoint nature of R6 accounts for PTs assimilation of this angle and not R1–R5 since these later angles can be viewed as additive compositions of reflex and convex angles. This suggests that additive compositions involving reflex angles are rare in the assimilatory domains of the PTs we studied. We do note that, as shown in Table 3, one PT stated that 12 angles were shown in the figure, which suggests all 12 possible convex and reflex angles noted in Figure 2; however, this PT was one of the six PTs who did not indicate any angles on their figures. Finally, we note that none of the 58 PTs indicated consideration of closed, straight, or full angles in their responses to this task.

Brief Conclusions and Considerations

In this report, we considered PTs' assimilatory domains of angularity through analyzing the angles they indicated when considering four segments with a mutually shared endpoint. In both interview and written settings, we found PTs were more likely to assimilate convex angles than reflex angles. Additionally, we found PTs were more likely to assimilate disjoint angles than angles formed via additive composition of disjoint angles. In particular, we found that PTs were unlikely to assimilate additive compositions involving reflex angles. Finally, we note that across the 64 PTs from both methods, only two PTs (Lori and Taneem) indicated consideration of a full angle and this consideration was occasioned through interviewer prompts. We also acknowledge that our results only reflect PTs assimilatory domains of angularity for static angle contexts and that dynamic angle models might indicate different assimilatory domains of angularity. Still, these results suggest purposeful interventions targeting angular operations may be necessary to support PTs conceptions of angularity.

As noted at the beginning of this report, angle measurement in degrees is typically introduced in Grade 4 by taking a full angle as a 360-unit composite. Considering the PTs we studied have had 12+ years of experience in school mathematics classrooms, we hypothesize that their assimilatory domains of angularity would tend to exceed those of the students they are to teach. Of course, more research is needed to verify this hypothesis. Nevertheless, given this hypothesis and our finding that PTs tended to assimilate angles in excess of 180° less frequently than the convex counterparts, future research is needed to investigate whether approaches that introduce degrees using other composite units (e.g., a right angle as a 90-unit composite) might better support students in productively quantifying angularity in the context of static angle models.

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