THE ROLE OF LEARNING PROGRESSIONS IN "DEMOCRATIZING" STUDENTS' ACCESS TO ALGEBRA

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Learning progressions have become an important construct in educational research, in part because of their ability to inform the design of coherent standards, curricula, assessments, and instruction. In this paper, I discuss how a learning progressions approach has guided our development of an early algebra innovation for the elementary grades and provide examples of how this approach can help challenge a settled mathematics learning status quo about the kind of algebra students can learn, when they can learn it, and how all students can be successful. Empirically derived learning progressions are an important part of designing early algebra innovations that can open new curricular pathways for teaching and learning algebra, creating accessible and effective avenues of learning for all students.

Keywords: Algebra and algebraic thinking, learning trajectories and progressions, elementary school education

Using a Learning Progressions Approach to Develop an Early Algebra Innovation

Over a decade ago, my research interests turned towards a question that I view as critically important in teaching and learning algebra: Does early algebra matter? Since I assume early algebra *does* matter, perhaps a better way to frame this question is to what extent does early algebra matter, in what ways does it matter, and how might we capture or measure this? There are deep implications for the answers to these questions. A truly effective integration of early algebra (or, algebraic thinking in the elementary grades) would entail significant costs because it requires "deep curriculum restructuring, changes in classroom practice and assessment, and changes in teacher education—each a major task" (Kaput, 2008, p. 6). Such costs highlight the need for carefully constructed models of early algebra instruction—models that have been missing from elementary grades mathematics. However, these models, as curricular roadmaps for developing children's algebraic thinking across elementary grades in a deep, systemic way, are essential to understanding early algebra's impact.

To build such a model, our research team turned to a *learning progressions approach*. Learning progressions have become an important construct in educational research (Clements & Sarama, 2004, 2009; Confrey et al., 2014; Simon, 1995; Stevens et al., 2009), in part because of their ability to inform the design of coherent standards, curricula, assessments, and instruction (Daro et al., 2011). We focused our work on the development of several core components aligned with this approach (e.g., Clements & Sarama, 2004; see also Fonger et al., 2018): (1) empirically-derived learning goals around algebraic thinking in elementary grades; (2) grade-level instructional sequences designed to address these learning goals; (3) validated assessments to measure students' understanding of core algebraic concepts and practices as they advance through the instructional sequences; and (4) progressions that specify increasingly sophisticated levels of thinking students exhibit about algebraic concepts and practices in response to an instructional sequence. A learning progressions approach served two purposes in our work. It provided an over-arching "large-grain-size-level" framework to guide our design of a Grades K–5 early algebra intervention from which we might measure the impact of early algebra on children's algebra readiness for middle grades. It also provided a theoretical mechanism for identifying "small-grain-size-level" cognitive foundations in children's algebraic thinking that, along with other existing research in the field, could inform the development of our intervention.

At a large-grain-size level, we used a learning progressions approach (e.g., Shin et al., 2009, Stevens et al., 2009) in the development of learning goals, grade-level instructional sequences, and grade-level assessments for K–5. Using Kaput's (2008) content analysis of algebra as a set of *key aspects* (ways of thinking algebraically) and *core strands* (content domains where algebraic thinking occurs), we identified core algebraic thinking practices of *generalizing*, *representing*, *justifying*, and *reasoning with* mathematical structure and relationships as an underlying conceptual framework for the design of our goals, sequences, and assessments (Blanton, Brizuela et al., 2018). We viewed a conceptual framework organized around algebraic thinking *practices* as critical to avoid designing instructional sequences based simply on the use of ubiquitous "algebra" tasks (e.g., solving equations) and not cohesively grounded in what it means to think algebraically.

To develop our learning goals, we analyzed the treatment of the core algebraic thinking practices in several dimensions: (1) empirical research on Grades K–8 students' algebraic thinking; (2) national curricular frameworks and standards such as the *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics [NCTM], 2000) and *Common Core State Standards* (National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010); (3) Grades K–8 curricular materials (e.g., *Everyday Mathematics*, *Singapore Math, Investigations)*; and (4) formal algebra content at both secondary and postsecondary levels. We then organized early algebra content to align with strands in Kaput's (2008) algebra content analysis, in particular, "the study of structures and systems abstracted from computations and relations" (Strand 1) and the "study of functions, relations, and joint variation" (Strand 2) (p. 11). These strands also aligned with core content around which the early algebra research base had coalesced. Based on this, we structured our findings on early algebra content within several "Big Ideas" (e.g., Stevens et al., 2009), or content domains, where core algebraic thinking practices can occur: Generalized Arithmetic; Equivalence, Expressions, Equations, and Inequalities; and Functional Thinking. We then unpacked the role of algebraic thinking practices within each Big Idea by delineating core algebraic *concepts* related to the practices, *claims* that specify the nature of skills or understandings expected of students regarding a specific concept, *evidence* in students' work that would indicate they had developed the skills or understandings specified in our claims, and research-based *difficulties and misconceptions* that students have with a concept (Shin et al., 2009). The concepts, claims, evidence, and difficulties or misconceptions associated with the algebraic thinking practices within each Big Idea informed the development of grade-specific learning goals.

Using our learning goals, we then constructed grade-level instructional sequences for each of Grades K–5 by designing specific task structures that connected the algebraic practices with related claims about the skills or understandings that might reasonably be expected of students. For example, for Generalized Arithmetic we designed sequences of tasks that created opportunities to *generalize* arithmetic relationships, to *represent* these relationships in different ways, to develop appropriate general arguments for *justifying* the arithmetic relationships students observed, and to use arithmetic generalizations students developed as objects (Sfard, 1991) for *reasoning with* novel problems or properties of arithmetic that make computational work more efficient. Grade-level learning goals were used to guide content for these task structures. For example, the types of arithmetic relationships addressed at a particular grade, the types of representations used to express them, or the nature of arguments students might develop to show relationships were valid, were guided by our learning goals. We then used design research (Cobb et al., 2003) to field-test and refine our proposed grade-level instructional sequences. Finally, we used these task structures to develop validated, grade-level assessments by which we could measure within-grade and across-grade (longitudinal) growth in children's algebraic thinking in order to understand early algebra's impact.

Our effort to design a broader, multi-year (Grades K–5) approach to developing students' early algebraic thinking—essentially, components (1)–(3) above—aligns with what is sometimes characterized as "learning progressions" (Stevens et al., 2009). At the same time, a smaller grain size approach was needed to fill in "gaps" in empirical research on our understanding of children's algebraic thinking (essentially, component (4) above), particularly in the early elementary grades where the research base was less developed than that for Grades 3–5. Because of its narrow scale in terms of a focus on specific algebraic concepts or practices within short instructional timelines (e.g., weeks), this second aspect of our work might be seen as more akin to learning trajectories (e.g., Stevens et al., 2009). This "small-grain-size-level" research was critical for identifying increasingly sophisticated levels in students' thinking about a particular practice or concept within a Big Idea. For example, we identified trajectories in students' thinking about concepts such as a relational understanding of the equal sign (Blanton, Otalora Sevilla et al., 2018) and variable and variable notation (Blanton et al., 2017), as well as for practices such as generalizing functional relationships (Blanton, Brizuela et al., 2015; Stephens et al., 2017), generalizing arithmetic relationships (Ventura et al., 2021), and justifying claims about arithmetic relationships (Blanton et al., 2021).

Regardless of the nomenclature used or whether focusing on "big" ideas over a broad span of time (e.g., multiple years) or "small" concepts in a narrow span of time (e.g., weeks), a learning progressions approach has provided a flexible theoretical paradigm with key features aligned with our core research goals: identifying increasingly sophisticated ways students come to think about an algebraic concept or practice in response to an instructional sequence (Duschl et al., 2007; Simon, 1995; Smith et al., 2006); attending to specific content domains rather than general cognitive structures in how we design our instructional sequences to study early algebra's impact (Baroody et al., 2004); organizing content within these domains to facilitate the development in students' understanding of algebraic concepts and practices over time (Smith, et al., 2006); and relying on classroom-based empirical research, rather than just a logical analysis of the discipline, to understand how students' early algebraic thinking develops (Stevens et al., 2009).

The value of this approach extends beyond its role as a research paradigm, however. One of the organizing questions of the PME-NA 2022 conference—*How does your work challenge a settled mathematics learning status quo?*—is at the heart of early algebra's goal of "democratizing" students' access to algebra. By democratizing access to algebra, we mean opening up pathways to students for whom the traditional "arithmetic-then-algebra" approach—teaching arithmetic in elementary grades, followed by formal algebra in secondary grades—has been unsuccessful (e.g., Hiebert et al., 2005) and has limited students' access to STEM career and workforce opportunities (e.g., LaCampagne, 1995; NCTM, 2000). These challenges have been particularly felt among students from historically underserved communities (e.g., Moses & Cobb, 2001; Museus et al., 2011). For example, elementary grades students from lower socioeconomic (SES) backgrounds are two times more likely to be deficient in mathematics than students from higher SES backgrounds (U.S. Dept. of Education, NCES, 2007). This makes such students especially vulnerable in later formal algebra courses, which impacts their chance for success in college (U.S Dept. of Education, 2008) and access to STEM-related disciplines and careers. The promise of early algebra, then, is to address existing inequities in school mathematics and broaden students' access to STEM disciplines. In what follows, I consider a few examples of how our work, built around learning progressions, has helped challenge the status quo of settled mathematics learning around the kind of algebra students can learn, when they can learn it, and how all students can be successful.

Representing Generalizations Using Variable Notation

The traditional "arithmetic-then-algebra" approach to teaching and learning algebra has entailed certain views about what kind of algebra content should be taught and when. One aspect of algebra that has historically been largely outside the purview of elementary grades is variable and variable notation. Our work takes the view that variable notation is a useful tool that children can begin to understand and use from early elementary grades. From this perspective, we have sought to understand trajectories in students' thinking about variable and the use of variable notation to represent arithmetic and functional relationships. While the act of symbolizing a generalization is central to algebraic thinking, the way in which a generalization is represented can vary. In elementary grades, non-conventional forms such as natural language and drawings—symbol systems whose meanings are already available to young children—have historically been prioritized as a more productive way to represent generalizations (Resnick, 1982).

Part of the hesitation for the use of variable notation with young children has likely been due to strict interpretations around Piaget's formal stages of development, along with the concern that premature formalisms (Piaget, 1964) might lead to meaningless actions on symbols (Blanton et al., 2017). It is reasonable to assume that the well-documented challenges adolescents have with the concept of variable and the use of variable notation (e.g., Knuth et al., 2011; Küchemann, 1981) would be even more prominent among younger, elementary grades children. Yet, unlike younger children, adolescents are expected to build a mathematical understanding of literal symbols to notate variable quantities *after* they have deeply developed ways of thinking about letters in linguistic contexts (e.g., Braddon et al., 1993). This suggests that difficulties with variable notation may be more related to conflicts generated by students' use of literal symbols in mathematical contexts that rely on the understandings they already have about literal symbols in non-mathematical contexts (McNeil et al., 2010).

Research, however, increasingly supports that variable notation can be a valuable tool for young learners (e.g., Blanton, Stephens et al., 2015; Brizuela et al., 2015; Carpenter et al., 2003; Cooper & Warren, 2011; Dougherty, 2008; Fuji & Stephens, 2008). Learning progressions have helped us better understand why this might be the case. In a recent study (Blanton et al., 2017), we explored a trajectory in first graders' understanding of variable and use of variable notation to represent functional relationships. The conceptual space of interest here is the level at which children do not yet understand the concept of variable quantity nor how to use variable notation to represent a variable quantity, a level we characterize as *pre-variable/pre-symbolic*. The close analysis involved in mapping out a learning trajectory helped surface a more nuanced view of students' understandings about variable and variable notation. In particular, we observed that students' whose thinking was "pre-variable" did not yet *perceive* a variable quantity in a mathematical situation. As such, they naturally searched for other tools and ways of understanding within their conceptual field to make sense of a situation involving an unknown.

How might this manifest in students' mathematical actions? When young learners at a pre-variable/pre-symbolic level of thinking encounter a situation with a variable quantity, they typically assign a numerical value to the quantity, either randomly or based on some numerical feature of the situation. They might also propose (hypothetically) that the quantity be measured or counted to determine a specific value, even though it cannot be. This is not an unreasonable approach, given that students' mathematical experiences are, typically, fully centered on arithmetic at this point, where quantities are known or can be counted or measured and represented by a numerical value. Moreover, a child whose thinking is pre-variable (that is, the child cannot imagine or does not "see" an unknown within a situation) would not reasonably be expected to *look for* literal symbols (or even non-conventional representations such as natural language) to symbolize a quantity. Instead, we would expect them to use the tools and ways of understanding already available to them, which are arithmetic in nature.

As students progress in their thinking, some pick up the use of algebraic notation before they can perceive variable quantities. Such cases are indicated by the use of literal symbols as labels or to represent objects rather than quantities. That is, students recognize that a literal symbol can be used to recognize *something*, but not a variable quantity, since this is outside of their conceptual field. However, once variable and variable notation co-emerge in children's thinking, children begin to use variable notation in meaningful ways to symbolize variable quantities (Blanton et al., 2017). Through our construction of a learning trajectory around variable and variable notation, we came to see that the challenge was not that young learners cannot understand (and, thus, should not be exposed to) variable notation. It is, rather, that they have learned to interpret mathematical situations through an arithmetic lens which leads to certain ways of (arithmetic) problem solving that are inadequate for situations involving variable quantities. Instead, if they first learn to perceive a variable quantity, this motivates the need for a symbol system—whether conventional or not—to represent the quantity. Once they perceive a variable quantity, the symbolic system—including the use of literal symbols—can be meaningfully used to represent unknowns.

Our findings elsewhere support that children can learn to use variable notation in meaningful ways. For example, in a large-scale, randomized (CRT) study in 46 elementary schools on the effectiveness of our Grades 3–5 instructional sequences (i.e., early algebra intervention), we found significant differences in how treatment students, who were taught the intervention as part of their regular curriculum, were able to represent functional relationships with variable notation in comparison to control students, who were taught only their regular curriculum. Figure 1 compares treatment and control students' use of variable notation and natural language (words) to represent a functional relationship they observed using data they had culled from a problem situation. Not only were treatment students significantly better able to represent a function with variable notation (as well as with words) than control students, treatment students were also significantly better able to use variable notation than words. (Even students in control schools were better able to use variable notation than words, although the differences were not significant).

The point here is that a learning progressions approach has enabled us to better understand potential challenges to young learners' understanding of variable and variable notation and to design instruction that can significantly improve their understanding, both of which call into question the long-held view that young learners should focus on representational tools that are already available to them (e.g., natural language, diagrams). While we strongly support young learner's use of representational systems such as natural language, we equally support the introduction of variable and variable notation in appropriate ways to young learners and view learning progressions research as a means to help shift perceptions that variable and variable notation is beyond the grasp of young children.

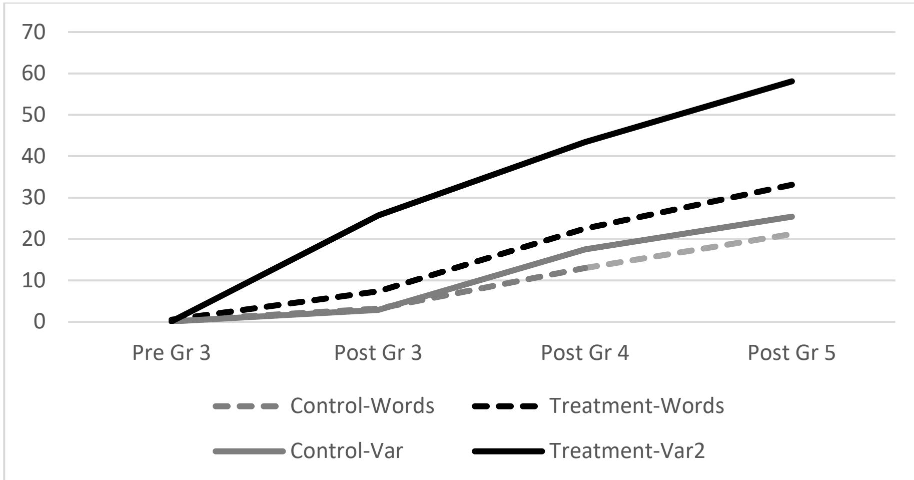

Figure 1. Comparison of treatment and control students use of variable notation to represent a functional relationship.

Developing Mathematical Arguments

Justifying claims about mathematical relationships is central to early algebraic thinking, yet the role of justifying or proving in school mathematics has historically been limited, particularly in elementary grades (Ball et al., 2002; Stylianides, 2016). However, studies suggest that deductive reasoning emerges in the elementary grades (Falmagne, 1980) and that with appropriate instruction, students can learn to use deductive—rather than empirical—reasoning to develop mathematical arguments (Stylianides & Stylianides, 2008). This challenges the long-held view that children's lack of ability to reason deductively is due to a developmental constraint and, instead, points to limited classroom opportunities as the more likely cause for their challenges with building good, grade-appropriate mathematical arguments (Stylianides, 2016; see also, e.g., Ball & Bass, 2003; Carpenter et al., 2003; Lampert, 1992; Maher & Martino, 1996). Studies suggest that the lack of argumentation in elementary grades has far-reaching implications in that it detaches students from sense-making (Staples et al., 2012) and can promote difficulties with proof and proving in high school (e.g., Coe & Ruthven, 1994; Knuth et al., 2002; Stylianides & Stylianides, 2008).

Stylianides and Stylianides (2008) and Stylianides (2007) suggest that research that details trajectories in children's thinking in response to instructional sequences focused on argumentation can help us understand how young children come to reason deductively in response to specific instructional conditions. Because justifying mathematical relationships is a core practice in our conceptual framework, we are interested in how students' come to build strong mathematical arguments in response to instruction. We recently conducted a study in which our goal was to identify progressions in Grades K–1 children's understanding of parity arguments and underlying concepts (e.g., even and odd numbers) in response to our Grade K and Grade 1 instructional sequences (Blanton et al., 2021). Our particular focus was on how children's understanding of representation-based proofs (Schifter, 2009)—versus empirical arguments—developed from the start of formal schooling, before their introduction to any parity concepts.

While our sample was small due to the design nature of our work, our findings were similar to those of other researchers who have conducted extensive work in this area (e.g., Schifter et al., 2009; Stylianides, 2007; Van Ness & Maher, 2019). For example, even in kindergarten, we found that students were able to construct informal structural parity arguments that did not rely on the use of specific or even generic numbers. What we found even more surprising was how rare empirical arguments were in these early grades, even though such arguments are a predominant proof strategy in secondary grades (Coe & Ruthven, 1994; Staples et al., 2012). For example, kindergarten students were unfamiliar with the concepts of pair and even and odd numbers prior to our instructional sequence in Grade K, but by Grade 1 pre-test (i.e., after the Grade K sequence), many students routinely used a pairs strategy (where numbers that can be represented as pairs of cubes are even and those that cannot be represented in this way are odd) to reason about the parity of numbers represented in concrete, visual, and abstract forms. More importantly, out of a group of 10 students interviewed, no student used an empirical argument to justify why the sum of an even and an odd would be odd. Six out of ten were able to correctly use a structural argument involving a pairs strategy (three students could not build either type of argument; one student was not asked this question).

In developing trajectories in children's thinking about particular algebraic concepts or practices, we have found that introducing algebraic thinking earlier—even as early as kindergarten—can help mitigate the development of misconceptions in students' thinking that can occur within an arithmetic-focused approach to instruction. We have observed this in students' understanding of variable quantity and variable notation, where engaging students in mathematical tasks that first help them perceive variable quantities can support their use of variable notation in meaningful ways. We have seen that the early introduction of representation-based arguments can help offset entrenched forms of empirical reasoning by providing students with an accessible, grade appropriate process for justifying their claims. We have seen early attention to functional relationships between quantities mitigate an ingrained focus on recursive patterns in function data that makes it difficult in later elementary grades to re-focus students' thinking on co-varying quantities. We have seen concrete and visual tools help students begin to think relationally about the equal sign even before symbols and equations are introduced, thereby disrupting the operational thinking that is often fostered through arithmetic work focused on standard forms of representations (Blanton, Otalora Sevilla et al., 2018). In this way, learning progressions in early algebra research has helped challenge historically settled notions (even some of our own) regarding what algebraic concepts should be addressed and when. But early algebra's mission to democratize access to algebra is broader than "what" or "when." It is ultimately about "who." In what follows, I briefly touch on some of our evidence around this.

Challenging Perceptions of *Who* Can Do Algebra

The underlying premise of early algebra is that developing children's informal notions about mathematical structure and relationships, beginning in kindergarten, will better prepare them for success in formal algebra in later grades. As described earlier, our work has focused on the use of a learning progressions approach to build the tools with which we could explore this premise, and we are starting to see the contours of an answer to the question of whether early algebra "matters." Our recent 3-year randomized (CRT) study on the effectiveness of the Grades 3–5 early algebra "intervention" (i.e., grade-level instructional sequences) offers some evidence of this. The study was conducted in 46 schools across urban, rural, and suburban settings, where the intervention was taught by classroom teachers as part of their regular mathematics instruction. We found overall that, at each of Grades 3–5, students who received the intervention as part of regular instruction significantly outperformed their peers who received only regular instruction in both their knowledge of algebraic concepts and practices and their use of algebraic strategies to solve tasks (Blanton et al., 2019). Moreover, treatment students maintained a significant advantage over control students in middle grades, one year after the intervention ended (Stephens et al., 2021).

Algebraic thinking in the elementary grades is now codified as essential to algebra education in frameworks such as the *Common Core State Standards for Mathematics* (NGA Center & CCSSO, 2010). Through the adoption of these standards, many states have elevated the role of algebra, leaving students potentially vulnerable to a persistent marginalization in school. With students from underserved communities already underrepresented in STEM professions (Oscos-Sanchez et al., 2008), the long-term implications for the inequity of access to educational on-ramps for students around early algebra are significant. As such, early algebra innovations should be designed to ensure that they are inclusive of learners across diverse classrooms. So, while our overall results are promising, it was (and continues to be) important for us to further unpack our findings for students in different demographics and learning conditions. One example of this was our comparison in the performance of a subset of participating treatment and control schools for which the majority of students were from underserved communities (i.e., 100% low SES, 94% underserved racial minorities). We found that, like our overall population, treatment students again significantly outperformed control students at each of Grades 3–5 on both their knowledge of algebraic concepts and practices and their use of algebraic strategies to solve tasks (Blanton et al., 2019). To visualize the significance of this finding, Figure 2 compares the performance of students in treatment and control schools for this subset of schools along with the performance of the overall control group. In addition, it also shows that, while treatment students from schools with underserved communities did underperform the overall control group prior to the intervention, by the end of Grade 3 they outperformed even the *overall* control group (although not significantly) and maintained this advantage throughout the end of the intervention in Grade 5.

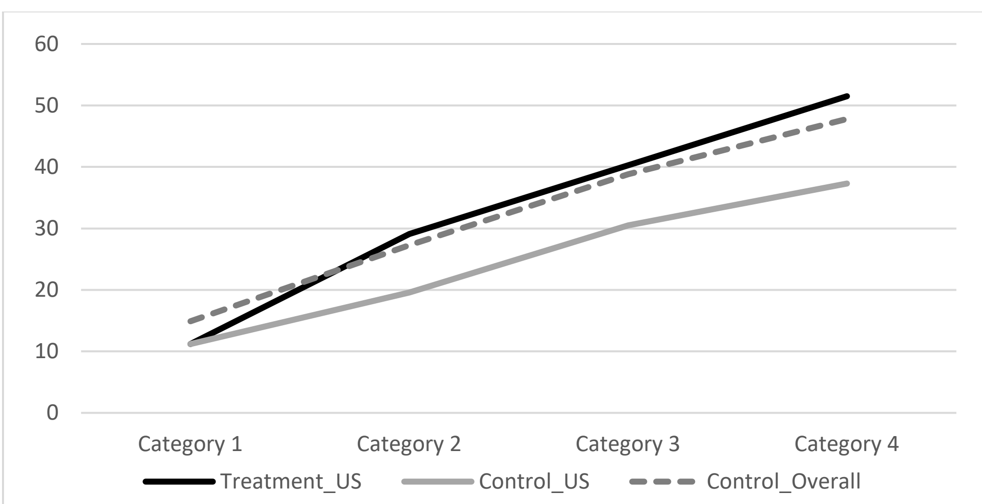

Figure 2. Comparison of performance for underserved students (US) in treatment and control schools with control students overall.

These promising results support findings from other studies that show providing students from historically underserved communities with more challenging learning environments in elementary grades can increase mathematics proficiency (Lee, 2011; Zilanawala et al., 2018). While it's difficult to tease out whether the role of a learning progressions approach was sufficient for these results, we can say that *when* a learning progressions approach was used, we have found a significant improvement in *all* learners' ability to think algebraically.

Building on this, we are continuing to explore how to develop a more inclusive early algebra intervention in Grades K–2. We expanded our recent design work around instructional sequences and assessments in Grades K–2 to include a more intentional focus on students with learning differences in order to understand how these students make sense of the algebraic concepts and practices in our intervention and how its features (e.g., concrete and visual tools) support learning. We have found that tools such as balance scales helped students analyze equations and reconsider unfamiliar equation forms (Stephens, Sung, Strachota et al., 2022), even prior to developing strong computational skills. Further, we have documented how these tools mediated the abilities of diverse learners to generalize and represent what they notice about structure and relationships involving parity concepts (Strachota et al., 2021)

While this work is ongoing, our Grades K–2 instructional sequences already show promise overall in developing children's algebraic thinking. We recently conducted a small, one-year cross-sectional study (n = 80) in each of Grades K–2 that examined the potential of each grade-level sequence when taught by classroom teachers. After only a one-year intervention in each grade, we found a marginally significant interaction between treatment condition and performance that showed gains favoring treatment students in their understanding of early algebra concepts such as the structure of evens and odds, mathematical equivalence and equations, properties of arithmetic, the representation of varying unknown quantities, and functional thinking [$F(1, 58) = 3.794$, p=.056] (Stephens, Sung, Blanton et al., 2022). Further, we found no significant three-way interaction among treatment condition, performance on the assessment, and grade level, suggesting that the impact of the intervention was similar across grade levels. With these findings for Grades K–2 and our more fully developed findings about the effectiveness of our Grades 3–5 intervention, we are optimistic about the innovation's potential to positively impact all students' experiences in algebra.

Conclusion

A learning progressions approach has been central in our efforts to develop an empirically grounded model for teaching and learning algebra that can increase all students' opportunities for success in algebra. It has provided both a framework for developing an innovation to measure early algebra's impact and a mechanism to examine fine-grained details in how children's algebraic thinking develops. It has also helped us think about how to reframe learning around an asset-based perspective in which instruction can build on the rich ways students think about algebraic concepts and practices, rather than from a deficit model built around students' misconceptions. Designing innovations from a perspective of what students *can* do (or, as Jim Kaput used to say, the "happy stories") can minimize the need to design for the purpose of "undoing" misconceptions in students' thinking that arise when elementary grades instruction does not attend to algebraic concepts and practices. Early algebra innovations based on learning progressions can open new curricular pathways for teachers and create effective avenues of learning that can democratize all students' access to algebra. We are hopeful that such models can continue to challenge the national discourse on teaching and learning algebra around the kind of algebra students can learn, when they can learn it, and how all students can be successful.

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