DECENTERING TO BUILD ASSET-BASED LEARNING TRAJECTORIES

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The development and use of learning trajectories is a body of research that has made enormous contributions to the field of mathematics education, offering insight into the teaching and learning of topics at all levels. Simultaneously, the work of building learning trajectories can benefit from explicitly adopting an anti-deficit stance, incorporating ways to center student voices from an asset-based perspective. In this paper I propose two related constructs to support this work: decentering and second-order models. In decentering, researchers work to set aside their own knowledge to understand students' reasoning as viable. This can support models of student mathematics that position student thinking as rational, powerful, and productive. I provide one example of the work of decentering and discuss ways to build learning trajectories that emphasize students' strengths and competencies.

Learning trajectories research has played a prominent role in the field of mathematics education, and it continues to exert influence on the teaching and learning of mathematics. In a recent plenary address to PME-NA, Steffe (2017) remarked that the construction of learning trajectories is "one of the most daunting but urgent problems facing mathematics education today" (p. 39). The influence of this sphere of research is evident in funding priorities at the NSF and the IES, in special journal issues (Duncan & Hmelo-Silver, 2009), in topics conferences (e.g., the learning trajectories panel held at the VARGA 100 Conference in 2019), and in special reports (Daro et al., 2011; Taguma & Barrera, 2019). For instance, the National Research Committee (NRC) issued a special report in 2009 identifying a set of goals for young children based on learning trajectories, which ultimately led to the use of learning trajectories as a foundation for the Common Core standards in mathematics (Clements et al., 2019). We also see the prominence of learning trajectories research for PME-NA as reflected in plenary paper topics (e.g., Battista, 2010; Confrey, 2012; Sarama, 2018; Steffe, 2017).

Researchers have defined and theorized learning trajectories in a variety of ways. Simon (1995) initially coined the term "hypothetical learning trajectory" to describe "the learning goal, the learning activities, and the thinking and learning in which students might engage" (p. 133). Clements and Sarama (2012) described a learning trajectory as a depiction of students' thinking and learning in a specific mathematics domain and a "related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children though a developmental progression of levels of thinking" (p. 83), and Confrey and Maloney (2010) described a learning trajectory as a progression of cognition that represents ordered, expected tendencies developed through empirical research aimed at identifying the likely steps students follow. There is variation in the degree to which researchers characterize learning trajectories as being (a) connected to particular task sequences, (b) influenced by specific teaching actions or other contextual factors, and (c) depictions of strategies, skills, or performances versus concepts and operations; for a more expanded discussion of these differences, see Battista, 2010, or Ellis et al., 2014. For my work, I have found Steffe's (2012) characterization to be particularly useful. He described a learning trajectory as a model of students' initial concepts and operations, an account of the observable changes in those concepts and operations as a result of students' interactive mathematical activity in the situations of learning, and an account of the mathematical interactions that were involved in the changes. I consider task sequences to be a part of mathematical interactions, but the emphasis is on the interactions themselves, including particular teaching moves, students' activity and conversation with one another, and students' interactions with tools, artifacts, and representations.

As a body of research, learning trajectories have made enormous contributions to the field. They have offered insight into major milestones of students' conceptual development for a variety of topics, including measurement (Battista, 2010; Clements & Sarama, 2009; Sarama et al. 2011), composition of geometric figures (Clements et al., 2012), fractions (Maloney & Confrey, 2010; Steffe, 2012b; Steffe & Olive, 2010; Wright, 2014), early algebra (Blanton et al., 2015; Hackenberg & Lee, 2015), geometry (Fitri & Prahmana, 2020), function (Ellis et al., 2016; Fonger et al., 2020), and probability (Rahmi et al., 2020; Wijaya & Doorman, 2021), among others. Learning trajectories research informs not only standards development, but also curriculum, pedagogical decision making, teacher noticing, and professional development (Clements, 2007; Confrey et al., 2014; Hackenberg & Sevinc, 2022; Liss, 2019; Meyers et al., 2015; Suh et al., 2021; Steffe, 2004). However, this body of research has also weathered critiques. These critiques include concerns about an overfocus on tasks, cautions about the need to better attend to variation in students' progression, scrutiny of the basis for the construction of learning trajectories, and calls to more explicitly address equity and inclusion.

An overfocus on tasks can occur when learning trajectories offer only task sequences paired with learning goals, without attending to the teaching actions and other contextual factors that are important for supporting students' development. Relatedly, learning trajectories can be construed as generalizable or transportable from one situation or context to the next, as if students, teachers, classrooms, and cultures were interchangeable. It is important to recognize that trajectories developed in one context may not always appropriately depict students' learning in a markedly different context. Additionally, not all students will progress in the same way throughout any given trajectory. Learning is more individualized, context-dependent, and idiosyncratic than what could ever be depicted in a neat, ladder-like sequence. Certainly, many researchers who construct learning trajectories are aware of these constraints. For instance, Clements and Sarama (2012) wisely reminded the reader that their task sequences are not necessarily the only or even the best path for learning and teaching, but are instead merely hypothesized to be "one fecund route" (p. 84). Nevertheless, learning trajectories have, at times, been interpreted in overly broad or simplified ways.

A more central issue that I would like to tackle in this paper is the models that constitute the basis of learning trajectories. In particular, it is worth considering the affordances and constraints of these models for developing and using learning trajectories to highlight students' competencies. In order to do so, I now turn to a consideration of first-order and second-order models, advocating for the use of second-order models to advance an asset-based perspective.

The Potential Pitfalls of Building Learning Trajectories from First-Order Models

Learning trajectories that are built on the foundation of the researcher's understanding of the discipline are based on what we call first-order models (Steffe & Olive, 2010). First-order models, or first-order knowledge, are the models that one constructs "to order, comprehend, and control his or her own experience" (ibid, p. xvi). There is robust evidence of reliance on researchers' first-order knowledge of mathematics in learning trajectories research. For instance, Clements and Sarama (2012) described a hypothetical learning trajectory as one involving conjectures about a possible learning route that aims at significant mathematical ideas, and a specific means to support and organize learning along this route. Those mathematical ideas are the researcher's ideas: "The trajectory is conceived of through a thought experiment in which the historical development of mathematics is used as a heuristic" (p. 82). To offer a few other examples, Confrey and colleagues (2014) used the term learning trajectory to refer to "clusters and sequences of standards and their related descriptors" (p. 720), Baroody et al. (2022) depicted the goals of a learning trajectory to be based on "the structure of mathematics, societal needs, and research on children's thinking about and learning of mathematics" (p. 195), and Andrews-Larson et al. (2017) described their hypothetical learning trajectory as content-specific documentation of common milestones and learning environments supporting students' progression across those milestones. Certainly, not all studies reporting on learning trajectory development conceive of learning trajectories in this manner. For instance, Confrey (2006) underscored the importance of the learner in guiding this work, emphasizing the centrality of students' voices and disciplinary perspectives, and others have published learning trajectories that reflect this aim (e.g., Fonger et al., 2020; Steffe, 2012; Steffe & Olive, 2010). Nevertheless, there remains a strong emphasis on learning trajectories that are based on researchers' own mathematics as starting points.

Building learning trajectories from first-order knowledge can offer important affordances. Such trajectories reflect the researcher's nuanced, in-depth understanding of the relevant content and key learning goals, as well as research-based knowledge of how to support student learning. At the same time, trajectories developed from first-order knowledge may also position students in terms of how they measure up against researchers' knowledge of the discipline. Moreover, this framing runs the risk of depicting students in terms of falling short. Adiredja (2019) characterized this stance as "epistemological violence", particularly towards minoritized students and students from marginalized communities, when the research we conduct positions students' knowledge as inferior or problematic. Furthermore, such a stance centers the perspective of the expert rather than that of the student. Certainly, many thoughtful scholars are careful to consider these issues in both their construction and use of learning trajectories, emphasizing the potential of learning trajectories to be asset-based models (e.g., Clements & Sarama, 2012; Hunt et al., 2020; Meyers et al., 2015; Suh et al., 2021). There is nothing inherent in a learning trajectory that requires it to be constructed as a deficit-based tool. Nevertheless, learning trajectories built from first-order models may fail to identify, sufficiently explore, or acknowledge the competence and brilliance of student thinking. In fact, as a field we run the risk of learning trajectories being used to bolster deficit stances towards minoritized and marginalized students, particularly when the trajectories over-privilege formal language, consistency in understanding, or straightforward and direct change in understanding (Adiredja, 2019). Adiredja pointed out that it is not that these mathematical goals are bad, but rather, an inflexible privileging of such goals can interact with deficit master-narratives to devalue the mathematical sensemaking of students, particularly students of color.

Learning trajectories built from first-order knowledge can also run the risk of encouraging teachers and other stakeholders to use them in a manner that places students on a continuum, with some positioned as more advanced and others positioned as deficient. Such an emphasis is reminiscent of the studies focused on achievement gaps, which allow researchers to "unconsciously normalize, the 'low achievement' of Black, Latina/Latino, First Nations, English language learners, and working-class students without acknowledging racism in society or the racialization of students in schools" (Gutiérrez, 2008, p. 359). Moreover, this treatment of learning trajectories may miss important nuances, not only about student thinking and reasoning, but also about the ways in which students may shift from one understanding to another based on complex, interrelated factors.

What, then, is the alternative? Researchers can instead psychologize students' mathematics by constructing second-order models, which are the hypothetical models observers construct of their students' knowledge in order to explain their observations of students' states and activities (Steffe & Olive, 2010). They are referenced to the researcher's first-order mathematics, as well as the researcher's conceptions and interpretations of the language and actions of students. These second-order models are sometimes referred to as the mathematics of students; students' first-order models (their own models of mathematics) are referred to as students' mathematics (Steffe, 2017).

Building Learning Trajectories from Second-Order Models

In building learning trajectories that are elaborations of second-order knowledge, we concern ourselves with identifying the mathematics of students and elaborating students' mathematical concepts and operations. I consider these learning trajectories to be *coproduced* by students and researchers (Steffe, 2012). Although initial hypothetical learning trajectories may be informed by a researcher's first-order mathematical knowledge, in combination with their knowledge of student thinking, these trajectories are nascent, ill-formed, and flexible. The learning trajectories that are consequently built out of teaching actions with students are accounts of students' initial concepts and operations, an account of the observable changes in those concepts and operations as a result of teaching and learning actions, and an account of the teaching and learning actions that led to the changes.

Building learning trajectories as second-order models encourages, or perhaps even requires, a different epistemology of mathematics, one that deviates from Western naïve realism traditions. Drawing on Piaget's epistemological beliefs, von Glasersfeld (1982) wrote that "The cognitive organism is first and foremost an organizer who interprets experience and, by interpretation, shapes it into a structured world" (p. 612). This one sentence conveys a radical departure, as von Glasersfeld put it, from traditional ideas of not only knowledge, but of reality itself. Knowledge is not a more or less accurate representation of reality. We construct our conceptions of reality through perception, not directly, and we cannot maintain a belief about knowledge being a reflection of reality by simply acknowledging that our reflection may not always be very accurate. This is not to say that von Glasersfeld denied reality; rather, he considered it to emerge only through bumping up against constraints. From this perspective, it does not make sense to judge knowledge based on its accuracy; in fact, this would be impossible, because it would require comparing one's knowledge to an independently existing reality and judging the closeness of the match. How can any human do this without direct access to that reality? Instead, knowledge is successful if it is viable, i.e., when it is not impeded by constraints.

Within this framing, there is no such thing as a mathematics that resides outside of human experience. The very concept of the second-order model is based on an epistemology that considers mathematics to be a product of the functioning of human intelligence. Students' mathematics *is* the mathematics. Certainly, we can compare our second-order model of the mathematics of a student to our first-order model of our own mathematics. In doing so, it is productive to understand that there are two mathematics, and both are legitimate. This requires a rejection of the Platonist knowledge traditions that frame mathematics as universal and objective. It also requires one to position students' ways of knowing and thinking as rational, rather than inferior when compared to standard strategies, procedures, and conventions (Louie et al., 2021). As L. Steffe explained, "students are 'never wrong' even though their thinking may not appear as viable with respect to certain situations or ways of thinking. Mistakes are always an observer's concept" (personal communication, October 5, 2022). Louie and colleagues argued that a failure to position students' reasoning as legitimate can discourage teachers from attending closely to unconventional ways of thinking and seeking to understand them, much less valuing or inviting them. I argue that this can also be true of researchers' treatment of students' ideas. In contrast, if we understand that students' mathematics *is* the mathematics, then we will be compelled to take students' reasoning and competencies as the starting point for building any learning trajectory.

One way to build second-order models is through the process of conceptual analysis, which is a process guided by the question, "What mental operations must be carried out to see the presented situation in the particular way one is seeing it?" (Steffe, 2017, p. 78). Thompson and Saldanha (2000) described conceptual analysis as articulating the conceptual operations that, "were people to have them, might result in them thinking the way they evidently do" (p. 315). Engaging in conceptual analysis draws on a researcher's ability to decenter, and can support the development of the epistemic student. Below I discuss each of these constructs in turn.

Decentering

Piaget (1955) introduced the idea of decentering to characterize the actions of an observer attempting to understand how an individual's perspective differs from their own (Teuscher et al., 2016). Piaget developed this idea to describe an aspect of a child's development: when a child learns to decenter, they begin to abandon egocentrism and develop the capacity to consider another's perspectives, thoughts, and feelings (Piaget & Inhelder, 1967). Steffe and Thompson (2000) then extended Piaget's construct to characterize a teacher's stance towards a student, particularly in terms of a teacher's ability to adjust their actions in order to understand a student's thinking.

Arcavi and Isoda (2007) described decentering as:

> the capacity to adopt the other's perspective, to 'wear her conceptual spectacles' (by keeping away as much as possible our own perspectives), to test in iterative cycles our understanding of what we hear, and possibly to pursue it and apply it for a while. Such a decentering involves a deep intellectual effort to be learned and exercised (p. 114).

Decentering is a stance that attends to both mathematical thinking and social interactions. It entails interacting with students reflectively, in a conscious attempt to set aside one's own knowledge to understand a student's reasoning as viable (Thompson, 2000). This reflective stance towards interactions with students is crucial for creating viable second-order models, and such efforts are hampered if a teacher – or a researcher – does not make concerted efforts to differentiate the mathematics of students from one's own mathematics. Steffe and Ulrich (2020), in distinguishing between responsive / intuitive interaction and analytic interaction, described the latter as a process of stepping out of analyzing students' thinking in ongoing interaction. All of the researcher's attention is absorbed in trying to think like the students, and produce and experience mathematical realities that are intersubjective with their own first-order models.

If researchers do not decenter, students' thinking and reasoning may not be considered worthwhile models of the environment in their own right, and instead may be positioned only in relation to standard models (i.e., the researcher's models) of mathematical knowledge. The construction of learning trajectories that are not a consequence of decentering may then position students as falling short, with insufficient attempt to understand or model students' thinking as viable, powerful, and potentially productive, even in times when it deviates from canonical mathematics. In contrast, a decentering researcher "always assumes that a student has some viable system of meanings that contribute to her or his actions" (Teuscher et al., 2016, p. 439).

Teuscher and colleagues went on to point out that the ideas of correct versus incorrect become largely irrelevant beyond informing one's future actions. This is not to suggest that correctness is unimportant. Rather, when engaged in the hard work of decentering, correctness is not a notion that contributes utility to building second-order models. A student does not position their own knowledge as incorrect, and decentering means seeing the world with the student's mathematical eyes.

The Epistemic Student

Hackenberg (2014) defined an epistemic student as an organization of schemes of action and operation that undergo change over time. The epistemic student is a model, one that is composed of the ways of operating common to all students at the same level of development "whose cognitive structures derive from the most general mechanisms of co-ordination of actions" (Beth & Piaget, 1966, p. 308). I see the epistemic student as a useful model of characteristic mathematical activity that is developmental, generalized, and dynamic (Ellis, 2014). It is an abstraction (Piaget, 1970), meant to explain some ways of operating that we suspect may be common across students.

The epistemic student is a helpful construct because students who share initial concepts and operations often respond in somewhat common ways to thoughtful instructional interactions. This does not mean that every student will respond identically, but typically there are a manageable number of ways of reasoning that bubble up repeatedly across participants and contexts. The epistemic student can be a useful model for trying to walk the tightrope between overgeneralization and over specificity. I acknowledge that it is not appropriate or even accurate to claim that my second-order models and resulting learning trajectories, which are developed from small numbers of students in specific contexts, would extend to all students in all contexts. To do so would ignore the variation in students' experiences, backgrounds, and positionalities, as well as the variation in classrooms, schools, and cultures. Simultaneously, the work of building learning trajectories necessarily entails a belief in the value of creating scientific (rather than experiential) models with the potential of being useful across different students and contexts.

Learning Trajectories Built from Second-Order Models Emphasize Anti-Deficit Stances

The body of learning trajectories research has been critiqued for not adequately considering equity or addressing student diversity (e.g., Zahner & Wynn, 2021). Some may even be used in ways that can reinforce deficit perspectives. A deficit perspective is "a propensity to locate the source of academic problems in deficiencies within students, their families, their communities, or their membership in social categories (such as race and gender)" (Peck, 2021, p. 941). In contrast, an anti-deficit perspective begins with the assumption that students are capable of reasoning mathematically and that they bring productive resources for learning mathematics. It acknowledges that learning is time-consuming, and that "imperfect articulations of mathematical ideas and some inconsistencies in the student's current conception are a natural part of the process" (Adiredja, 2019, pp. 416-417). Furthermore, adopting an anti-deficit perspective means locating the source of students' academic challenges within the racist, sexist, and ableist institutional structures that restrict, or even actively oppose, access to high-quality educational opportunities. When considering student thinking, a researcher considers and identifies the assets and competencies that students possess, rather than what students lack.

A goal of learning trajectory construction must be to position student thinking as rational, powerful, and viable, and from that position, seek to understand why students reason the way they do. It is our job, as researchers, to construct second-order models that reflect a value that student thinking is sensible and intelligent. In constructing learning trajectories, we must begin with that stance, identify student concepts and operations so that we can meet students where they are, and then consider productive teaching interactions that can support students' shifts from one way of thinking to the next. In doing so, we must also understand and acknowledge that these shifts may be idiosyncratic, time-consuming, and messy, as is learning itself. By starting from a model of the mathematics of students, we can then construct models for how teachers might interact with students to bring forth productive changes in their concepts and operations.

The learning trajectories that my colleagues and I produce (e.g., Ellis et al., 2016; Fonger et al., 2020) emphasize students' strengths and competencies, even when student thinking differs from canonical mathematics. We see an important outcome of our learning trajectory work to be that of highlighting those strengths and competencies with stakeholders. The goal of our work is to understand why students reason the way they do, and to show how students can and do think in ways that are thoughtful, reasonable, and nuanced, even if, at first glance, one might only see an incorrect answer or a puzzling strategy. Like many others (e.g., Clements & Sarama, 2012), our learning trajectories provide multiple viable paths and do not claim to represent the only (or even the best) route to learning. Centering the mathematics of students is explicit in our theoretical framing and constitutes the starting point for creating and refining trajectories.

An Example of Building a Learning Trajectory from Second-Order Models

Our learning trajectories are depictions of concepts and mental operations, in concert with teaching interactions and in relation to task sequences, set in specific teaching and learning concepts. The concepts and mental operations *are* the mathematics in our trajectories. As an example, my colleagues and I constructed a learning trajectory of students' understanding of exponential growth from a covariation perspective (Ellis et al., 2016). That trajectory in its entirety is beyond the scope of this paper, but I will highlight here four of the operations we identified: (1) Explicit coordination of change in y-values for 1-unit change in x-values, (2) Coordination of change in y-values for multiple-unit changes in x-values: repeated multiplication imagery, (3) Coordination of change in y-values for multiple-unit changes in x-values: exponentiation imagery, and (4) Coordination of change in y-values for any unit change in x-values, for any Δx. Mathematically, from our perspective, these are all the same operation. For an exponential function $y = ab^x$, it is possible to coordinate the change of any two y-values with any two corresponding x-values according to the relation $\frac{y_2}{y_1} = b^{x_2 - x_1}$, and the value of Δx is immaterial. Conceptually, however, these are not the same operation. In our work with students, my colleagues and I found that coordinating changes in x-values and corresponding y-values for unit changes in x is different from coordinating for large changes in x. Additionally, one can engage in coordination for large changes in x either by appealing to repeated multiplication imagery, or by appealing to a different set of stretching or scaling images. Furthermore, managing this type of coordination for cases when the change in x is less than 1 draws on a different set of concepts, and students can engage in operations (1) – (3) long before they can do operation (4).

What students can do and how they can reason is always rational from their perspective. Not yet being able to engage in operation (4) is something that a researcher would describe as a constraint for the student. But, from the student's perspective, there does not exist a more "advanced" way to coordinate exponential growth. Students are always reasoning with their available conceptual operations. For teachers, then, it is advantageous to understand how students may be operating, so that they do not impose ways of thinking on the students that run counter to the students' reasoning. Curricular treatments of exponential growth, however, to the extent that they might address a coordination approach at all, do not distinguish these forms of reasoning, because they can all be handled with the same formula. In contrast, my teaching interactions with students revealed that it is sensible for students to draw on different imagery when constructing these operations, and that transitioning from one form of reasoning to another can be effortful and may require specific instructional support. In short, such a transition is a significant intellectual achievement. Without this knowledge, teachers and textbooks will not distinguish them, and students may consequently experience challenges in making sense of expressions such as $2^{(1/7)}$; after all, when the meaning of exponents presented to students in school is only that of repeated multiplication what does it mean to engage in such multiplication 1/7 times? Now that we are aware that these operations are mathematically different for students, we can help improve the teaching and learning of exponential growth ideas.

As an example of the decentering work that supported our understanding of the mathematics of students, consider a task in which students are provided with a table of height values at certain times for a special plant called a Jactus, which grows exponentially (Figure 1).

This is a table for a Jactus that doubles every week. The entries are approximate. How much taller will the plant grow in a quarter of a week?

Week	Height
0	1"
0.25	???
0.5	1.414214"
0.75	1.681793"
1	2"
1.25	2.378414"
1.5	2.828427"

Figure 1: Table of Week and Height Values for a Doubling Jactus

If I were to solve this problem, I would take the ratio of any two consecutive height values in the table that were a quarter of a week apart. That ratio is approximately 1.189, and so I can divide the height at week 0.5 by 1.189 to find the missing height value. The question is written in an unusual way, because it asks students about how the plant grows in a quarter of a week, but the table has an empty spot for the height value at a specific time, 0.25 weeks, which is actually a slightly different question. This was a “serendipitous mistake” (Tasova et al., 2021), because it enabled us to identify a form of reasoning about which we had been unaware prior to students encountering the task. Our initial intention was to support the idea that the ratio of height values for any quarter-week gap will always be the same.

When working with 8th-grade participants who had never before had school instruction on exponential growth, we initially expected that they would use a strategy like the one I described, because they had already used that strategy with prior tables. For instance, when encountering tables with uniform gaps of 1 week, 2 weeks, or 5 weeks, our students had divided height values to determine the plant’s growth for the corresponding amount of time. But in this case, they did not leverage this strategy. For instance, consider the work of one of our participants, Uditi (a self-chosen pseudonym). In describing herself, Uditi discussed her experiences as an immigrant to the midwestern United States from India. She shared that she enjoyed mathematics and science, which was why she had volunteered for our research study, and she preferred expressing her ideas in small groups rather than with the whole class. When encountering the task in Figure 1, Uditi wrote the expression "$1 \times \_\_\_\_^{0.5}$". She then proceeded to use a cumbersome guess and check strategy to determine the missing value that would go in the blank to yield the plant's height at 0.5 weeks, which she knew had to be approximately 1.414. By doing this, Uditi determined that the growth factor was 2, and then laughed ruefully as she saw that the task description had already told her that the Jactus doubled each week. She then wrote the equation, "Height = $1 \bullet 2^{\text{week}}$", and then substituted 0.25 for the exponent to determine the height at 0.25 weeks.

Uditi's strategy was correct, and it was also creative. It revealed an understanding of many important ideas, including the idea that she could write a correspondence relation of the form $y = 1b^x$ because the initial height at Week 0 was 1 inch. Her strategy, however, also surprised me and my colleagues, because it was different from what she had done before, and it was also more cumbersome and difficult than just dividing. Moreover, Uditi was not the only student who approached the problem in this surprising way. Other students across two different teaching experiments did as well, which suggested to us that there was an important conceptual issue with that task that we had not anticipated. In combination with other students' responses to similar tasks, we began to realize that the value of Δx was critical. If one week is the period of time for the plant to double its growth, then it became clear that asking students to determine what happened *within* a week was a conceptually different task than asking students what happened across a span of multiple weeks – even though, from our perspective as researchers, the two tasks were mathematically identical.

Part of the job of creating second-order models is to engage with students reflectively, attempting to decenter in order to understand why their behavior and reasoning is sensible. Uditi and other students could already write expressions such as $y = x^{0.25}$, therefore presumably using decimal and fractional exponents to determine a fixed height value. What puzzled us was that they could also divide two height values to determine an amount of growth for a given time span. Why, then, did Uditi not do so with this task? Our goal was to now try to understand Uditi's reasoning that drove the unanticipated strategy. In doing so, we hypothesized that it was because Uditi could attribute two meanings to an equation such as height = 2^3, but only one meaning to an equation such as height = $2^{0.25}$. The expression 2^3 in the first equation meant two things to Uditi: It could be a static height value, such as the plant's height at 3 weeks, or it could be a measure of growth, i.e., how many times larger the plant grows in height for a time span of 3 weeks. But the expression $2^{0.25}$, we hypothesized, could be the plant's static height value at 0.25 weeks, but not how many times larger the plant would grow in a time span of 0.25 weeks. We suspected two reasons for this, both related to students' meanings for multiplication. The first is that determining growth across multiple weeks entails generalizing the operation of multiplication to an exponential context. This is fairly easy to do, as many students hold an expectation that multiplication makes bigger (Greer, 1987). They can extend their meaning of doubling by mentally repeating the operation multiple times across multiple weeks. In contrast, 0.25 weeks is less than a week, and there is no easy way to extend the operation of doubling to a fraction of a week. Furthermore, Uditi and the other students we worked with had all received school instruction that described exponential growth as repeated multiplication. The image of repeated multiplication does not easily lend itself to decimal exponents, as it is difficult to imagine such an operation.

Our efforts to construct a second-order model of Uditi's mathematics via her activity with this and related tasks led us to realize that constructing exponents less than 1 as a representation of growth is its own separate mathematical concept. This is not something I understood prior to working with Uditi and other students. We learned that in order to make sense of non-natural exponents as representations of growth, it is useful to invite students to shift to images that do not entail repeated multiplication. For our participants, this meant creating an image of change in the plant's height between weeks that represented an action of stretching, or scaling. For instance, Pei's drawing (Figure 2) shows a Jactus stretching as it doubled from Week 1 to Week 2 to Week 3, on the right, and then he was able to reverse his doubling operation to imagine halving the Jactus's height to see how tall it would be at Week ½ on the left. Once Uditi developed a similar scaling image, she was then able to answer the following question: "Say a plant grows 3 times as tall every week. How many times taller will it grow in 1 day?". Uditi wrote "$3^{.14}$" and explained, "There are seven day(s) in a week. So, I divided one week into seven parts, which represent one day." She could now conceive of an expression such as $3^{.14}$ to represent a measure of growth, not just a static height value.

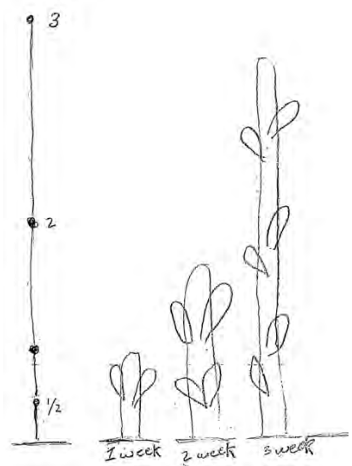

Figure 2: Pei's Drawing of a Doubling Jactus

Before my teaching interactions with Uditi, I was unaware that an expression such as $y = ab^x$ could represent two different ideas, a static value or an expression of growth. Certainly, this is not a big idea, and its truth is obvious to me now. Nevertheless, the concept was not originally part of my own first-order knowledge, nor was I aware that the mental imagery needed to undergird the second idea would need to be different from the first in order to accommodate non-natural exponents. Thus, my participants' mathematics served as a source of novel mathematics for me as a researcher, as it could also do for teachers who make use of the learning trajectory.

This approach toward the creation of learning trajectories shifts mathematical authority to the students. Uditi's mathematics served as a source of new mathematics for me. As a researcher, it was my job to understand her mathematics, why it made sense conceptually, and then determine ways to support her to create the meanings and images that would be productive for fostering an understanding of nonnatural exponents. This work centers student thinking as the core of our activity. Moreover, a productive stance to aid in my own decentering is to ask the question why. Why did Uditi's activity make sense? Why did she use the strategy that she did? Researchers must ask these questions with the assumption that there is always a sensible reason driving the student's activity. We simply need to be careful enough in our own research to find it. Moreover, in doing that work, we as researchers can grow in our own first-order knowledge: knowledge of the mathematics itself, of student thinking about mathematics, of the concepts and operations needed to make sense of particular ideas, and of the kinds of tasks and teaching moves that can support the development of those ideas.

I would like to close this extended example by pointing out that this model of a learning trajectory, with the four operations I shared, differs from learning trajectories that are (a) a proposal of the kinds of reasoning we should expect from students based on content analysis, (b) a specification of target performances, (c) a set of strategies, (d) a network of constructs that one might encounter through curriculum and/or instruction, or (e) a set of tasks that could be provided as stand-alone problems. An elaboration of a learning trajectory built from second-order models will identify student concepts and operations in relation to both tasks and teaching actions. I do not mean to denigrate or minimize the incredibly valuable contributions that prior learning trajectories have made, but rather, to articulate and clarify my vision of what a learning trajectory could be when it is built from second-order models.

Learning Cannot be Separated from Activity and Context

Helping our students develop stretching and scaling images for exponential growth turned out to be a productive route for their learning. Furthermore, because we saw the type of reasoning Uditi demonstrated in other students, our construction of the epistemic student from Uditi and her peers supported a model in which one may need explicit support to shift from a repeated multiplication image to an alternate image. Sharing second-order models in this manner can also help mathematically experienced adults, such as curriculum authors and teachers, understand and appreciate a different mathematics from the one they already know. In this manner, learning trajectories research can play a role not only in helping the field better understand how to support student learning of particular mathematics topics in the curriculum, it can – and should – determine what mathematics should be in the curriculum to begin with.

Nevertheless, emphasizing stretching and scaling images may not necessarily be a universally productive route. The degree to which it proves to be fruitful for other students in other contexts is an open question. As I alluded to above, researchers have raised concerns about the need to attend more explicitly, and more theoretically, to the role that teaching interactions play in influencing student learning (Empson, 2011; Simon et al., 2010). This body of work challenges the assumption that features of learning are transportable, or that it is possible to study effective teaching and learning in a particular context and tease out some key aspects that can then be generalizable to other contexts. Mathematics learning occurs in interaction, not only via teaching actions, but also via tasks, tool use, student discourse, classroom norms, school and community settings, and in relation to students' identities, histories, and positionalities (Nasir et al., 2008). Because mathematics learning does not occur in isolation from these sociocultural contexts, it is wise to avoid overly strong claims about the transportability of any particular finding. Learning trajectories research is a worthwhile endeavor not just for its potential to broadly improve the teaching and learning of particular topics (Baroody et al., 2022), but also to develop a set of contextualized, specific case studies of the types of reasoning that can exist and can be supported in particular ways.

Build Learning Trajectories that are Engines of Equity

I have advocated for the usefulness of the epistemic student as a construct that can help researchers navigate a balance between over generalizability and over specificity. In what might seem like an odd turn, I am now going to argue against my own argument, or, at least, consider an alternate stance. That stance is this: If the epistemic student is an idealized abstraction, a model composed of common ways of operating, this leads me to question what type of student we imagine when evoking the epistemic student. Analyzing student reasoning without attending to sociocultural diversity runs the risk of reinforcing deficit narratives about minoritized students and students from marginalized communities (Zahner & Wynn, 2021). And yet, the construct of the epistemic student may encourage this form of analysis.

Studies of cognition and equity are frequently positioned as separate areas of research in mathematics education (Adiredja, 2019). For instance, Adiredja has pointed out that racial and gender inequities are seldom considered in cognition studies, and, furthermore, engaging in analysis that does not include these positionalities of the students we study does not make our research apolitical: “Rather, it has the impact of maintaining the status quo that is the dominant master-narrative about White male exclusive membership in mathematics and centering education around their needs and concerns” (ibid, p. 426). One way to begin to address this limitation can be to extend the notion of the epistemic student to understand the identities and positionalities of our research participants who contribute to the epistemic student model. We can invite studies highlighting the powerful reasoning of marginalized and minoritized students, including being deliberate about who we include as participants in research opportunities, being thoughtful about the ways in which we engage our participants in research, and being explicit about our participants’ positionalities.

We must build learning trajectories that are explicitly and theoretically organized from an asset-based perspective. Such trajectories can begin with efforts to understand our students’ cultural competencies, and by drawing on our students’ backgrounds and out-of-school knowledge and practices, rather than ignoring or even excluding them. Deliberately creating learning environments that leverage students’ cultural and linguistic strengths supports their mathematical reasoning (Abdulrahim & Orosco, 2020). This means, then, broadening our starting point for the construction of hypothetical learning trajectories. Rather than beginning only with our first-order knowledge of mathematics, combined with research and pilot studies about student learning, we can also incorporate (a) hypothesized second-order models, (b) research on students’ funds of knowledge relative to the topic at hand, and (c) information about our participants’ values, interests, and knowledge. In this manner, learning trajectories research can meaningfully center student voices, and can serve as a bridge connecting research on cognition and equity.

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