HOW THE TEACHER AND STUDENTS IMPACT THE UNFOLDING OF MATHEMATICAL IDEAS ACROSS A LESSON

Amanda Huffman
Purdue University
huffma33@purdue.edu

Leslie Dietiker
Boston University
dietiker@bu.edu

Andrew Richman
Boston University
asrich@bu.edu

By highlighting the curriculum modifications that lead to maintaining, or enhancing, the mathematical quality of an algebra lesson introducing the substitution method for solving systems of equations from an algebra textbook, we present an analysis of how a teacher and her students impact how the mathematical ideas unfold across the lesson and how they are experienced. Using a narrative-based analytical approach to write the stories of the written and enacted lessons, we found key similarities and differences in the lessons. In comparing the mathematical plots, we found evidence of how the teacher and students alter the unfolding story with the incorporation of more jamming than seen in the text and more questions developed based on the students' needs and their responses.

Keywords: Curriculum; Teaching Practice; Algebra and Algebraic Thinking

Research has demonstrated that teachers combine pedagogical practice, contextual factors, and the curriculum when teaching a lesson (e.g., Remillard & Heck, 2014). Teachers make pedagogical and contextual decisions (e.g., restricting calculators) based on the needs of their students, the context of learning, and their own prior experiences. While the text offers resources for the lesson and can certainly impact the teacher's instruction and students' learning (Remillard et al., 2014), what happens in the classroom often appears quite different than what the textbook authors intended since the actions of both teachers and students are influenced by a variety of prior experiences, goals, and perspectives. Ben-Peretz (1990) refers to the set of intended and unintended curricular uses of written curriculum materials as *curriculum potential* and describes the subset of these that are viewed favorably by its authors in the *curriculum envelope*. This theoretical framing implies there are a range of appropriate and beneficial interpretations and uses of all written curriculum materials.

However, there is little understanding of the ways in which lessons can vary within the curriculum envelope. Therefore, in this study, we illustrate how complex interactions between teachers and students in an enacted lesson shifts how the lesson unfolds while still drawing from the elements of the textbook lesson and maintaining its overall intentions. We present the case of one enacted algebra lesson implemented to a group of students from a set of written materials that were explicitly designed to increase the mathematical quality of courses by providing students access to rich, conceptually connected mathematical ideas. In this case, the enacted lesson maintained (or even enhanced) the mathematical quality of the lesson as written and thus is within the curriculum envelope. By highlighting the curriculum modifications that lead to maintaining, or enhancing, the mathematical quality of the written materials (e.g., its aesthetic opportunities, rigor), we begin to address the research question: *In an enacted lesson that is within the curriculum envelope, in what ways can the teacher and students impact how the mathematical ideas unfold across the lesson and how they are experienced?* We present a new methodology to comparing enacted and written curriculum that focuses not only on content, but also on how the content unfolds across the lesson. This comparison allows us to see how enacted lessons can maintain or even enhance the mathematical quality of the lesson for students and could reveal potential strategies for taking advantage of the design of the curriculum.

Theoretical Framework

To compare mathematics lessons how mathematical ideas unfold, we interpret the lessons as a form of narrative. Similar to how literary stories can captivate a reader by the withholding and revelation of information (Nodelman and Reimer, 2003), mathematics lessons can similarly structure how information emerges, enabling curiosity and encouraging a reader to build interest in learning more (Dietiker, 2015). A *mathematical story* is a metaphorical interpretation of how the mathematical ideas unfold across a lesson, connecting the beginning with the end. Note, however, that mathematical stories are not limited to contextualized word problems, but rather describe the twists and turns of the mathematical revelations throughout a lesson. As with literary stories, mathematical stories have characters (i.e., mathematical objects, such as algebraic expressions) that are manipulated and changed through *action* (i.e., mathematical transformations, such as substituting an expression to make an equivalent equation). In addition, these mathematical characters and actions play out in one or more mathematical *settings* (i.e., the representational space, such as symbols on paper or manipulatives) (Dietiker, 2015).

In a literary story, the text communicates via a narrator, and its interpretation focuses on how the text communicates to a reader in ways that are both logical (i.e., what makes sense) and aesthetic (i.e., what it makes a reader feel) (Bal, 2009). Mathematical stories in written curriculum materials operate similarly; the text narrates the story, and it can be read for how it communicates to potential readers—who in this case are teachers and students. Reading a written mathematical story for how it communicates requires identifying how the mathematical ideas, which are assumed to be unknown from the start, emerge and change as the story proceeds. For example, if a new definition or a theorem is presented that has not yet entered the story, it is interpreted as a new revelation even if it is already familiar to the researcher.

Enacted mathematical stories (i.e., those that unfold within classroom) also have narrators and can be interpreted by how the mathematical ideas emerge and change across a lesson. However, in this case, the narration occurs both by the written curriculum (e.g., when a task is read from a worksheet) or by a human (e.g., when a teacher introduces a topic or when a student describes how they solved a problem). Thus, enacted mathematical stories are more like guerilla theater, where the teachers and students are both actors and audience members contributing to and experiencing the story (Dietiker et al., in progress). The teachers in enacted mathematical stories can also have a role in *storytelling*, as they may have specific intentions for how the mathematical story will progress, which impacts their decisions in the moment. For example, when a student raises a new mathematical idea that alters what is known at that point in a story, the teacher can choose what to do with that information (e.g., suppress, elaborate) based on how they want the mathematical story to progress.

The way a mathematical story unfolds has a potential aesthetic impact on its readers, what we refer to as its *mathematical plot*. A mathematical plot "describes the aesthetic response of a reader as he or she experiences a mathematical story, perceives its structure, and anticipates what is ahead" (Dietiker, 2015, p. 298). The way in which the mathematical ideas play out over time can spur curiosity in a reader, leading them to pose new questions (e.g., asking *how can that be?*) and seek information (e.g., recognizing that two equations are equivalent) that answer them. As new facts are revealed, a reader has an opportunity to make progress on what is known about the questions they have adopted at that point in the story. According to literary theory, stories that offer numerous sustained questions simultaneously are more compelling (Barthes, 1974). The transition of what is known about a question, from when it opens to how it is answered is referred to as a *story arc*. A story arc can be short-lived when a question is answered quickly after being asked, or can be longer when a question remains unanswered and is still in consideration throughout portions of the lesson. Since some questions may never be resolved, some story arcs may remain open at the end of the story. Others may be abandoned when it is clear to a reader that the story has moved on. With multiple story arcs open, a student may sense a growing mystery, while a decrease may cause a reader to sense relief (Dietiker, 2015).

Methods

This study is part of a larger exploratory study that initially examined the different ways experienced (i.e., more than five years of experience) Algebra I teachers enacted the same textbook lessons. Using the mathematical story framework, this study compares one teacher's enacted lesson juxtaposed with the written lesson through a narrative-based analytical approach. In this section, we describe the context of the written and enacted lessons and describe how we interpreted and compared their mathematical plots. Note that we share our interpretations and they may vary from other groups completing this same process. Therefore, a key to this analysis is that both the written and enacted stories were analyzed using a consistent process. Our interpretation of these mathematical stories represents a consensus of a diverse mixture of mathematics education researchers, practicing teachers, mathematicians, and graduate students, and thus were informed by a mixture of perspectives regarding curriculum, mathematics, and teaching. Our interpretations represent a *potential* interpretation of a novice learner, as we took into consideration the story's previous revelations while setting aside our prior knowledge.

Sources of Data

The written lesson in this study, Lesson 4.2.1 from the *CPM Core Connections Algebra* textbook (Dietiker et al., 2006), introduces the substitution method for solving a system of linear equations. All elements of the lesson as provided in the teacher's edition were analyzed, which includes the student-facing materials (i.e., tasks, explanations) and suggestions to the teacher.

The enacted lesson, which was based on lesson 4.2.1 from the *CPM Core Connections Algebra* textbook (Dietiker et al., 2006), was observed in Ms. Turner's (all names are pseudonyms) Southern United States high school Algebra I course for special education students during the Fall of 2015. Ms. Turner's classroom consisted of six students, all of whom were Black, seated in small groups. The lesson was videotaped using three cameras: one facing the front board and two facing the students. In addition, audio recorders were placed on student desks about the room. The main video recording facing the front board, along with the audio recorders, was used to build the lesson transcript.

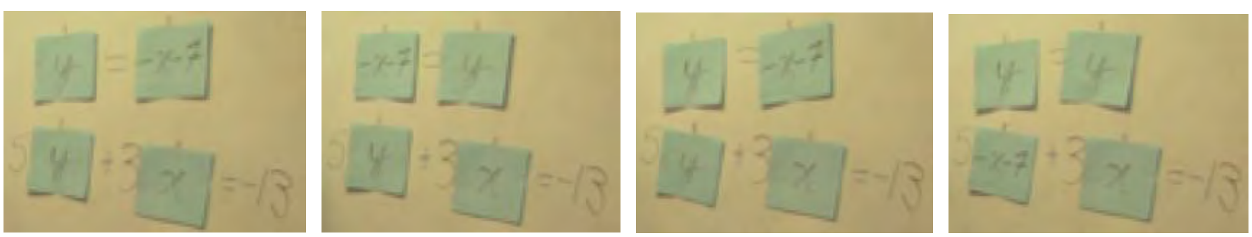
Ms. Turner, who is White, was in her 19th year of teaching and was selected to be observed for this study based on her expertise. For six years prior to this study, Ms. Turner led professional development for mathematics teachers in addition to coaching fellow teachers in her district for the previous seven years. Prior to the observation of this lesson, Ms. Turner was interviewed for information on her school demographics, her goals for the lesson, and her anticipated challenges during the lesson. According to Ms. Turner, from the pre-interview, the learning goal for this lesson was for students to rewrite the two equations with two variables as a single equation in a single variable by using a suitable substitution expression. After the observation, she was interviewed again about her reflections and was asked to discuss her curricular decisions that were made throughout the lesson.

The Written Mathematical Story

The lesson begins with the students solving the system $y = -x - 7$ and $5y + 3x = -13$ using a method that was introduced in a prior lesson, namely, the *equal values method* (i.e., a process that involves solving each of two equations for the same variable and then setting the two resulting expressions equal to each other). After providing the students a few minutes to solve the system of equations, the teacher is encouraged to stop the students and launch a discussion about whether it would be useful to have an easier method. A task then introduces the substitution method for the same system of equations, and the teacher is encouraged to use strips of paper with parts of the expressions to build the equations to aid in a discussion of the logic of the new method. Four more systems of equations are then given for students to solve using this new method. One of these results in a new situation (i.e., no solution). The lesson concludes with the task prompting students to figure out a way to know if the solution of a fictional student is correct or not.

The Enacted Mathematical Story

Ms. Turner started the lesson by reviewing the equal values method. She then prompted students to solve a system of equations (i.e., $y = -x - 7$ and $5y + 3x = -13$) with this method. After the students struggled to solve the problem for a few minutes, Ms. Turner stopped the students and offered a new method to the students with a reasoning of why this new method was beneficial (i.e., you can avoid ugly numbers). Using the paper switching method described in the teacher's guide, Ms. Turner introduced the students to the substitution method. Figure 1 shows the paper switching process with (a) the original system of equations formed by cards and symbols, (b) the system after switching the cards in the top equation, (c) the same systems with the *y* cards switched in the top and bottom equations, and (d) the same system after switching the bottom *y* card with the $-x - 7$ card in the top. Before Ms. Turner even started through the process one student indicated insight, but she quickly quieted the student. She continued to guide the students through the substitution method with the paper switching visual. After "showing her (Ms. Turner's) magic" (i.e., the substitution method), the students practiced solving two more systems of equations using their new method.



(a) (b) (c) (d)

Figure 1: Paper switching process.

Interpreting the Mathematical Story and Plot

In order to analyze and compare the mathematical plots of the written and enacted lessons, the researchers analyzed each portion of both stories for opportunities for new understanding. To interpret the stories, we analyzed the text of the story; the text of the written lesson included the suggested components of the lesson in the teacher's guide including all tasks and statements, while the text of the enacted lesson was the transcript from the classroom.

Analyzing the mathematical plots required three passes through each text. Each pass was first performed individually and then the group of researchers met to resolve differences. First, the researchers divided the text into agreed upon *acts*, where each act represented a portion of the story during which the mathematical story advanced and different acts were marked by changes in the mathematical characters, actions, or settings. Second, the researchers identified all the questions raised, either explicitly or implicitly, by the curriculum, teacher, or students in the text.

Third, we used Barthes' (1974) codes to describe the transition from question to answer and any suspense or surprise in between: *question formulation, promise of an answer, snare* (misleading information), *jamming* (unanswerable question), *suspension* (delayed answers), *partial answer* (progress), and *disclosure of the answer* (endorsing the answer).

With both the written and enacted mathematics lessons coded for their mathematical plots, the researchers analyzed how the mathematical plots of the written and enacted stories were similar and different.

Findings

The mathematical plots, which reveal how the stories unfolded in the lessons, are provided in Figure 2 (as written) and Figure 3 (as enacted). The acts (in columns) are presented in order from left to right, while the mathematical questions are listed in the order they emerged in the left column. The shaded cells, representing the story arcs, contain letters for Barthes' (1974) plot codes, where a: Question by Teacher or Environment, b: Question by Student, c: Promise, d: Progress by Teacher or Environment, e: Progress by Student; f: Snare, g: Jamming, h: Suspension, and i: Disclosure.

	ACT	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	How do you solve systems using the substitution method?	a	c		c	de	de	ed	d	e		e	e	e	
2	Why do we need another method for solving systems of equations?	a	d	e	di										
3	What makes a system of equations too messy for the equal values method?		a	e	i										
4	What is the solution to the system of equations (y=-x-7 and 5y+3x=-17)?			aeg		ed	de		d	i					
5	How does the substitution method eliminate the fractions that arise when using the equal values method?				a	e	de		i						
6	Can you switch the y with the -x-7 in the first equation? Why or why not?					aei									
7	Can we switch the y in the second equation with the -x-7? Why or why not?						aei								
8	Could we switch the -x-7 with the x in the second equation? Why or why not?							aei							
9	How do I decide what to substitute for?							ade					e	e	
10	What is the solution to 4-33a (y=3x and 2y-5x=4)?										aei				
11	What is the solution to 4-33b (x-4=y and -5y+8x=29)?											aei			
12	What is the solution to 4-33c (2x+2y=18 and x=3-y)?												aeg		
13	What does it mean to get a false equation?												be		
14	What is the solution to 4-33d (c=-b-11 and 3c+6=6b)?													aei	
15	How do you know that Mei's solution is correct?														ae

Figure 2: Mathematical plot of the *Core Connections Algebra* textbook Lesson 4.2.1.

	ACT	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1	What is the substitution method?	ad				ceg	de	de	ed	ded	ed	de	de					
2	What makes a system too messy to solve with equal values?	a			e	d						d						
3	How do you solve a system of equations with the equal values method?	a	d	de		g												
4	How do you solve the system y=-x-7 and 5y+3x=-13 using the equal values method?			ade	ed	gd												
5	What is the solution to the system of equations y=-x-7 and 5y+3x=-13?			ade	ed	e			ed	ed	ed	dei						
6	Can I set these two equations equal to each other and start solving?			aei														
7	What do we need to do to one of these equations to solve with equal values method?			aei														
8	How do we isolate y in the equation 5y+3x=-13?				ae	gd												
9	What should I do with -13 and -3x, and why?				aei													
10	How do we use algebra tiles to help us solve equations?				af	gh				d	d							
11	What makes this new method easier?					aeg			d	d		i						
12	What does that student mean by "you put it in the place of y?"					bg	d		di									
13	What does the "=" symbol represent?					aed	de		e									
14	Is y=-x-7 equivalent to -x-7=y? Why?						aei											
15	Is the system equivalent if we switch the [y] terms? Why?							aei										
16	Can the y in the bottom equation be switched with the -x-7 in the top equation? Why?								aei									
17	Is the equation y=y true?								ae									
18	How do we solve this new one-variable equation?									aed	ei							
19	Why do we have to use the distributive property?									ae								
20	How do I multiply 5 by -x?									bei								
21	How do I solve -2x=22?										aei							
22	Once I have a value for one variable, how can I find the other variable?											aed			def			
23	What mistake is she predicting?											adei						
24	What is -x when x is negative?											bei						
25	What is the solution for the system x=2y+4 and 3x+2y=-28 when using substitution?												aed	ed	dei			
26	What is the solution to the system y=3x and 2y-5x=4 when using substitution (4-33a)?															adei		
27	How can you tell if the solution x=4 and y=2 is correct for the system without solving?																aegc	
28	What is the solution of 3(3x+5)=42?																	aed

Figure 3: The Mathematical Plot of the Enacted Lesson.

In comparing the mathematical plots, we found evidence of how the teacher and students alter an unfolding story within the curriculum envelope for the benefit of student experiences. Specifically, we discuss three shifts in the enacted mathematical story: (a) additional jamming by Ms. Turner's added questions and interactions with her students, (b) Ms. Turner's students' increased engagement through her verbal declaration of knowing what mistake her student was about to make and evidence of her understanding of her students and the rapport she has built with them, and (c) added questions asked by the teacher in response to her students' needs while addressing the algebraic processes of solving the systems of equations.

Added Jamming in the Enacted Story

In the enacted lesson, specifically in Act 5, Ms. Turner's actions of posing the problem to solve a system of equations (i.e., $y = -x - 7$ and $5y + 3x = -13$) using the equal values method created multiple instances of *jamming*, or an experience where you think you are going to get an answer and then the story threatens and drops hints that you may not get the answer. Both the written lesson Question 5 and the enacted lesson Question 4 (*What is the solution to the system of equations* $y = -x - 7$ and $5y + 3x = -13$*?*) created a similar instance of jamming as the posed problem became too messy to solve. However, Ms. Turner added additional questions around this problem which added more jamming to the enacted story:

3	How do you solve a system of equations with the equal values method?
8	How do we isolate *y* in the equation $5y + 3x = -13$?
10	How do we use algebra tiles to help us solve equations?
12	What does that student mean by "you put it in the place of the *y*"?

That is, these additional questions (i.e., Question 3, Question 8, Question 10) offered support for her students as they tried to solve the given system of equations using the equal values method. Then, when Ms. Turner stopped the class as the problem became too messy, abandoning the equal values method to solve the given system of equations, this simultaneously disrupted Question 3, Question 4, and Question 8. As Ms. Turner moved on to introduce the substitution method, Question 12 was introduced by a student, but this question was quickly jammed in order for Ms. Turner to walk through the entire process of the substitution method step-by-step with the paper switching method.

During the paper switching demonstration, beginning in Act 5, another element of jamming occurred as one student quickly picked up on what to do. Ms. Turner stopped the student from sharing his complete thought, to continue explaining the process and to ensure the other students followed the substitution method. Ms. Turner enabled this aesthetic moment in her classroom by deciding in the moment to quiet her students' *aha* moment, as the other students in the class would benefit from a guided introduction to substitution. After Ms. Turner explained a little more, she returned to the student who was eager to share and quickly saw how the substitution method worked when doing the paper switching visual provided by the curriculum. These added elements of jamming and Ms. Turner's deviations enhanced the lesson for her students while providing her students access to rich, conceptually connected mathematical ideas as the author(s) intended in the written lesson.

Predicting a Student's Error in the Enacted Story

In Act 11, Ms. Turner's response, "I guarantee you've made a mistake and I'll eat my words if you didn't," while the students are solving for *y* to finish solving the system of equations after as a class they found the value for *x* created question 23. Question 23, *What mistake is she predicting?* is not in the written lesson, but enhanced the lesson as Ms. Turner knew what her student was going to do. This student had mixed up the double negative in the problem, and this student action led to the formulation of Question 26, *What is -x when x is negative?* These two questions were added to the enacted story based on the teacher's predictions and the student's mistake. Again, these questions were created in the moment and based on the context of the lesson and Ms. Turner's students, something not seen in the written story.

Additional Questions to Solve the System in the Enacted Story

In the enacted story, Ms. Turner also added questions not found in the written story. These added questions supported the students as they worked to understand the substitution method and solve systems of equations, as the students needed reminders and guidance in solving the equation for one variable once they made the substitution (e.g., reviewing the distributive property). These additional questions demonstrate how Ms. Turner's pedagogical practice and contextual factors impacted the enacted lesson. The first occurrence of added questions is at the very beginning as Ms. Turner guided the students in recalling the equal values method from the previous lesson. Ms. Turner added the following questions:

6	Can I set these two equations equal to each other and start solving?
7	What do we need to do to one of these equations to solve with equal values?
8	How do we isolate y in the equation $5y + 3x = -13$?
9	What should I do with -13 and -3x, and why?
10	How do we use algebra tiles to help us solve equations?

These additional questions helped students to start solving the system of equations. These questions were not raised when the written substitution lesson was analyzed, since both the equal values method and how to solve an equation for one variable were presumed prior knowledge. Seeing Ms. Turner's introduction of these questions indicates the need for the students' prior knowledge to be recalled as a class. This also demonstrates how Ms. Turner knew her students and the context of the lesson and made adjustments to meet them where they were in supporting their development of the substitution method.

This is not the only time Ms. Turner added additional questions to the enacted lesson. The next occurrence was seen when guiding the students through the substitution method. Ms. Turner added additional questions (i.e., Questions 18 - 21) to assist the students in solving the system of equations using substitution. Ms. Turner even referred back to a visual representation, algebra tiles, and also discussed the distributive property while solving the equation for *x* after substituting in the expression for *y*. These added questions made the lesson fit the students and allowed the students to engage in the rich mathematical task.

18	How do we solve this new one variable equation?
19	Why do we have to use the distributive property?
20	How do I multiply 5 by -*x*?
21	How do I solve -2*x*=22?

Discussion

Our findings reveal how, in the enacted lesson, the teacher's questioning and the questions and responses of the students altered the story of the written lesson while still maintaining its overall intentions. The students' needs were known by the teacher and the students' responses and actions (i.e., "I already know what *y* is", forgetting a negative) enhanced the lesson. Ms. Turner's use of the introductory problem, the paper switching visual, and the practice problems showed her use of the curriculum when teaching the lesson, while the added jamming, prediction of a students' error, and added questioning demonstrated Ms. Turner's use of pedagogical practice and contextual factors when teaching the lesson. These experiences enhanced the lesson for these students and allowed these students to complete the mathematical task and learning goal envisioned by the authors. Ms. Turner's crafting and improvisation of the lesson in the moment supported the written lesson.

Through our narrative-analytical approach, we illustrated how complex interactions between a teacher and students in an enacted lesson shifted how the lesson unfolded while the lesson still drew from the elements of the textbook lesson. We demonstrated how one enacted lesson maintained, and even enhanced, the lesson for the students. Additional research is needed to highlight other enacted lessons also maintaining, and possibly even enhancing, the written lesson. Additionally, further research is needed to explore how to support teachers in recognizing the potential strategies of implementation that take advantage of the design of the curriculum materials so that more enacted lessons maintain or enhance the mathematical quality for students.

Acknowledgments

An author of this paper receives research funding from CPM Educational Program, which sells products involved in this research. This author also receives book royalties from this publisher for the textbooks used in the study. The terms of this arrangement have been reviewed and approved by Boston University in accordance with its conflict of interest policies.

References

Bal, M. (2009). Narratology: Introduction to the theory of narrative (C. Van Boheemen, Trans.; 3rd ed.). University of Toronto Press.

Barthes, R. (1974). *S/Z* (R. Miller, Trans.) Macmillan.

Ben-Peretz, M. (1990). *The teacher-curriculum encounter: Freeing teachers from the tyranny of texts.* SUNY Press.

Boston, M. D., & Smith, M. S. (2009). Transforming secondary mathematics teaching: Increasing the cognitive demands of instructional tasks used in teachers' classrooms. *Journal for Research in Mathematics Education, 119-156.* https://www.jstor.org/stable/40539329

Dietiker, L., Miller, E. R., Brakoniecki, A., & Richman, A. S. (in progress). "I'm Feelin' It!" Curricular contributions for aesthetic experiences in mathematics.

Dietiker, L. (2015). Mathematical story: A metaphor for mathematics curriculum. *Educational Studies in Mathematics, 90*(3), 285-302. https://doi.org/10.1007/s10649-015-9627-x

Dietiker, L., Kysh, J., Thomas Sallee, G., & Hoey, B. (2006). *Core connections: Algebra.* CPM Educational Program.

Nodelman, P., & Reimer, M. (2003). *The pleasures of children's literature* (3rd ed.). Allyn and Bacon.

Remillard, J. T., Harris, B., & Agodini, R. (2014). The influence of curriculum material design on opportunities for student learning. *ZDM Mathematics Education, 46,* 735-749. https://doi.org/10.1007/s11858-014-0585-z

Remillard, J. T., & Heck. (2014). Conceptualizing the curriculum enactment process in mathematics education. *ZDM Mathematics Education, 46,* 705-718. https://doi.org/10.1007/s11858-014-0600-4

Stein, M. K., Smith, M. S., Henningsen, M. A., & Silver, E. A. (2000). *Implementing standards-based mathematics instruction: A casebook for professional development.* Teachers College Press.