THE GEOMETRY OF SUNLIGHT: CONTINUOUS MULTIPLICATION WITH NATURALLY OCCURRING PARALLEL LINES

Justin K. Dimmel
University of Maine
justin.dimmel@maine.edu

Eric A. Pandiscio
University of Maine
ericp@maine.edu

Camden Bock
University of Maine
camden.bock@maine.edu

Emma Reedman
University of Maine
Emma.reedman@maine.edu

We report the design of an analog technology, what we refer to as a SunRule, that uses sunlight to model multiplication. Physical models that explore multiplication are fixtures in elementary mathematics classrooms. Our interest in physical models of multiplication was driven by an overarching design problem: How could a physical tool realize a continuous model of multiplication? That is, how could we represent continuous, variable quantities with physical things? We identify specific challenges the SunRule was designed to solve. We explain the mathematical underpinnings of the device and report a teaching experiment during which pre-service teachers explored the device in small, socially-distanced groups. We consider how explorations with the SunRule create opportunities for mathematically rich instructional activities that are also essentially connected to being outside.

Keywords: Technology, Measurement, Geometry and Spatial Reasoning, Design Experiments.

Introduction

Sunlight provides an abundant, renewable, accessible source of naturally occurring parallel lines. It is the rare example of a mathematical contextualization with which nearly all children are familiar. Despite its familiarity and universality, sunlight plays almost no part in K-12 mathematics classrooms. Furthermore, while sunlight is among the closest physical realizations to the Euclidean ideal for parallel lines, there is scant research about how K-12 students might use sunlight and the real-world parallel rays it provides to engage in mathematical activities. But in the shadow of the global pandemic, when so much of schooling has moved to screens, there is an urgent need for outdoor, socially-distance-able activities that have robust mathematical designs – i.e., designs where mathematical concepts are intrinsic to the activity. To respond to this need, we report the design of an analog technology, what we refer to as a *SunRule*, that uses sunlight to model multiplication. We explain the mathematical underpinnings of the device and report an initial teaching experiment where pre-service teachers explored the device in groups. We consider how investigating multiplication with the *SunRule* can challenge familiar notions of contextualized mathematics.

Background & Design Problem

The sun shadows phenomenon

Sunlight and what has been described as the *sun shadows phenomenon* was used as a tool by Garuti and Boero (1992) to investigate geometric proportionality as a physical phenomenon with 11 and 12 year olds. This study offered promise that embedding problem situations in a context in which directly experiencing the geometrical-physical aspect is paramount may move students from an additive model to a multiplicative one. Building on this early success, the Genoa Group

(Boero, Garuti, & Mariotti, 1996a; Douek, 1999) used the overarching approach of examining heights of objects and the lengths of the shadows they cast to pursue a wide range of research questions within the sun shadows field-of-experience (Boero, 1989). Topics included argumentation, conjecture, proof construction, and angle concepts, among others. They all shared the structure of teaching experiments that capitalized on shadows cast by the sun (either imagined or observed and recorded experimentally) to explore problem situations in different dynamical ways (Boero, Garuti & Mariotti, 1996b). Students produced, through open problem solving situations, meaningful conjectures from a space geometry point of view. Douek (1999) also demonstrated the link between context-related arguments, mathematical modelling, and conceptualization of geometric ideas. The present study extends this work to use geometric proportionality of shadows cast by the parallel rays of the sun (Decamp & Hosson, 2012) to generate products of real numbers.

Models of multiplication

Various physical and visual aids are used to model multiplication in elementary classrooms (Kosko, 2019). There are discrete models that involve arranging things, such as playing chips, into equal-sized groups. For example, the problem (2)(3) could be represented as two groups of three things each or else three groups of two things each. Discrete models frame multiplication as a kind of repeated addition, and this is one of the most widely-used models to conceptually define multiplication (Hurst, 2015; Vest, 1985). But discrete models are harder to physically realize with fractions and decimals. Visual models that use area to represent multiplication are an alternative. For example, the numbers to be multiplied could be arranged as the length and width of a rectangle, and the area of the rectangle would be the product (Reys et al, 2014; National Governors Association 2010, 25). An advantage of this continuous model is that it applies to any of the kinds of numbers children encounter in school (Kosko, 2019). A drawback is that it models unidimensional numbers—that is, single points on a number line—as areas, thereby misrepresenting products as two-dimensional (McLoughlin & Droujkova, 2013).

Physical models are pedagogically compelling because they can create diverse avenues for exploration and learning (Clements, 2000; Domino, 2010). Our interest in physical models of multiplication has been driven by an overarching design problem: How could a physical, manipulable tool realize a continuous model of multiplication? That is, how could we represent continuous, variable quantities with physical things?

Design Framework

Diagrammatic multiplication

Our answer to this question was inspired by a geometric interpretation of multiplication that is predicated on the following observation:

> the hypotenuse of the right triangle determined by an object and its shadow must be parallel to the hypotenuse of any other object and its shadow. Hence, knowing the shadow of one object (we call this object the unit) gives us a way to deduce the shadow of any other object. (McLoughlin & Droujkova, 2013, p. 2)

From this observation, McLoughlin and Droujkova (2013) developed a diagrammatic definition that models multiplication as *continuous directed scaling—i.e.*, the length of one segment is a positive or negative multiplier that stretches the length of another segment in the positive or negative direction (Dimmel & Pandiscio, 2020). We initially realized this geometric definition of multiplication in a dynamic diagram that had draggable points (see Figure 1).

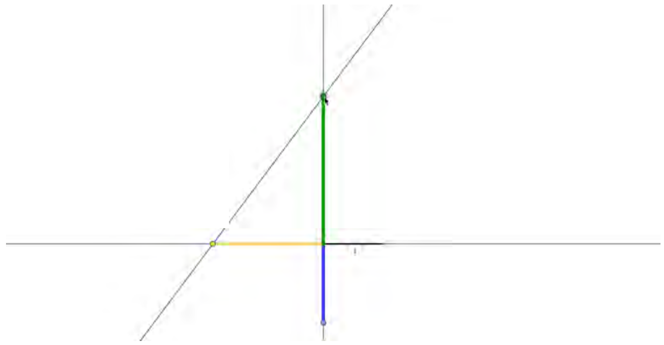

Figure 1. A dynamic diagram that realizes a geometric definition of multiplication. The yellow and blue points can be dragged along their respective axes to increase or decrease the lengths of the yellow and blue segments. The line that intersects the vertical axis determines the product of these lengths, which is represented by the green point.

In these diagrams, the "parallel to the hypotenuse" condition described above was satisfied by constructing a parallel line whose point of intersection with the *y*-axis would specify the product. The dynamic diagram we developed allowed students to use continuous transformations to explore ranges of products, such as products for pairs of numbers that are between 0 and 1 (Thompson & Saldanha, 2003). Because physical models for exploring arithmetic are common in elementary mathematics classrooms (e.g., Base Ten blocks, Cuisenaire rods, Pattern Blocks, Unifix cubes, chips and counters), we sought to design a physical embodiment of the multiplication diagram.

The variable altitude and variable length design problems

The keystone of the geometric definition of multiplication is parallel lines. Fortunately, sunlight offers a readily available, renewable, and abundant supply of naturally occurring parallel rays. The problem with using the sun as the source for parallel lines is that, at any time, the sun appears in one (and only one) position in the sky, and this position determines the proportion between an object's height and the length of its shadow (Douek, 1999). Thus, to multiply numbers in general requires control over the position of the sun. We refer to this as the *variable altitude* design problem.

Of course the sun cannot be moved, but there is nevertheless a solution to the variable altitude problem: We can change the *apparent* altitude of the sun by varying the angle of inclination of a surface onto which shadows are cast. By increasing the angle of inclination of a surface (i.e., the *shadow plane*), we decrease the lengths of any shadows falling upon it; by decreasing the angle of inclination, we increase the lengths of those shadows. Thus, by varying angles of inclination, it is possible to control the apparent altitude of the sun from 90 degrees (directly overhead, no shadow) to 0 degrees (sun on the horizon, undefined/infinite shadow).

The inclined plane provides control over the multiplier in a multiplication product – by varying the angle of inclination of a shadow plane, it is possible to stretch or shrink the length of the shadow of whatever object has been determined to be the multiplicative unit. What remains is a means to vary the multiplicand. This requires some method for increasing/decreasing length. We refer to this as the *variable length* design problem. The historical solution to this problem was the *slide rule,* an arithmetic aid that reigned from the 17th century until it was abandoned for electronic calculators in the 1970s (Cajori, 1909; Tympas, 2017, 7-8). We adapted the sliding

Olanoff, D., Johnson, K., & Spitzer, S. (2021). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Philadelphia, PA.

action of a slide rule, though not its logarithmic scales, to solve the variable length design problem.

Figure 2 shows a prototype of a device that embodies solutions to the *variable length* and *variable altitude* design problems. We refer to this device as a *SunRule*. It is not, strictly speaking, a combination of a sundial and slide rule; however the name is apt because it combines essential elements of each device (e.g., gnomons[1], adjustable scales) in novel ways.

Figure 2. A *SunRule* consists of a ruled board (shadow plane) and rods that are orthogonal to the ruled board. By changing the angle of the shadow plane, one changes the length of the shadow of the shorter rod, which serves as the multiplicative unit. The height of the longer gnomon represents the multiplicand. The device works because rays from the sun are parallel.

Method

Our initial plan was to analyze how pairs of elementary mathematics teacher candidates explored the *SunRule*. That plan is on hold until it becomes safe for pairs of students to interact in close proximity. In an effort to persevere through the challenge of data collection during the pandemic, we developed a handheld version of a *SunRule* that could be constructed from common household items (Figure 3). Thus, multiple devices could be built, which allowed students to interact at safe distances.

Figure 3. A handhels *SunRule*, constructed by elementary teacher candidates. The *SunRule* shows that (3)(4) = 12. Photo by Meg Pandiscio (2020).

In Figure 3, there is a longer gnomon (bottom) and a shorter gnomon (top). The shorter gnomon functions as a unit length. The unit length and the factor by which its shadow is stretched define a multiplier; in this case, that multiplier is (3), since the device has been inclined so that the length of the unit shadow extends (3) units. The height of the longer gnomon can be adjusted by sliding it up or down; the height of this gnomon specifies the multiplicand, which in this case is (4). The product is (12), shown here by the length of the shadow of the adjustable gnomon.

We report here an initial teaching activity where elementary teacher candidates built *SunRules* and used them to explore multiplication. We frame the initial teaching activity as a teaching experiment (Steffe & Thompson, 2000), where the second author was in the role of researcher-teacher. The purpose of the experiment was to generate hypotheses about how interacting with a *SunRule* creates opportunities for pre-service teachers to explore multiplication conceptually.

Context

During Fall, 2020, the second author taught an elementary mathematics methods course that convened in a hybrid in-person/online format. To comply with limits on indoor gatherings, the in-person students were split across two, five-student sections of the course that met on different days. The *SunRule* activity was planned as a two-class lesson that would allow elementary mathematics teacher candidates to explore a physical model of multiplication. For the first part of the activity, students worked with the second author to build *SunRules*. For the second part of the activity, students explored the *SunRules* outside, in small groups, while wearing masks and maintaining social distance. Both in-person sections of the course completed the first part of the activity. Students were told that the device had something to do with mathematics and that it needed to be used outside, on a sunny day. Figure 4 shows a selection of student-constructed *SunRules*.

Figure 4. *SunRules* constructed by elementary mathematics teacher candidates. Photo by Meg Pandiscio (2020).

Data collection

For the second part of the activity, five students from one section of the course[2] explored the *SunRules* in groups of two and three. The students within each group maintained social distancing throughout the activity, and the groups were separated by approximately twenty feet. Fixed video cameras recorded the activity of each group. The second author moved back and forth between the groups to facilitate their explorations of the device, following a semi-structured protocol. The protocol was designed to provide gradually more directed guidance to the groups of students. An example of a minimally directed question is, "What does the tool do?" An example of a more directed question is, "What are the ways that the lengths of the shadows of the gnomons could be varied?" The second author posed questions from the protocol to each group, as needed, to keep the students from getting stuck and spur them toward investigations of its mathematical opportunities. Below, we describe two episodes that capture how students explored and interacted with the *SunRule*. Episodes were identified by reviewing the video records of the teaching experiment and looking for instances where the tile of the shadow plane or the height of the gnomon was adjusted.

Episode 1: Sara's initial encounter with the *SunRule*

One group consisted of two students, Zak and Sara[3]. The second author launched the exploration activity for them by asking, "Any idea what this box does?" Sara replied, "Not yet", though as she said this, she had positioned the *SunRule* so that it was aligned with the azimuth of the sun, which caused the shadows of the gnomons to fall in parallel along its ruled surface (Figure 5).

Figure 5. While Sara declares that she does not know what the device does, she has oriented the device the way that it was designed to be oriented.

In this instance, Sara has guessed – in the technical sense of Wobbrock et al (2005) – how to interact with the device. She may not know what it does, but she already knows how it must be positioned in order to do it. Her next moves were to change the angle of inclination of the device. She tilted the device toward and then away from the sun, which caused the shadows of the gnomons to shorten and then lengthen (Figures 6, 7). As she varied the angle of inclination, she and Zak speculated that the device indicated a relationship between the sun and the shadows.

Figure 6. Sara inclines the device more toward the sun, which increases the sun's apparent altitude and causes the shadows of the gnomons to shorten.

Figure 7. Sara inclines the device less toward the sun, which decreases the sun's apparent altitude and causes the shadows of the gnomons to lengthen.

Sara noted the significance of the angle of inclination to the length of the shadows, "It really depends on how you hold it, like, if you tilt it towards (*sic*) the sun, then the shadows become very short, if you tilt it away from the sun the shadows get a lot longer." These initial interactions that varied the lengths of the shadows by changing the angle of inclination are the core of the mathematical design of the *SunRule*. This feature was salient for Sara almost immediately and suggests that the grounding predicate for the geometric definition of multiplication (block quote, above) is a natural and potentially powerful embodiment for a continuous scaling conception of multiplication.

Episode 2: Modeling division with the *SunRule*

After 10 minutes of open-ended exploration, both groups had zeroed in on the idea of the shadows varying in a constant ratio as the angle of inclination of the device was increased or decreased. As neither group had connected their observations about ratio to the operation of multiplication, the second author assembled the groups in a socially-distanced semicircle. He summarized the ratio ideas each group had discussed, and then stated that a mathematical operation the device could model is multiplication. Sara then demonstrated how the device could be used to show that (2)(3) = 6. The second author adjourned the groups to their respective places and asked them to continue exploring how the device could be used to model products.

In their discussion of multiplication, Zak and Sara realized that the device could also be used to represent division. Zak demonstrated this idea, which he attributed to Sara, by showing how the multiplication problem (5)(2) = 10 could be interpreted as the division problem (10)/(5) = 2 (Figure 8).

Figure 8. Zak positions the *SunRule* to show the quotient that (10)/(5) = 2.

To multiply with the *SunRule*, the angle of tilt varies the length of the shadow of the unit gnomon. This increase/decrease in the length of the unit shadow amounts to a scale factor that is applied to the length of the shadow of the other gnomon. To complete the product, one sets the height of the longer gnomon equal to the number that is being multiplied. The shadow of this gnomon is the answer. Zak and Sara realized that to represent division with the *SunRule* would require reversing this process; or, as Zak said: “there is an inverse relationship between multiplication and division.”

To use the *SunRule* to divide two numbers, set the height of the adjustable gnomon to be the divisor. Then, vary the angle of inclination of the *SunRule* so that the length of the adjustable gnomon’s shadow is the number that is being divided. The quotient will then be given by the length of the unit gnomon’s shadow. Sara made the connection to division within moments of demonstrating how the device represented multiplication. She and Zak conjectured that division should be possible to represent with the device and then worked out how that was possible. Zak demonstrated this, while narrating his *SunRule* manipulations. Zak pointed to the long shadow and said “10 divided by 5” and then pointed to the adjustable gnomon. He then said “equals 2” as he pointed to the shadow cast by the unit gnomon. He further noted that “2 is then the answer and it's the short shadow.” Sara and Zak's explorations of the connection between multiplication and division underscore the rich pedagogical opportunities of the SunRule.

Discussion and Reflections

The teaching experiment reported here documented pre-service teachers’ initial encounters with a physical device for modeling multiplication through continuous movements – e.g., tilting the device more or less, sliding the gnomon up or down. The movements made by Sara to vary the angle of inclination of the shadow plane and Sara and Zak’s linking of multiplication to division offer preliminary indications that the device worked as it was designed to work. Zak’s and Sara’s explorations of the *SunRule* suggested that it can be used to explore how multiplication and division are conceptually linked; we plan to develop and explore this hypothesis in follow up teaching experiments.

The *SunRule’s* connection to the real world is immediate, rather than applied or abstracted. The *SunRule* doesn’t apply mathematics to explain the world, rather, it uses an affordance of the world (sunlight) to model a mathematical operation (multiplication). Simultaneously, it shares a mathematically valid and robust representation of multiplication that is often missing in elementary school classrooms—that of multiplication as continuous scaling (Dimmel & Pandiscio, 2020; Kosko, 2019). By using a feature of the world to build a robust mathematical model, the *SunRule* represents an inversion of what is typically encountered in authentic/contextualized/real world mathematics.

The COVID-19 pandemic has triggered a reconsideration of how we gather. For schools, this has meant adapting instruction to remote, hybrid, or outdoor modalities, among other innovations, some of which will (hopefully) endure even when it is safe again to gather indoors. The *SunRule* provides a concrete material context for doing a mathematical activity outside—not simply for the sake of being outside, but because being outside is essential to use the device to do mathematical work. It provides a variable, tangible device for modeling families of multiplication problems and probing their mathematical structure. Beyond arithmetical utility, activities with the *SunRule* could pull students away from screens and create opportunities for students and teachers to reflect on how the geometry of sunlight is integrated with its design. These would be enviable outcomes at any time, and they are especially urgent in the face of the disruptions to teaching and learning brought on by the pandemic.

Notes

[1] This is the name for the part of a sundial that casts a shadow.
[2] The other section's opportunity was precluded by inclement weather.
[3] All names pseudonyms.

Acknowledgments

The research reported in this study was supported by grants from the Spencer Foundation and the NSF (award number 1822800) to the first author. The views expressed in this report are the author's and do not necessarily reflect the views of the Spencer Foundation or the NSF.
An expanded version of this research report was published in the July (2021) issue of the *Journal of Mathematics Education, Teachers College* (Dimmel, Pandiscio, & Bock, 2021).
The authors thank Adam Godet, of Godet Woodworking, for building prototypes of the *SunRule*.

References

Boero, P. (1989). Mathematical literacy for all: Experiences and problems. In *Proceedings of the 13th Annual Conference of the International Group for the Psychology of Mathematics Education* (pp. 62-75). Paris, France.

Boero, P., Garuti, R., & Mariotti, M.A. (1996). Challenging the traditional school approach to theorems: A hypothesis about the cognitive unity of theorems. In L. Puig & A. Gutierrez (Eds.), *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education* (pp. 113-120), Valencia, Spain.

Boero, P., Garuti, R., & Mariotti, M.A. (1996). Some dynamic mental processes underlying producting and proving conjectures. In L. Puig & A. Gutierrez (Eds.), *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education* (pp. 121-128), Valencia, Spain.

Cajori, F. (1909). *A history of the logarithmic slide rule and allied instruments*. New York: Engineering News Publishing Company.

Clements, D. (2000). 'Concrete' manipulatives, concrete ideas. *Contemporary Issues in Early Childhood* 1(1), 45-60.

Decamp, N., & Hosson, C. (2012). Implementing Eratosthenes' discovery in the classroom: Educational difficulties needing attention. *Science & Education, 21*(6), 911-920.

Dimmel, J. K., & Pandiscio, E. A. (2020). Continuous directed scaling: How could dynamic multiplication and division diagrams be used to cross mathematical borders? In Radakovic, N. and Jao, L. (Eds.), *Borders in mathematics pre-service teacher education* (pp. 21-45). Cham, Switzerland: Springer.

Dimmel, J.K., Pandiscio, E.A., & Bock, C.G. (2021). Multiplication by Sunlight: How can a geometric definition be realized in a physical tool? *Journal of Mathematics Education, Teachers College, 12*(1), 9 - 16

Domino, J. (2010). The effects of physical manipulatives on achievement in mathematics in grades k–6: A meta-analysis (Doctoral dissertation). State University of New York at Buffalo, Buffalo, NY.

Douek, N. (1999). Argumentation and conceptualization in context: a case study on sunshadows in primary school. *Educational Studies in Mathematics, 39*(1), 89-110.

Fuson, K. C. (2003). Developing mathematical power in whole number operations. In J. Kilpatrick, W.G. Martin & D. Schifter (Eds), *A research companion to principles and standards for school mathematics* (pp. 68-94). Reston, VA: National Council of Teachers of Mathematics.

Garuti, R., & Boero, P. (1992). A sequence of proportionality problems: An exploratory study. In W. Geesliln & K. Graham (Eds.) *Proceedings of the 16th Conference of the International Group for the Psychology of Mathematics Education* (pp. 225-232). Durham, NH.

Hiebert, J. (Ed.). (2013). *Conceptual and procedural knowledge: The case of mathematics*. New York: Routledge.

Hurst, C. (2015). The multiplicative situation. *Australian Primary Mathematics Classroom 20*(3), 1-16.

Kosko, K. W. (2019). Third-grade teachers' self-reported use of multiplication and division models. *School Science and Mathematics,* 119(5), 262-274.

Lampert, M. (1986). Knowing, doing, and teaching multiplication. *Cognition and Instruction 3*(4) 305-342.

McLoughlin, P. F., & Droujkova, M. (2013). A geometric approach to defining multiplication. arXiv preprint, 1301.6602.

National Governors Association. (2010). *Common core state standards.* Washington, DC.: Author.

Reys, R., Lindquist, M., Lambdin, D. V. & Smith, N. L. (2014). *Helping children learn mathematics*. New York: John Wiley & Sons.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. *Handbook of research design in mathematics and science education*, 267-306.

Tympas, A. (2017). The delights of the slide rule. In A. Typmas (Ed.), *Calculation and computation in the pre-electronic era* (pp. 7-38). London: Springer.

Thompson, P., & Saldanha, L. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, W.G. Martin & D. Schifter (Eds), *A research companion to principles and standards for school mathematics* (pp. 95-113). Reston, VA: National Council of Teachers of Mathematics.

Vest, F. (1985). Episodes with several models of multiplication. *Mathematics in School 14*(4), 24-26.

Olanoff, D., Johnson, K., & Spitzer, S. (2021). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Philadelphia, PA.