DYNAMIC SPATIAL DIAGRAMS AND SOLID GEOMETRIC FIGURES

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This paper reports on a study of learners' use of immersive spatial diagrams to make arguments about three-dimensional geometric figures. Immersive spatial diagrams allow learners to use the movement of their bodies to control their point of view, while immersed in three-dimensional digital renderings. We present analysis of two pairs of pre-service elementary teachers' argumentation about the shearing of pyramids, using the ck¢-enriched Toulmin Model of Argumentation (Pedemonte & Balacheff, 2016) to link the affordances of immersive spatial diagrams to the learners' mathematical reasoning. We share how one pair of learners took points of view bending beside and standing within the pyramid to describe how the space inside is transformed without reference to one- or two-dimensional components of the representation.

Keywords: Measurement, Technology, Geometry and Spatial Reasoning

Diagrammatic Representations of Three-Dimensional Figures

Three-dimensional geometric figures are often represented with diagrams on two-dimensional canvases in school geometry (Clements et al., 2017; Dimmel & Herbst, 2015; Dorko & Speer, 2015, 2013; Duval, 2006; Pittalis & Christou, 2010; Stevens et al., 2015), mediated through projection or cross-section. These frozen perspectives split learners' attention between spatially navigating the diagram and attending to the features of the mathematical figure. While 3D dynamic geometry software (*e.g.*, Cabri3D) allows learners to make visual observations from many points of view (Mithalal & Balacheff, 2019), these points of view are often controlled by two-dimensional (*e.g.,* touch, mouse) or keystroke-based input systems. Learners have difficulty working with two-dimensional representations of three-dimensional figures without spatial observation of the 3D shape (Pittalis & Christou, 2010).

Further, it is often impracticable to change the perspective *while* continuously manipulating the figure. Continuous manipulations of a diagram are important because they support learners' reasoning in school geometry. The continuous manipulation of dragging can support learners noticing the spatial properties of the diagram that are mathematically necessary (Clements, 2003; Laborde, 2005) and dragging can also allow for geometric transformations to be represented as "continuous and temporal" processes (Ng & Sinclair, 2015, p. 85). Observations of two-dimensional representations of three-dimensional figures may focus learners' struggles on navigation and manipulation of the diagram rather than on discerning which spatial properties of the diagram are incidental or mathematically necessary.

Physical spatial inscriptions (*e.g.*, 3D pens) are one alternative to two-dimensional renderings of three-dimensional figures. Using physical materials (*e.g.,* extruded plastics), diagrams can take up space and be manipulated by the learners' grasp (Ng & Sinclair, 2018). Further, learners can vary their point of view as they might with any other physical object – by walking, turning and bending their body and turning their head. However, physical spatial inscriptions have material constraints and are not generally able to be manipulated continuously with nonrigid transformations.

Immersive spatial diagrams are digitally rendered diagrams that share the learners' spatial environment, like physical spatial inscriptions, but also offer the digitally rendered flexibility of

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3D dynamic geometry software (Bock & Dimmel, 2020; Dimmel et al., 2020). Immersive spatial diagrams can be rendered using various consumer-ready devices: virtual-reality head-mounted-displays (*e.g.,* HTC Vive, Oculus Rift*),* augmented-reality head-mounted-displays (*e.g.,* Microsoft Hololens*),* and mixed-reality head-mounted-displays (*e.g.,* Varjo XR-3). These diagrams bring learners into a world where space-occupying objects can have the dynamic properties of digital renderings or bring those dynamic spatial objects onto the learners' physical world. Immersive spatial diagrams offer learners an opportunity to explore the properties of spatial representations governed by mathematical laws rather than the laws of physics.

By combining embodied control over point of view, continuous transformations, and a three-dimensional visual experience of the diagram, immersive spatial diagrams offer learners new modes of interactions with representations of three-dimensional figures. In this study, we explored how learners interacted with an immersive spatial diagram — a dynamic, digital, three-dimensional representation of a pyramid bound between parallel planes, focused on the affordances that allow learners to use their body to access multiple points of view. We asked: How do the points of view that learners take while immersed in a spatial diagram shape their argumentation about geometric transformations?

Theoretical Framework: ck¢-enriched Toulmin Model of Argumentation

We used the conception-knowing-concept (ck¢) enriched Toulmin model of argumentation (Pedemonte & Balacheff, 2016) to analyze the arguments that learners constructed while using an immersive spatial diagram. The ck¢-enriched Toulmin model of argumentation situates Balacheff & Gaudin's (2010) conception-knowing-concept models' rich description of learners' reasoning about a mathematical context within the Toulmin (1958) model's transformation of observed data into a claim through inference. The ck¢ model describes mathematical conceptions in terms of observable components of the interactions between learners and their environment (Balacheff & Gaudin, 2002, 2010; DeJarnette, 2018; Herbst, 2005; Mithalal & Balacheff, 2019). We chose this model to highlight how the points of view available to learners, as a constraint on their observations used for data in their argument, shaped their claims and inferences with a rich mathematical characterization. Components of the (ck¢) enriched Toulmin model are explained in greater detail with application to spatial diagrams in Bock and Dimmel (2020).

Mathematical Context: Shearing of Pyramids

The mathematical context for the study was the *shearing* of a pyramid between parallel planes. In plane figures, a transformation is a shearing transformation if a figure can be bound between parallel lines such that the lengths of the parallel cross-sections of the figure are preserved by the transformation (Ng & Sinclair, 2015). Shearing can be extended as a volume-preserving transformation of 3-dimensional figures bound between parallel planes. Consider a pyramid whose apex is bound to a plane parallel to its base (Figure 1: *ABCDE*).

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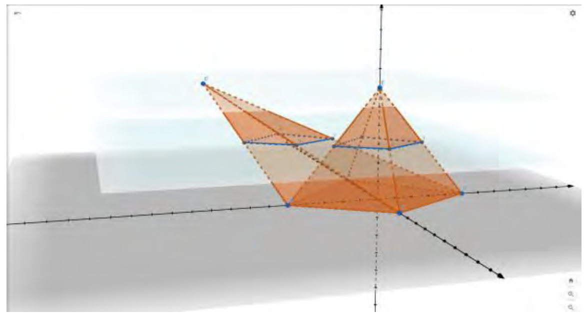

Figure 1: Pyramid *ABCDE* is sheared to form Pyramid *ABCDE'*

Any non-empty intersection of a pyramid and a plane parallel to its base is either the apex of the pyramid or a dilated copy of the base of the pyramid. In the case of a dilated copy of the base (Figure 1: *FGHI*), this cross-section is translated as the apex of the pyramid moves along a plane parallel to its base (Figure 1: *F'G'H'I'*). Then the area of the cross-section and the volume of the pyramid are preserved under the shearing transformation.

Figure 1 illustrates some difficulties with visual observations of three-dimensional geometric transformations mediated through a two-dimensional canvas. It is not immediately obvious if the cross-sections of the pyramid are (at least approximately) congruent. While a diagram, on its own, is not sufficient to prove geometric relationships, accurate diagrams are powerful heuristics that can suggest what relationships one ought to try to prove (Larkin & Simon, 1987). However, the two-dimensional diagram of a pyramid is caught in a conflict between seeing and knowing (Parzysz, 1988): perspectives that allow the pyramid to be *seen* as a figure that occupies space distort the polygonal cross-sections of the pyramid. As a result, it is difficult to show how the cross-section is transformed while showing how the cross-section relates to the volume of the pyramid.

Design of the Virtual Environment

We designed a virtual environment (Bock et al., 2020) where learners could explore the shearing of a pyramid with an immersive spatial diagram (Figure 2A). Learners could use pinch, drag and throw gestures to manipulate the apex of the pyramid, which was bound within a plane parallel to its base (Figure 2B). An open-palm gesture, parallel to the pyramid's base controlled the position of a cross-secting plane (Figure 2C). Instead of offering numeric representations of measure, a cube with volume equal to the volume of the pyramid and a square with area equal to the surface area of the pyramid could be loaded into the environment (Figure 2D). Finally, the participants had previously explored an analogous case of a triangle bound to parallel lines (described in Bock & Dimmel, 2020), which was also available for reference.

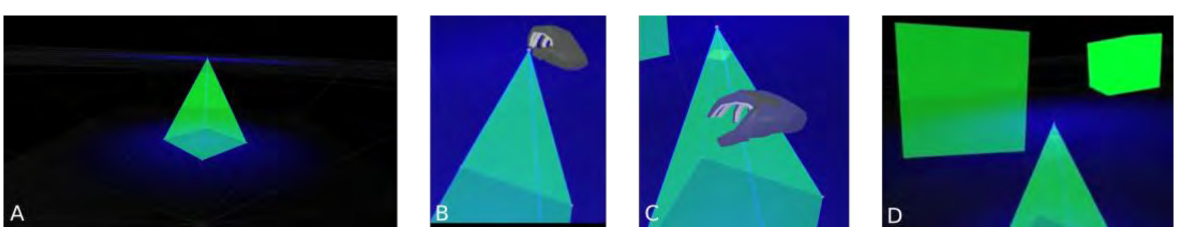

Figure 2: The Virtual Environment

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Pedagogical Rationale

The environment was designed to provide immersed participants with direct embodied control over their point of view; it featured a gesture-based interface that allowed immersed participants to move the apex of the pyramid and investigate its cross-sections. By excluding numeric measures, we hoped to discourage learners from using routine empirical calculations (*e.g.,* calculating the area of the base as the square of the length of the sides, calculating the volume as the product of the area of the base and the height) to reason about the effects of the shearing transformation on the pyramid. We hypothesized that learners would instead make arguments using the congruence of stacks of cross-secting planes to make sense of the shearing of the pyramid – either as units of area, or as approximations to small units of volume.

Methods

This study used a case-study methodology (Yin, 2012), where a set of arguments made using a diagram was the unit for a case. We analyzed a set of arguments constructed by pre-service elementary teachers using an immersive spatial diagram at a public university in the United States. Participants worked collaboratively with asymmetric roles. One participant was immersed in the environment, via a head-mounted display, the other participant, who was not immersed in the virtual environment, viewed a real-time projected video of the immersed participants' interactions on a television screen. Though the learners' views of and roles in the environment were different, collaborative interactions have been analyzed in other settings with similar immersed and non-immersed roles where the non-immersed learner's view was mediated through a two-dimensional projection (Price et al., 2020, p. 216). We considered their co-constructed arguments as the unit of analysis.

Participants

Below, we analyze the argumentation of two pairs of participants. All four participants identified as female. The first pair of participants were a junior and senior pre-service elementary teacher with concentrations in mathematics and art, respectively. The second pair of participants were two first-year pre-service elementary teachers without selected concentrations. Each participant is referred to using a pseudonym.

We archived the participants' experiences using first person composite, mixed-reality composite (BluePrint Reality, 2017; Sheftel & Williams, 2019), and third-person physical views as well as a microphone for recording dialog between participants and interviewers (see Bock & Dimmel, 2020, p. 13). The mixed-reality view blends together the virtual with the actual, offering an observer's perspective on how the immersed participant navigated the immersive environment. We used these video records to identify episodes where participants made geometric arguments about the effects of shearing on the measures of the pyramid. We then used these episodes to construct ck¢-enriched Toulmin models of each argument.

Example of Analysis

We analyzed three arguments from two pairs of participants using the ck¢-enriched Toulmin model of argumentation. We report here on one excerpt of one of those analyses, as a means of illustrating how we applied the ck¢-enriched Toulmin Model. Each of the components of the ck¢-enriched Toulmin models are developed from the video records and transcriptions. Figure 3 shows an enriched Toulmin model for an argument made by Emily and Olivia.

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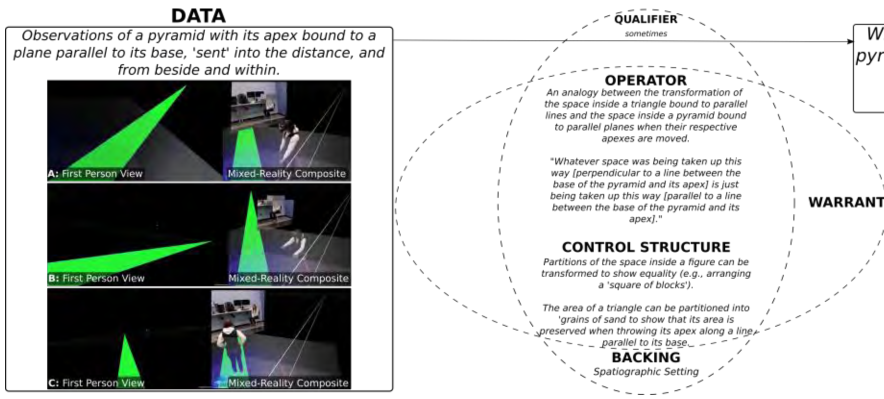

Figure 3. Model of Emily and Olivia's Second Argument

This excerpt explains the *data* element of the model in Figure 3, beginning after Emily and Olivia had experimented with manipulating the pyramid but had not begun developing an argument. Emily "sent" the apex of the pyramid into the distance by pinching, grasping and throwing the vertex. Emily waved goodbye as the apex was thrown along a line in a plane parallel to the pyramid's base.

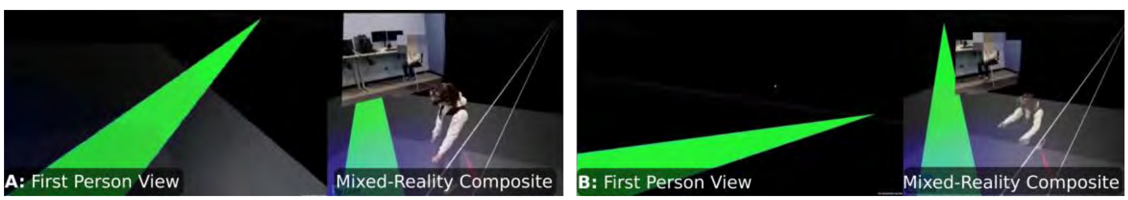

Figure 4: Emily's Points of View

As the apex continued to move away, Emily remarked that the pyramid "looks like its getting bigger" while her gaze looked along the length of the pyramid's nearest face (Figure 4A). Emily then walked beside the pyramid (Figure 4B), bent down, and remarked that "if you look at it...then it's getting so thin?". While in that position, Emily explained "whatever space was being taken up this way...it's just being taken up this way [gesturing along the length of the pyramid's face]." In this excerpt, Emily describes two visual observations: the pyramid "looks like it's getting bigger" and "if you look at it...then it's getting so thin." Emily's explanation of how the space inside the pyramid is being "taken up" showed that these observations serve as *data* (feedback from the virtual environment) to be transformed into their *claim* about the pyramid (see Figure 3).

Results

For each argument, a ck¢-enriched Toulmin model of argumentation was developed; these models are presented below and are accompanied by brief narratives. In our analysis, we were interested in how the feedback from the learner's environment – the *data* – shaped the mathematically rich descriptions of their conceptions in the *operator* and *control structure*. The data component of the model informs how the participants' interactions with the environment might have shaped their argumentation. The operator and control structure help to understand whether the learners used their interactions with the diagram to understand the shearing transformation differently then they might in other contexts.

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A Point of View Within the Spatial Diagram: Emily and Olivia's Arguments

Emily and Olivia made two arguments about the shearing of a pyramid. In their first argument Emily and Olivia claimed that when you "send" or throw the apex of the pyramid into the distance within its plane, its volume will be conserved. Emily wore the head-mounted-display, while Olivia observed a mixed-reality third-person view on a large television screen and took three points of view within the virtual environment – standing beside the pyramid and dragging the apex of the pyramid locally (Figure 3A), bending down alongside the pyramid after the apex had been thrown (Figure 3B), and standing with her legs intersecting the pyramid (Figure 3C). While standing inside the pyramid, Emily remarked "this kind of looks like a road, I feel like Dorothy... I wish I could see my legs and see them being chopped off by the planes [faces of the pyramid]." Emily and Olivia used these observations from these three points of view to develop an argument that explained how it would be plausible that the space inside the pyramid is redistributed by the shearing transform such that the volume is conserved.

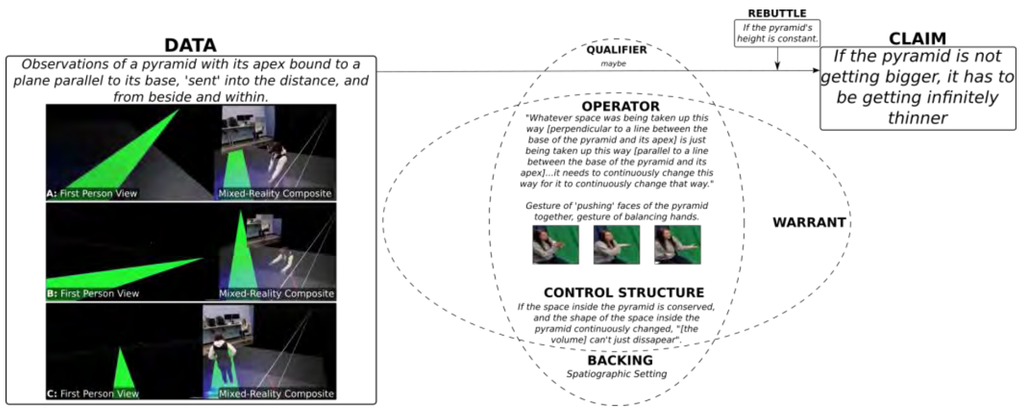

Figure 5. Model of Emily and Olivia's First Argument

After Emily and Olivia constructed this argument, the interviewers prompted: "is there anything else about the pyramid?... is there anything else about the pyramid changing?" Emily noted that the "length" [altitude] of the sides is becoming "super, super long... this looks infinitely long." While not infinite, the apex of the pyramid continued to move indefinitely into the distance. The interviewers prompted "so are you saying that a pyramid with infinitely long sides [faces] can have a finite volume?" Emily and Olivia then constructed another argument (Figure 5) to describe how the space inside the pyramid would need to change if the pyramid's volume is constant. Olivia described how "it would have to also get infinitely thin [as it is sheared], if it's not flattening out then I don't know where the space inside would like go," repeating gesture where she had her palms facing together and then pushed her palms together while tilting horizontally (Figure 5A). In this argument, Emily and Olivia reframed the warrant and control to be in terms of continuous and temporal transformations – describing how the pyramid is "flattening out", "getting infinitely thinner", and would need to "continuously change this way for it to continuously change that way". Emily and Olivia added a rebuttal that the height of the pyramid must be constant, however it was not clear why they attended to this measure.

Points of View from Above and Beside the Diagram: Abigail and Madison's Arguments

Abigail and Madison made an argument about the unbounded shearing of a pyramid where the only points they used were above and beside the pyramid. Abigail and Madison's argument

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included an argument about the area and perimeter preserving properties of shearing on a triangle to conclude that the volume and surface area of the pyramid would be analogously preserved under shearing (Figure 6). Abigail and Madison used a set of visual observations of the measures of the angles of the vertices at the base of the triangle and their line segments (Figure 6B) and visual observations of the behavior of the faces of the pyramid (Figure 6A) as they sheared each of the figures. While Abigail and Madison's warrant would not be understood to support their claim in a school mathematics context, the ck¢-enriched Toulmin model situates their conception within the feedback – or data – from their environment. During these observations, Abigail and Madison did not 'send' the triangle or the pyramid, so they did not have disconfirming feedback for their conclusions.

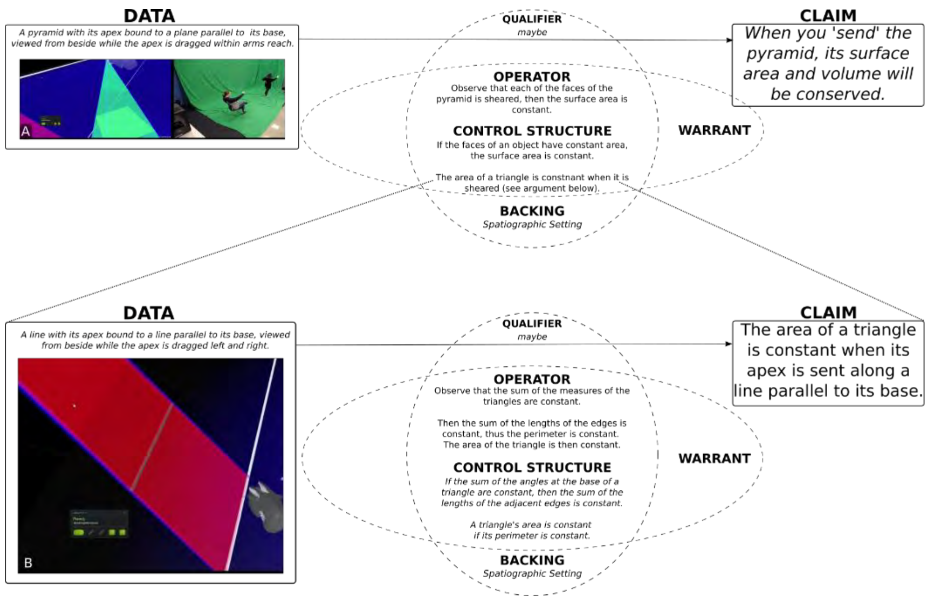

Figure 6. Model of Abigail and Madison's Argument

Discussion

In a school mathematics setting, the arguments developed by each pair of participants might feel incomplete – their arguments would need to be refined to be a rigorous explanation of the properties of the shearing transformation. With the lens of the ck¢-enriched Toulmin model we can look past a superficial evaluation of correctness to understand how the affordances of immersive spatial diagrams and the environment design supported their arguments, and the contexts where these diagrams might be useful in a less exploratory pedagogical setting.

Points of View

Both pairs of participants used their control over the point of view in the environment in ways that would be impracticable to replicate outside of immersive spatial diagrams: they used gestures to manipulate diagrammatic representations of pyramids and triangles while walking, bending, and turning their heads to make visual observations. Emily and Olivia took two points of view that would be difficult to replicate with two-dimensional diagrams: bending down beside and standing inside the pyramid. Emily took these points of view in order to share visual observations with Olivia and the interviewers as they constructed their argument. While we anticipated that participants might put their heads inside the pyramid, we did not anticipate the use of these points of view in the environment design. In contrast, Abigail and Madison engaged

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with the diagram using each of the novel affordances of immersive spatial diagrams but their visual observations of the triangle and pyramid are practicable to recreate with a two-dimensional dynamic representation of the figures.

Dimensionality

Emily and Olivia's argument had another unique feature – they described the continuous transformations of the space inside the pyramid and the triangle without describing lower-dimensional quantities. The points of view Emily shared beside and within the pyramid and Emily and Olivia's description of cross-sections of the pyramid through Emily's legs suggest that immersive spatial diagrams offer learners an opportunity to engage with three-dimensional mathematical figures without reconstruction from two-dimensional or one-dimensional elements. In contrast, Abigail and Madison's argument might have been better supported by the environment if measurement tools for angles, lengths and areas were available in the environment. Measurement of the triangle's angles or segments might have suggested that they do not correlate as Abigail and Madison suspected; measure of the faces of the pyramids might have suggested a changing surface area as the pyramid was sheared. This is a constraint of the environment design – not of immersive spatial diagrams – but also a feature that is easily accessible in many traditional dynamic geometric environments.

Existing research on immersive spatial diagrams has focused on representing mathematical figures with numeric representations of length and symbolic representations of area and volume (Lai et al., 2016), rigid transformations of static shapes (Gecu-Parmaksiz & Delialioglu, 2019), and gesture-based construction (Dimmel & Bock, 2017). This study was designed to explore how learners used points of view with immersive spatial diagrams to reason – and struggle with – the properties of continuous geometric transformations of three-dimensional figures. The results of this study explore two cases where learners investigated the shearing of three-dimensional figures, an extension of research on how learners reason about the shearing in plane geometry (Bock & Dimmel, 2020; Ng & Sinclair, 2015). In one case, the pair of participants identified mathematically relevant spatial invariants (volume of the pyramid, height of the pyramid) and described how these properties might relate to the shape of the pyramid for the spatial invariants to be plausible. However, the participants' argument did not relate properties of the figure to explain why the shearing transformation *necessarily* preserves volume. This is consistent with expectations from learners use of two-dimensional dynamic geometry environments, where dragging affordances have been linked to identification of spatial invariants (Clements, 2003). This process of "learning [to] identif[y] of visually relevant spatiographic invariants attached to geometrical invariants" is an important to the learning of geometry, alongside deductive reasoning from theoretical statements (Laborde, 2005, p. 177).

Conclusion

Emily and Olivia struggled productively to describe continuous transformations of volume without reducing to lower-dimensional elements, confidently reasoning from visual observations. This addresses a key constraint of two-dimensional representations of three-dimensional geometrical objects — that the figures must be analyzed through reconstruction from lower-dimensional components of the representation (Mithalal & Balacheff, 2019). Further research is needed to explore how spatial diagrams can be designed for learners to *see* or attend to one-, two-, or three-dimensional elements of figures, analogous to diagrammatic representations of two-dimensional figures (Duval, 2006, p. 116). Finally, there is an opportunity to explore how learners' analysis of three-dimensional figures without reconstruction from one- and two-

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dimensional elements might be designed into geometric construction environments and where this might best support learners' reasoning in school geometry.

References

Balacheff, N., & Gaudin, N. (2002). *Students conceptions: An introduction to a formal characterization* (Les Cahiers Leibniz, N65). https://telearn.archives-ouvertes.fr/hal-00190425/document

Balacheff, N., & Gaudin, N. (2010). Modeling students' conceptions: The case of function. *CBMS Issues in Mathematics Education*, *16*, 207–234.

BluePrint Reality. (2017). *MixCast* (1.3) [OSVR]. Blueprint Reality Inc. https://mixcast.me/mixcast-download/

Bock, C. G., Dimmel, J., Emerson, C., Gazey, N., Antonellis, G., Haney, J., & Bruce, T. (2020). *Shearing Lab*. Zenodo. https://doi.org/10.5281/zenodo.4116602

Bock, C. G., & Dimmel, J. K. (2020). Digital Representations without Physical Analogues: A Study of Body-Based Interactions with an Apparently Unbounded Spatial Diagram. *Digital Experiences in Mathematics Education*. https://doi.org/10.1007/s40751-020-00082-4

Clements, D. (2003). Teaching and Learning Geometry. In *A research Companion to Principles and Standards for School Mathematics* (pp. 151–178). National Council of Teachers of Mathematics.

Clements, D., Samara, J., & Van Dine, D. W. (2017). Volume. In J. Barett, D. Clements, & J. Sarama (Eds.), *Children's Measurement: A Longitudinal Study of Children's Knowledge and Learning of Length, Area, and Volume* (Vol. 16, pp. 3–24). National Council of Teachers of Mathematics.

DeJarnette, A. F. (2018). Students' Conceptions of Sine and Cosine Functions When Representing Periodic Motion in a Visual Programming Environment. *Journal for Research in Mathematics Education*, *49*(4), 390–423. https://doi.org/10.5951/jresematheduc.49.4.0390

Dimmel, J., & Bock, C. (2017). *HandWaver: A gesture-based virtual mathematics making environment*. 13th International Conference on Technology in Mathematics Teaching, Lyon, France.

Dimmel, J., & Herbst, P. G. (2015). The Semiotic Structure of Geometry Diagrams: How Textbook Diagrams Convey Meaning. *Journal for Research in Mathematics Education*, *46*(2), 147–195. https://doi.org/10.5951/jresematheduc.46.2.0147

Dimmel, J., Pandiscio, E., & Bock, C. (2020). The Geometry of Movement: Encounters with Spatial Inscriptions for Making and Exploring Mathematical Figures. *Digital Experiences in Mathematics Education*. https://doi.org/10.1007/s40751-020-00078-0

Dorko, A., & Speer, N. (2015). Calculus Students' Understanding of Area and Volume Units. *Investigations in Mathematics Learning*, *8*(1), 23–46. https://doi.org/10.1080/24727466.2015.11790346

Dorko, A., & Speer, N. M. (2013). Calculus Students' Understanding of Volume. *Investigations in Mathematics Learning*, *6*(2), 48–68. https://doi.org/10.1080/24727466.2013.11790332

Duval, R. (2006). A Cognitive Analysis of Problems of Comprehension in a Learning of Mathematics. *Educational Studies in Mathematics*, *61*(1), 103–131. https://doi.org/10.1007/s10649-006-0400-z

Gecu-Parmaksiz, Z., & Delialioglu, O. (2019). Augmented reality-based virtual manipulatives versus physical manipulatives for teaching geometric shapes to preschool children. *British Journal of Educational Technology*, *0*(0). https://doi.org/10.1111/bjet.12740

Herbst, P. (2005). Knowing About "Equal Area" While Proving A Claim About Equal Areas. *Recherches En Didactique Des Mathematiques*, *25*(1), 11–56.

Laborde, C. (2005). The Hidden Role of Diagrams in Students' Construction of Meaning in Geometry. In J. Kilpatrick, C. Hoyles, O. Skovsmose, & P. Valero (Eds.), *Meaning in Mathematics Education* (pp. 159–179). Springer US. https://doi.org/10.1007/0-387-24040-3_11

Lai, C., McMahan, R. P., Kitagawa, M., & Connolly, I. (2016). Geometry Explorer: Facilitating Geometry Education with Virtual Reality. *Virtual, Augmented and Mixed Reality*, 702–713. https://doi.org/10.1007/978-3-319-39907-2_67

Larkin, J. H., & Simon, H. A. (1987). Why a Diagram is (Sometimes) Worth Ten Thousand Words. *Cognitive Science*, *11*(1), 65–100. https://doi.org/10.1016/S0364-0213(87)80026-5

Mithalal, J., & Balacheff, N. (2019). The instrumental deconstruction as a link between drawing and geometrical figure. *Educational Studies in Mathematics*, *100*(2), 161–176. https://doi.org/10.1007/s10649-018-9862-z

Ng, O.-L., & Sinclair, N. (2015). "Area Without Numbers": Using Touchscreen Dynamic Geometry to Reason About Shape. *Canadian Journal of Science, Mathematics and Technology Education*, *15*(1), 84–101. https://doi.org/10.1080/14926156.2014.993048

Articles published in the Proceedings are copyrighted by the authors.

Ng, O.-L., & Sinclair, N. (2018). Drawing in Space: Doing Mathematics with 3D Pens. In *Uses of Technology in Primary and Secondary Mathematics Education* (pp. 301–313). Springer.

Parzysz, B. (1988). "Knowing" vs "Seeing". Problems of the Plane Representation of Space Geometry Figures. *Educational Studies in Mathematics*, *19*(1), 79–92.

Pedemonte, B., & Balacheff, N. (2016). Establishing links between conceptions, argumentation and proof through the ck¢-enriched Toulmin model. *The Journal of Mathematical Behavior*, *41*, 104–122. https://doi.org/10.1016/j.jmathb.2015.10.008

Pittalis, M., & Christou, C. (2010). Types of reasoning in 3D geometry thinking and their relation with spatial ability. *Educational Studies in Mathematics*, *75*(2), 191–212. https://doi.org/10.1007/s10649-010-9251-8

Price, S., Yiannoutsou, N., & Vezzoli, Y. (2020). Making the Body Tangible: Elementary Geometry Learning through VR. *Digital Experiences in Mathematics Education*, *6*(2), 213–232. https://doi.org/10.1007/s40751-020-00071-7

Sheftel, B. J., & Williams, T. (2019). *Mixed reality cinematography using remote activity stations* (United States Patent No. US10497182B2). https://patents.google.com/patent/US10497182B2/en

Stevens, I. E., Hobson, N. L. F., Moore, K. C., Paoletti, T., LaForest, K. R., & Mauldin, K. D. (2015, November). *Changing Cones: Themes in Students' Representations of a Dynamic Situation*. https://eric.ed.gov/?id=ED584218

Toulmin, S. E. (1958). *The uses of argument*. University Press.

Yin, R. K. (2012). Case study methods. In *APA handbook of research methods in psychology, Vol 2: Research designs: Quantitative, qualitative, neuropsychological, and biological* (pp. 141–155). American Psychological Association. https://doi.org/10.1037/13620-009

Olanoff, D., Johnson, K., & Spitzer, S. (2021). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Philadelphia, PA.

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