PRESERVICE SECONDARY TEACHERS' REASONING ABOUT STATIC AND DYNAMIC REPRESENTATIONS OF FUNCTION

Demet Yalman Ozen
Middle Tennessee State University
dy2i@mtmail.mtsu.edu

Nina G. Bailey
University of North Carolina Charlotte
nbaile15@uncc.edu

Samantha Fletcher
Middle Tennessee State University
scf3a@mtmail.mtsu.edu

Hamid Reza Sanei
North Carolina State University
hrsanei@ncsu.edu

Allison W. McCulloch
University of North Carolina Charlotte
amccul11@uncc.edu

Jennifer N. Lovett
Middle Tennessee State University
jennifer.lovett@mtsu.edu

Charity Cayton
East Carolina University
caytonc@ecu.edu

This study aims to describe how preservice secondary mathematics teachers (PSMTs) reason about different function representations. The study focuses on two PSMTs' reasonings across static and dynamic representations of functions. Sfard's (2008) Theory of Commognition guided our analysis. Findings indicate that while static representations restrict attention given to covariation, dynamic representations support PSMTs' reasoning about covariation including making connections to how covariation is represented in static graphs.

Keywords: Mathematical representations, algebra and algebraic thinking, technology

The concept of function permeates all levels of mathematics and is a large focus of the high school curriculum. Central to the treatment of functions in high school is attention to characteristics of families of functions given their usefulness for mathematical modeling. This attention means that significant emphasis is placed on graphical representations of functions (i.e., static graphs on a Cartesian plane). Research has shown that when analyzing graphical representations of functions, students and teachers alike often attend to perceptual cues rather than the relationships between the variables the perceptual cues are representing (e.g., Moore & Thompson, 2015; Sinclair et al., 2009). The coordination of two quantities represented in a graph of a function and the ways they change in relation to each other is called *covariational reasoning* and has been identified as foundational for mathematical modeling as well as many calculus concepts (e.g., Carlson et al., 2002). Given the role that functions play in the high school curriculum, it is essential that preservice secondary mathematics teachers (PSMTs) develop covariational reasoning skills. Carlson et al. pointed to the potential of dynamic technologies to support those learning to apply covariational reasoning. Recent work with a particular dynamic representation of functions in one dimension, the dynagraph (Goldenberg et al., 1992), has pointed to its potential to elicit student reasoning about the ways in which independent and dependent variables vary and covary (e.g., Antonini et al., 2020; Sinclair et al., 2009). To that end, the purpose of this study was to examine the similarities and differences in the ways PSMTs reasoned about different representations of functions–static and dynamic.

Olanoff, D., Johnson, K., & Spitzer, S. (2021). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Philadelphia, PA.

Background Literature

Reasoning about Static Graphs

Static representations of functions include tables, lists of ordered pairs, equations, and graphs. There is evidence that students' limited experience with graphical representations constrains them from making meaningful connections among algebraic and graphical representations (Knuth, 2000). Static graphical representations also conceal the dynamic aspects of function, such as the rate of change and relative position, that are essential for forming a robust understanding of function (e.g., Antonini et al., 2020; Carlson, 1998; Confrey & Smith, 1995; Ng, 2016). Research has shown that when reasoning about static graphs, it is not unusual to pay attention to shape and perceptual cues rather than the ways the graph represents how the variables change together (e.g., Moore & Thompson, 2015; Oehrtman et al., 2008; Weber, 2012). Moore and Thompson (2015) have also shown that it is not uncommon for a graph to be interpreted as the function itself, rather than a representation of the function. They distinguished *static shape thinking* from *emergent shape thinking* and described static shape thinking as considering a graph as an object, reacting to perceptual cues and the perceived shape of the graph rather than perceiving a graph as a trace and a representation of covarying quantities. Students with emergent shape thinking interpret features of the graph as properties of covariation. Thompson and Carlson (2017) noted that covariational reasoning happens "most strongly when a person is strategizing how to keep track of quantities' values simultaneously" (p. 438). Thus, dynamic representations where one can act on one quantity and track the concurrent relationship with another quantity might support the development of covariational reasoning skills.

Reasoning about Dynamic Representations of Functions

Digital technology provides many affordances related to the ways in which one can interact with functions and their graphs (Drijvers, 2015). Such technologies allow for the use of tables, expressions, and graphs to be dynamically linked to animated motion (e.g., Johnson et al., 2020; Kaput Center, 2016). There is evidence that engaging in activities of these types can support the development of reasoning about varying quantities and the ways in which they are represented in graphs (e.g., Johnson et al., 2020). Another way of leveraging the dynamic affordances of digital technologies to represent function is to use parallel axes rather than perpendicular axes, typically referred to as a dynagraph. Goldenberg et al. (1992) introduced dynagraphs to draw attention to dynamic function behavior and to help students focus on the function by eliminating complex information shown in a Cartesian graph. Students can test their conjectures of the relationship between input and output by dragging the input and observing the resulting change in the output. With the movement of this interactive representation, identification of invariants and covariation become central to one's exploration. Antonini et al. (2020) described dynagraphs as dynamic interactive mediators because students can engage in discourse with a dynagraph, "asking" questions of the tool and engaging to receive an "answer". Research has shown that use of dynagraphs can support the teaching of functions (e.g., Antonini et al., 2020) and can foster covariational reasoning by eliciting attention to movement, time, and space (Lisarelli, 2017).

There is little research comparing student reasoning when engaging with static and dynamic representations of function (exceptions include Antonini et al., 2020; Ng, 2016, and Sinclair et al., 2009). Given the potential of dynagraphs to support the development of reasoning about variables both separately and together, it is of interest to compare the ways in which one would reason about static and dynamic representations of the same functions. Our aim is to describe how PSMTs reason about functions with different visual mediators (i.e., static and dynamic).

Olanoff, D., Johnson, K., & Spitzer, S. (2021). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Philadelphia, PA.

Theoretical Framework

Sfard's (2008) Theory of Commognition unites cognitive and communicational processes to explain student thinking as "an individualized version of interpersonal communication" (p. 81). For Sfard, communication is a back and forth (action/reaction) that includes all communication, including with oneself. From this perspective, Antonini et al. (2020) explained "doing mathematics means engaging in the type of communication defined as mathematics and learning means becoming able to access and express this discourse" (p. 4). So, to study students' learning, we must attend to the words, visual mediators, narratives, and routines that form their discourse.

Mathematical discourse is characterized by specific words that are used by experts in specific ways. In the context of this study, that might include words like function, quadratic, increasing, rate of change, domain, or it might include informal language that is clear enough for an expert to understand the mathematics one is referring to. Visual mediators are objects that can be seen and operated on in the communication process. In mathematics, visual mediators can include, but are not limited to, symbols and graphs. Visual mediators of this type are static in that they can be seen and operated on, but not interacted with. In contrast, Antonini et al. (2020) introduced a dynamic interactive visual mediator (referred to as a DIM) described as a mediator that is "both *dynamic* - they change over time - and *interactive* - they respond to a person's manipulations" (p. 5). A dynagraph is an example of a DIM. In this study, we attended to the ways students communicate with and about different types of visual mediators (i.e., static vs. DIM).

Context of the Study

Since our goal was to compare and contrast the ways in which PSMTs reasoned about static graphical representations of functions and dynamic interactive representations of functions, asking students to compare and contrast within each representation type met our needs. To this end, we decided to use the instructional routine of Which One Doesn't Belong (WODB) (Danielson, 2016). A typical WODB task includes four objects and students are simply asked, which one doesn't belong? The main characteristic of a WODB task is that all of the options within the task can be considered correct which shifts students' focus away from trying to obtain the "correct" answer to distinguishing attributes of the objects presented in each option. This study used two WODB tasks, one with four static graphs of functions – each from a different function family (referred to going forward as the static task) and one with four dynagraphs of the same four functions (going forward referred to as the dynamic task).

In the static task, PSMTs considered four graphs (Figure 1) of functions and were asked to decide WODB and why. Once the PSMTs explained their choice, they were then asked to explain why someone else might argue that each of the other remaining graphs does not belong.

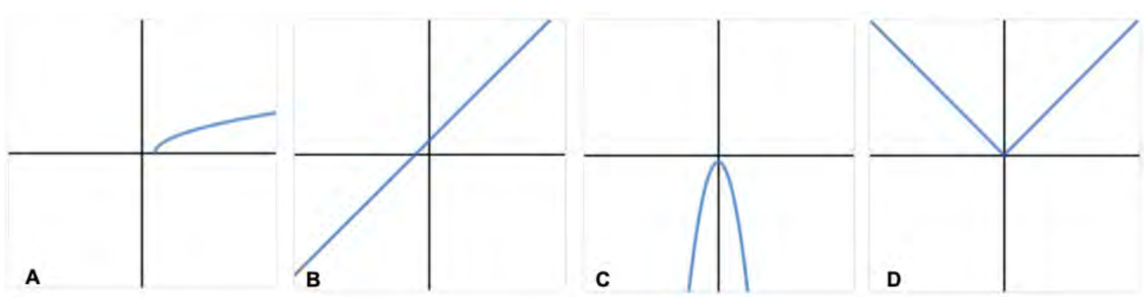

Figure 1: The static WODB task

Olanoff, D., Johnson, K., & Spitzer, S. (2021). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Philadelphia, PA.

In the dynamic task, the same four functions were presented (in a different order) but this time represented using dynagraphs (Note: PSMTs were not told they were the same). As a reminder, a dynagraph consists of two parallel number lines (function input on one, output on the other), and as the input is dragged, corresponding changes to the output will result (Figure 2). Just like the static task, PSMTs were asked to decide and explain WODB. Then they were asked why someone else might argue that each of the other remaining dynagraphs does not belong.

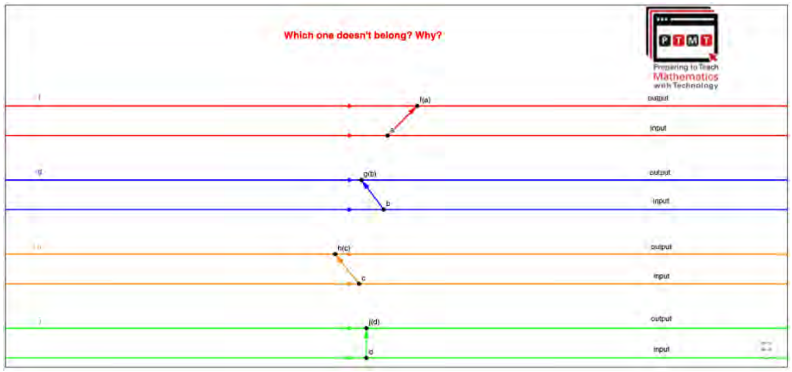

Figure 2: The dynamic WODB task–linear, square root, quadratic, and absolute value functions from top to bottom (https://www.geogebra.org/m/wjecnfev)

Methodology

This study was situated within the context of a larger study investigating how PSMTs reasoned across static and dynamic representations of function. Here we used a multiple case study design (Yin, 2017) to explore two cases, where each case was defined by the type of visual mediator (i.e., static and DIM) with which the students interacted. Our overarching research question was: What is the nature of students' discourse about function as they interact with different visual mediators (i.e., static and dynamic interactive mediators)?

The full study included 25 PSMTs attending six universities. Here we focus on two female participants; neither had experience with dynagraphs before. They were secondary mathematics education majors attending different universities. Both participants were enrolled in a math methods course, prior to student teaching, at the time of the study.

Video screen capture recordings of semi-structured interviews (Goldin, 2000) served as the main data source. One interview posed the static visual mediator first, and the second began with the DIM. Interviews were transcribed verbatim and uploaded in Atlas.ti to assist with coding.

Similar to Antonini et al. (2020), we used Sfard's (2008) Theory of Commognition to guide our analysis. We attended to words, discourses, and narratives to code the PSMTs' comparing and contrasting of the mediators presented in the tasks. In our consideration of the scholarly mathematical discourse (words), we were specifically interested in the characteristics of function that were elicited. We read the transcripts for the specific characteristics of function being described (e.g., domain, range, increasing, decreasing, maximum), created quotations for each chunk of transcript referring to a specific characteristic, and applied labels. A full list of these characteristics is included in the findings.

Next, we coded for the discourse about and with each mediator. To do so, all team members watched each video and reviewed the transcript to become familiar with each participant's discourse. This was followed by full team meetings to develop short codes describing the discourse using the protocol set forth by DeCuir-Gunby et al. (2011). We compared the data and emerging codes using a constant comparative method to create our final version of the codebook. From there two researchers coded each interview, and any discrepancies were discussed amongst them to come to a consensus. Finally, the researchers read the coded data again, condensing codes into categories and then identifying themes in the participants' narratives about the different types of mediators. These themes are presented in the findings section.

Findings

The PSMTs engaged with two different visual mediators, a set of four static Cartesian graphs and a set of four dynagraphs. Regardless of the order of engagement (i.e., static or DIM first), the characteristics of function they attended to when comparing and contrasting the functions in each activity were similar. The characteristics elicited included: differentiability, domain and range, function families, function vs. non-function, increasing and decreasing, independent and dependent variables, local and global extrema, rate of change, and symmetry. Representative examples from our data for the most commonly noted characteristics are shown in Table 1. The only characteristics not included in the communication with both mediators were function/non-function and symmetry. In both cases, these characteristics were mentioned by only one of the two PSMTs and on only one of the four functions being compared. The other seven characteristics were routinely included by both PSMTs in their discourse about both visual mediators. Given the similarities in the mathematical focus of their discourse related to both visual mediators, we next present findings related to the nature of the discourse related to the mathematical focus for each of the two visual mediators.

Table 1: Representative examples of discourse related to characteristics of functions

	Representative Examples	
	Static	DIM
Domain/Range	PSMT 1: "The domain [of A] is only from one to infinity."	PSMT 1: "The domain [of h(c)] is going to go on forever here."
Increasing/ Decreasing	PSMT 1: "[A's] just going to keep going up."	PSMT 2: "And *b*, I'm noticing that as you increase the input of *b*, more and more like the like *g(b)* increases less."
Rate of Change	PSMT 1: "The slope of the lines [of absolute value function] was one or negative one."	PSMT 2: "And it looks like so as *d* is approaching negative infinity *j(d)* increases at the same rate that *d* is decreasing."

Nature of Discourse with and about Static Visual Mediators

When engaging with the static version of the WODB task, the nature of the PSMTs' discourse was routinely focused on using formal mathematical language (though not always precisely) and what we referred to as "describing the image". Representative examples for each of these types of narratives are provided in Table 2.

As they compared and contrasted the four static graphs, the PSMTs consistently used the formal language to typically describe characteristics of functions – e.g., domain, range, increasing, decreasing, slope, concave up. For example, PSMT 2 described graph A using the term "restricted domain" (see row 1 of Table 2). This is to be expected given the years of experience they have of being asked to identify characteristics of functions based on their graphs. Of course, their discourse would include the words used by the mathematicians they have learned from over the years. However, the PSMTs did not always use these words in precise ways. For example, when trying to describe the non-constant rate of change of the function in graph C, PSMT 1 refers to it as "average rate of change" with uncertainty, as she knows it is not constant, but is not sure what to call it.

Table 2: Representative examples of discourse about the static visual mediators

Features of PSMTs' narratives	Representative Examples
Use of formal and precise mathematical language	PSMT 2: "[A's] the only one that has like a restricted domain. Because it like, it doesn't have any inputs, that work for the, they give, like a real value lower than one."
Use of formal and imprecise mathematical language	PSMT 1: "The slope [of D] from zero to infinity and the range is going to be positive one. But I wouldn't say it's that's not really an av- (pause) that's not really an average rate of change."
Describing the image	PSMT 1: "Well, [C's] concave down, it's opening down. ... [D's] the only one that goes through the origin. ... B's the only one that doesn't belong because it's the only one that touches all or touches three quadrants while the rest touch one or two." PSMT 2: "[C's] only one where it doesn't go above zero. It doesn't have an output above zero, right."

Whether their formal mathematical language was precise or not, the PSMTs consistently compared and contrasted the static graphs by describing the presented images as if they were pictures. For example, while PSMT 1 (see row 3 of Table 2) correctly described graph C as "concave down", the follow up phrase "it's opening down" reveals she is describing the image rather than the increasing and then decreasing rates of change that the term concavity is intended to describe. The routine of describing the image in the static graphs is consistent with what Moore and Thompson (2015) refer to as "static shape thinking" or "treating a graph as a piece of wire ... attending to perceptual cues and the perceptual shape of a graph" (p. 784). The attention to perceptual cues (e.g., "goes through the origin", "touches three quadrants") and shape (e.g., "doesn't go above zero") is evident in both examples presented in row 3 of Table 2.

Nature of Discourse with and about Dynamic Interactive Mediators

When engaging with the dynamic version of the WODB task, the nature of the PSMTs' discourse was routinely focused on describing relative direction, distance, and/or speed and connecting their noticing to imagined graphs of known functions. Representative examples for each of these routines are provided in Table 3.

The PSMTs interacted with each dynagraph by dragging the independent variable and examining the resulting reaction of the dependent variable. As they explored dynamically, they described the dynamic characteristics they noticed. Both PSMTs consistently noted the variables' relative direction and speed. For example, as PSMT 2 dragged d to the far left, she saw $j(d)$ move to the right and stated, "as d is approaching negative infinity, $j(d)$ increases". In addition, she noted that "$j(d)$ is increasing constantly or very close to the same amount that d is". Relative distance and speed were also often discussed. For example, when describing her exploration of *g* alongside of *h*, PSMT 1 noted their relative speed but used distance to make sense of it, referring to the length of the connector between *c* and *h(c)* to describe that distance, she explains "so, the outputs, the *g(b)* aren't moving as fast as this one" and then describing *h* she said, "it's like the arrow from the inputs, the outputs, gets larger and larger and larger." The arrow getting "larger" corresponded to the increasing rate of change. In the PSMTs' discourse about the relative speed, direction, and distance, there is evidence of attending to not only how each of the input and output are varying, but also the ways in which they are covarying.

Table 3: Representative examples of discourse about the dynamic interactive mediators

Features of PSMTs' narratives	Representative Examples
Describing relative speed, direction, and/or distance	PSMT 1: "So the outputs, the *g(b)* aren't moving as fast as if like if I move this one [talking about *h*]. And I move it, it's like the arrow from the inputs, the outputs, gets larger and larger and larger. But here [talking about *g*] it does get larger. But it's like it takes a longer time for it to get as long, like cause see, like here I have to move *b* all the way to the right for it to get really long."
	PSMT 2: "It looks like *j(d)* is increasing constantly or very close to the same amount that *d* is and then when *d* is less than zero *j(d)* is increasing, oooh. And it looks like so as *d* is approaching negative infinity, *j(d)* increases at the same rate that *d* is decreasing. And so that reminds me of the absolute value function."
Imagining a Cartesian graphical representation of a known function	PSMT 2: "Because, so like, if I'm picturing, like, the function, like absolute value of *x* and to the right of zero, *d* would be, or to the right of zero, *x* and *f(x)* would be the same or *d* and *j(d)* would be the same. But then to the left of zero, as *d* approaches negative infinity *x* would approach or I feel like a mixing of other notation like as *x* approaches negative infinity, *f(x)* would approach infinity at the same rate. If that makes sense, like it's just the *x* function on both sides. But. Yeah."
	PSMT 1: "[h(c)'s] probably like (pause) a quadratic that is facing down or maybe even an absolute value, because if I move. The *x* values from negative to positive, the *y* values look like they're staying. (pause) They're kind of repeating themselves, the outputs are similar."

Another routine that emerged in the PSMTs' discourse was connecting the relationships they were observing among the variables with Cartesian graphs of function families with which they were familiar. They tended to describe images of static graphs of functions on Cartesian planes they envisioned as sharing characteristics with the dynamic representations they were investigating. For example, as PSMT 2 dragged *d* from the far right to the far left and noticed *j(d)* moving at a constant speed and the same direction until a certain point and continuing at a constant speed but moving the opposite direction she said, "like if I'm picturing, like, the

Olanoff, D., Johnson, K., & Spitzer, S. (2021). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Philadelphia, PA.

function, like absolute value of *x* and to the right of zero, *d* would be, or to the right of zero, *x* and *f(x)* would be the same ... as *x* approaches negative infinity, *f(x)* would approach infinity at the same rate. If that makes sense, like it's just the *x* function on both sides." She is connecting what she is seeing in the dynagraph with a Cartesian graph that she is familiar with. PSMT 1's description is similar; she starts by connecting the relative direction of *d* and *j(d)* to "probably like (pause) a quadratic that is facing down", but then considers the constant relative distance and says "maybe even an absolute value ... *y* values look like they're staying. They're kind of repeating themselves". This routine of connecting the dynamic movement of the dynagraph representation to an imagined Cartesian graph of a known function is consistent with what Moore and Thompson (2015) referred to as emergent shape thinking, "understanding a graph *simultaneously* as what is made (a trace) and how it is made (covariation)" (p. 785). Here the PSMTs were imagining the one-dimensional trace as a two-dimensional trace and in doing so demonstrated their understanding of the ways in which the quantities were covarying.

Discussion and Conclusion

In this study, we investigated the similarities and differences in the ways PSMTs reasoned about static and dynamic representations of functions. Findings from this study suggest that static representations of function limit students' attention to covariation; this is consistent with prior research that showed students pay attention to shape and perceptual cues rather than the ways the graph represents how the variables change together (Moore & Thompson, 2015; Oehrtman et al., 2008; Weber, 2012). On the other hand, we found evidence of emergent shape thinking when students engaged with the DIM. They were imagining how their action would be represented in a static graph (a trace) which led them to reason covariationally. This attempt to make a connection raises a question: Do PSMTs not naturally tend to reason covariationally when presented with static representations in the first place or did the DIM support this connection making? Further research is needed to gain more insight to the reason behind this connection.

Results also suggest that different representations of function influenced PSMTs' use of mathematical language. The DIM seemed to disrupt PSMTs' reliance on formal mathematical language since they did not have a formal language to attach to the non-traditional function representations (i.e., dynagraphs). This is consistent with Ng's (2016) findings that students demonstrated increased reliance on verbs of motion and less reliance on formal mathematical language in the dynamic environment. In both representations, PSMTs described dynamic situations; however, they used dynamic language only with the DIM. It is possible that PSMTs had to think differently than they are used to, and thus did not have an expert's discourse at hand to describe what they observed. Whereas the familiarity of the static representation caused them to draw upon formal language (sometimes imprecisely), either because they are accustomed to doing so or because they felt it was expected. The fact that the DIM elicited more informal language may benefit the development of precise use of formal language. PSMTs tend to attend to dynamic features of the function as they are describing what they see. This will eventually evolve into formal mathematical language with the support of mathematics teacher educators.

Given the promise of the use of dynagraphs in supporting PSMTs' use of dynamic language and expressing emerging shape thinking, we plan to scale up the study with more PSMTs to determine if the patterns we saw here are consistent. In addition, we plan to consider how the use of DIMs might support the development of emergent shape thinking about static representations.

Olanoff, D., Johnson, K., & Spitzer, S. (2021). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Philadelphia, PA.

Acknowledgments

This work was partially supported by the National Science Foundation (NSF) under grant DUE 1820998 awarded to Middle Tennessee State University, DUE 1821054 awarded to University of North Carolina at Charlotte, DUE 1820967 awarded to East Carolina University, and DUE 1820976 awarded to NC State University. Any opinions, findings, and conclusions or recommendations expressed herein are those of the principal investigators and do not necessarily reflect the views of the NSF.

References

Antonini, S., Baccaglini-Frank, A., & Lisarelli, G. (2020). From experiences in a dynamic environment to written narratives on functions. *Digital Experiences in Mathematics Education, 6*, 1–29. https://doi.org/10.1007/s40751-019-00054-3

Carlson, M. (1998). A cross-sectional investigation of the development of the function concept. In E. Dubinsky, A. H. Schoenfeld, & J. J. Kaput (Eds.), *Research in collegiate mathematics education, III. Issues in Mathematics Education* (Vol. 7, pp. 114–162). American Mathematical Society. https://doi.org/10.1090/cbmath/007/04

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, *33*(5), 352–378.

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education, 26*(1), 66–86.

Danielson, C. (2016, September 26). The power of having more than one right answer: Ambiguity in math class. https://www.nctm.org/Publications/TCM-blog/Blog/The-power-of-having-more-than-one-right-answer_-Ambiguity-in-math-class/

DeCuir-Gunby, J. T., Marshall, P. L., & McCulloch, A. W. (2011). Developing and using a codebook for the analysis of interview data: An example from a professional development research project. *Field methods, 23*(2), 136–155.

Drijvers, P. (2015). Digital technology in mathematics education: Why it works (or doesn't). In S. J. Cho (Ed.), *Selected regular lectures from the 12th International Congress on Mathematical Education* (pp. 1–17). Springer, Cham.

Goldenberg, P., Lewis, P., & O'Keefe, J. (1992). Dynamic representation and the development of a process understanding of function. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 235–260). MAA.

Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A. Kelly & R.A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 517–545). Lawrence Erlbaum Associates.

Johnson, H. L., McClintock, E. D., & Gardner, A. (2020). Opportunities for reasoning: Digital task design to promote students' conceptions of graphs as representing relationships between quantities. *Digital Experiences in Mathematics Education*, *6*(3), 340–366.

Kaput Center (2016). *SimCalc MathWorlds Curriculum*. *http://kaputcenter.org/products/curriculm/*

Knuth, E. J. (2000). Student understanding of the cartesian connection: An exploratory study. *Journal for Research in Mathematics Education*, *31*(4), 500–508. https://doi.org/10.2307/749655

Lisarelli, G. (2017). Exploiting potentials of dynamic representations of functions with parallel axes. In G. Aldon & J. Trgalova (Eds.), *Proceedings of the 13th International Conference on Technology in Mathematics Teaching* (pp. 144–150). ICTMT.

Moore, K. C., & Thompson, P. W. (2015). Shape thinking and students' graphing activity. In T. Fukawa-Connelly, N. E. Infante, K. Keene, & M. Zandieh (Eds.), *Proceedings of the 18th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 782–789). The Special Interest Group of the Mathematics Association of America (SIGMAA) for Research in Undergraduate Mathematics Education. http://sigmaa.maa.org/rume/RUME18v2.pdf

Ng, O. (2016). Comparing calculus communication across static and dynamic environments using a multimodal approach. *Digital Experiences in Mathematics Education, 2*, 115–141. https://doi.org/10.1007/s40751-016-0014-8

Oehrtman, M., Carlson, M. P., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In M. P. Carlson & C. L. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education* (pp. 27–42). Mathematical Association of America.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. Cambridge University Press.

Sinclair, N., Healy, L., & Sales, C. O. R. (2009). Time for telling stories: Narrative thinking with dynamic geometry. *ZDM – The International Journal on Mathematics Education, 41*(4), 441–452.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421–456). National Council of Teachers of Mathematics.

Weber, E. D. (2012). *Students' ways of thinking about two-variable functions and rate of change in space* [Doctoral dissertation, Arizona State University]. ProQuest LLC. https://www.proquest.com/docview/1009080743

Yin, R. K. (2017). *Case study research and applications: Design and methods*. Sage publications.

Olanoff, D., Johnson, K., & Spitzer, S. (2021). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Philadelphia, PA.