MOVES TEACHERS USE TO RESPOND TO STUDENTS' NON-CANONICAL APPROACHES FOR SOLVING EQUATIONS

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A historical review of mathematics textbooks suggests a canonical method to solving equations that teachers often see as "the" way to solve equations. In this paper, we examine data from a nationally-distributed sample of 524 secondary mathematics teachers who responded to scenario-based survey items that represent the instructional situation of solving equations. The items featured scenarios in which students presented non-canonical solution methods and asked participants to share how they would respond. Using a framework that draws on systemic functional linguistics, we describe the linguistic resources teachers used. While closed moves are frequently used to avoid discussion of non-canonical solutions, our results suggest that teachers find ways to make regular use of: (1) closed moves for accommodating non-canonical solutions and (2) open moves when steering the conversation back to the canonical method.

Keywords: Algebra and Algebraic Thinking, Classroom Discourse, Research Methods

Background and Framework

While policy documents have been crafted to provide numerous visions for mathematics instruction in the U.S. (NCTM, 1991, 2014)—such visions have yet to become a regular state of affairs in actual classrooms. This is nowhere less true than teachers' instructional practices of responding to students' mathematical contributions (Milewski & Strickland, 2016) where teachers tend to be overly evaluative and propagate standard teaching routines—praising only those contributions that correctly carry out previously-demonstrated procedures while dismissing contributions that do not use expected methods even if they present correct solutions (Ball, 1997; Crespo, 2002). Furthermore, when teachers demonstrate a stalwart commitment to a single procedure, they cue students to learn rotely—undermining the development of conceptual understanding and flexible thinking (Hiebert & Carpenter, 1992).

In the case of solving equations in Algebra 1, a historical review of the mathematics textbooks suggests a long-standing canonical method (Buchbinder et al., 2015) that teachers expect students to use to solve equations (Buchbinder et al., 2019a). This method has been described by scholars as containing the following steps: (1) use the distributive property to clear out grouping symbols (when applicable), (2) simplify expressions on each side of the equation, (3) use the addition and subtraction properties of equality to isolate the variable from the constants, and (4) use the multiplication and division properties of equality to solve for the unknown variable (Buchbinder et al., 2015; Star & Seifert, 2006).

While many teachers prefer to spend class time on the canonical method (Buchbinder et al., 2019a), they sometimes have to make on-the-spot decisions about how to handle non-canonical solutions offered by students (Mason, 2015; Schoenfeld, 2008). This study investigates the

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linguistic resources teachers use when responding to non-canonical solutions in the instructional situation of solving equations: including those responses that manage to make use of students' alternative contributions as well as those that do not. In this paper, we examine data collected from a nationally-distributed sample of 524 secondary mathematics teachers who responded to a set of scenario-based survey items that each featured an embedded, rich-media representation of the instructional situation of solving equations (Chazan & Lueke, 2009). Within these items, teachers were asked to share how they would respond to scenarios in which a student presents a non-canonical solution for an equation on the board if such a situation would occur in their class.

Theoretical Framework

While teachers' instructional decisions are commonly modeled as expressions of individual characteristics, such as a teacher's resources, orientations, and goals (Schoenfeld, 2010), other factors need to be taken into consideration. Phenomena such as cultural scripts (Hiebert & Stigler, 2000) and lesson signatures (Givvin et al., 2005) provide evidence that the norms of teaching can be distinguished across cultural lines, which suggests that teaching is as much a socially-shaped activity as it is individual.

The theory of practical rationality (Herbst & Chazan, 2012) accounts for teachers' decision making using both individual and social resources. It does this using the two primary building blocks of (1) Brousseau's (1997) notion of didactical contract, and (2) Herbst's (2006) notion of instructional situation. Brousseau's concept of didactical contract identifies relationships between the teacher, their students, and the content in ways that tacitly regulate the ways that the teacher and students are expected to act within instructional exchanges (Herbst, 2003). Author's notion of instructional situation takes note of the way the didactical contract is shaped within the set of recurring situations within a course of study. For example, the theory posits the set of norms for solving equations in algebra differs from the set of norms for doing proofs in geometry and these differences impact both the teachers' and students' understanding of what kind of work is necessary for the teacher to claim the student has learnt what is expected of them (Herbst, 2006; Herbst & Chazan, 2012). In this way, the normative and routine nature of these instructional situations create a stable social resource that can be used by teachers and students to know how to act within a given situation.

In the case of the instructional situation of solving equations, the canonical method represents or activates the norms of the situation (Buchbinder et al., 2019a; Chazan & Lueke, 2009). To be clear, the norms of the situation are not deterministic, even for teachers with strong preferences for the canonical method. For example, when faced with the circumstance of having a shy student at the board presenting a non-canonical solution, a teacher who might normally feel quite strongly about adhering to the situational norms may respond in ways that accommodate the student's work to avoid embarrassing the student. Teachers have resources they can use to navigate such circumstances. For example, at least some portion of the reform literature has aimed to delineate specific linguistic resources teachers can use to shift their practices of responding to supporting students' mathematical contributions (e.g., O'Connor & Michaels, 2019).

Research Questions

In our prior work (Buchbinder et al., 2019b), we have shown that when confronted with non-canonical student solutions in the instructional situation of solving equations, teachers' responses can be parsed into one of three broad categories—those responses where the teacher: a) ***complies with the norm*** by finding a way to move quickly back to the canonical method , b) ***repairs the task*** by finding a way to make slight accommodations for a non-canonical solution, for example, by ensuring each step of the students' solution was justified before moving on, and, c) ***repairs the situation*** by making large accommodations for a non-canonical solution such as switching the focus of the lesson towards that solution. In this paper, we ask: what are the various linguistic resources teachers use to: (a) ***comply*** with the norms of that situation?, (b) ***repair the task***?, and (c) ***repair the situation***?

Methodology and Data Sources

Participants

Data used in this paper come from a nationally-distributed sample of 524 secondary mathematics teachers from 47 states who were invited by email and received an honorarium for participation. The sample of teachers included 59.6% female, 40.1% male, and 0.36% other or no answer; 83.58% White, 7.3% Black, 2% Hispanic, 2.8% Asian, 0.89% Other. The teachers had an average of 14.32 years of experience (*SD*=8.68) ranging from 1 to 40 years. Participants were invited to partake in a total of 27 open-ended scenario-based instruments—one of which, the Algebra-Equations Decision Instrument, we focus on here.

Instrument

As part of their participation in the Algebra-Equations Decision Instrument, each participant was provided with four rich-media, scenario-based items; each containing a classroom scenario that played out across several storyboard frames. Such multimedia representations have been found effective at gauging participant teachers' decision-making (Herbst & Chazan, 2015). Each scenario begins with a teacher posing a solving equations task and includes a moment in which a student is called to the board to share their work and the student subsequently describes a solution. In all cases, the students' solution was both mathematically correct and non-canonical.

For example, in one item, the teacher poses the problem 4x + 2 = 5x - 3, and a student volunteer approaches the board to share their solution where they *solve by graphing* (see Figure 1a). In another item, the teacher poses the problem 5(x + 2) = 56 - 2 (x +2), and a student shares a solution in which they *attended to structure* of the equation, meaning that the student solves by treating the term (x+2) as a quantity, instead of distributing first (see Figure 1b).

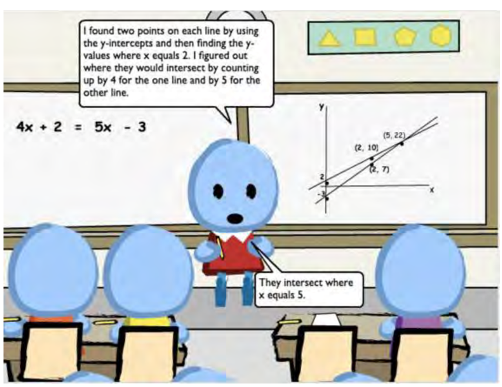

Figure 1a.* A frame from one of the items where a student elects to *solve by graphing

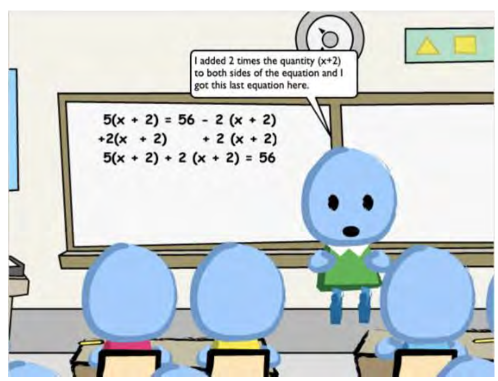

***Figure 1b.* A frame from one of the items where a student elects to attend to structure.**

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After viewing each scenario, study participants were asked to respond to the following open-ended prompt: “Please describe the action you would do next and your reasons for doing this action”. Participants’ open-ended responses are the focus of our analysis for this paper.

Data Corpus and Analytical Method

In total, the corpus contains 2,087 participant responses: some included a single “next action” (n=1,530), while others included a more detailed sequence of moves (n=463), or no action (n=94), i.e. restating of the scenario but not addressing the prompt. Among the single actions responses, some avoided addressing the students’ solution (n=251) by naming some other action such as, “I would apologize to the class for my poor time management.”. The present analysis focused on those responses that managed to address the students’ solution with a single “next action” (n=1,279) and proceeded in two parallel phases. In phase one, we coded responses according to the degree the participant indicated they would direct the class towards the canonical method or towards the offered non-canonical solution provided by the student in each scenario: (a) ***comply*** with the norms of that situation?, (b) ***repair the task***?, and (c) ***repair the situation***?.

In phase two, we used a previously-established coding scheme that augments a framework developed by teachers, who were conducting action research, (Authors, 2020) with functional classifications drawn from the linguistic framework by Eggins and Slade (2005). The Eggins and Slade framework comprises two functional systems of choice which organize responding moves according to how they shape the discourse. The first functional system of choice (***open/close)*** distinguishes between moves that prolong or curtail the discussion of the prior contribution; the second distinguishes between moves that demonstrate a willingness to accept the contribution (***support***, ***confront***) or defers responsibility for responding by asking other students to react to the contribution (***invite***).

Altogether, the combination of these systems of choice produce the following six codes for actual utterances: curtail the interaction by supporting the student contribution (***close-support***), curtail the interaction by confronting the contribution (***close-confront***), defer responsibility for responding by suggesting other students curtail the interaction (***close-invite***), extend the interaction by supporting the contribution (***open-support***) extend the interaction by confronting the contribution (***open-confront***) and defer responsibility for responding by suggesting other students prolong the interaction (***open-invite***). Details about the first and second phases of the coding can be found in our earlier work (Buchbinder et al., 2019b & Milewski & Strickland, 2020), but will also be illustrated with examples in the results section. After both phases of coding were complete, we examined patterns in the frequency of overlap of codes to help answer the research questions.

Analysis and Results

From the 1,279 responses we coded, 599 (47%) contained descriptions of actions that ***comply with the norms*** of the situation—finding ways to move quickly back to the canonical method. Of these 599 responses, the majority (n=404, 67%) represent actions that could be coded as ***close-confront***. Some of these *close-confront* responses took on the form of telling (e.g., *I would work through it using another method that is more routine*) while others took the form of a negative evaluation (e.g., *Since the bell rang I would make a note to bring up the same problem next class period and start off by solving it the right way -- meaning the way the students were used to*). Still others took a softer form, soliciting the class for a different solution (e.g., *I would ask if anyone in the class solved the problem a different way so that we could discuss the more* *traditional method).* Of course, ***close-confront*** moves are not the only way that teachers can manage to comply with the norm (see Table 1).

Table 1.* Examples of responses distinct from *closed-confront* that teachers used to *comply with the situation

Linguistic Code	Participant Response Example	% of responses
Close-Support	*I would explain while that works there's a much simpler way to solve the equation.*	*17%*
Close-Invite	*Have someone else share their method and show how it shows the same thing as what orange just did*	*5%*
Open-Support	*I would ask [the student]: 'why did you not divide (x+5) by 9 also on the right side?' ((common mistake))…*	*5%*

From the 1,279 total responses, 430 (34%) contained descriptions of actions that represent mild breaches of the norms of the instructional situation (repair the task)—providing some slight accommodations for the student's non-canonical solution. Nearly a third of those 430 responses (n=156, 36%) fit into the linguistic category of open-support. Some represented the teacher asking the student to clarify or justify aspects of the student's non-canonical solution (e.g., *Have the student explaining reiterate the step and make sure the class understands*) while others represented the teacher resolving the uncertainty in the room by some reassurance about the mathematical appropriateness of the method (e.g., *I would explain that as long as the same action is performed to each side of the equation that method is valid*). That said, teachers sometimes found other ways, beyond open-support moves, to repair the task (see Table 2).

Table 2.* Examples of responses distinct from *open-support* that teachers used to *repair the task

Linguistic Function	Participant Response Example	% of responses
Open-Invite	*It's not clear what 'dividing everything by 9' means so prompt students to ask questions of the student.*	*27%*
Close-Support	*Go over how each term changes when you divide it by 9.*	*22%*
Close-Invite	*I would ask the students for homework to write down whether or not they thought the solution on the board was correct and if they could get the same solution algebraically.*	*7%*

From the 1,279 responses, the remaining 250 (20%) responses contained descriptions of actions that implied breaches of the instructional situation (repair the situation)—making large accommodations for the student's non-canonical solution. Nearly half (n=135, 54%) of those 250 responses were coded as close-invite. Some of these responses represented the teacher asking other students to evaluate the contribution (e.g., *I would ask the students to discuss at their tables what was on the board and see if they agree or disagree with what is on the board*) while others represented the teacher requesting other students or the class take up the strategy on another problem (e.g., *I would give them another problem similar to the one [that student] did and see if they can duplicate the process*). Again, not all of the responses describing actions that breach the situation were categorized as close-invite (see Table 3).

Table 3.* Examples of responses distinct from *close-invite* that teachers used to *repair the situation

Linguistic Function	Participant Response Example	% of responses
Open-Invite	*I would have students discuss in pairs what they think Blue did.*	*20%*
Close-Support	*I would answer the students questions about why certain procedures were done in the problem.*	*14%*
Open-Support	*I would ask the student (with help from the class) to justify using mathematical properties or concepts each step.*	*7%*

In this section, we have shown that the modal teacher response to students' non-canonical solutions ***comply with norms*** of the situation (47%) and the preponderance of those responses take up the form of moves that could be coded as ***close-confront*** (67%). We have also shown that teachers sometimes elect to make small accommodations for students' non-canonical solutions (***repair the task***, 34%), and when they manage to do so they tend to use moves that were coded as ***open-support*** (36%). That said, nearly half of the responses that ***repaired the task*** were accomplished with moves that were coded as ***open-invite*** (27%) or ***close-support*** (22%). Finally, in 20% of the responses, we see teachers make sweeping accommodations for students' non-canonical solutions by ***repairing the situation***; and in the majority of those responses, teachers elected to use moves that could be coded as ***close-invite*** (54%).

Discussion, Conclusion and Significance

Despite reformers' calls for teachers to embrace the open discussion of multiple students' solutions, our research has reported that teachers favor canonical solution methods over non-canonical one. The theory of instructional situations and practical rationality has suggested teachers are often operating in contexts in which they feel responsible for maintaining the norms of the situation, which favors the canonical method. That said, we see in this data some promise in that a small majority of teachers' responses (54%) deviate from the norms of the situation by making some kind of accommodations for students' non-canonical methods. Yet, teachers' willingness to use ***open*** and/or ***supportive*** moves is mostly restricted to those instances when they are making only slight accommodations of students' non-canonical solutions (***repair the task***). In contrast, when a teacher takes the risk of making a significant accommodation for a students' non-canonical solution (***repair the situation***), they tend to use ***closed*** moves—albeit they often elect to use ***closed invitations.*** Yet even the invitational nature of these more-accommodating moves allow the teacher to maintain some semblance of control of the situation by sanctioning a narrow platform from which students can react to the non-canonical solution presented (e.g., requesting students evaluate, add on to, or replicate the method). These results support our prior hypotheses (Chazan & Lueke, 2009) that even when teachers are willing to engage with students' non-canonical solutions, there are important tensions in doing that.

While the analysis we have reported herein focuses on the response set as a whole, we have reason to believe that the breaches to the instructional situation represented across these items, i.e., the types of non-canonical student solutions, are not equivalent in terms of their likeliness to be perceived by teachers as reasonable approaches to take up in whole class discussion (Buchbinder et al., 2019a). Drawing from a recent use of the instrument administered to a set of secondary teachers prior to their involvement in professional development focused on facilitating whole class discussion, we have noticed that when aggregating teachers' responses according to item, some items (such as the type of solution featured in Figure 1b) seemed to also have greater numbers of closed responding moves than others (such as the type of solution featured in Figure 1a). Further, such items also contained more comments like the following, in which teachers remark on the represented method in ways that suggest they have concerns about it.

> The approach [the student represented in Figure 1b] took may be a bit confusing for students (such as [those who used] order of operations) and may lead to more anxiety and apprehension ... I think [the teacher] did a nice job hearing [the student] out, but should also show the [order of operation] approach ...and see if that helps to clarify some confusion.

To further explore teachers' rationality about particular kinds of non-canonical work, future work could interrogate patterns that exist when looking across teachers' responses to different items.

In closing, one of the primary ways that reformers have sought to further teachers' openness towards student-generated solutions is by suggesting alternative discursive moves that encourage teachers to use more open or invitational responding moves. The results from the analysis of the second and third parts of the research question cast some suspicion on the efficacy of such prescriptions. These results suggest that teachers can and do find ways to make regular use of ***closed*** moves to make accommodations for the students' non-canonical solutions (***repair the situation***)—in which they, in some serious way, take the risk of abandoning the canonical solution method. These results also suggest that teachers make regular use of ***open*** moves to ***repair the task***—steering the conversation back to the canonical solution method. These findings are reminiscent of earlier work in the field that looked critically at reform recommendations (Chazan & Ball, 1999; Cohen, 1990). In closing, we suggest that more work is needed to understand teachers' practical rationality in order to better understand which suggestions teachers may be more inclined to take up.

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